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$\aleph_k$-free separable groups with prescribed endomorphism ring

by

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Abstract. We will consider unital rings A with free additive group, and want to construct (in ZFC) for each natural number k a family of $\aleph_k$-free A-modules G which are separable as abelian groups with special decompositions. Recall that an A-module G is $\aleph_k$-free if every subset of size $< \aleph_k$ is contained in a free submodule (we will refine this in Definition 3.2); and it is separable as an abelian group if any finite subset of G is contained in a free direct summand of G. Despite the fact that such a module G is almost free and admits many decompositions, we are able to control the endomorphism ring $\operatorname{End} G$ of its additive structure in a strong way: we are able to find arbitrarily large G with $\operatorname{End} G = A \oplus \operatorname{Fin} G$ (so $\operatorname{End} G/\operatorname{Fin} G = A$, where $\operatorname{Fin} G$ is the ideal of $\operatorname{End} G$ of all endomorphisms of finite rank) and a special choice of A permits interesting separable $\aleph_k$-free abelian groups G. This result includes as a special case the existence of non-free separable $\aleph_k$-free abelian groups G (e.g. with $\operatorname{End} G = \mathbb{Z} \oplus \operatorname{Fin} G$), known until recently only for $k = 1$.

1. Introduction. Methods for constructing in ZFC $\aleph_1$-free modules (so every countably generated submodule is contained in a free submodule) with prescribed endomorphism algebra using the Black Box combinatorial principle (due to S. Shelah) were developed thirty years ago. This follows from classical work in A. L. S. Corner and R. Göbel [1] and standard references given there.

The importance of these constructions lies in the fact that they can be used to solve various kinds of problems like the realization of E-rings, which are crucial in algebraic topology, or the existence of negative answers to Kaplansky's Test Problems.

Another interesting application leads to constructions of $\aleph_1$-free modules G which are *separable* torsion-free abelian groups with prescribed endomorphism ring $\operatorname{End} G/\operatorname{Fin} G$, where $\operatorname{Fin} G$ is the ideal of $\operatorname{End} G$ of all

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endomorphisms of finite rank (see [4]). Separable groups G are defined by the property that every finite subset of G is contained in a free direct summand of G. In the case of torsion-free abelian groups they are characterized as pure subgroups of a cartesian product of copies of the integers. Separable $\aleph_1$-free groups have been treated at many places; for a survey on this topic, see [10].

It is natural to ask if we can strengthen the 'torsion-free Main Theorem (5.2)' from [1] to obtain $\aleph_k$-free modules. Can we, in particular, find in ZFC $\aleph_k$-free modules for $k > 1$ which are separable torsion-free abelian groups with prescribed endomorphism ring $\operatorname{End} G/\operatorname{Fin} G$? We will provide a definitive positive answer to this question in Main Theorem 5.3.

Classical preparatory work in this direction was done by P. Griffith [12], P. Hill [15], P. Eklof and S. Shelah [6]. Assuming V = L (or a weak version of the axiom of constructibility, see Eklof and Mekler [5]), the construction of such modules G can be carried out fairly easy; see Dugas and Göbel [3]. In recent years, methods have been developed for constructing in ZFC $\aleph_k$-free structures with prescribed endomorphism ring using the Easy Black Box combinatorial principle from [8, 19] (cf. [9, 14]). The original constructions in [7, 18] use so-called triple modules, which required a highly elaborate setting. Furthermore, an application of Shelah's Strong Black Box was needed. Following the lines in [13] and [7], we will add certain new closure conditions to avoid additional complications. Thus we can work exclusively with the Easy Black Box and its algebraic version. Fortunately, this Black Box (see Section 5) needs almost no further adjustments.

We will consider a separably realizable ring A which is p-reduced for a previously fixed prime number p (as explained in the next section) and a free A-module B defined on a special set Λ_* satisfying certain cardinal conditions. Let $\mathfrak{F}$ be a family of elements of the p-completion $\widehat{B}$ of B. This family is obtained from *branch-like elements*, which are specific elements of $\widehat{B}$ modified by *correcting terms*. Here is where the Easy Black Box is required, since it allows us to properly choose these correcting terms in order to eliminate unwanted endomorphisms. The desired $\aleph_k$-free A-module will be the p-pure closure of the A-module generated by B and $\mathfrak{F}$. We are now ready to describe the tools for Main Theorem 5.3, which is stated in Section 5.

2. Preliminaries. Let A be a ring with 1 having free additive structure $A^+ = \bigoplus_{\alpha<\kappa} \mathbb{Z}e_\alpha$ such that

$$\overline{A} = \widehat{A} \cap \prod_{\alpha<\kappa} \mathbb{Z}e_\alpha$$

is an A-module, where $\widehat{A}$ denotes the p-completion of A for some prime p. Such rings are called *separably realizable*. It was shown at different times and independently that $\overline{A}$ is a ring for $\kappa = \aleph_0$ by Goodearl, Menal and Moncasi [11], Corner and Göbel [2] and Nielsen [17]. For more details, see Göbel and Trlifaj [10].

We would like to construct a free A-module B, which serves as a starting point of our construction. For that purpose, let $k > 1$ be a fixed integer throughout this work. We recursively construct a sequence of infinite cardinals $\overline{\lambda} = \langle \lambda_1, \ldots, \lambda_k \rangle$ as follows. Consider the cardinal $\lambda_0 = |A|$ as an initial step. Now suppose we have constructed λ_m for some $0 \le m < k$. Choose some cardinal μ_{m+1} such that $\mu_{m+1}^{\lambda_m} = \mu_{m+1}$ and let $\lambda_{m+1} = \mu_{m+1}^+$. For example, we could use a slight modification of the beth numbers: put $\mu = \mu_1 = 2^{\lambda_0}$, $\beth_0^+(\mu) = \mu^+$ and $\beth_{n+1}^+(\mu) = (2^{\beth_n^+(\mu)})^+$, which is the successor cardinal of the cardinality of the powerset of $\beth_n^+(\mu)$. Hence, we can take $\lambda_m = \beth_m^+(\mu)$ for all $1 \le m \le k$.

For an infinite cardinal λ, we denote the set of all *order preserving* maps $\eta : \omega \to \lambda$ by ${}^{\omega\uparrow}\lambda$, while ${}^{\omega\uparrow>}\lambda$ denotes the set of all *order preserving* maps $\eta : n \to \lambda$ with $n < \omega$.

We associate with $\overline{\lambda}$ two sets Λ and Λ_*. The set Λ is defined as

$$\Lambda = {}^{\omega\uparrow}\lambda_1 \times \cdots \times {}^{\omega\uparrow}\lambda_k.$$

For Λ_*, we first define the set Λ_{m*}, which is obtained by replacing the mth coordinate ${}^{\omega\uparrow}\lambda_m$ of Λ by ${}^{\omega\uparrow>}\lambda_m$, namely,

$$\Lambda_{m*} = {}^{\omega\uparrow}\lambda_1 \times \cdots \times {}^{\omega\uparrow>}\lambda_m \times \cdots \times {}^{\omega\uparrow}\lambda_k.$$

Then, let

$$\Lambda_* = \dot{\bigcup_{1 \le m \le k}} \Lambda_{m*}.$$

Definition 2.1. Let $1 \le m \le k$ and $n < \omega$.

(a) If $\eta \in {}^{\omega\uparrow}\lambda_m$, then the *support* of η is the set

$$[\eta] = \{\eta{\restriction}n \mid n < \omega\}.$$

(b) If $\overline{\eta} = (\eta_1, \ldots, \eta_k) \in \Lambda$, then $\overline{\eta}{\upharpoonright}\langle m, n\rangle$ denotes the element of Λ_{m*} obtained from $\overline{\eta}$ by replacing its component η_m by $\eta_m{\restriction}n$, i.e.

$$\overline{\eta}{\upharpoonright}\langle m, n\rangle = (\eta_1, \ldots, \eta_{m-1}, \eta_m{\restriction}n, \eta_{m+1}, \ldots, \eta_k).$$

(c) For every $\overline{\eta} \in \Lambda$, consider the sets

$$[\overline{\eta}{\upharpoonright}m]_n = \{\overline{\eta}{\upharpoonright}\langle m, n'\rangle \mid n \le n' < \omega)\}, \qquad [\overline{\eta}]_n = \dot{\bigcup_{1 \le m \le k}} [\overline{\eta}{\upharpoonright}m]_n.$$

If $n = 0$, then we simply write $[\overline{\eta}{\upharpoonright}m]$ and $[\overline{\eta}]$ instead. The set $[\overline{\eta}]$ is called the *support* of $\overline{\eta}$.

(d) For $\overline{\nu} = (\eta_1, \ldots, \eta_k) \in \Lambda_{m*}$, we define the *length* $\ell(\overline{\nu})$ of $\overline{\nu}$ as the domain of η_m.

(e) It will be useful to introduce the following notation:

$$\Lambda_{m*}^{\geq n} = \{\overline{\nu} \in \Lambda_{m*} \mid \ell(\overline{\nu}) \geq n\}.$$

Given a subset $X_* \subseteq \Lambda_*$, we consider the free A-module

$$B_{X_*} = \bigoplus_{\overline{\nu} \in X_*} Ae_{\overline{\nu}}.$$

The basic A-module on which we base the final construction is the free A-module

$$B = \bigoplus_{\overline{\nu} \in \Lambda_*} Ae_{\overline{\nu}}.$$

Moreover, let

$$\overline{B} = \widehat{B} \cap \prod_{\overline{\nu} \in \Lambda_*} Ae_{\overline{\nu}}$$

where $\widehat{B}$ denotes the p-completion of B.

Definition 2.2.

(a) For every $b = \sum_{\overline{\nu} \in \Lambda_*} a_{\overline{\nu}} e_{\overline{\nu}} \in \widehat{B}$ with $a_{\overline{\nu}} \in \widehat{A}$, the Λ_*-*support* of b is the set

$$[b] = \{\overline{\nu} \mid a_{\overline{\nu}} \neq 0\}.$$

Note that $[b]$ is at most countable.

(b) We write $[b]_{\overline{\nu}} = a_{\overline{\nu}}$ for all $\overline{\nu} \in \Lambda_*$, and $[b]_{X_*} = \sum_{\overline{\nu} \in X_*} a_{\overline{\nu}} e_{\overline{\nu}}$ for all $X_* \subseteq \Lambda_*$.

(c) If $S \subseteq \widehat{B}$, then the Λ_*-*support* of S is the set $[S] = \bigcup_{b \in S} [b]$.

3. $\aleph_k$-free A-modules

Definition 3.1. If S is a set and κ is a cardinal, then $[S]^{\leq \kappa}$ denotes the set of all $X \subseteq S$ such that $|X| \leq \kappa$. Analogously we define $[S]^{<\kappa}$ and $[S]^{\kappa} = \{X \subseteq S \mid |X| = \kappa\}$.

For non-hereditary rings, it is necessary to modify the notion of κ-free modules. The following definition of κ-freeness is due to Göbel, Herden and Shelah [7], and is a slightly stronger version of the one in Eklof and Mekler [5].

Definition 3.2. If κ is a regular uncountable cardinal, we say that an A-module M is κ-*free* if there is a family $\mathcal{C}$ of p-pure A-submodules of M satisfying:

(a) Every element of $\mathcal{C}$ is $<\kappa$-generated and free.

(b) Every element of $[M]^{<\kappa}$ is contained in an element of $\mathcal{C}$.

(c) $\mathcal{C}$ is closed under unions of well-ordered chains of length $< \kappa$.

Definition 3.3.

(a) For $\eta \in {}^{\omega\uparrow>}\lambda_k \cup {}^{\omega\uparrow}\lambda_k$, we define the *norm* $\|\eta\|$ of η as

$$\|\eta\| = \sup_{n<\ell(\eta)} (n\eta + 1) \in \lambda_k;$$

in particular, $\|\alpha\| = \alpha + 1$ for $\alpha \in \lambda_k$ and $\|\emptyset\| = 0$.

Recall that a map f (like η above) acts on X via xf for any $x \in X$.

(b) For $\overline{\eta} \in \Lambda \cup \Lambda_*$, define $\|\overline{\eta}\| = \|\eta_k\|$.

(c) For $X \subseteq \Lambda$, put $\|X\| = \sup_{\overline{\eta}\in X} \|\overline{\eta}\|$. Similarly, $\|X\| = \sup_{\overline{\nu}\in X} \|\overline{\nu}\|$ if $X \subseteq \Lambda_*$.

(d) If $b \in \widehat{B}$, then $\|b\| = \|[b]\|$, and for $S \subseteq \widehat{B}$, let $\|S\| = \sup_{b\in S} \|b\|$.

Definition 3.4. A sequence of elements $(b_{\overline{\eta}n})_{n<\omega} \subseteq \overline{B}$ is called *regressive* with respect to $\overline{\eta} \in \Lambda$ if:

(i) $\|b_{\overline{\eta}0}\| < 0\eta_k$.

(ii) $b_{\overline{\eta}n} - pb_{\overline{\eta}\,n+1} \in B$ for all $n < \omega$, i.e. $(b_{\overline{\eta}n})_{n<\omega}$ is a *divisibility chain.*

(iii) $[b_{\overline{\eta}n}] \subseteq [b_{\overline{\eta}0}]$ for all $n < \omega$.

Evidently, every element $b_{\overline{\eta}} \in \overline{B}$ allows for a suitable sequence $(b_{\overline{\eta}n})_{n<\omega}$ of elements $b_{\overline{\eta}n} \in \overline{B}$ such that conditions (ii) and (iii) hold with $b_{\overline{\eta}0} = b_{\overline{\eta}}$. Furthermore, we may fix in advance such a sequence for each $b_{\overline{\eta}} \in \overline{B}$.

Definition 3.5. For $\overline{\eta} \in \Lambda$ and $n < \omega$, we define the *branch element* associated with $\overline{\eta}$ and n as

$$y_{\overline{\eta}n} = \sum_{i=n}^{\infty} p^{i-n}\Big(\sum_{m=1}^{k} e_{\overline{\eta}\restriction\langle m,i\rangle}\Big).$$

We write $y_{\overline{\eta}}$ for $y_{\overline{\eta}0}$. Choose an element $b_{\overline{\eta}} \in \overline{B}$ with a regressive sequence $(b_{\overline{\eta}n})_{n<\omega} \subseteq \overline{B}$. We define the *branch-like element* associated with $\overline{\eta}$ and n as

$$y'_{\overline{\eta}n} = b_{\overline{\eta}n} + y_{\overline{\eta}n}.$$

We also write $y'_{\overline{\eta}}$ for $y'_{\overline{\eta}0}$.

Definition 3.6.

(a) A triple $(X_*, X, \mathfrak{F})$ is called Λ-*closed* if:

  (i) $X \subseteq \Lambda$ and $X_* \subseteq \Lambda_*$.

  (ii) For all $\overline{\eta} \in X$ there exists some (minimal) $N_{\overline{\eta}} < \omega$ such that $[\overline{\eta}]_{N_{\overline{\eta}}} \subseteq X_*$.

  (iii) $\mathfrak{F} = \{y'_{\overline{\eta}} = b_{\overline{\eta}} + y_{\overline{\eta}} \mid \overline{\eta} \in X, b_{\overline{\eta}} \in \overline{B}_{X_*}\}$ is a *regressive* family of branch-like elements, i.e. $\|b_{\overline{\eta}}\| < 0\eta_k$ for all $\overline{\eta} \in X$.

(b) If $(X_*, X, \mathfrak{F})$ is Λ-closed, we define the A-module

$$G_{X_*X} = \langle B_{X_*}, Ay'_{\overline{\eta}n} \mid \overline{\eta} \in X,\ n \geq N_{\overline{\eta}}\rangle = \langle B_{X_*}, Ay'_{\overline{\eta}N_{\overline{\eta}}} \mid \overline{\eta} \in X\rangle_*.$$

Recognition Lemma 3.7. *For* $g \in \widehat{B}$, $1 \le m \le k$ *and* $n < \omega$, *if* $g \in \langle B, Ay'_{\overline{\eta}} \mid \overline{\eta} \in \Lambda\rangle$ *for a suitable choice of* $\mathfrak{F} = \{y'_{\overline{\eta}} = b_{\overline{\eta}} + y_{\overline{\eta}} \mid b_{\overline{\eta}} \in \overline{B}, \overline{\eta} \in \Lambda\}$ *with regressive sequences* $(b_{\overline{\eta}i})_{i<\omega}$, *then:*

(i) *There exist some unique* $b \in B$ *and* $a_{\overline{\eta}} \in A$ $(\overline{\eta} \in \Lambda)$ *with* $g = b + \sum_{\overline{\eta}\in\Lambda} a_{\overline{\eta}} y'_{\overline{\eta}}$.
(ii) *If* $a_{\overline{\eta}} \ne 0$, *then* $\|\overline{\eta}\| \le \|g\|$.
(iii) *The elements* $a_{\overline{\eta}}$ $(\overline{\eta} \in \Lambda)$ *are entirely determined by* $[g]_{\Lambda^{\ge n}_{m*}}$ *(see Definition* 2.1(e)) *and for* $\|\overline{\eta}\| = \|g\|$ *the coefficient* $a_{\overline{\eta}}$ *is independent of the choice of* $\mathfrak{F}$.

Proof. Observe that $[b]$, $[b_{\overline{\eta}}] \cap [\overline{\eta}]$, $[b_{\overline{\eta}}] \cap [\overline{\eta}']$ and $[\overline{\eta}] \cap [\overline{\eta}']$ are finite for $\overline{\eta} \ne \overline{\eta}' \in \Lambda$ with $\|\overline{\eta}\| \le \|\overline{\eta}'\|$, while $[\overline{\eta}] \cap \Lambda^{\ge n}_{m*}$ and $[\overline{\eta}'] \cap \Lambda^{\ge n}_{m*}$ are infinite. Therefore:

(a) If $a_{\overline{\eta}} = 0$ for all $\overline{\eta} \in \Lambda$, then $[g] = [b]$ is finite.

(b) Otherwise, $[g]$ and $[g] \cap \Lambda^{\ge n}_{m*}$ are both infinite. Furthermore, if $\overline{\eta} \in \Lambda$ with $\|\overline{\eta}\| = \|g\|$, then $a_{\overline{\eta}} \ne 0$ if and only if $[y_{\overline{\eta}i}] \subseteq [g]$ for some $i < \omega$.

Now, (ii) is an immediate consequence of (b), while (i) and (iii) are proven by transfinite induction over $\|g\|$: If $[g] \cap \Lambda^{\ge n}_{m*}$ is finite, then $a_{\overline{\eta}} = 0$ for all $\overline{\eta} \in \Lambda$ by combining (a) and (b), and we are done. If $[g] \cap \Lambda^{\ge n}_{m*}$ is infinite, then using (b) we can read off in a first step all those $\overline{\eta}$ with $\|\overline{\eta}\| = \|g\|$ and $[y_{\overline{\eta}i}] \cap \Lambda^{\ge n}_{m*} \subseteq \big[[g]_{\Lambda^{\ge n}_{m*}}\big]$, and then determine $a_{\overline{\eta}}$ accordingly from the coefficients appearing in $\big[[g]_{\Lambda^{\ge n}_{m*}}\big]_{[y_{\overline{\eta}i}]}$ (see Definition 2.2(b)). After that, proceed with

$$g' = g - \sum_{\substack{\overline{\eta}\in\Lambda \\ \|\overline{\eta}\| = \|g\|}} a_{\overline{\eta}} y'_{\overline{\eta}}\,.$$

Furthermore, as this first step does not use the correction elements $b_{\overline{\eta}}$, it is independent of the particular choice of $\mathfrak{F}$. ■

A well-defined Λ-support is a consequence of Recognition Lemma 3.7.

Definition 3.8. Let $(X_*, X, \mathfrak{F})$ be Λ-closed. For $g \in G_{X_*X}$, define the Λ-*support* $[g]_\Lambda$ of g to be the set of elements of Λ that contribute to the representation of g. More precisely, if $p^m g = b + \sum_{\overline{\eta}\in\Lambda} a_{\overline{\eta}} y'_{\overline{\eta}}$ for some $m \ge 0$, where $b \in B$ and $a_{\overline{\eta}} \in A$ for all $\overline{\eta} \in \Lambda$, then

$$[g]_\Lambda = \{\overline{\eta} \in \Lambda \mid a_{\overline{\eta}} \ne 0\}.$$

Obviously, $[g]_\Lambda$ is finite. For $H \subseteq G_{X_*X}$, we define $[H]_\Lambda = \bigcup_{g\in H} [g]_\Lambda$.

The A-modules G_{X_*X} just introduced can be shown to be $\aleph_k$-free. For this purpose, we need the so-called Freeness Proposition, which allows us to enumerate subsets of Λ in a convenient way so that we can prove linear independence in the constructed A-modules.

Definition 3.9. A function $F : \Lambda \to [\Lambda_*]^{\leq \aleph_0}$ is called *regressive* if $\|\overline{\eta} F\| < 0\eta_k$ for all $\overline{\eta} \in \Lambda$.

Freeness Proposition 3.10. *Let $F : \Lambda \to [\Lambda_*]^{\leq \aleph_0}$ be a regressive map, $1 \leq f \leq k$, $\Omega \in [\Lambda]^{\aleph_{f-1}}$ and $\langle u_{\overline{\eta}} \mid \overline{\eta} \in \Omega \rangle$ be a family of subsets of $\{1, \ldots, k\}$ such that $|u_{\overline{\eta}}| \geq f$. Then there exists a bijective enumeration $\{\overline{\eta}^\alpha \mid \alpha < \zeta\}$ of Ω for some $\omega_{f-1} \leq \zeta < \omega_f$ such that, for all $\alpha < \zeta$, there exist $\ell_\alpha \in u_{\overline{\eta}^\alpha}$ and $1 \leq n_\alpha < \omega$ with the property that, for all $n \geq n_\alpha$,*

$$\overline{\eta}^\alpha \upharpoonright \langle \ell_\alpha, n \rangle \notin \{\overline{\eta}^\beta \upharpoonright \langle \ell_\alpha, n \rangle \mid \beta < \alpha\} \cup \bigcup \Omega_\alpha F$$

where $\Omega_\alpha = \{\overline{\eta}^\beta \mid \beta \leq \alpha\}$.

Proof. We proceed by induction on f. If $f = 1$, then $|\Omega| = \aleph_0$ and $u_{\overline{\eta}} \neq \emptyset$ for all $\overline{\eta} \in \Omega$. For all $\alpha < \lambda_k$, define $U_\alpha = \{\overline{\eta} \in \Omega \mid 0\eta_k = \alpha\}$. Let $N = \{\alpha < \lambda_k \mid U_\alpha \neq \emptyset\}$ and enumerate it $N = \{\alpha_\beta \mid \beta < \delta\}$ for some $\delta < \omega_1$ in such a way that $\alpha_\beta < \alpha_\gamma$ if and only if $\beta < \gamma < \delta$. Put $\gamma_\beta = |U_{\alpha_\beta}|$ and $\sigma_\beta = \sum_{\alpha < \beta} \gamma_\alpha$. We enumerate $U_{\alpha_\beta} = \{\overline{\eta}^\alpha \mid \sigma_\beta \leq \alpha < \sigma_\beta + \gamma_\beta\}$. This results in a bijective enumeration $\{\overline{\eta}^\alpha \mid \alpha < \zeta\}$ of Ω such that $\omega \leq \zeta < \omega_1$ and, for all $\alpha < \zeta$,

$$0\eta_k^\alpha \leq 0\eta_k^{\alpha+1} < 0\eta_k^{\alpha+\omega}.$$

Choose $\ell_\alpha \in u_{\overline{\eta}^\alpha}$ arbitrarily. If $\overline{\eta}^\alpha \in U_\gamma$ and β_0 is the minimal ordinal such that $\overline{\eta}^{\beta_0} \in U_\gamma$, then we can find some $1 \leq n_{\alpha,\beta} < \omega$ such that $\overline{\eta}^\alpha \upharpoonright \langle \ell_\alpha, n \rangle \neq \overline{\eta}^\beta \upharpoonright \langle \ell_\alpha, n \rangle$ for all $\beta_0 \leq \beta < \alpha$ and $n \geq n_{\alpha,\beta}$. Put

$$n_\alpha = \max_{\beta_0 \leq \beta < \alpha} n_{\alpha,\beta}.$$

Then, for all $n \geq n_\alpha$, $\overline{\eta}^\alpha \upharpoonright \langle \ell_\alpha, n \rangle \notin \{\overline{\eta}^\beta \upharpoonright \langle \ell_\alpha, n \rangle \mid \beta_0 \leq \beta < \alpha\}$. Moreover,

$$\overline{\eta}^\alpha \upharpoonright \langle \ell_\alpha, n \rangle \notin \{\overline{\eta}^\beta \upharpoonright \langle \ell_\alpha, n \rangle \mid \beta < \beta_0\} \cup \bigcup \Omega_\alpha F$$

since F is regressive and $0\eta_k^\beta < 0\eta_k^\alpha$ for all $\beta < \beta_0$.

Now suppose the assertion is true for some $1 \leq f < k$. Let $\Omega \in [\Lambda]^{\aleph_f}$ and $\langle u_{\overline{\eta}} \mid \overline{\eta} \in \Omega \rangle$ with $|u_{\overline{\eta}}| \geq f + 1$. Choose an $\aleph_f$-filtration $\{\Omega^\alpha \mid \alpha < \omega_f\}$ of Ω such that $\Omega^0 = \emptyset$ and $|\Omega^{\alpha+1} \setminus \Omega^\alpha| = \aleph_{f-1}$ for all $\alpha < \omega_f$. The next crucial idea comes from [19] and is based on the construction of elementary submodels: We can assume that this filtration is *coordinatewise-closed*, meaning that for all $\overline{\eta} \in \Omega^{\alpha+1}$, if there exist $\overline{\eta}', \overline{\eta}'' \in \Omega^\alpha$ such that

$$\begin{aligned} \{\eta_m \mid 1 \leq m \leq k\} \subseteq \{&\eta'_m, \eta''_m \mid 1 \leq m \leq k\} \\ &\cup \{\nu_m \mid \overline{\nu} \in \overline{\eta}' F \cup \overline{\eta}'' F,\ 1 \leq m \leq k\}, \end{aligned}$$

then $\overline{\eta} \in \Omega_\alpha$. For every $\overline{\eta} \in \Omega^{\alpha+1} \setminus \Omega^\alpha$, consider

$$\begin{aligned} u^*_{\overline{\eta}} = \big\{&1 \leq m \leq k \mid \\ &\exists \overline{\eta}' \in \Omega^\alpha,\ n < \omega\ (\overline{\eta} \upharpoonright \langle m, n \rangle = \overline{\eta}' \upharpoonright \langle m, n \rangle \text{ or } \overline{\eta} \upharpoonright \langle m, n \rangle \in \overline{\eta}' F)\big\}. \end{aligned}$$

It follows that $|u^*_{\overline{\eta}}| \le 1$, since $|u^*_{\overline{\eta}}| > 1$ would imply that $\overline{\eta} \in \Omega^\alpha$. Put $u'_{\overline{\eta}} = u_{\overline{\eta}} \setminus u^*_{\overline{\eta}}$ and observe that $|u'_{\overline{\eta}}| \ge f$. We apply the induction hypothesis on each of the sets $\Omega^{\alpha+1} \setminus \Omega^\alpha$ together with the family $\langle u'_{\overline{\eta}} \mid \overline{\eta} \in \Omega^{\alpha+1} \setminus \Omega^\alpha \rangle$ to obtain an enumeration $\Omega^{\alpha+1} \setminus \Omega^\alpha = \langle \overline{\eta}^\beta \mid \beta < \zeta \rangle$ for some $\omega_{f-1} \le \zeta < \omega_f$ with the required property. We induce an enumeration on Ω with the desired property by ordering these enumerations lexicographically. ∎

We now present the main result of this section. It is an immediate application of the Freeness Proposition 3.10 to the regressive function $F : \Lambda \to [\Lambda_*]^{\le \aleph_0}$ given by $\overline{\eta}F = [b_{\overline{\eta}}]$. We refer the reader either to [7, Freeness Lemma 3.7] or [13, Freeness Lemma 3.6] for a proof.

Theorem 3.11. *If $(X_*, X, \mathfrak{F})$ is Λ-closed, then G_{X_*X} is an $\aleph_k$-free A-module.*

4. The Step Lemma. We now introduce the Step Lemma, which is the central piece of the final construction. It allows us to choose the corrections for the branch elements in order to get rid of unwanted endomorphisms.

Definition 4.1. Let $(X^1_*, X^1, \mathfrak{F}^1)$ and $(X^2_*, X^2, \mathfrak{F}^2)$ be Λ-closed triples.

(a) We write $(X^1_*, X^1, \mathfrak{F}^1) \subseteq (X^2_*, X^2, \mathfrak{F}^2)$ if $X^1_* \subseteq X^2_*$, $X^1 \subseteq X^2$ and $\mathfrak{F}^1 \subseteq \mathfrak{F}^2$.

(b) If $b^1_{\overline{\eta}} = b^2_{\overline{\eta}}$ for all $\overline{\eta} \in X^1 \cap X^2$, then
- $(X^1_*, X^1, \mathfrak{F}^1) \cap (X^2_*, X^2, \mathfrak{F}^2) = (X^1_* \cap X^2_*, X^1 \cap X^2, \mathfrak{F}^1 \cap \mathfrak{F}^2)$,
- $(X^1_*, X^1, \mathfrak{F}^1) \cup (X^2_*, X^2, \mathfrak{F}^2) = (X^1_* \cup X^2_*, X^1 \cup X^2, \mathfrak{F}^1 \cup \mathfrak{F}^2)$

denote the canonically induced Λ-closed triples.

Similarly, further notation for Λ-closed triples $(X_*, X, \mathfrak{F})$ can be defined componentwise.

Definition 4.2. Let $(X_*, X, \mathfrak{F}) \subseteq (Y_*, Y, \mathfrak{G})$ be Λ-closed triples.

(a) For $\overline{\eta} \in \Lambda$, let
$$u_{\overline{\eta}}(X_*) = \{1 \le m \le k \mid \exists n < \omega\ ([\overline{\eta} \upharpoonright m]_n \subseteq X_*)\}.$$

(b) For $1 \le f \le k$ and $\alpha \in \lambda^o_k = \{\gamma \in \lambda_k \mid \mathrm{cf}(\gamma) = \omega\}$, let
$$Y_{X_*Xf\alpha} = \{\overline{\eta} \in Y \setminus X \mid |u_{\overline{\eta}}(X_*)| \ge f \text{ and } \|\overline{\eta}\| = \alpha\}.$$
We write Y_{X_*X} instead of $Y_{X_*Xf\alpha}$ if the pair (f, α) is clear from the context.

(c) Let $1 \le f \le k$ and $\alpha \in \lambda_k$. Then the triple $(X_*, X, \mathfrak{F})$ is called (f, α)-*closed with respect* to $(Y_*, Y, \mathfrak{G})$ if the following is satisfied:
$$\text{if } \overline{\eta} \in Y \text{ with } |u_{\overline{\eta}}(X_*)| \ge f \text{ and } \|\overline{\eta}\| \ne \alpha, \text{ then } \overline{\eta} \in X.$$

(d) If $(Y_*, Y) = (\Lambda_*, \Lambda)$, then we omit $(Y_*, Y, \mathfrak{G})$ and just say that $(X_*, X, \mathfrak{F})$ is (f, α)-*closed.*

As a remarkable result involving these definitions we have the following simple but crucial lemma.

LEMMA 4.3. *If* $(X_*, X, \mathfrak{F})$ *is* (k,α)*-closed with respect to* $(Y_*, Y, \mathfrak{G})$*, and*

$$[\overline{\eta}]_{N_{\overline{\eta}}} \subseteq X_* \quad \text{and} \quad [b_{\overline{\eta}}] \subseteq X_* \quad \text{for all } \overline{\eta} \in Y_{X_*X},$$

then $G_{Y_*Y} \cap \overline{B}_{X_*} = G_{X_*X \cup Y_{X_*X}}$.

Proof. The inclusion $G_{X_*X \cup Y_{X_*X}} \subseteq G_{Y_*Y} \cap \overline{B}_{X_*}$ is immediate. Let g be in $G_{Y_*Y} \cap \overline{B}_{X_*}$. If $|[g]_\Lambda| = 0$, then $g \in B_{X_*} \subseteq G_{X_*X \cup Y_{X_*X}}$. If $|[g]_\Lambda| > 0$, then $\overline{\eta} \in X \cup Y_{X_*X}$ follows from the Recognition Lemma 3.7(ii) for all $\overline{\eta} \in [g]_\Lambda$ with $\|\overline{\eta}\| = \|g\|$. Thus, g can be reduced by these elements $y'_{\overline{\eta}}$ modulo $G_{X_*X \cup Y_{X_*X}}$ and the induction step applies. ∎

Before introducing the Step Lemma, which is the main result of this section, we need some more definitions.

DEFINITION 4.4.

(a) For $g \in \widehat{B}$ and $\varepsilon \in \{0,1\}$, we define

$$[g]^\varepsilon = \{\overline{\nu} \in [g] \mid \ell(\overline{\nu}) \equiv \varepsilon \bmod 2\}.$$

(b) For $0 \le f < k$ and $\overline{\xi} = \langle \xi_{f+1}, \ldots, \xi_k \rangle \in {}^{\omega\uparrow}\lambda_{f+1} \times \cdots \times {}^{\omega\uparrow}\lambda_k$, let

$$\Lambda^{\overline{\xi}} = \{\overline{\eta} \in \Lambda \mid \eta_m = \xi_m \text{ for all } f+1 \le m \le k\},$$
$$\Lambda_*^{\overline{\xi}} = [\Lambda^{\overline{\xi}}] \cap \bigcup_{1 \le m \le f} \Lambda_{m*},$$
$$\Lambda^{\overline{\xi}*} = [\Lambda^{\overline{\xi}}] \cap \bigcup_{f+1 \le m \le k} \Lambda_{m*}.$$

In case $f = k-1$ and $\overline{\xi} = \langle \eta \rangle$, we simply write $\overline{\xi} = \eta$, $\Lambda_*^{\overline{\xi}} = \Lambda_*^\eta$ and $\Lambda^{\overline{\xi}*} = \Lambda^{\eta*}$.

DEFINITION 4.5.

(a) For $\overline{\nu} \in \Lambda_* \cup \Lambda$, we define the *ordinal content*

$$\operatorname{orco} \overline{\nu} = \bigcup_{1 \le m \le k} \operatorname{Im} \nu_m.$$

(b) If $S \subseteq \Lambda_* \cup \Lambda$, then $\operatorname{orco} S = \bigcup_{\overline{\nu} \in S} \operatorname{orco} \overline{\nu}$.
(c) If $S, T \subseteq \lambda_k$ and $\tau : S \to T$ is a bijection, then τ extends canonically to a bijection $\tau : {}^{\omega\ge}S \to {}^{\omega\ge}T$, and for $\overline{\nu} \in \Lambda_* \cup \Lambda$ with $\operatorname{orco} \overline{\nu} \subseteq S$ we define $\overline{\nu}\tau = (\nu_1\tau, \ldots, \nu_k\tau)$.
(d) If $X_* \subseteq \Lambda_*$, then we say that a bijection $\tau : S \to T$ is X_*-*admissible* if $\operatorname{orco} X_* \subseteq S$ and $X_*\tau \subseteq \Lambda_*$.

(e) If $\tau : S \to T$ is an X_*-admissible bijection, then τ extends canonically to an A-module monomorphism $\tau : \widehat{B}_{X_*} \to \widehat{B}_{\Lambda_*} = \widehat{B}$, which we call a *shift-isomorphism* (onto its image).

X_*-admissible maps are compatible with many of the notions already introduced, like Λ- and (f,α)-closeness. We refer the reader to D. Herden [13] for more details.

Step Lemma 4.6. *Let the following be given:*

(i) $0 \le f < k$, $\overline{\xi} \in {}^{\omega\uparrow}\lambda_{f+1} \times \cdots \times {}^{\omega\uparrow}\lambda_k$ *with* $\alpha = \|\xi_k\|$.
(ii) *A countable* $C_* \subseteq \Lambda_*$ *such that* $\|C_*\| < 0\xi_k$ *and* $[C_*] = [C_*]^\delta$ *for some* $\delta \in \{0,1\}$.
(iii) $(Y_*, Y, \mathfrak{G})$ *with* $\mathfrak{G} = \{y'''_{\overline{\eta}} = b''_{\overline{\eta}} + y_{\overline{\eta}} \mid \overline{\eta} \in Y\}$ *is a* Λ*-closed triple with* $C_* \subseteq Y_*$.
(iv) *A homomorphism* $\varphi : B_{X'_*} \to G_{Y_*Y}$ *for which there exists some* $z \in \overline{B}_{C_*}$ *such that* $z\varphi \notin \langle G_{Y'_*Y'}, Az\rangle_*$ *for all* $Y'_* \subseteq \Lambda_*$, $Y' \subseteq \Lambda$, *where* $X'_* = C_* \cup \Lambda^{\overline{\xi}*}$.

If we take $X = \Lambda^{\overline{\xi}}$ *and* $X_* = C_* \cup [\Lambda^{\overline{\xi}}]$, *then, for all* $\overline{\eta} \in X$, *we can choose an element* $\varepsilon_{\overline{\eta}} \in \{0,1\}$ *such that the triple* $(X_*, X, \mathfrak{F})$ *with* $\mathfrak{F} = \{y'_{\overline{\eta}} = \varepsilon_{\overline{\eta}} z + y_{\overline{\eta}} \mid \overline{\eta} \in X\}$ *is* Λ*-closed and* G_{X_*X} *satisfies the following condition:*

If $(Z_*, Z, \mathfrak{H})$ *is a* Λ*-closed triple with* $\mathfrak{H} = \{y''_{\overline{\eta}} = b'_{\overline{\eta}} + y_{\overline{\eta}} \mid \overline{\eta} \in Z\}$ *and* τ *is a* Y_**-admissible bijection such that*

(v) $\tau \restriction \mathrm{orco}(z \cup z\varphi) = \mathrm{Id}$;
(vi) $(Y_*, Y, \mathfrak{G})\tau \subseteq (Z_*, Z, \mathfrak{H})$ *so* $G_{Y_*\tau Y\tau} \subseteq G_{Z_*Z}$;
(vii) $(Y_*, Y, \mathfrak{G})\tau$ *is* $(k-f,\alpha)$*-closed with respect to* $(Z_*, Z, \mathfrak{H})$;
(viii) $b'_{\overline{\eta}} \in \{0, z\}$ *for all* $\overline{\eta} \in Z_{Y_*\tau Y\tau}$;

then $\varphi\tau : B_{X'_*} \to G_{Y_*\tau Y\tau}$ *does not extend to a homomorphism from* G_{X_*X} *to* G_{Z_*Z}.

Proof. We proceed by induction on f, although we only discuss the initial step $f = 0$ here. See R. Göbel, D. Herden and S. Shelah [7] or D. Herden [13] for a proof of the induction step.

If $f = 0$, then $\overline{\xi} \in \Lambda$, $X = \{\overline{\xi}\}$ and $X'_* = X_* = C_* \cup [\overline{\xi}]$. For $\varepsilon \in \{0,1\}$, let $y^\varepsilon_{\overline{\xi}} = \varepsilon z + y_{\overline{\xi}}$ and $G^\varepsilon = \langle B_{X_*}, y^\varepsilon_{\overline{\xi}}\rangle_*$. Towards a contradiction, suppose that there exist Y_*-admissible bijections τ^ε, Λ-closed triples $(Z^\varepsilon_*, Z^\varepsilon, \mathfrak{H}^\varepsilon)$ containing $(Y_*, Y, \mathfrak{G})\tau^\varepsilon$, $G_{Z^\varepsilon_* Z^\varepsilon} = \langle B_{Z^\varepsilon_*}, Ay'_{\overline{\eta}N^\varepsilon_{\overline{\eta}}} \mid \overline{\eta} \in Z^\varepsilon\rangle_*$ for some integers $N^\varepsilon_{\overline{\eta}} \ge 0$ and homomorphisms

$$\psi^\varepsilon : G^\varepsilon \to G_{Z^\varepsilon_* Z^\varepsilon}$$

extending $\varphi\tau^\varepsilon$. We take some $s < \omega$ such that for both $\varepsilon \in \{0,1\}$,

$$p^s y_{\overline{\xi}}^\varepsilon \psi^\varepsilon = b^\varepsilon + \sum_{\substack{\overline{\eta}\in Z^\varepsilon \\ y''_{\overline{\eta}}\in\mathfrak{H}^\varepsilon}} a_{\overline{\eta}}^\varepsilon y''_{\overline{\eta}N_{\overline{\eta}}} \in \langle B_{Z_*^\varepsilon}, Ay''_{\overline{\eta}N_{\overline{\eta}}} \mid y''_{\overline{\eta}} \in \mathfrak{H}^\varepsilon\rangle$$

for suitable elements $b^\varepsilon \in B_{Z_*^\varepsilon}$, and for all $\overline{\eta} \in Z^\varepsilon$ we have $a_{\overline{\eta}}^\varepsilon \in A$ and $N_{\overline{\eta}} = \max(N_{\overline{\eta}}^0, N_{\overline{\eta}}^1)$.

Consider the unique extensions (simply written as) $\varphi : \widehat{G}^\varepsilon \to \widehat{G}_{Y_*Y}$ and $\tau^\varepsilon : \widehat{G}_{Y_*Y} \to \widehat{G}_{Y_*\tau^\varepsilon Y\tau^\varepsilon}$. Then $\varphi\tau^\varepsilon\restriction G^\varepsilon = \psi^\varepsilon$ and $y_{\overline{\xi}}^\varepsilon\psi^\varepsilon \in G_{Z_*^\varepsilon Z^\varepsilon} \cap \widehat{G}_{Y_*\tau^\varepsilon Y\tau^\varepsilon}$. Since $(Y_*, Y, \mathfrak{G})\tau^\varepsilon$ is (k,α)-closed with respect to $(Z_*^\varepsilon, Z^\varepsilon, \mathfrak{H}^\varepsilon)$, by putting $\Delta^\varepsilon = Z_{Y_*\tau^\varepsilon Y\tau^\varepsilon}^\varepsilon$, it follows similarly to Lemma 4.3 that

$$G_{Z_*^\varepsilon Z^\varepsilon} \cap \widehat{G}_{Y_*\tau^\varepsilon Y\tau^\varepsilon} \subseteq G_{Y_*\tau^\varepsilon\, Y\tau^\varepsilon\cup\Delta^\varepsilon}.$$

There are elements $b^\varepsilon \in B_{Y_*\tau^\varepsilon\cup[Y\tau^\varepsilon\cup\Delta^\varepsilon]}$ and suitable coefficients from A such that

$$p^s y_{\overline{\xi}}^\varepsilon\psi^\varepsilon = p^s y_{\overline{\xi}}^\varepsilon\varphi\tau^\varepsilon = b^\varepsilon + \sum_{\substack{\overline{\eta}\in Y\tau^\varepsilon \\ y''_{\overline{\eta}}\in\mathfrak{H}^\varepsilon}} a_{\overline{\eta}}^\varepsilon y''_{\overline{\eta}} + \sum_{\substack{\overline{\eta}\in\Delta^\varepsilon \\ y''_{\overline{\eta}}\in\mathfrak{H}^\varepsilon}} a_{\overline{\eta}}^\varepsilon y''_{\overline{\eta}}.$$

By applying $(\tau^\varepsilon)^{-1}$ to the corresponding equation and subtracting them we obtain

$$p^s z\varphi = b + \sum_{\overline{\eta}\in Y}(a_{\overline{\eta}\tau^1}^1 - a_{\overline{\eta}\tau^0}^0)y'''_{\overline{\eta}} + \sum_{\overline{\eta}\in\Delta^1} a_{\overline{\eta}}^1 y_{\overline{\eta}(\tau^1)^{-1}} - \sum_{\overline{\eta}\in\Delta^0} a_{\overline{\eta}}^0 y_{\overline{\eta}(\tau^0)^{-1}} + az$$

where $b = b^1(\tau^1)^{-1} - b^0(\tau^0)^{-1}$ and $az = \sum_{\overline{\eta}\in\Delta^1} a_{\overline{\eta}}^1 b'_{\overline{\eta}(\tau^1)^{-1}} - \sum_{\overline{\eta}\in\Delta^0} a_{\overline{\eta}}^0 b'_{\overline{\eta}(\tau^0)^{-1}}$. If we take $Y' = Y\cup\Delta^0(\tau^0)^{-1}\cup\Delta^1(\tau^1)^{-1}$ and $Y'_* = \Lambda_*$, then $z\varphi \in \langle G_{Y'_*Y'}, Az\rangle_*$, which is a contradiction to condition (iv). Hence, we can choose an $\varepsilon \in \{0,1\}$ that proves our claim. ∎

In the end, we want to construct a group G in such a way that $\operatorname{End} G = A \oplus \operatorname{Fin} G$, where $\operatorname{Fin} G$ is the ideal of endomorphisms of G with finite rank image. For this purpose, we need to eliminate all other possible homomorphisms. This will result from combining the following crucial lemma and condition (iv) of Step Lemma 4.6. Recall that A is a ring with free additive structure $A^+ = \bigoplus_{\alpha<\kappa}\mathbb{Z}e_\alpha$.

Lemma 4.7. *Let $V \subseteq \Lambda$ and $G = G_{\Lambda_*V}$ for a suitable regressive family $\mathfrak{F}$ of branch-like elements. If $\varphi \in \operatorname{End} G \setminus (A \oplus \operatorname{Fin} G)$, then there exists an element $z \in \overline{B}$ with $[z] = [z]^\delta$ for some $\delta \in \{0,1\}$ such that $z\varphi \notin \langle G', Az\rangle_*$, whenever $G' = G_{\Lambda_*V'}$ for some $V' \subseteq \Lambda$ with accompanying regressive family $\mathfrak{F}'$ of branch-like elements.*

Proof. We will examine the images $e_{\overline{\nu}}\varphi$ of the generators of B in order to construct the element z. It will be necessary to consider several cases. In each of them, we assume that the previous cases do not apply. In each case, we will assume the existence of an infinite subset $\mathcal{I} \subseteq \Lambda_*$ satisfying, without loss of generality, $\mathcal{I} = [\mathcal{I}]^\delta$ for some $\delta \in \{0, 1\}$.

(i) *There is an infinite subset $\mathcal{I} \subseteq \Lambda_*$ such that $\bigcap_{\overline{\nu} \in \mathcal{I}} [e_{\overline{\nu}}\varphi] \neq \emptyset$.* Consider an element $\overline{\gamma} \in \bigcap_{\overline{\nu} \in \mathcal{I}} [e_{\overline{\nu}}\varphi]$. To construct z, we choose elements of $\mathcal{I}$ inductively. Choose elements $\overline{\nu}_n \in \mathcal{I} \setminus \{\overline{\nu}_0, \ldots, \overline{\nu}_{n-1}\}$ with $[e_{\overline{\nu}_n}\varphi]_{\overline{\gamma}} = a_n \in p^{s_n} A \setminus p^{s_n+1} A$ (see Definition 2.2(b)), where $0 \leq s_0 \leq \cdots \leq s_n$. If there is an $\alpha < \kappa$ such that $\alpha \in \bigcap_{n<\omega} [a_n]$ when comparing supports with respect to $A^+ = \bigoplus_{\alpha<\kappa} \mathbb{Z} e_\alpha$, then we consider the sums $\sum_{n<\omega} \varepsilon_n p^n e_{\overline{\nu}_n}$ with $\varepsilon_n \in \{0, 1\}$ for all $n < \omega$, and choose z to be one of these sums such that $\big[[z\varphi]_{\overline{\gamma}}\big]_\alpha \in J_p \setminus \mathbb{Q}$. This is possible, since there are $2^{\aleph_0}$ such sums. Otherwise, we can choose the elements $\overline{\nu}_n$ in such a way that the coefficients a_n have disjoint supports in A^+. Put $z = \sum_{n<\omega} p^n e_{\overline{\nu}_n}$. Then $[z\varphi]_{\overline{\gamma}}$ is an element of $\overline{A} \setminus A$.

(ii) *There is an infinite subset $\mathcal{I} \subseteq \Lambda_*$ and an element $\overline{\eta} \in \Lambda$ such that for all $\overline{\nu} \in \mathcal{I}$, $\overline{\eta} \in [e_{\overline{\nu}}\varphi]_\Lambda$ is of maximal norm.* Choose a strictly increasing sequence $\{n_i\}_{i<\omega}$ and elements $\overline{\nu}_i \in \mathcal{I}$ such that $[e_{\overline{\nu}_i}\varphi]_{\overline{\eta}\restriction\langle m,n\rangle} = p^{n-n_i} a_i$ for all $i < \omega$, $n \geq n_i$ and $[e_{\overline{\nu}_i}\varphi]_{\overline{\eta}\restriction\langle m,n\rangle} = 0$ for all $0 < i < \omega$, $n \leq n_{i-1}$. Take $z = \sum_{n<\omega} p^n e_{\overline{\nu}_n}$. The Recognition Lemma 3.7 will not be able to recognize a coefficient of $y_{\overline{\eta}}$ from $[z\varphi]^{1-\delta}$, despite $[y_{\overline{\eta}}]_{n_0} \subseteq [z\varphi]$ and $\|z\varphi\| = \|\overline{\eta}\|$.

(iii) *There are some countably infinite subsets $\{\overline{\nu}_n \mid n < \omega\} \subseteq \mathcal{I}$ and $\{\overline{\eta}_n \mid n < \omega\} \subseteq \Lambda$ such that for all $n \neq m < \omega$, $\overline{\eta}_n \neq \overline{\eta}_m$, $\|\overline{\eta}_n\| = \|\overline{\eta}_m\|$ and $\overline{\eta}_n \in [e_{\overline{\nu}_n}\varphi]_\Lambda$ is of maximal norm.* Without loss of generality, we may assume $\overline{\eta}_m \notin [e_{\overline{\nu}_n}\varphi]$ for all $m \neq n$. Take $z = \sum_{n<\omega} p^n e_{\overline{\nu}_n}$. By the Recognition Lemma 3.7, it is not possible to read off finitely many branch summands of maximal norm from $[z\varphi]^{1-\delta}$.

(iv) *There are some countably infinite subsets $\{\overline{\nu}_n \mid n < \omega\} \subseteq \mathcal{I}$ and $\{\overline{\eta}_n \mid n < \omega\} \subseteq \Lambda$ such that for all $n < \omega$, $\|\overline{\eta}_n\| < \|\overline{\eta}_{n+1}\|$ and $\overline{\eta}_n \in [e_{\overline{\nu}_n}\varphi]_\Lambda$ is of maximal norm.* Take z as in the previous case and the Recognition Lemma 3.7 once again fails to read off a branch summand of maximal norm from $[z\varphi]^{1-\delta}$.

If φ does not comply with these cases, then it means that $e_{\overline{\nu}}\varphi \in B$ for almost all $\overline{\nu} \in \Lambda_*$.

(v) *For all $\overline{\nu} \in \mathcal{I}$, $e_{\overline{\nu}}\varphi \in B$ and $[e_{\overline{\nu}}\varphi] \setminus \{\overline{\nu}\} \neq \emptyset$.* For all $n < \omega$, we choose inductively elements $\overline{\nu}_n \in \mathcal{I}$ and $\overline{\nu}'_n \in [e_{\overline{\nu}_n}\varphi]$ such that $\overline{\nu}_n \neq \overline{\nu}'_n$ for all $n < \omega$, and $[e_{\overline{\nu}_n}\varphi] \cap [e_{\overline{\nu}_m}\varphi] = \emptyset$ for all $n \neq m$. For all $n < \omega$, take $z_n = p^{s_n} e_{\overline{\nu}_n}$, where s_n is chosen inductively such that $p^{n+s_n+\ell(\overline{\nu}'_m)}[e_{\overline{\nu}_n}\varphi]_{\overline{\nu}'_n} \neq p^{m+s_m+\ell(\overline{\nu}'_n)}[e_{\overline{\nu}_m}\varphi]_{\overline{\nu}'_m}$ for all $n \neq m$. Put $z = \sum_{n<\omega} p^n z_n$. We obtain the same conclusion as in the second case.

(vi) *There are countably infinite subsets $\{\overline{\nu}_i \mid i < \omega\} \subseteq \mathcal{I}$ and $\{a_i \mid i < \omega\} \subseteq A$ such that for all $i \neq j < \omega$, $a_i \neq a_j$ and $e_{\overline{\nu}_i}\varphi = a_i e_{\overline{\nu}_i}$.* Take

$z = \sum_{i<\omega} p^i e_{\overline{\nu}_i}$ and suppose that $z\varphi \in \langle G_{Y'_*Y'}, Az\rangle_*$ for some Λ-closed triple $(Y'_*, Y', \mathfrak{F}')$. Then there exist $n > 0$, $b \in B_{Y'_*}$ and $a \in A$ such that

$$p^n z\varphi = b + \sum_{\overline{\eta}\in Y'} a_{\overline{\eta}} y'_{\overline{\eta}N_{\overline{\eta}}} + az.$$

Since $[z]^{1-\delta} = \emptyset$, we have $a_{\overline{\eta}} = 0$ for all $\overline{\eta} \in Y'$. It follows that $p^{n+i}a_i = [p^n z\varphi]_{\overline{\nu}_i} = p^i a$ for all $\overline{\nu}_i \notin [b]$, so $a = p^n a_i$, which in turn implies that $a_i = a_j$ for almost all $i, j < \omega$, a contradiction.

(vii) *There are a countably infinite subset* $\{\overline{\nu}_i \mid i < \omega\} \subseteq \mathcal{I}$ *and two elements* $a_0, a_1 \in A$ *such that* $a_0 \neq a_1$ *and for all* $i < \omega$, $e_{\overline{\nu}_{2i}}\varphi = a_0 e_{\overline{\nu}_{2i}}$ *and* $e_{\overline{\nu}_{2i+1}}\varphi = a_1 e_{\overline{\nu}_{2i+1}}$. Proceed in a similar way to the previous case.

If φ does not comply with these last cases either, then there exist a finite subset $S \subseteq \Lambda_*$ and an $a \in A$ such that $e_{\overline{\nu}}\varphi = ae_{\overline{\nu}}$ for all $\overline{\nu} \in \Lambda_* \setminus S$, so that $\varphi = \varphi' + a$, where $\varphi' \in \operatorname{Fin} G$ with $e_{\overline{\nu}}\varphi' = e_{\overline{\nu}}\varphi - ae_{\overline{\nu}}$ for all $\overline{\nu} \in \Lambda_*$. ∎

5. The main result. In this section we realize the main construction of an $\aleph_k$-free A-module G with prescribed endomorphism ring $\operatorname{End} G = A \oplus \operatorname{Fin} G$ by means of the Easy Black Box principle and the Step Lemma 4.6. We will actually introduce an *algebraic* version of the Easy Black Box, properly arranged for this construction. Such prediction principles need the notion of *traps* for capturing the objects to be predicted.

Definition 5.1. A quintuple $p = (\eta, V_*, V, \mathfrak{F}, \varphi)$ is a *trap for the Easy Black Box* if:

(i) $\eta \in {}^{\omega\uparrow}\lambda_k$.
(ii) $(V_*, V, \mathfrak{F})$ is Λ-closed.
(iii) $|V_*|, |V| \leq \lambda_{k-1}$.
(iv) $\Lambda^{\eta *} = \{\overline{\nu} \in \Lambda_{k*} \mid \nu_k \in [\eta]\} \subseteq V_*$.
(v) $\|\overline{\nu}\| < \|\eta\|$ for all $\overline{\nu} \in V_*$, and $\|\overline{\eta}\| < \|\eta\|$ for all $\overline{\eta} \in V$.
(vi) $\varphi : G_{V_*V} \to G_{V_*V}$ is an endomorphism.

We denote by $\|p\| = \|\eta\| = \|V_*\|$ the *norm of the trap* p.

Recall that for an infinite cardinal λ, $\lambda^o = \{\alpha \in \lambda \mid \operatorname{cf}(\alpha) = \omega\}$.

The Easy Black Box 5.2. *Let* $|A| \leq \theta < \lambda = \lambda^\theta$ *with* λ *a regular cardinal. If* E *is a stationary subset of* λ^o, *then there are an ordinal* $\lambda \leq \lambda^* < \lambda^+$ *and a list of traps*

$$\langle p_\alpha = (\eta_\alpha, V_{\alpha *}, V_\alpha, \mathfrak{F}_\alpha, \varphi_\alpha) \mid \alpha < \lambda^*\rangle$$

with the following properties:

(i) $\|p_\alpha\| \in E$ *for all* $\alpha < \lambda^*$.
(ii) $\|p_\alpha\| \leq \|p_\beta\|$ *for all* $\alpha < \beta < \lambda^*$.
(iii) $\eta_\alpha \neq \eta_\beta$ *for all* $\alpha < \beta < \lambda^*$.

(iv) THE PREDICTION: *For any Λ-closed triple $(\Lambda_*, V, \mathfrak{F})$ with $G = G_{\Lambda_* V}$, any homomorphism $\varphi \in \operatorname{End} G$ and any set $S \subseteq \Lambda_*$ with $|S| \le \theta$, the set of ordinals $\alpha \in E$ for which there is a $\beta < \lambda^*$ with*

$$\|p_\beta\| = \alpha,\ \|S\| < 0\eta_\beta,\ (V_{\beta*}, V_\beta,\ \mathfrak{F}_\beta) \subseteq (\Lambda_*, V, \mathfrak{F}),\ \varphi_\beta \subseteq \varphi,\ S \subseteq V_{\beta*}$$

is stationary.

MAIN THEOREM 5.3. *If A is a ring with free additive group $A^+ = \bigoplus_{\alpha<\kappa} \mathbb{Z}e_\alpha$ such that $\overline{A} = \widehat{A} \cap \prod_{\alpha<\kappa} \mathbb{Z}e_\alpha$ is an A-module, $|A| \le \mu$, $1 \le k < \omega$ and $\lambda = \beth_k^+(\mu)$, then it is possible to construct an $\aleph_k$-free A-module G of cardinality λ such that G is separable as an abelian group and* $\operatorname{End} G = A \oplus \operatorname{Fin} G$.

Proof. Since the case $k = 1$ is a classical result due to M. Dugas and R. Göbel (see [4]), we assume $k > 1$.

Consider the stationary subset $\lambda_k^o = \{\alpha < \lambda_k \mid \operatorname{cf}(\alpha) = \omega\}$ of $\lambda_k = \mu_k^+$. By Solovay's Theorem (Jech [16, p. 95]), we can decompose λ_k^o into λ_k disjoint stationary subsets, say $\lambda_k^o = \dot{\bigcup}_{\alpha<\lambda_k} E_\alpha$. Since $|\overline{B} \setminus B| = \lambda_k$, we are allowed to write

$$\lambda_k^o = \dot{\bigcup_{z \in \overline{B} \setminus B}} E_z.$$

For each E_z, the Easy Black Box provides us with a family of traps

$$p_\alpha^z = (\eta_\alpha^z, V_{\alpha*}^z, V_\alpha^z, \mathfrak{F}_\alpha^z, \varphi_\alpha^z)$$

for $\alpha < \lambda^{*z} < \lambda_k^+$. We gather all these traps and order them according to the norm of their first component, namely $\|\eta_\alpha\| \le \|\eta_\beta\|$ for all $\alpha < \beta < \lambda^* < \lambda_k^+$.

Let $V = \bigcup_{\alpha<\lambda^*} \Lambda^{\eta_\alpha}$. We will construct a regressive family $\mathfrak{F} = \{y'_{\overline{\eta}} = b_{\overline{\eta}} + y_{\overline{\eta}} \mid \overline{\eta} \in V\}$ of branch-like elements by choosing for all $\alpha < \lambda^*$ and all $\overline{\eta} \in \Lambda^{\eta_\alpha}$ an element $b_{\overline{\eta}} \in \overline{B}$, and define $G = G_{\Lambda_* V}$. Suppose that when considering the trap $p_\alpha = (\eta_\alpha, V_{\alpha*}, V_\alpha, \mathfrak{F}_\alpha, \varphi_\alpha)$ with $\mathfrak{F}_\alpha = \{y'''_{\overline{\eta}} = b''_{\overline{\eta}} + y_{\overline{\eta}} \mid \overline{\eta} \in V_\alpha\}$ and the unique $z \in \overline{B} \setminus B$ such that $\|\eta_\alpha\| \in E_z$, we get

(i) $V_\alpha \subseteq \bigcup_{\beta<\alpha,\, \|\eta_\beta\|<\|\eta_\alpha\|} \Lambda^{\eta_\beta}$,
(ii) $y'''_{\overline{\eta}} = y'_{\overline{\eta}}$ for all $\overline{\eta} \in V_\alpha$,
(iii) $[z] \subseteq V_{\alpha*}$,
(iv) $[z] = [z]^\varepsilon$ for some $\varepsilon \in \{0, 1\}$,
(v) $\|z\| < 0\eta_\alpha$,
(vi) $z\varphi \notin \langle G', Az\rangle_*$ whenever $G' = G_{\Lambda_* V'}$ for some $V' \subseteq \Lambda$ and a suitable regressive family $\mathfrak{F}'$.

Then we let

$$Y_{\alpha *} = \{\overline{\nu} \in \Lambda_* \mid \|\overline{\nu}\| < \|\eta_\alpha\|\},$$

$$Y_\alpha = \bigcup_{\substack{\beta<\alpha \\ \|\eta_\beta\|<\|\eta_\alpha\|}} \Lambda^{\eta_\beta} \quad \text{and} \quad \mathfrak{G}_\alpha = \{y'_{\overline{\eta}} \mid \overline{\eta} \in Y_\alpha\}.$$

It is easy to verify that the assumptions of Step Lemma 4.6 hold for

$$f = k-1,\ \overline{\xi} = \eta_\alpha,\ C_* = [z],\ z,\ (Y_*, Y, \mathfrak{G}) = (Y_{\alpha *}, Y_\alpha, \mathfrak{G}_\alpha),\ \varphi = \varphi_\alpha \restriction B_{X'_*},$$

where $X'_* = [z] \cup \Lambda^{\eta_\alpha *}$. Applying Step Lemma 4.6 we will obtain correction elements $b_{\overline{\eta}} = \varepsilon_{\overline{\eta}} z$ for $\overline{\eta} \in \Lambda^{\eta_\alpha}$. If any of the conditions (i) to (vi) fail, we set $\varepsilon_{\overline{\eta}} = 0$ instead. In this way, all elements $\varepsilon_{\overline{\eta}}$ ($\overline{\eta} \in V$) are chosen and the construction of $G = G_{\Lambda_* V}$ is finished.

To obtain a contradiction, suppose there is some $\psi \in \operatorname{End} G \setminus (A \oplus \operatorname{Fin} G)$. We apply Lemma 4.7 to G to obtain an element $z \in \overline{B}$ such that $[z] = [z]^\varepsilon$ for some $\varepsilon \in \{0,1\}$ and $z\psi \notin \langle G', Az\rangle_*$, whenever $G' = G_{\Lambda_* V'}$ for some $V' \subseteq \Lambda$ and a suitable regressive family $\mathfrak{F}'$. Applying the Easy Black Box 5.2 to the stationary set $E_z \subseteq \lambda^o$ we deduce for $G = G_{\Lambda_* V}$, ψ and $S = [z]$ that the set of ordinals $\gamma \in E_z$ for which there is an $\alpha < \lambda^{*z}$ with

$$\|p^z_\alpha\| = \gamma,\quad \|z\| < 0\eta^z_\alpha,\quad (V^z_{\alpha *},\ \ V^z_\alpha,\ \ \mathfrak{F}^z_\alpha) \subseteq (\Lambda_*, V, \mathfrak{F}),\quad \varphi^z_\alpha \subseteq \psi,\quad [z] \subseteq V^z_{\alpha *}$$

is stationary.

In particular, there is some $\alpha < \lambda^*$ such that

$$\|p_\alpha\| \in E_z,\quad \|z\| < 0\eta_\alpha,\quad (V_{\alpha *}, V_\alpha, \mathfrak{F}_\alpha) \subseteq (\Lambda_*, V, \mathfrak{F}),\quad \varphi_\alpha \subseteq \psi,\quad [z] \subseteq V_{\alpha *}.$$

Notice that

$$V_\alpha \subseteq V \cap \{\overline{\eta} \in \Lambda \mid \|\overline{\eta}\| < \|\eta_\alpha\|\} = \bigcup_{\substack{\beta<\alpha \\ \|\eta_\beta\|<\|\eta_\alpha\|}} \Lambda^{\eta_\beta}.$$

Thus, the non-trivial case of the construction applies, so the $b_{\overline{\eta}} \in \{0, z\}$ ($\overline{\eta} \in \Lambda^{\eta_\alpha}$) were chosen according to Step Lemma 4.6. In order to derive the desired contradiction, we set

$$f = k-1,\ \overline{\xi} = \eta_\alpha,\ C_* = [z],\ z,\ (Y_*, Y, \mathfrak{G}) = (Y_{\alpha *}, Y_\alpha, \mathfrak{G}_\alpha),\ \varphi = \varphi_\alpha \restriction B_{X'_*},$$
$$X = \Lambda^{\overline{\xi}},\ X_* = C_* \cup [\Lambda^{\overline{\xi}}],\ (Z_*, Z, \mathfrak{H}) = (\Lambda_*, V, \mathfrak{F}),\ \tau = \mathrm{Id}_{\mathrm{orco}\, Y_{\alpha *}}$$

and verify for this choice the missing conditions (v) to (viii) of Step Lemma 4.6:

(v) Immediate.

(vi) $(Y_*, Y, \mathfrak{G})\tau = (Y_{\alpha *}, Y_\alpha, \mathfrak{G}_\alpha) \subseteq (\Lambda_*, V, \mathfrak{F}) = (Z_*, Z, \mathfrak{H})$ by definition.

(vii) By definition, $(Y_*, Y, \mathfrak{G})\tau = (Y_{\alpha *}, Y_\alpha, \mathfrak{G}_\alpha)$ is $(1, \|\eta_\alpha\|)$-closed with respect to $(\Lambda_*, V, \mathfrak{F})$ since from $\overline{\eta} \in V$ with $|u_{\overline{\eta}}(Y_{\alpha *})| \geq 1$ and $\|\overline{\eta}\| \neq \|\eta_\alpha\|$ it follows that

$$\overline{\eta} \in V \cap \{\overline{\eta} \in \Lambda \mid \|\overline{\eta}\| < \|\eta_\alpha\|\} = \bigcup_{\substack{\beta<\alpha \\ \|\eta_\beta\|<\|\eta_\alpha\|}} \Lambda^{\eta_\beta} = Y_\alpha.$$

(viii) We have

$$\begin{aligned} Z_{Y_{*\tau}Y_\tau} = V_{Y_{\alpha*}Y_\alpha} &= \{\overline{\eta} \in V \setminus Y_\alpha \mid |u_{\overline{\eta}}(Y_{\alpha*})| \geq 1 \text{ and } \|\overline{\eta}\| = \|\eta_\alpha\|\} \\ &= V \cap \{\overline{\eta} \in \Lambda \mid \|\overline{\eta}\| = \|\eta_\alpha\|\} = \bigcup_{\substack{\beta<\lambda^* \\ \|\eta_\beta\|=\|\eta_\alpha\|}} \Lambda^{\eta_\beta}. \end{aligned}$$

In particular, $\|\overline{\eta}\| = \|\eta_\alpha\| = \|p_\alpha\| \in E_z$, and therefore $b_{\overline{\eta}} \in \{z, 0\}$ for all $\overline{\eta} \in Z_{Y_{*\tau}Y_\tau}$.

The existence of $(Z_*, Z, \mathfrak{H})$, τ and ψ with $\varphi\tau = \varphi \subseteq \psi$ contradicts the choice of $\varepsilon_{\overline{\eta}}$ ($\overline{\eta} \in \Lambda^{\eta_\alpha}$) made during the construction of G by means of Step Lemma 4.6 for $f = k - 1$, $\overline{\xi} = \eta_\alpha$, $C_* = [z]$, z, $(Y_*, Y, \mathfrak{G}) = (Y_{\alpha*}, Y_\alpha, \mathfrak{G}_\alpha)$, $\varphi = \varphi_\alpha \restriction B_{X'_*}$. ■

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