FUNDAMENTA
MATHEMATICAE
231 (2015)

Lelek fan from a projective Fraïssé limit

by

Dana Bartošová (São Paulo) and
Aleksandra Kwiatkowska (Los Angeles, CA)

Abstract. We show that a natural quotient of the projective Fraïssé limit of a family that consists of finite rooted trees is the Lelek fan. Using this construction, we study properties of the Lelek fan and of its homeomorphism group. We show that the Lelek fan is projectively universal and projectively ultrahomogeneous in the class of smooth fans. We further show that the homeomorphism group of the Lelek fan is totally disconnected, generated by every neighbourhood of the identity, has a dense conjugacy class, and is simple.

1. Introduction

1.1. Lelek fan. A *continuum* is a compact and connected metric space. Let C denote the Cantor set. The *Cantor fan* F is the cone over the Cantor set, that is, $C \times [0,1]/\sim$, where $(a,b) \sim (c,d)$ if and only if either $a = c$ and $b = d$, or $b = d = 0$. Recall that an *arc* is a homeomorphic image of the closed unit interval $[0,1]$. If X is a space and $h\colon [0,1] \to X$ is a homeomorphism onto its image, we call $h(0) = a$ and $h(1) = b$ the *endpoints* of the arc given by h and denote this arc by ab. An *endpoint* of a continuum X is a point e such that for every arc ab in X, if $e \in ab$, then $e = a$ or $e = b$. Finally, a *Lelek fan* L is a non-degenerate subcontinuum of the Cantor fan with a dense set of endpoints.

In the literature, a Lelek fan is often defined as a smooth fan with a dense set of endpoints. However, smooth fans are exactly fans that can be embedded into the Cantor fan (see [CC, Proposition 4]). We give the definition of a smooth fan in Subsection 2.2.

A Lelek fan was constructed by Lelek [L]. Several characterizations of a Lelek fan were collected in [CCM, Theorem 12.14]. A remarkable property of a Lelek fan is its uniqueness, which was proved independently by

2010 *Mathematics Subject Classification*: 03E15, 37B05, 54F15, 03C98.
Key words and phrases: Lelek fan, Fraïssé limits, homeomorphism groups.

DOI: 10.4064/fm231-1-4

Bula–Oversteegen [BO] and by W. Charatonik [C]: any two non-degenerate subcontinua of the Cantor fan with a dense set of endpoints are homeomorphic. We can therefore speak about “the” Lelek fan.

A very interesting and well-studied by many people is the space E of endpoints of the Lelek fan L. This space is a dense G_δ set in L, therefore it is separable and completely metrizable. It is homeomorphic to the complete Erdős space, to the set of endpoints of the Julia set of the exponential map, to the set of endpoints of the separable universal $\mathbb{R}$-tree; see Kawamura–Oversteegen–Tymchatyn [KOT] for more details. Since the complete Erdős space is 1-dimensional, so is E.

Dijkstra–Zhang [DZ] showed that the space of Lelek fans, endowed with the Vietoris topology, in the Cantor fan is homeomorphic to the separable Hilbert space.

Here we introduce some notation that we will need later on. By v we denote the *top* $v = (0,0)/\sim$ of the Cantor fan. For a point $x \in F$, let $[v,x]$ denote the closed line with endpoints v and x. If x is in the line segment $[v,y]$, we denote by $[x,y]$ the line segment $([v,y] \setminus [v,x]) \cup \{x\}$. Points in F will be denoted by (c,y), where $c \in C$ and $y \in [0,1]$. Let $\pi_1 \colon F \setminus \{v\} \to C$, $\pi_1(c,x) = c$, and $\pi_2 \colon F \to [0,1]$, $\pi_2(c,x) = x$, be projections. Let E be the set of endpoints of the Lelek fan L, and let $H(L)$ be the group of all homeomorphisms of the Lelek fan.

1.2. Projective Fraïssé limits. Given a language $\mathcal{L}$ that consists of relation symbols r_i with arity m_i, $i \in I$, and function symbols f_j with arity n_j, $j \in J$, a *topological $\mathcal{L}$-structure* is a compact Hausdorff zero-dimensional second-countable space A equipped with closed relations $r_i^A \subseteq A^{m_i}$ and continuous functions $f_j^A \colon A^{n_j} \to A$, $i \in I$, $j \in J$. A continuous surjection $\phi \colon B \to A$ is an *epimorphism* if it preserves the structure, that is, for a function symbol f in $\mathcal{L}$ of arity n and $x_1, \ldots, x_n \in B$ we require

$$f^A(\phi(x_1), \ldots, \phi(x_n)) = \phi(f^B(x_1, \ldots, x_n)),$$

and for a relation symbol r in $\mathcal{L}$ of arity m and $x_1, \ldots, x_m \in A$ we require

$$\begin{aligned} &r^A(x_1, \ldots, x_m) \\ &\quad \Leftrightarrow \ \exists y_1, \ldots, y_m \in B \ \big(\phi(y_1) = x_1, \ldots, \phi(y_m) = x_m, \text{ and } r^B(y_1, \ldots, y_m)\big). \end{aligned}$$

By an *isomorphism* we mean a bijective epimorphism.

For the rest of this section fix a language $\mathcal{L}$. Let $\mathcal{G}$ be a family of finite topological $\mathcal{L}$-structures. We say that $\mathcal{G}$ is a *projective Fraïssé family* if it is countable and the following two conditions hold:

(JPP) (the joint projection property) for any $A, B \in \mathcal{G}$ there are $C \in \mathcal{G}$ and epimorphisms from C onto A and from C onto B;

(AP) (the amalgamation property) for $A, B_1, B_2 \in \mathcal{G}$ and any epimorphisms $\phi_1 \colon B_1 \to A$ and $\phi_2 \colon B_2 \to A$, there exist $C \in \mathcal{G}$, $\phi_3 \colon C \to B_1$, and $\phi_4 \colon C \to B_2$ such that $\phi_1 \circ \phi_3 = \phi_2 \circ \phi_4$.

A topological $\mathcal{L}$-structure $\mathbb{G}$ is a *projective Fraïssé limit* of $\mathcal{G}$ if the following three conditions hold:

(L1) (the projective universality) for any $A \in \mathcal{G}$ there is an epimorphism from $\mathbb{G}$ onto A;
(L2) for any finite discrete topological space X and any continuous function $f \colon \mathbb{G} \to X$ there are $A \in \mathcal{G}$, an epimorphism $\phi \colon \mathbb{G} \to A$, and a function $f_0 \colon A \to X$ such that $f = f_0 \circ \phi$;
(L3) (the projective ultrahomogeneity) for any $A \in \mathcal{G}$ and any epimorphisms $\phi_1 \colon \mathbb{G} \to A$ and $\phi_2 \colon \mathbb{G} \to A$ there exists an isomorphism $\psi \colon \mathbb{G} \to \mathbb{G}$ such that $\phi_2 = \phi_1 \circ \psi$.

We will often use the following immediate consequence of (L2).

Remark 1.1. Let $\mathbb{G}$ be the projective Fraïssé limit of $\mathcal{G}$. Then every finite open cover can be *refined by an epimorphism*, i.e. for every open cover $\mathcal{U}$ of $\mathbb{G}$ there is an epimorphism $\phi \colon \mathbb{G} \to A$, for some $A \in \mathcal{G}$, such that for every $a \in A$, $\phi^{-1}(a)$ is contained in some open set in $\mathcal{U}$.

Remark 1.2. In the projective Fraïssé theory, a projective Fraïssé family has properties dual to the joint embedding property and to the amalgamation property from the (injective) Fraïssé theory. We do not have a condition that corresponds to the hereditary property. Nevertheless, we can think of (L2) as a dualization of a "cofinal hereditary property": if $\mathbb{K}$ is the Fraïssé limit of a Fraïssé family $\mathcal{K}$, then for any finite $X \subseteq \mathbb{K}$, there is $A \in \mathcal{K}$ with $X \subseteq A \subseteq \mathbb{K}$.

Theorem 1.3 (Irwin–Solecki [IS]). *Let $\mathcal{G}$ be a projective Fraïssé family of finite topological $\mathcal{L}$-structures. Then:*

(1) *there exists a projective Fraïssé limit of $\mathcal{G}$;*
(2) *any two projective Fraïssé limits of $\mathcal{G}$ are isomorphic.*

We will frequently use the following property of the projective Fraïssé limit, called the *extension property*.

Proposition 1.4. *If $\mathbb{G}$ is the projective Fraïssé limit of $\mathcal{G}$, the following condition holds: Given $A, B \in \mathcal{G}$ and epimorphisms $\phi_1 \colon B \to A$ and $\phi_2 \colon \mathbb{G} \to A$, there is an epimorphism $\psi \colon \mathbb{G} \to B$ such that $\phi_2 = \phi_1 \circ \psi$.*

1.3. Summary of results. In Section 2, we construct the Lelek fan L as a natural quotient of the projective Fraïssé limit of a family of finite ordered trees. In fact, we show that we can restrict our attention to a subclass $\mathcal{F}$ of simple trees called fans. We then use this construction to show projective universality and projective ultrahomogeneity of the Lelek fan in the family of all smooth fans (Theorem 2.12). In particular, we deduce that every smooth fan is a continuous image of the Lelek fan.

In Section 3, we prove that the homeomorphism group of the Lelek fan, $H(L)$, has the following properties:

(1) $H(L)$ is totally disconnected (Proposition 3.1).
(2) $H(L)$ is generated by every neighbourhood of the identity (Corollary 3.3).
(3) $H(L)$ has a dense conjugacy class (Theorem 3.8).
(4) $H(L)$ is simple (Theorem 3.18).

To prove properties (2) and (3), we use our projective Fraïssé limit construction. For a detailed discussion of motivation, connections with other known results, etc., of each of these four properties, we refer to Section 3.

Lewis–Zhou [LZ, Question 5] asked whether every homeomorphism group of a continuum which is generated by every neighbourhood of the identity has to be connected. As $H(L)$ satisfies properties (1) and (2) above, the answer to this question is negative.

We were recently informed by Megrelishvili that results in this paper together with results due to Ben Yaacov and Tsankov [BYT, Corollary 4.10] give a positive answer to a question posed by Glasner and Megrelishvili [Me, Question 6.14] and [GM, Question 10.5(1)]: Is it true that there exists a non-trivial Polish group G which is reflexively trivial but does not contain $H_+[0,1]$, the group of increasing homeomorphisms of $[0,1]$? Indeed, $H(L)$ provides an example of such a group. As properties (1) and (2) above hold for $H(L)$ and since $\mathrm{Aut}(\mathbb{L})$, where $\mathbb{L}$ is the projective Fraïssé limit of the family of finite rooted reflexive fans discussed below, is an oligomorphic group (see [BKn]), Corollary 4.10 from [BYT] implies that $H(L)$ is reflexively trivial. Since $H(L)$ is totally disconnected, it does not contain $H_+[0,1]$.

2. Lelek fan as a quotient of a projective Fraïssé limit

2.1. Construction of the Lelek fan. Let T be a *finite tree*, that is, an undirected simple graph which is connected and has no cycles. We will only consider rooted trees, i.e. trees with a distinguished element $r_T \in T$. On a rooted tree T there is a natural partial order $\le_T$: for $t, s \in T$ we let $s \le_T t$ if and only if s belongs to the path connecting t and the root. We say that t is a *successor* of s if $s \le_T t$, $t \neq s$. It is an *immediate successor* if additionally there is no $p \in T$, $p \neq s, t$, with $s \le_T p \le_T t$. A *chain* is a rooted tree T on which the order $\le_T$ is linear. A *branch* of a rooted tree T is a maximal chain in $(T, \le_T)$. If b is a branch in T, we will sometimes write $b = (b(0), b(1), \ldots, b(n))$, where $b(0)$ is the root of T and $b(i)$ is an immediate successor of $b(i-1)$ for every $i = 1, \ldots, n$. We denote by $B(T)$ the set of all branches of T.

Let R be a binary relation symbol. Consider the language $\mathcal{L} = \{R\}$. For $s, t \in T$ we write $R^T(s,t)$ if $s = t$ or t is an immediate successor of s. Let $\mathcal{T}$ be the family of all finite rooted trees, viewed as topological $\mathcal{L}$-structures, equipped with the discrete topology.

A function $\phi\colon (S, R^S) \to (T, R^T)$ is a *homomorphism* if for every $s_1, s_2 \in S$, whenever $R^S(s_1, s_2)$ then $R^T(\phi(s_1), \phi(s_2))$.

Remark 2.1. Notice that $\phi\colon (S, R^S) \to (T, R^T)$ is an epimorphism if and only if it is a surjective homomorphism.

Let $\mathcal{F}$ be the family of *finite rooted reflexive fans*, that is, the family that consists of rooted trees $T \in \mathcal{T}$ such that for every $s, t \in T$ which are incomparable in $\leq_T$, if $p \neq s, t$ is such that $R^T(p,s)$ and $R^T(p,t)$, then p is the root of T, and moreover all branches of T have the same length.

Remark 2.2. The family $\mathcal{F}$ is *coinitial* in $\mathcal{T}$, that is, for every $T \in \mathcal{T}$ there are $S \in \mathcal{F}$ and an epimorphism $\phi\colon S \to T$.

Proposition 2.3. *The family $\mathcal{T}$ is a projective Fraïssé family.*

Proof. JPP: Take $S_1, S_2 \in \mathcal{T}$. Then the tree T equal to the disjoint union of S_1 and S_2 with their roots identified, together with the natural projections from T onto S_1 and from T onto S_2, witness the JPP.

AP: Take $P, Q, S \in \mathcal{T}$ together with epimorphisms $\phi_1\colon Q \to P$ and $\phi_2\colon S \to P$. Without loss of generality, as $\mathcal{F}$ is coinitial in $\mathcal{T}$, Q and S are in $\mathcal{F}$.

Let b be a branch in Q, and let $a = \phi_1(b)$. Note that a is an initial segment of a branch of P. Consider any branch c in S such that $a \subseteq \phi_2(c)$. Take a chain d_b and R-preserving maps ψ_1 and ψ_2 defined on d_b (we do not require them to be surjective) such that $\psi_1(d_b) = b$, $\psi_2(d_b) \subseteq c$, and for every $t \in d_b$, $\phi_1 \circ \psi_1(t) = \phi_2 \circ \psi_2(t)$.

We get d_b for every branch b in Q, and we get d_b for every branch b in S. Without loss of generality, all chains d_b are of the same length. Let $T \in \mathcal{F}$ be the disjoint union of chains d_b with their roots identified for b a branch in Q or in S. The functions ψ_1 and ψ_2 are well defined on T, ψ_1 is onto Q, ψ_2 is onto S, and $\phi_1 \circ \psi_1 = \phi_2 \circ \psi_2$. ∎

By Theorem 1.3, there exists a unique Fraïssé limit of $\mathcal{T}$, which we denote by $\mathbb{L} = (\mathbb{L}, R^{\mathbb{L}})$.

The following remark justifies that we can work only with the family $\mathcal{F}$.

Remark 2.4. From Remark 2.2 and Proposition 2.3, it follows that $\mathcal{F}$ is a projective Fraïssé family, and by Theorem 1.3, the projective Fraïssé limit of $\mathcal{F}$ is isomorphic to the one of $\mathcal{T}$.

For a topological $\mathcal{L}$-structure $\mathbb{X}$, we define $R_S^{\mathbb{X}}$ to be the *symmetrization* of $R^{\mathbb{X}}$, that is, $R_S^{\mathbb{X}}(s,t)$ if and only if $R^{\mathbb{X}}(s,t)$ or $R^{\mathbb{X}}(t,s)$, for every $s,t \in \mathbb{X}$.

THEOREM 2.5. *The relation $R_S^{\mathbb{L}}$ is an equivalence relation which has only one-element and two-element equivalence classes.*

Proof. To show that $R_S^{\mathbb{L}}$ is reflexive, take $x \in \mathbb{L}$. From (L2) in the definition of the projective Fraïssé limit it follows that for every clopen $U \subseteq \mathbb{L}$ such that $x \in U$, there is $T \in \mathcal{F}$ and an epimorphism $\phi\colon \mathbb{L} \to T$ refining the partition $\{U, \mathbb{L} \setminus U\}$. By the definition of an epimorphism, there are $x_U, y_U \in U$ such that $R^{\mathbb{L}}(x_U, y_U)$. Since $R^{\mathbb{L}}$ is closed in $\mathbb{L} \times \mathbb{L}$, it follows that $R^{\mathbb{L}}(x,x)$, and therefore $R_S^{\mathbb{L}}(x,x)$.

Clearly, $R_S^{\mathbb{L}}$ is symmetric.

To finish the proof of the theorem, it suffices to show that for any p,q,r, pairwise different, we cannot have both $R_S^{\mathbb{L}}(p,q)$ and $R_S^{\mathbb{L}}(p,r)$. Suppose the opposite. Since each member of $\mathcal{F}$ is a tree, it cannot happen that $R^{\mathbb{L}}(q,p)$ and $R^{\mathbb{L}}(r,p)$. Therefore either $R^{\mathbb{L}}(p,q)$ and $R^{\mathbb{L}}(p,r)$, or $R^{\mathbb{L}}(q,p)$ and $R^{\mathbb{L}}(p,r)$. Consider a clopen partition P of $\mathbb{L}$ such that p,q,r are in different clopens of P. Using (L2), take $T \in \mathcal{F}$ and an epimorphism $\psi_1 : \mathbb{L} \to T$ refining P. Then $p' = \psi_1(p)$, $q' = \psi_1(q)$, and $r' = \psi_1(r)$ are pairwise different, and we have $R^T(p',q')$ (or $R^T(q',p')$, respectively) and $R^T(p',r')$. Take S which is equal to T with p' "doubled", i.e. let $S = T \cup \{\bar{p}'\}$, $R^S(\bar{p}',\bar{p}')$, $R^S(p',\bar{p}')$, $R^S(\bar{p}',r')$, and for $x,y \in T$, $(x,y) \neq (p',r')$, we let $R^S(x,y)$ if and only if $R^T(x,y)$. Then $\phi\colon S \to T$ that sends $\bar{p}'$ to p', and other points to themselves, is an epimorphism. Using the extension property, we get an epimorphism $\psi_2 : \mathbb{L} \to S$ such that $\psi_1 = \phi \circ \psi_2$. Then either $\psi_2(p) = p'$ or $\psi_2(p) = \bar{p}'$. Either option leads to a contradiction. ∎

Take the quotient $\mathbb{L}/R_S^{\mathbb{L}}$ and denote it by L. Let $\pi\colon \mathbb{L} \to L$ be the quotient map.

THEOREM 2.6. *The space L is the Lelek fan.*

In order to prove Theorem 2.6, we will show that L is a continuum, it embeds into the Cantor fan F, and has a dense set of endpoints.

LEMMA 2.7. *The space L is Hausdorff, compact, second-countable, and connected.*

Proof. Since $\mathbb{L}$ and $R_S^{\mathbb{L}}$ are compact and π is continuous, it follows that L is Hausdorff, compact, and second-countable, since $\mathbb{L}$ is such.

Suppose towards a contradiction that L is not connected. Let U be a clopen non-empty subset of L such that $L \setminus U$ is also non-empty. Let $V = \pi^{-1}(U)$. Let $T \in \mathcal{F}$ and let $\phi\colon \mathbb{L} \to T$ be an epimorphism refining the partition $\{V, \mathbb{L} \setminus V\}$. It follows that there are $x \in V$ and $y \in \mathbb{L} \setminus V$ such that $R^{\mathbb{L}}(x,y)$. Since $\pi(x) = \pi(y)$, we get a contradiction. ∎

We call a sequence (T_n, f_n) an *inverse sequence* if $T_n \in \mathcal{F}$ and $f_n\colon T_{n+1} \to T_n$ are epimorphisms for every n. We will denote by f^n_m the composition $f_m \circ \cdots \circ f_{n-1}$ whenever $m < n$, and $f^m_m = \mathrm{Id}_{T_m}$. If $\mathbb{T}$ is the inverse limit of (T_n, f_n), then there is a sequence of epimorphisms $f^\infty_n\colon \mathbb{T} \to T_n$ such that $f^n_m \circ f^\infty_n = f^\infty_m$. If (T_n, f_n) and (S_n, g_n) are two inverse sequences with inverse limits $\mathbb{T}$ and $\mathbb{S}$ respectively, and for every n there is an injective homomorphism $\iota_n\colon T_n \to S_n$ such that $\iota_n \circ f_n = g_n \circ \iota_{n+1}$, then there is a continuous homomorphic embedding $\iota_\infty\colon \mathbb{T} \to \mathbb{S}$ satisfying $\iota_n \circ f^\infty_n = g^\infty_n \circ \iota_\infty$.

Following the proof of [IS, Theorem 2.4], we can write $\mathbb{L}$ as the inverse limit of an inverse sequence (T_n, f_n) satisfying the following properties:

(1) For any $T \in \mathcal{F}$ there is an n and an epimorphism from T_n onto T.
(2) For any m, any $S, T \in \mathcal{F}$, and epimorphisms $\phi_1\colon T_m \to T$ and $\phi_2\colon S \to T$, there exists $m < n$ and an epimorphism $\phi_3\colon T_n \to S$ such that $\phi_1 \circ f^n_m = \phi_2 \circ \phi_3$.

For $T \in \mathcal{F}$ let as before $B(T)$ denote the set of branches of T.

By passing to a subsequence, we can assume that (T_n, f_n) moreover satisfies:

(3) For every $b \in B(T_{n+1})$ and $x \in b$, there is $x' \in b$, $x' \neq x$, such that $f_n(x) = f_n(x')$.
(4) For every $b \in B(T_n)$ there are $b_1 \neq b_2 \in B(T_{n+1})$ such that $f_n(b_1) = f_n(b_2) = b$.

Any sequence (T_n, f_n) that satisfies properties (1)–(4) above will be called a *Fraïssé sequence*.

Our goal now is to show the following proposition.

Proposition 2.8. *The continuum L can be embedded into the Cantor fan F.*

Let $\mathbb{I}$ be the inverse limit of any inverse sequence (I_n, h_n), where I_n is a finite chain and $h_n\colon I_{n+1} \to I_n$ is an epimorphism such that for every $x \in I_{n+1}$, there is $x' \in I_{n+1}$, $x' \neq x$, with $h_n(x) = h_n(x')$. Then it is easily seen that $R^{\mathbb{I}}_S$ has only one-element and two-element equivalence classes, and $\mathbb{I}/R^{\mathbb{I}}_S$ is homeomorphic to the unit interval $[0, 1]$.

The inverse limit of an inverse sequence (C_n, e_n), where C_n is a finite set and $e_n\colon C_{n+1} \to C_n$ is a surjection such that for every $x \in C_{n+1}$ there is $x' \in C_{n+1}$, $x' \neq x$, with $e_n(x) = e_n(x')$, is clearly homeomorphic to the Cantor set.

It follows that if $\mathbb{F}$ is the inverse limit of an inverse sequence (S_n, g_n) satisfying conditions (3) and (4) in the definition of a Fraïssé sequence and condition (5) below, then $R^{\mathbb{F}}_S$ has only one-element and two-element equivalence classes and $\mathbb{F}/R^{\mathbb{F}}_S$ is homeomorphic to the Cantor fan F.

(5) For every $b \in B(S_n)$ and $b' \in B(S_{n+1})$ such that $g_n(b') \subseteq b$, we have $g_n(b') = b$.

We will find an injective, continuous homomorphism $h\colon \mathbb{L} \to \mathbb{F}$, which will induce a topological embedding from L into F.

LEMMA 2.9. *Suppose that* (T_n, f_n) *is a Fraïssé sequence. Then there is an inverse sequence* (S_n, g_n) *satisfying* (3)–(5) *above such that* $T_n \subset S_n$ *and* $g_n \restriction T_{n+1} = f_n$ *for every* n. *In particular, the inclusions induce a continuous injective homomorphism* h *from the inverse limit* $\mathbb{L}$ *of* (T_n, f_n) *to the inverse limit* $\mathbb{F}$ *of* (S_n, g_n).

Proof. Let $S_0 = T_0$. Suppose that S_k and g_{k-1} have been constructed for $k \leq n$. We will construct S_{n+1} from T_{n+1} by adding nodes and branches, and we will define $g_n\colon S_{n+1} \to S_n$ to be equal to f_n when restricted to T_{n+1}. For every $b \in B(T_{n+1})$, let $b' \in B(S_n)$ be the branch such that $f_n(b) \subset b'$. Let e, e' denote the endpoints of b, b' respectively, and let $m_{b'} = f_n(e)$. For every $x \in b'$ such that $m_{b'} <_{S_n} x$, we will put two points $x_1 \neq x_2$ into S_{n+1} and set $R^{S_{n+1}}(x_1, x_2)$, $R^{S_{n+1}}(x_i, x_i)$, and $g_n(x_i) = x$ for $i = 1, 2$. If $R^{S_n}(m_{b'}, x)$, then $R^{S_{n+1}}(e, x_1)$. If $m_{b'} <_{S_n} x <_{S_n} y$ and $R^{S_n}(x, y)$, then $R^{S'_{n+1}}(x_2, y_1)$. Finally, for every branch c in $S_n \setminus T_n \cup \{r_{T_n}\}$, we will add two branches c_1, c_2 to S_{n+1} that map onto c under g_n and such that for every $x \in c$ there are $x' \neq x'' \in c_i$ such that $g_n(x') = g_n(x'') = x$ for $i = 1, 2$. ∎

Proof of Proposition 2.8. The continuous injective homomorphism h from Lemma 2.9 induces a continuous embedding between the respective quotients L and F. ∎

Finally, we show the density of endpoints of L. Let A be a topological $\mathcal{L}$-structure. We say that $K \subseteq A$ is *R-connected* if for any two non-empty, disjoint clopen subsets K_1, K_2 in K such that $K_1 \cup K_2 = K$, there are $x \in K_1$ and $y \in K_2$ such that $R^A(x, y)$ or $R^A(y, x)$. We again consider $\mathbb{L}$ as the inverse limit of a Fraïssé sequence (T_n, f_n). Let $r_n = r_{T_n}$ denote the root of T_n, and $r = (r_n)$ the top of $\mathbb{L}$. Recall that $\pi\colon \mathbb{L} \to L$ is the quotient map.

PROPOSITION 2.10. *The set* E *of all endpoints in* L *is dense in* L.

Proof. Let $U \subseteq L$ be open and non-empty. We will find an endpoint in U. Let $V = \pi^{-1}(U)$. Take n_1 such that there is $e_{n_1} \in T_{n_1}$ with $(f^{\infty}_{n_1})^{-1}(e_{n_1}) \subseteq V$. Let $T \in \mathcal{F}$, $\psi_1\colon T \to T_{n_1}$, and $x \in T$ be such that $\psi_1(x) = e_{n_1}$ and x is an endpoint of T (i.e. for no $y \in T$, $y \neq x$, do we have $R^T(x, y)$). Using the fact that (T_n, f_n) is a Fraïssé sequence, find n_2 and $\psi_2\colon T_{n_2} \to T$ with $f^{n_2}_{n_1} = \psi_1 \circ \psi_2$. Pick any endpoint $e_{n_2} \in T_{n_2}$ in the preimage of x by ψ_2. For $n > n_2$ inductively pick an endpoint e_n in T_n such that $f^n_{n-1}(e_n) = e_{n-1}$ and for $n < n_2$ let $e_n = f^{n_2}_n(e_{n_2}) = e_n$. Then $e = (e_n) \in V$, and therefore $\pi(e) \in U$. Moreover, e is not the root of $\mathbb{L}$ as e_{n_2} is not the root of T_{n_2}.

By property (2) in the definition of a Fraïssé sequence, $\pi^{-1}(\pi(r)) = \{r\}$ for r the root of $\mathbb{L}$. Consequently, $\pi(e) \neq \pi(r)$.

We show that $\pi(e) \in E$. Let $i\colon [0,1] \to L$ be a homeomorphic embedding such that $\pi(e) \in i(I)$. Suppose towards a contradiction that $\pi(e) \neq i(0)$ and $\pi(e) \neq i(1)$. Without loss of generality, $\pi(r) \notin i(I)$. Denote $X = \pi^{-1}(i([i^{-1}(\pi(e)), 1]))$, $Y = \pi^{-1}(i([0, i^{-1}(\pi(e))]))$, and $Z = \pi^{-1}(i([0,1]))$. All three sets X, Y, Z are compact, R-connected in $\mathbb{L}$, and $e \in X \cap Y$. Let $X_n = f_n^\infty(X)$, $Y_n = f_n^\infty(Y)$, and $Z_n = f_n^\infty(Z)$. All sets X_n, Y_n, Z_n are R-connected in T_n. Since $\pi(r) \notin i(I)$, there is $N > n_2$ such that whenever $n > N$, Z_n (and so Y_n and X_n) is contained in a single branch of T_n.

Let $x = (x_n) \in X \setminus Y$ and let $y = (y_n) \in Y \setminus X$. We notice that $e = (e_n) \in X \cap Y$. Then either for every $n > N$, $r_n <_{T_n} x_n <_{T_n} y_n <_{T_n} e_n$, or for every $n > N$, $r_n <_{T_n} y_n <_{T_n} x_n <_{T_n} e_n$. Without loss of generality, we may assume that the former holds. Since $x_n, e_n \in X_n$ for every n, R-connectivity of each X_n implies $y_n \in X_n$ for $n > N$. Therefore $y \in X$, which is a contradiction. ∎

2.2. Properties of the Lelek fan: projective universality and projective ultrahomogeneity. The main goal of this subsection is to prove Theorem 2.12. This is an analog of [IS, Theorem 4.4].

Let $\mathrm{Aut}(\mathbb{L})$ be the group of all automorphisms of $\mathbb{L}$, that is, of all homeomorphisms of $\mathbb{L}$ that preserve the relation $R^{\mathbb{L}}$. This is a topological group when equipped with the compact-open topology inherited from $H(\mathbb{L})$, the group of all homeomorphisms of the Cantor set underlying the structure $\mathbb{L}$. Since $R^{\mathbb{L}}$ is closed in $\mathbb{L} \times \mathbb{L}$, the group $\mathrm{Aut}(\mathbb{L})$ is closed in $H(\mathbb{L})$.

We will denote by $H(L)$ the group of homeomorphisms of the Lelek fan with the compact-open topology.

Remark 2.11.

(1) Every $h \in \mathrm{Aut}(\mathbb{L})$ induces a homeomorphism $h^* \in H(L)$ satisfying $h^* \circ \pi(x) = \pi \circ h(x)$ for $x \in \mathbb{L}$. The map $h \to h^*$ is injective and we will frequently identify $\mathrm{Aut}(\mathbb{L})$ with the corresponding subgroup $\{h^*\colon h \in \mathrm{Aut}(\mathbb{L})\}$ of $H(L)$.
(2) The group $\mathrm{Aut}(\mathbb{L})$ is non-trivial by projective ultrahomogeneity, which immediately implies that $H(L)$ is non-trivial.
(3) The compact-open topology on $\mathrm{Aut}(\mathbb{L})$ is finer than the topology on $\mathrm{Aut}(\mathbb{L})$ that is inherited from the compact-open topology on $H(L)$.

A continuum is *hereditarily unicoherent* if the intersection of any two subcontinua is connected. A *dendroid* is a hereditarily unicoherent and arcwise connected continuum. A point x of a dendroid X is a *ramification point* if there are $a, b, c \in X$ and arcs ax, bx, cx such that $ax \cap bx = \{x\}$, $bx \cap cx = \{x\}$, and $ax \cap cx = \{x\}$. A *fan* is a dendroid that has exactly one ramification point, called the *top*. A *smooth fan* X is a fan such that whenever $t_n \to t$, $t_n, t \in X$, then the sequence of arcs $t_n w$ converges to the arc tw (in the Hausdorff metric), where w is the top point of X. Smooth fans are exactly fans that can be embedded into the Cantor fan (see [CC, Proposition 4]). These are exactly non-degenarate subcontinua of the Cantor fan F that are not homeomorphic to the interval $[0,1]$.

We say that a continuous surjection $f\colon L \to X$, where X is a smooth fan, is *monotone on segments* if $f(v) = w$, where v is the top of L and w is the top of X, and for every $x, y \in L$ such that $x \in [v, y]$, we have $f(x) \in [w, f(y)]$.

Theorem 2.12.

(1) *Each smooth fan is a continuous image of the Lelek fan L via a map that is monotone on segments.*
(2) *Let X be a smooth fan with a metric d. If $f_1, f_2\colon L \to X$ are two continuous surjections that are monotone on segments, then for any $\epsilon > 0$ there exists $h \in \mathrm{Aut}(\mathbb{L})$ such that for all $x \in L$, we have $d(f_1(x), f_2 \circ h^*(x)) < \epsilon$.*

In order to prove Theorem 2.12, we will represent every smooth fan as a quotient of an inverse limit of elements in $\mathcal{F}$, and apply the following proposition by Irwin and Solecki.

Proposition 2.13 ([IS, Proposition 2.6]). *Let $\mathcal{G}$ be a projective Fraïssé family of finite topological $\mathcal{L}$-structures, and let $\mathbb{G}$ be its projective Fraïssé limit. Let X be a topological $\mathcal{L}$-structure such that any open cover of X is refined by an epimorphism onto a structure in $\mathcal{G}$. Then there is an epimorphism from $\mathbb{G}$ onto X.*

Moreover, we will show that the epimorphism between the limits as in Proposition 2.13 induces a continuous surjection monotone on segments between the respective continua.

Lemma 2.14. *Let $\epsilon > 0$. Let X be a smooth fan with the top w. Then there is $A \in \mathcal{F}$ and an open cover $\{U_a\}_{a\in A}$ of X such that*

(C1) *for each $a \in A$, $\mathrm{diam}(U_a) < \epsilon$,*
(C2) *for each $a, a' \in A$, if $U_a \cap U_{a'} \neq \emptyset$ then $R^A_S(a, a')$,*
(C3) *for each $x, y \in X$ with $y \in [w, x]$, if $y \in U_a$ and $x \in U_b$, but $\{x, y\} \not\subset U_a \cap U_b$ unless $a = b$, then $a \leq_A b$,*
(C4) *for every $a \in A$, there is $x \in X$ such that $x \in U_a \setminus \bigcup\{U_{a'} : a' \in A, a' \neq a\}$.*

Remark 2.15. Note that (C3) implies $w \in U_{r_A}$, where r_A is the root of A, and that if $a, a' \in A$ satisfy $R^A_S(a, a')$ then $U_a \cap U_{a'} \neq \emptyset$.

Proof of Lemma 2.14. We first show that the lemma holds for the Cantor fan F. Let $\{O_1, \ldots, O_n\}$ be an open $(\epsilon/2)$-cover of the unit interval $I = [0,1]$ such that for every i, j, $O_i \cap O_j \neq \emptyset$ if and only if $|i - j| \leq 1$, and for $x \in O_i \setminus O_j$ and $y \in O_j$ with $i < j$ we have $x < y$. Moreover we require $O_i \setminus O_j \neq \emptyset$ whenever $i \neq j$. Let $\{V_1, \ldots, V_m\}$ be a clopen $(\epsilon/2)$-cover of the Cantor set C. Then $\{O_i \times V_j : i = 1, \ldots, n,\ j = 1, \ldots, m\}$ is an open ϵ-cover of $C \times I$. Let $O \subseteq F$ be an open neighbourhood of the top w of F of the form $O = \bigcup_{j=1}^{m} O_1 \times V_j/\sim$, where $(a,b) \sim (c,d)$ if and only if either $a = c$ and $b = d$, or $b = d = 0$. The desired cover is then $\mathcal{V} = \{O\} \cup \{O_i \times V_j : i = 2, \ldots, n,\ j = 1, \ldots, m\}$ with $A = \{r\} \cup \{(i,j) : i = 2, \ldots, n,\ j = 1, \ldots, m\}$, where $R^A(r, (i,j))$ if and only if $j = 2$, and for $(i,j), (i',j') \in \{2, \ldots, n\} \times \{1, \ldots, m\}$, $R^A((i,j),(i',j'))$ if and only if $i = i'$ and $0 \leq j' - j \leq 1$.

If X is a smooth fan, we think of X as embedded in F and obtain the desired cover as $\{V \cap X : V \in \mathcal{V}\}$, and the structure A from the one defined for F. We can further arrange that all branches of A have the same length. ∎

Proposition 2.16. *For every smooth fan X, there exists a topological $\mathcal{L}$-structure $(\mathbb{X}, R^{\mathbb{X}})$ such that $R_S^{\mathbb{X}}$ has one-element and two-element equivalence classes and X is homeomorphic to $\mathbb{X}/R_S^{\mathbb{X}}$. Moreover, every finite open cover of $\mathbb{X}$ can be refined by an epimorphism onto a fan in $\mathcal{F}$.*

Proof. Let X be a smooth fan viewed as a subfan of the Cantor fan F. While proving Proposition 2.8, we already described how to obtain the Cantor fan as a quotient of a topological $\mathcal{L}$-structure.

Let C be the Cantor set viewed as the middle third Cantor set. Each point of C can be expanded in a ternary sequence $0.a_1a_2a_3\ldots$, where $a_i \in \{0,2\}$ for each i. Similarly, each point of $[0,1]$ can be expanded in a binary sequence $0.a'_1a'_2a'_3\ldots$, where $a'_i \in \{0,1\}$ for each i. Let $f : C \to [0,1]$ be given by $f(0.a_1a_2a_3\ldots) = 0.a'_1a'_2a'_3\ldots$, where $a'_n = 0$ when $a_n = 0$, and $a'_n = 1$ when $a_n = 2$. Consider $\mathrm{Id} \times f/\sim : C \times C/\sim \to F$, where $(a,b) \sim (c,d)$ if and only if either $a = c$ and $b = d$, or $b = d = 0$.

Let $\mathbb{X} = (\mathrm{Id} \times f/\sim)^{-1}(X)$. Set $R^{\mathbb{X}}((a,b),(c,d))$ if and only if $a = c$ and $b = d$, or $a = c$ and (b,d) is an interval removed from $[0,1]$ in the construction of C. Then $\mathbb{X} = (\mathbb{X}, R^{\mathbb{X}})$ is a topological $\mathcal{L}$-structure. Observe that $\mathbb{X}/R_S^{\mathbb{X}} = X$.

To prove the "moreover" part, observe that the following claim is true and can be proved analogously to Lemma 2.14.

Claim 2.17. *For every $\epsilon > 0$ there exist $A \in \mathcal{F}$ and an epimorphism $\phi : \mathbb{X} \to A$ such that for each $a \in A$, $\mathrm{diam}(\phi^{-1}(a)) < \epsilon$, where the diameter is taken with respect to some fixed compatible metric on $\mathbb{X}$.*

Now, if we have an open cover of $\mathbb{X}$, then since $\mathbb{X}$ is compact, by the Lebesgue covering lemma we can find an $\epsilon > 0$ such that the epimorphism guaranteed by Claim 2.17 is as required. ∎

Proof of Theorem 2.12. (1) Let X be a smooth fan and let $\mathbb{X}$ be as in Proposition 2.16. By Proposition 2.13, there is an epimorphism $f\colon \mathbb{L} \to \mathbb{X}$. This epimorphism induces a continuous surjection $\bar{f}$ from $L = \mathbb{L}/R_S^{\mathbb{L}}$ onto $X = \mathbb{X}/R_S^{\mathbb{X}}$. It remains to show that $\bar{f}$ is monotone on segments. Let $\pi : \mathbb{L} \to L$ be the quotient map. Clearly, $\bar{f}(v) = w$, where v and w are the tops of L and X respectively. Let $x, y \in \mathbb{L}$ be such that $\pi(x) \in [v, \pi(y)]$. We show that $\bar{f}(\pi(x)) \in [w, \bar{f}(\pi(y))]$. Let $T \in \mathcal{F}$ and let $\phi\colon \mathbb{X} \to T$ be an epimorphism that separates $f(x)$ and $f(y)$ and such that if $[w, \bar{f}(\pi(x))] \cap [w, \bar{f}(\pi(y))] = \{w\}$, then $\phi \circ f(x)$ and $\phi \circ f(y)$ are in different branches of T. Since $\pi(x) \in [v, \pi(y)]$ and $\phi \circ f$ is an epimorphism, we have $\phi \circ f(x) \leq_T \phi \circ f(y)$. Now, since ϕ is an epimorphism, we conclude that $\bar{f}(\pi(x)) \in [w, \bar{f}(\pi(y))]$.

(2) Take $A \in \mathcal{F}$ and an open ϵ-cover $\{U_a\}_{a\in A}$ of X as in Lemma 2.14. Using the Lebesgue covering lemma, find δ such that for every $M \subseteq X$ with $\operatorname{diam}(M) < \delta$ there exists $a \in A$ such that $M \subseteq U_a$. Since $f_1 \circ \pi$ and $f_2 \circ \pi$ are uniformly continuous on $\mathbb{L}$, there is $B \in \mathcal{F}$ and epimorphisms $\phi_i\colon \mathbb{L} \to B$, $i = 1, 2$, such that for $b \in B$, $\operatorname{diam}(f_i\pi\phi_i^{-1}(b)) < \delta$. Let A and $\psi_i\colon B \to A$ be defined as follows: $\psi_i(b) = a$ if and only if $f_i\pi\phi_i^{-1}(b) \subseteq U_a$ and whenever $f_i\pi\phi_i^{-1}(b) \subseteq U_{a'}$, then $R^A(a', a)$.

We show that ψ_i is an epimorphism for $i = 1, 2$. Firstly, ψ_i is onto. That follows from the fact that $\{U_a\colon a \in A\}$ and $\{f_i\pi\phi_i^{-1}(b)\colon b \in B_i\}$ are covers of X, and from (C4).

Secondly, let $b, b' \in B$ be such that $R^B(b, b')$. Since ϕ_i is an epimorphism, $\pi\phi_i^{-1}(b) \cap \pi\phi_i^{-1}(b') \neq \emptyset$, and consequently $f_i\pi\phi_i^{-1}(b) \cap f_i\pi\phi_i^{-1}(b') \neq \emptyset$; therefore $U_{\psi_i(b)} \cap U_{\psi_i(b')} \neq \emptyset$. By (C2), $R^A(\psi_i(b), \psi_i(b'))$ or $R^A(\psi_i(b'), \psi_i(b))$. We will show that only the former is possible whenever $\psi_i(b) \neq \psi_i(b')$.

Suppose on the contrary that $R^A(\psi_i(b'), \psi_i(b))$. By the definition of ψ_i, there exists $x_i \in f_i\pi\phi_i^{-1}(b') \setminus U_{\psi_i(b)} \subseteq U_{\psi_i(b')} \setminus U_{\psi_i(b)}$. Let $s_i, s_i' \in \mathbb{L}$ be such that $s_i \in [r, s_i']$, where r is the top of $\mathbb{L}$, $\phi_i(s_i) = b$, $\phi_i(s_i') = b'$, and $f_i\pi(s_i') = x_i$. It follows that $\pi(s_i) \in [v, \pi(s_i')]$, and since f_i is monotone on segments, also $f_i\pi(s_i) \in [w, f_i\pi(s_i') = x_i]$. This however contradicts (C3) as $f_i\pi(s_i) \in U_{\psi_i(b)}$ and $f_i\pi(s_i') \in U_{\psi_i(b')}$.

We proved that ψ_i's are surjective homomorphisms. By Remark 2.1, they are automatically epimorphisms.

Finally, by (L3), there exists $h \in \operatorname{Aut}(\mathbb{L})$ such that $\psi_1 \circ \phi_1 = \psi_2 \circ \phi_2 \circ h$. This shows that for each $y \in \mathbb{L}$, there is $a \in A$ with $f_1 \circ \pi(y), f_2 \circ \pi \circ h(y) \in U_a$. Hence for all $x \in L$, $d(f_1(x), f_2 \circ h^*(x)) < \epsilon$. ∎

Corollary 2.18. *The group* $\operatorname{Aut}(\mathbb{L})$ *is dense in* $H(L)$.

Proof. In (2) of Theorem 2.12 take $X = L$, an arbitrary $f_1 \in H(L)$, and $f_2 = \mathrm{Id}$. ∎

A metric space X is *uniformly pathwise connected* if there exists a family P of paths in X such that

(1) for $x, y \in X$ there is a path in P joining x and y, and
(2) for every $\epsilon > 0$ there is a positive integer n such that each path in P can be partitioned into n pieces of diameter at most ϵ.

Kuperberg [K] showed that the continuous images of the Cantor fan are precisely the uniformly pathwise connected continua.

Since the Lelek fan is clearly uniformly pathwise connected, it is a continuous image of the Cantor fan, and since the Cantor fan is a continuous image of the Lelek fan (by the first part of Theorem 2.12), we obtain the following corollary.

Corollary 2.19. *The continuous images of the Lelek fan are precisely the uniformly pathwise connected continua.*

3. The homeomorphism group of the Lelek fan

3.1. Connectivity properties of $H(L)$. We show that $H(L)$, the homeomorphism group of the Lelek fan L, is totally disconnected (Proposition 3.1) and is generated by every neighbourhood of the identity (Corollary 3.3). A topological space X is *totally disconnected* if for any $x, y \in X$ there is a clopen set $U \subseteq X$ such that $x \in U$ and $y \in X \setminus U$. Note that this implies that every subspace of X containing more than one element is not connected (in the literature, the latter property is often used as a definition of being totally disconnected).

We say that a metric group (G, d) is *generated by every neighbourhood of the identity* if for every $\epsilon > 0$ and $h \in G$ there are homeomorphisms $f_1, \ldots, f_n \in G$ such that $d(f_i, \mathrm{Id}) < \epsilon$ for every i, and $h = f_1 \circ \cdots \circ f_n$. The definition of being generated by every neighbourhood of the identity naturally extends to topological groups, but we will only need it in the context of metric groups.

Lewis [Le1] showed that the homeomorphism group of the pseudo-arc is generated by every neighbourhood of the identity. However, it is not known whether that group is totally disconnected (see [Le2, Question 6.14]). There are examples of totally disconnected *Polish groups* (separable and completely metrizable topological groups) that are generated by every neighbourhood of the identity. The first such example, solving Problem 160 in the Scottish Book (see [M]), posed by Mazur, asking whether a complete metric group that is generated by every neighbourhood of the identity must be connected, was given by Stevens [S]; another example was presented by Hjorth [H].

The groups constructed by Stevens and Hjorth are algebraically subgroups of the additive group of real numbers. Our example is different. The group $H(L)$ is non-abelian, because it is non-trivial (Remark 2.11(2)) and has non-trivial conjugacy classes (Theorem 3.8). Moreover, it is explicitly given as the homeomorphism group of a continuum. Lewis–Zhou [LZ, Question 5] asked whether the homeomorphism group of a continuum that is generated by every neighbourhood of the identity has to be connected. Our example shows that the answer is negative.

As in Subsection 1.1, let C be the Cantor set, let F be the Cantor fan with the top point v, and let $\pi_1\colon F \setminus \{v\} \to C$ be the projection.

Proposition 3.1. *The group $H(L)$ is totally disconnected.*

Proof. Let $h_1 \neq h_2 \in H(L)$. We show that there is a clopen set A in $H(L)$ such that $h_1 \in A$ and $h_2 \notin A$. Since the set of endpoints E is dense in L and $h_1 \neq h_2$, there is $e \in E$ such that $h_1(e) \neq h_2(e)$. Let U_0 be a clopen set in C such that $\pi_1(h_1(e)) \in U_0$ and $\pi_1(h_2(e)) \notin U_0$, and let $U = \pi^{-1}(U_0) \cap E$. Then U is a clopen set in E. Since $H(L) \to E$, $h \mapsto h(e)$, is continuous, $A = \{h \in H(L)\colon h(e) \in U\}$ is a clopen set in $H(L)$ such that $h_1 \in A$ and $h_2 \notin A$. ∎

Fix a compatible metric d on L. Denote the corresponding supremum metric on $H(L)$ by d_{sup}. A homeomorphism $h \in H(L)$ is called an *ϵ-homeomorphism* if $d_{\mathrm{sup}}(h, \mathrm{Id}) < \epsilon$.

Theorem 3.2. *For every $\epsilon > 0$ and $h \in \mathrm{Aut}(\mathbb{L})$ there exist $f_1, \ldots, f_n$ in $\mathrm{Aut}(\mathbb{L})$ such that $h = f_1 \circ \cdots \circ f_n$ and $f_0^*, \ldots, f_n^*$ are ϵ-homeomorphisms.*

Proof. Let $\mathcal{B}_0$ be an open cover of L that consists of sets of diameter $< \epsilon/2$. Let $\mathcal{B} = \{\pi^{-1}(B)\colon B \in \mathcal{B}_0\}$ be an open cover of $\mathbb{L}$, where $\pi\colon \mathbb{L} \to L$ is the quotient map. Let $S \in \mathcal{F}$ and $\alpha\colon \mathbb{L} \to S$ be an epimorphism that refines $\mathcal{B}$. Note that for $s, s' \in S$,

$$(\triangle) \qquad R^S(s, s') \to \mathrm{diam}(\pi \circ \alpha^{-1}(s) \cup \pi \circ \alpha^{-1}(s')) < \epsilon$$

since $\pi \circ \alpha^{-1}(s) \cap \pi \circ \alpha^{-1}(s') \neq \emptyset$.

Using the uniform continuity of h and the Lebesgue covering lemma, find a finite open cover $\mathcal{U}$ of $\mathbb{L}$ refining $\mathcal{C}$ such that $h(\mathcal{U}) = \{h(U)\colon U \in \mathcal{U}\}$ also refines $\mathcal{C}$. Let $T \in \mathcal{F}$ and let $\gamma\colon \mathbb{L} \to T$ be an epimorphim refining $\mathcal{U}$. Then also $\mathcal{D} = \{\gamma^{-1}(t)\colon t \in T\}$ and $h(\mathcal{D}) = \{h \circ \gamma^{-1}(t)\colon t \in T\}$ both refine $\mathcal{C} = \{\alpha^{-1}(s)\colon s \in S\}$. Denote by β the surjection from T onto S such that $\alpha = \beta \circ \gamma$. We see that β is an epimorphism, since α and γ are.

Let $\beta_0 = \alpha \circ h \circ \gamma^{-1}$ and let $\gamma_0 = \gamma \circ h^{-1}$. Note that β_0 is an epimorphism and $\alpha = \beta_0 \circ \gamma_0$.

Without loss of generality, we can assume the following property:

$(*)$ For every branch in S there are at least $k+1$ branches in T that map onto the given branch under β_0.

If the original fan T does not have this property, we take T' and $\phi\colon T' \to T$ such that for every branch b in T there are $k+1$ branches in T' that are mapped by ϕ onto b. We apply the extension property to ϕ and γ_0, and get $\psi\colon \mathbb{L} \to T'$ such that $\gamma_0 = \phi \circ \psi$. We replace T by T', γ_0 by ψ, β_0 by $\beta_0 \circ \phi$, γ by $\psi \circ h$, and β by $\alpha \circ h^{-1} \circ \psi^{-1}$.

It is enough to construct epimorphisms $\beta_1, \ldots, \beta_n = \beta\colon T \to S$, for some n, such that for every $0 \le i < n$ and for every $t \in T$, $R^S(\beta_i(t), \beta_{i+1}(t))$ or $R^S(\beta_{i+1}(t), \beta_i(t))$. Then using the extension property, we find $\gamma_1, \ldots, \gamma_n = \gamma$ such that $\alpha = \beta_i \circ \gamma_i$, $i = 1, \ldots, n$, while projective ultrahomogeneity then provides us with $h = h_0, h_1, \ldots, h_{n-1}, h_n = \mathrm{Id} \in \mathrm{Aut}(\mathbb{L})$ such that $\gamma = \gamma_i \circ h_i$. For each automorphism h_i, let h_i^* denote the corresponding homeomorphism of L, let $f_i = h_{i-1}^* \circ (h_i^*)^{-1}$, $i = 1, \ldots, n$. Clearly, the composition $f_1 \circ \cdots \circ f_n$ is equal to h, and each f_i is an ϵ-homeomorphism. Indeed, for every $x \in \mathbb{L}$ and $i = 0, 1, \ldots, n-1$, we have

$$R^S(\alpha \circ h_i(x), \alpha \circ h_{i+1}(x)) \quad \text{or} \quad R^S(\alpha \circ h_{i+1}(x), \alpha \circ h_i(x)),$$

since

$$\alpha \circ h_i(x) = \beta_i \circ \gamma_i \circ h_i(x) = \beta_i \circ \gamma(x)$$

and

$$R^S(\beta_i(t), \beta_{i+1}(t)) \quad \text{or} \quad R^S(\beta_{i+1}(t), \beta_i(t))$$

for every $t \in T$. By $(\triangle)$, for every $x \in \mathbb{L}$ we get

$$\mathrm{diam}\big(\pi \circ \alpha^{-1}(\alpha \circ h_i(x)) \cup \pi \circ \alpha^{-1}(\alpha \circ h_{i+1}(x))\big) < \epsilon,$$

and therefore

$$d_{\sup}(h_i^*, h_{i+1}^*) = d_{\sup}(h_i^* \circ (h_{i+1}^*)^{-1}, \mathrm{Id}) < \epsilon.$$

Enumerate all branches in S as $c_1, \ldots, c_k$ and all branches in T as $d_1, \ldots, d_l$ in such a way that

$(**)$ For every $1 \le i \le k$, $\beta \restriction d_i$ is onto c_i.

Let $\beta_0(d_1) = (c(0), c(1), \ldots, c(m_1)) \subseteq c$ and $\beta(d_1) = (c_1(0), c_1(1), \ldots, c_1(m_2)) \subseteq c_1$. We construct $\beta_1, \ldots, \beta_{n_1}$ for $n_1 = m_1 + m_2$. For $i = 1, \ldots, m_1$, let

$$\beta_i(t) = \begin{cases} c(m_1 - i) & \text{if } t \in d_1 \text{ and } \beta_{i-1}(t) = c(m_1 - i + 1), \\ \beta_{i-1}(t) & \text{otherwise.} \end{cases}$$

For $i = 1, \ldots, m_2$, let

$$\beta_{m_1+i}(t) = \begin{cases} c_1(i) & \text{if } t \in d_1 \text{ and } \beta(t) \in \{c_1(i), \ldots, c_1(m_2)\}, \\ \beta_{m_1+i-1}(t) & \text{otherwise.} \end{cases}$$

We continue in the same manner for $2, \ldots, l$ and construct $\beta_{n_1+1}, \ldots, \beta_{n_2}$, $\ldots, \beta_{n_{l-1}+1}, \ldots, \beta_{n_l}$. By $(*)$ and $(**)$, each β_i is onto. All β_i's are epimorphisms and they satisfy the required condition: for every $0 \leq i < n$ and for every $t \in T$, we have $R^S(\beta_i(t), \beta_{i+1}(t))$ or $R^S(\beta_{i+1}(t), \beta_i(t))$. ∎

Theorem 3.2 immediately yields the following corollaries. To obtain the first one, we also use Corollary 2.18, which says that $\mathrm{Aut}(\mathbb{L})$ is dense in $H(L)$.

Corollary 3.3. *The group $H(L)$ is generated by every neighbourhood of the identity.*

Corollary 3.4. *The group $H(L)$ has no proper open subgroup.*

A Polish group is *non-archimedean* if it contains a basis at the identity that consists of open subgroups. This class of groups is equal to the class of automorphism groups of countable model-theoretic structures.

Corollary 3.5. *The group $H(L)$ is not a non-archimedean group.*

The following is a classical theorem about locally compact groups.

Theorem 3.6 (van Dantzig; see [HR, (7.7)]). *A totally disconnected locally compact group admits a basis at the identity that consists of compact open subgroups.*

Since $H(L)$ is totally disconnected (by Proposition 3.1), the theorem above implies the following corollary.

Corollary 3.7. *The group $H(L)$ is not locally compact.*

3.2. Conjugacy classes of $H(L)$. The main result of this subsection is the following theorem.

Theorem 3.8. *The group of all homeomorphisms of the Lelek fan, $H(L)$, has a dense conjugacy class.*

This will follow from Theorem 3.9.

Theorem 3.9. *The group of all automorphisms of $\mathbb{L}$, $\mathrm{Aut}(\mathbb{L})$, has a dense conjugacy class.*

Let us first see how Theorem 3.9 implies Theorem 3.8.

Proof of Theorem 3.8. As noticed in Remark 2.11, $\mathrm{Aut}(\mathbb{L})$ can be identified with a subgroup of $H(L)$ and its topology is finer than the one inherited from $H(L)$. From Corollary 2.18, $\mathrm{Aut}(\mathbb{L})$ is a dense subset of $H(L)$. From these observations, a set which is dense in $\mathrm{Aut}(\mathbb{L})$ is also dense in $H(L)$. ∎

To show Theorem 3.9, we use the criterion stated in Proposition 3.10 below. The proof of this criterion is given in [Kw, Theorem A1], and it is an analog of a theorem due to Kechris–Rosendal [KR] in the context of (injective) Fraïssé theory.

Let $\mathcal{G}$ be a projective Fraïssé family of finite $\mathcal{L}_0$-structures, for some language $\mathcal{L}_0$, with the limit $\mathbb{G}$. Let s be a binary relation symbol and let $\mathcal{L}_1$

be the language $\mathcal{L}_0 \cup \{s\}$. Define a class $\mathcal{G}^+$ of finite $\mathcal{L}_1$-structures as follows:

$$\mathcal{G}^+ = \{(A, s^A) \colon A \in \mathcal{G} \text{ and there are } \phi \colon \mathbb{G} \to A \text{ and } f \in \mathrm{Aut}(\mathbb{G}) \\ \text{such that } \phi \colon (\mathbb{G}, \mathrm{graph}(f)) \to (A, s^A) \text{ is an epimorphism}\},$$

where $\mathrm{graph}(f)$ is viewed as a closed relation on $\mathbb{G}$: $\mathrm{graph}(f)(x, y)$ if and only if $f(x) = y$.

As in Subsection 1.2, say that $\mathcal{G}^+$ has the joint projection property (JPP) if and only if for every $(A, s^A), (B, s^B) \in \mathcal{G}^+$ there is $(C, s^C) \in \mathcal{G}^+$ and epimorphisms from (C, s^C) onto (A, s^A) and from (C, s^C) onto (B, s^B).

Proposition 3.10 ([Kw]). *The group* $\mathrm{Aut}(\mathbb{G})$ *has a dense conjugacy class if and only if* $\mathcal{G}^+$ *has the JPP.*

The lemma below is a general fact of the projective Fraïssé theory.

Lemma 3.11. *Let* $\mathcal{G}$ *be a projective Fraïssé family with the limit* $\mathbb{G}$. *Then* $(T, s^T) \in \mathcal{G}^+$ *if and only if there are* $S \in \mathcal{G}$ *and epimorphisms* $p_1 \colon S \to T$ *and* $p_2 \colon S \to T$ *such that* $s^T = \{(p_1(x), p_2(x)) \colon x \in S\}$.

Proof. ($\Leftarrow$) Let S, p_1, p_2 be as in the hypothesis. Let $\phi \colon \mathbb{G} \to S$ be any epimorphism (it exists by the universality property (L1)). Let $\phi_1 = p_1 \circ \phi$ and let $\phi_2 = p_2 \circ \phi$. Using the projective ultrahomogeneity (L3), get $f \in \mathrm{Aut}(\mathbb{G})$ such that $\phi_1 \circ f = \phi_2$. Then $\phi_1 \colon (\mathbb{G}, \mathrm{graph}(f)) \to (T, s^T)$ is an epimorphism. So $(T, s^T) \in \mathcal{G}^+$.

($\Rightarrow$) Let $(T, s^T) \in \mathcal{G}^+$. Let $\psi \colon (\mathbb{G}, f) \to (T, s^T)$ be an epimorphism. Denote $\phi_1 = \psi$ and $\phi_2 = \phi_1 \circ f$. Let X be the common refinement of the partitions $\phi_1^{-1}(T)$ and $\phi_2^{-1}(T)$. Let $\alpha \colon \mathbb{G} \to S$, $S \in \mathcal{G}$, be an epimorphism refining the partition X. Then $p_1 \colon S \to T$ satisfying $\phi_1 = p_1 \circ \alpha$ and $p_2 \colon S \to T$ satisfying $\phi_2 = p_2 \circ \alpha$ are as required. ∎

Every fan in $\mathcal{F}$ is specified by its height and its width. Recall that we assumed that all branches in a given fan have the same length. The *height* of a fan is the number of elements in a branch minus one (we do not count the root). The *width* of a fan is the number of its branches. Let T be a fan of height k and width n. If b is a branch in a fan T of height k, we denote by $b(j)$ the jth element of b for $j = 0, 1, \ldots, k$ (where $b(0)$ is the root). We say that a binary relation s^T on T is *surjective* if for every $t \in T$ there are $r, s \in T$ such that $s^T(t, r)$ and $s^T(s, t)$. Let s^T be a surjective relation on T. Let $b_1, \ldots, b_n$ be the list of all branches of T, and let r_T be the root of T. We say that $(x_1, y_1) \in T^2$ is s^T-*adjacent* to $(x_0, y_0) \in T^2$ if $R^T(x_0, x_1)$, $R^T(y_0, y_1)$, $s^T(x_0, y_0)$, and $s^T(x_1, y_1)$. We say that (c, d) is s^T-*connected* to (a, b) if there are l and $(x_0, y_0), (x_1, y_1), \ldots, (x_l, y_l) \in T^2$ such that $(x_0, y_0) = (a, b)$, $(x_l, y_l) = (c, d)$, and for each i, (x_{i+1}, y_{i+1}) is s^T-adjacent to (x_i, y_i).

LEMMA 3.12. *We have* $(T, s^T) \in \mathcal{F}^+$ *if and only if* s^T *is surjective,* $s^T(r_T, r_T)$*, and for every* (x, y)*, whenever* $s^T(x, y)$*, then* (x, y) *is* s^T*-connected to* (r_T, r_T)*.*

Proof. $(\Leftarrow)$ We define S, p_1, p_2 as in Lemma 3.11. Let k be the height of T. For every (x, y) such that $s^T(x, y)$ we pick a chain of length $2k+2$ and denote it by $b_{(x,y)}$. Let S be the disjoint union of all chains $b_{(x,y)}$ with their roots identified. Now we define p_1 and p_2. Fix (x, y) such that $s^T(x, y)$. Fix a sequence $(r_T, r_T) = (x_0, y_0), (x_1, y_1), \dots, (x_l, y_l) = (x, y)$ witnessing that (x, y) is s^T-connected to (r_T, r_T). We let $p_1(b_{(x,y)}(i)) = x_i$ and $p_2(b_{(x,y)}(i)) = y_i$ whenever $i \leq l$. We let $p_1(b_{(x,y)}(i)) = x$ and $p_2(b_{(x,y)}(i)) = y$ whenever $i > l$.

$(\Rightarrow)$ Let $(T, s^T) \in \mathcal{F}^+$ and let S, p_1, p_2 be as in Lemma 3.11. Clearly $s^T(r_T, r_T)$. Take (x, y) such that $s^T(x, y)$, and let $s \in S$ be such that $(x, y) = (p_1(s), p_2(s))$. Let b be a branch in S connecting r_S to s, i.e. $r_S = s_0 = b(0), s_1 = b(1), \dots, s_l = b(l)$. Then the sequence $(r_T, r_T) = (p_1(s_0), p_2(s_0)), (p_1(s_1), p_2(s_1)), \dots, (p_1(s_l), p_2(s_l)) = (x, y)$ witnesses that (x, y) is s^T-connected to (r_T, r_T). ∎

PROPOSITION 3.13. *The family* $\mathcal{F}^+$ *has the JPP.*

Proof. Let $(T_1, s^{T_1}), (T_2, s^{T_2}) \in \mathcal{F}^+$. For the JPP, take T to be the disjoint union of T_1 and T_2 with their respective roots identified. For $x, y \in T$ we let $s^T(x, y)$ if and only if either $x, y \in T_1$ and $s^{T_1}(x, y)$, or $x, y \in T_2$ and $s^{T_2}(x, y)$. Then, using Lemma 3.12, we conclude that $(T, s^T) \in \mathcal{F}^+$. Moreover, $\phi_1 \colon (T, s^T) \to (T_1, s^{T_1})$ such that $\phi_1 \restriction T_1 = \mathrm{Id}_{T_1}$ and $\phi_1 \restriction T_2$ map onto the root, and $\phi_2 \colon (T, s^T) \to (T_2, s^{T_2})$ such that $\phi_2 \restriction T_2 = \mathrm{Id}_{T_2}$ and $\phi_2 \restriction T_1$ map onto the root, are epimorphisms. ∎

3.3. Simplicity of $H(L)$. A group is *simple* if it has no non-trivial proper normal subgroups. In this subsection, we show that the group $H(L)$ is simple. Anderson [A] gave a criterion for groups of homeomorphisms that implies their simplicity. Anderson's criterion is satisfied for instance by the homeomorphism group of the Cantor set, the homeomorphism group of the universal curve, and by the group of all orientation-preserving homeomorphisms of S^2. As we will see, a modification of that criterion applies to $H(L)$.

There are various recent results concerning simplicity of topological groups. Tent–Ziegler [TZ1] showed that the isometry group of the bounded Urysohn metric space is simple, and Macpherson–Tent [MT] gave a general result on simplicity of automorphism groups of countable ultrahomogeneous structures whose classes of finite substructures have the free amalgamation property. This last result was later generalized by Tent–Ziegler [TZ2], who also showed that the isometry group of the Urysohn space modulo the normal subgroup of bounded isometries is a simple group.

Recall from Subsection 1.1 that E denotes the set of endpoints of L, C is the Cantor set, F is the Cantor fan, and $\pi_1\colon F\setminus\{v\}\to C$, $\pi_2\colon F\to[0,1]$ are projections. Let v denote the top of L. Define

$$\mathcal{K}=\{K\subseteq L\colon \text{both } K \text{ and } (L\setminus K)\cup\{v\} \text{ are closed and different from } L\}.$$

The properties listed below follow immediately from the definition of $\mathcal{K}$.

REMARK 3.14.

(1) Let $K\in\mathcal{K}$. Then for any $e\in E$, we have either $[v,e]\subseteq K$ or $[v,e]\subseteq(L\setminus K)\cup\{v\}$.
(2) Whenever $K\in\mathcal{K}$ and $g\in H(L)$, then $g(K)\in\mathcal{K}$.
(3) If $K\in\mathcal{K}$, then $K\setminus\{v\}$ is an open non-empty set in L. Moreover $\{K\setminus\{v\},L\setminus K\}$ is a clopen decomposition of $L\setminus\{v\}$.
(4) If $K\in\mathcal{K}$, then $(L\setminus K)\cup\{v\}\in\mathcal{K}$. If $K,K'\in\mathcal{K}$ are such that $K\cup K'\neq L$, then $K\cup K'\in\mathcal{K}$. If $K,K'\in\mathcal{K}$ are such that $K\cap K'\neq\{v\}$, then $K\cap K'\in\mathcal{K}$.
(5) If $X\subseteq C$ is a clopen set such that $\pi_1^{-1}(X)\cap L$ and $\pi_1^{-1}(C\setminus X)\cap L$ are non-empty, then $(\pi_1^{-1}(X)\cap L)\cup\{v\}\in\mathcal{K}$.

Let G^0 denote the subgroup of $H(L)$ consisting of those $g\in H(L)$ that are the identity when restricted to some $K\in\mathcal{K}$. We say that $g\in G^0$ is *supported* on $K\in\mathcal{K}$ if $g\restriction(L\setminus K)$ is the identity. For $K\in\mathcal{K}$ let $E(K)$ denote the set of endpoints of K. Observe that $E\cap K=E(K)$ by Remark 3.14(1).

LEMMA 3.15. *The family $\mathcal{K}$ satisfies the following properties:*

(1) *each $K\in\mathcal{K}$ is homeomorphic to L,*
(2) *for every $h\neq\mathrm{Id}\in H(L)$ there is $K\in\mathcal{K}$ such that*

$$K\cap(h(K)\cup h^{-1}(K))=\{v\}.$$

Proof. (1) Let $K\in\mathcal{K}$. To show that K is homeomorphic to L, it is enough to show that $E(K)$ is dense in K. Let $x\in K\setminus\{v\}$. There is a sequence (e_i) of endpoints of L that converges to x. By passing to a subsequence, we can assume that either every e_i is in K, or every e_i is in $L\setminus K$. Since $(L\setminus K)\cup\{v\}$ is closed and $x\neq v$, the latter possibility cannot be true. Therefore, since $E\cap K=E(K)$, (e_i) is a sequence of endpoints of K and it converges to x. This shows that $E(K)$ is dense in $K\setminus\{v\}$. However, $\overline{K\setminus\{v\}}=K$, so $E(K)$ is also dense in K.

(2) Since E is dense in L, there is $e\in E$ such that $h(e)\neq e$. Consequently, $h([v,e])\cap[v,e]=\{v\}$ and $h^{-1}([v,e])\cap[v,e]=\{v\}$. Let $M_1,M_2,M_3\in\mathcal{K}$ be such that $M_1\cap M_2=\{v\}$, $M_1\cap M_3=\{v\}$, $e\in M_1$, $h(e)\in M_2$, and $h^{-1}(e)\in M_3$. Let $K=h^{-1}(M_2)\cap M_1\cap h(M_3)$. Then $K\in\mathcal{K}$, $K\subseteq M_1$, $h(K)\subseteq M_2$, and $h^{-1}(K)\subseteq M_3$. Therefore $K\cap(h(K)\cup h^{-1}(K))=\{v\}$. ∎

For $K \in \mathcal{K}$, define the *height* of K to be $\max(\pi_2(K))$. We say that a sequence $(K_i)_{i\in\mathbb{Z}}$ of elements of $\mathcal{K}$ is a *β-sequence* if (1) $\bigcup_{i\in\mathbb{Z}} K_i \in \mathcal{K}$ and $K_i \cap K_j = \{v\}$ for $i \neq j$, and (2) $\lim_{i\to\infty} \mathrm{ht}(K_i) = 0 = \lim_{i\to-\infty} \mathrm{ht}(K_i)$.

LEMMA 3.16. *For every $K \in \mathcal{K}$ there exist a β-sequence (K_i) with $\bigcup K_i = K$ and $\rho_1, \rho_2 \in G^0$ supported on K such that*

1. *$\rho_1(K_i) = K_{i+1}$ for each i;*
2. *$\rho_2\restriction K_0 = \rho_1\restriction K_0$, $\rho_2\restriction K_{2i} = \rho_1^{-2}\restriction K_{2i}$ for $i > 0$, and $\rho_2\restriction K_{2i-1} = \rho_1^2\restriction K_{2i-1}$ for $i > 0$;*
3. *if $\phi_i \in G^0$ is supported on K_i, for each i, then there exists $\phi \in G^0$ supported on K such that $\phi\restriction K_i = \phi_i\restriction K_i$ for every i.*

Proof. Given $K \in \mathcal{K}$, we first inductively construct a sequence $(K'_i)_{i\in\mathbb{N}}$ of elements of $\mathcal{K}$ such that $\bigcup_{i\in\mathbb{N}} K'_i = K$, $K'_i \cap K'_j = \{v\}$ for $i \neq j$, and $\lim_{i\to\infty} \mathrm{ht}(K'_i) = 0$. Fix a compatible metric on the Cantor set C such that $\mathrm{diam}(C) \leq 1$.

To construct K'_0, pick $e \in E(K)$ such that $\pi_2(e) < 2^{-1}$. Let $X \subseteq C$ be a clopen such that $\pi_1(e) \in X$ and $\mathrm{ht}(M) < 2^{-1}$, where $M = (\pi_1^{-1}(X) \cap L) \cup \{v\}$. Let $K'_0 = (K \setminus M) \cup \{v\}$. Then $K'_0 \in \mathcal{K}$ and $K \setminus K'_0 \neq \emptyset$ since $e \in K \setminus K'_0$. Note that $\mathrm{ht}((K \setminus K'_0) \cup \{v\}) = \mathrm{ht}(M) < 2^{-1}$.

Suppose that we have constructed $K'_0, K'_1, \ldots, K'_n$ such that (a) for every $i \neq j$, $i, j \leq n$, $K'_i \cap K'_j = \{v\}$ and $K \setminus \bigcup_{j\leq i} K'_j \neq \emptyset$, (b) for every $i \leq n$, $\mathrm{ht}(K'_i) < 2^{-i}$ and $\mathrm{ht}((K\setminus\bigcup_{j\leq i} K'_j)\cup\{v\}) < 2^{-(i+1)}$, and (c) for every $i \leq n$, $\mathrm{diam}(\pi_1(K \setminus \bigcup_{j\leq i} K'_j)) < 2^{-(i-1)}$.

Now we construct K'_{n+1} such that conditions (a)–(c), with n replaced by $n+1$, are fulfilled: Using that $(K \setminus \bigcup_{j\leq n} K'_j) \cup \{v\} \in \mathcal{K}$ and consequently $K\setminus\bigcup_{j\leq n} K'_j$ is open in L, pick $e \in E(K\setminus\bigcup_{j\leq n} K'_j)$ so that $\pi_2(e) < 2^{-(n+2)}$. Further let $X \subseteq C$ be a clopen such that $\pi_1(e) \in X$ and $\mathrm{ht}(M) < 2^{-(n+2)}$, where $M = (\pi_1^{-1}(X)\cap L)\cup\{v\}$. By shrinking M if necessary, we can assume $(M \cap K) \cup \bigcup_{j\leq n} K'_j \neq K$ and $\mathrm{diam}(\pi_1(M \setminus \{v\})) < 2^{-n}$. Let $K'_{n+1} = (K \setminus (\bigcup_{j\leq n} K'_j \cup M)) \cup \{v\}$. Then $K'_{n+1} \in \mathcal{K}$ is as required. In particular, $K \setminus \bigcup_{j\leq n+1} K'_j \neq \emptyset$, $\mathrm{ht}((K \setminus \bigcup_{j\leq n+1} K'_j) \cup \{v\}) = \mathrm{ht}(M) < 2^{-(n+2)}$, and $\mathrm{diam}(\pi_1(K \setminus \bigcup_{j\leq n+1} K'_j)) \leq \mathrm{diam}(\pi_1(M \setminus \{v\})) < 2^{-n}$.

The sequence $(K_i)_{i\in\mathbb{Z}}$ such that $K_0 = K'_0$, $K_{-i} = K'_{2i}$ for $i = 1, 2, \ldots,$ and $K_i = K'_{2i-1}$ for $i = 1, 2, \ldots$, is a β-sequence satisfying $\bigcup_{i\in\mathbb{Z}} K_i = K$.

We first show that (3) holds. Let ϕ_i be as in the assumptions. Let ϕ be such that $\phi\restriction K_i = \phi_i\restriction K_i$ and ϕ is the identity outside K. We want to show that ϕ is a homeomorphism. Clearly ϕ is a bijection. Since L is compact, it is enough to show that ϕ is continuous. Let $x \in L$. If $x \neq v$, then either $x \in K_i \setminus \{v\}$ for some i, or $x \in L \setminus K$. Since each of $K_i \setminus \{v\}$ and $L \setminus K$ is open, whenever (x_n) converges to x, then eventually $x_n \in K_i \setminus \{v\}$ for some

i or $x_n \in L \setminus K$, respectively. Therefore, eventually $\phi(x_n) \in K_i \setminus \{v\}$ for some i, or $\phi(x_n) \in L \setminus K$, respectively. Since each ϕ_i is continuous, $\phi(x_n)$ converges to $\phi(x)$. Now let $x = v$ and let (x_n) converge to v. We show that $\phi(x_n)$ converges to $v = \phi(v)$. Fix an open neighbourhood U of v. Since $\mathrm{ht}(K_i) \to 0$ both for $i \to \infty$ and for $i \to -\infty$, we can find $i_0 > 0$ such that when $i > i_0$ or $i < -i_0$, then $K_i \subseteq U$. By continuity of ϕ_i, find n_0 such that whenever $n > n_0$ and x_n is in one of K_i, $-i_0 \le i \le i_0$, or in $L \setminus K$, then $\phi(x_n) = \phi_i(x_n) \in U$, or $\phi(x_n) = x_n \in U$, respectively. Then since $\phi_i(K_i) \subseteq K_i$ for each i, whenever $n > n_0$ we have $\phi(x_n) \in U$. This shows the continuity of ϕ at v.

To show (1) we let $\rho_1^i\colon K_i \to K_{i+1}$ be any homeomorphism, which exists as all K_i's are homeomorphic to the Lelek fan. Let ρ_1 be such that $\rho_1{\restriction}K_i = \rho_1^i$, $i \in \mathbb{Z}$, and let ρ_1 be the identity outside K. Then similarly to the proof of (3), we can show that ρ_1 is a homeomorphism of L.

Having defined ρ_1, we set ρ_2 on each K_i, $i \ge 0$, as in (2), and we let ρ_2 be the identity otherwise. Then again as in the proof of (3), we show that ρ_2 is a homeomorphism of L. ■

Remark 3.17. Anderson [A] showed that whenever G is a group of homeomorphisms of a space X, and there exists a family $\mathcal{K}$ of closed sets that satisfies conditions similar to those given in Remark 3.14 and in Lemmas 3.15 and 3.16, then G is a simple group. He assumes that the sets K_i in the definition of a β-sequence are disjoint, and that for every open non-empty set $U \subseteq X$ there exists $K \in \mathcal{K}$ such that $K \subseteq U$, which is false in our situation. Nevertheless, it is enough to substitute this condition by (2) of Lemma 3.15.

Theorem 3.18. *The group of all homeomorphisms of the Lelek fan, $H(L)$, is simple.*

The proof of Theorem 3.18 will go along the lines of the proof of simplicity of homeomorphism groups studied by Anderson. We sketch it here for the reader's convenience, and for the details we refer to [A].

We need the following lemma (analogous to [A, Theorem I]).

Lemma 3.19. *Let $h \neq \mathrm{Id} \in H(L)$. Then every $g \in G^0$ is the product of four conjugates of h and h^{-1} (appearing alternately).*

Proof. Since any two elements of $\mathcal{K}$ are homeomorphic via a homeomorphism of L, it is enough to show that there exists $K_0 \in \mathcal{K}$ such that for any $g_0 \in G^0$ supported on K_0, g_0 is the product of four conjugates of h and h^{-1}.

By Lemma 3.15(2), there is $K \in \mathcal{K}$ such that $K \cap (h(K) \cup h^{-1}(K)) = \{v\}$. Let (K_i) be a β-sequence such that $\bigcup_i K_i = K$, and let ρ_1 and ρ_2 be as in (1) and (2) of Lemma 3.16. We show that K_0 is as required. Let $g_0 \in G^0$ be supported on K_0. For $i \ge 0$, let $\phi_i = \rho_1^i g_0 \rho_1^{-i}$, and let ϕ_i be the identity on K_i when $i < 0$. Take ϕ as in (3) of Lemma 3.16. Take $f = h^{-1}\phi^{-1}h\phi$. Note that f is supported on $K \cup h^{-1}(K)$, $f\restriction K = \phi\restriction K$, and $f\restriction(h^{-1}(K)) = (h^{-1}\phi^{-1}h)\restriction(h^{-1}(K))$. Take $\rho = h^{-1}\rho_2 h\rho_1^{-1}$. Note that ρ is supported on $K \cup h^{-1}(K)$, $\rho\restriction K = \rho_1^{-1}\restriction K$, and $\rho\restriction(h^{-1}(K)) = (h^{-1}\rho_2 h)\restriction(h^{-1}(K))$. Let $w = \rho^{-1}f^{-1}\rho f$. Then $w = (\rho^{-1}\phi^{-1}h^{-1}\phi\rho)(\rho^{-1}h\rho)(h^{-1})(\phi^{-1}h\phi)$, therefore it is a product of four conjugates of h and h^{-1}. Unraveling the definitions of ϕ, f, ρ, and w, as is done in [A], we get $w = g_0$. ∎

Proof of Theorem 3.18. Let $g \in H(L)$ and let $h \in H(L)$, $h \neq \mathrm{Id}$. We will show that g is the product of eight conjugates of h and h^{-1}. This will immediately imply that $H(L)$ is simple.

Let $K \in \mathcal{K}$ be such that $g(K) \cap K = \{v\}$ and $g(K) \cup K \neq L$. Take $\alpha \in H(L)$ such that $\alpha\restriction K = g\restriction K$, $\alpha\restriction g(K) = g^{-1}\restriction g(K)$, and α is equal to the identity outside $g(K) \cup K$. Notice that $\alpha, (\alpha^{-1}g) \in G^0$ and $g = \alpha(\alpha^{-1}g)$. By Lemma 3.19, g is the product of eight conjugates of h and h^{-1}. ∎

Remark 3.20. As in [A], one can modify the proof of Theorem 3.18 to show that whenever $g \in H(L)$ and $h \in H(L)$, $h \neq \mathrm{Id}$, then g is the product of six conjugates of h and h^{-1}.

Acknowledgements. A large portion of this work was done during the trimester program on 'Universality and Homogeneity' at the Hausdorff Research Institute for Mathematics in Bonn. We would like to thank the organizers: Alexander Kechris, Katrin Tent, and Anatoly Vershik for the opportunity to participate in the program. We would also like to thank the anonymous referee for numerous detailed suggestions that considerably helped us to improve the presentation of the paper.

References

[A] R. D. Anderson, *The algebraic simplicity of certain groups of homeomorphisms*, Amer. J. Math. 80 (1958), 955–963.

[BKn] D. Bartošová and A. Kwiatkowska, Aut($\mathbb{L}$) *is an oligomorphic group*, http://www.math.ucla.edu/~akwiatk2/.

[BYT] I. Ben Yaacov and T. Tsankov, *Weakly almost periodic functions, model-theoretic stability, and minimality of topological groups*, arXiv:1312.7757 (2013).

[BO] W. Bula and L. Oversteegen, *A characterization of smooth Cantor bouquets*, Proc. Amer. Math. Soc. 108 (1990), 529–534.

[C] W. Charatonik, *The Lelek fan is unique*, Houston J. Math. 15 (1989), 27–34.

[CC] J. J. Charatonik and W. J. Charatonik, *Images of the Cantor fan*, Topology Appl. 33 (1989), 163–172.

[CCM] J. J. Charatonik, W. J. Charatonik, and S. Miklos, *Confluent mappings of fans*, Dissertationes Math. 301 (1990).

[DZ] J. J. Dijkstra and L. Zhang, *The space of Lelek fans in the Cantor fan is homeomorphic to Hilbert space*, J. Math. Soc. Japan 62 (2010), 935–948.

[GM] E. Glasner and M. Megrelishvili, *New algebras of functions on topological groups arising from G-spaces*, Fund. Math. 201 (2008), 1–51.

[HR] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis. Vol. I*, 2nd ed., Grundlehren Math. Wiss. 115, Springer, Berlin, 1979.

[H] G. Hjorth, *A new zero-dimensional Polish group*, http://www.math.ucla.edu/~greg/research.html.

[IS] T. Irwin and S. Solecki, *Projective Fraïssé limits and the pseudo-arc*, Trans. Amer. Math. Soc. 358 (2006), 3077–3096.

[KOT] K. Kawamura, L. Oversteegen, and E. D. Tymchatyn, *On homogeneous totally disconnected 1-dimensional spaces*, Fund. Math. 150 (1996), 97–112.

[KR] A. S. Kechris and C. Rosendal, *Turbulence, amalgamation, and generic automorphisms of homogeneous structures*, Proc. London Math. Soc. 94 (2007), 302–350.

[K] W. Kuperberg, *Uniformly pathwise connected continua*, in: Studies in Topology (Charlotte, NC, 1974; dedicated to Math. Sect. Polish Acad. Sci.), Academic Press, New York, 1975, 315–324.

[Kw] A. Kwiatkowska, *Large conjugacy classes, projective Fraïssé limits, and the pseudo-arc*, Israel J. Math. 201 (2014), 75–97.

[L] A. Lelek, *On plane dendroids and their end points in the classical sense*, Fund. Math. 49 (1960/1961), 301–319.

[Le1] W. Lewis, *Most maps of the pseudo-arc are homeomorphisms*, Proc. Amer. Math. Soc. 91 (1984), 147–154.

[Le2] W. Lewis, *The pseudo-arc*, Bol. Soc. Mat. Mexicana (3) 5 (1999), 25–77.

[LZ] W. Lewis and Y. C. Zhou, *Continua whose homeomorphism groups are generated by arbitrarily small neighborhoods of the identity*, Topology Appl. 126 (2002), 409–417.

[MT] D. Macpherson and K. Tent, *Simplicity of some automorphism groups*, J. Algebra 342 (2011), 40–52.

[M] R. D. Mauldin (ed.), *The Scottish Book*, Birkhäuser, Boston, 1981.

[Me] M. Megrelishvili, *Topological transformation groups: selected topics*, in: Open Problems in Topology II, E. Pearl (ed.), Elsevier, Amsterdam, 2007, 423–437.

[S] T. C. Stevens, *Connectedness of complete metric groups*, Colloq. Math. 50 (1986), 233–240.

[TZ1] K. Tent and M. Ziegler, *The isometry group of the bounded Urysohn space is simple*, Bull. London Math. Soc. 45 (2013), 1026–1030.

[TZ2] K. Tent and M. Ziegler, *On the isometry group of the Urysohn space*, J. London Math. Soc. 87 (2013), 289–303.

Dana Bartošová
Instituto de Matematica e Estatística
Universidade de São Paulo
São Paulo, Brazil
E-mail: dana@ime.usp.br

Aleksandra Kwiatkowska
Department of Mathematics
University of California
Los Angeles, CA, U.S.A.
E-mail: akwiatk2@math.ucla.edu

Received 14 January 2014;
in revised form 27 October 2014 and 2 February 2015