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On splitting infinite-fold covers

by

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Abstract. Let X be a set, κ be a cardinal number and let $\mathcal{H}$ be a family of subsets of X which covers each $x \in X$ at least κ-fold. What assumptions can ensure that $\mathcal{H}$ can be decomposed into κ many disjoint subcovers?

We examine this problem under various assumptions on the set X and on the cover $\mathcal{H}$: among other situations, we consider covers of topological spaces by closed sets, interval covers of linearly ordered sets and covers of $\mathbb{R}^n$ by polyhedra and by arbitrary convex sets. We focus on problems with κ infinite. Besides numerous positive and negative results, many questions turn out to be independent of the usual axioms of set theory.

1. Introduction. Let X be a set, κ and λ be cardinal numbers, and let $\mathcal{H}$ be a family of subsets of X which covers each $x \in X$ at least κ-fold. What assumptions on $\mathcal{H}$ can ensure that $\mathcal{H}$ can be decomposed into λ many disjoint subcovers? That is, which κ-fold covers can be split into λ many subcovers?

Depending on personal taste, every mathematician can readily formulate the "most relevant" context for the splitting problem; therefore splitting covers has a long-standing tradition. Unarguably, the most studied version of the problem is when X is a topological space and $\mathcal{H}$ is an open cover with special combinatorial properties. We do not attempt to summarize the vast amount of results in this direction, the interested reader is referred to [19] and the references therein. Nevertheless, we note that the literature on combinatorial properties of open covers is mainly concerned with how the combinatorics of open covers is related to topological properties of the underlying space. Therefore a strong topological motivation for considering the given special classes of open covers is always present, and splitting is concerned as far as one is looking for "nice" disjoint subcovers of a "not

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so nice" open cover. Moreover, for most of the problems discussed in these papers the open covers are automatically countable. In the present paper we do not work with open covers, and as we will see, we treat the problem of splitting covers from a more set-theoretic point of view.

Another well-understood variant of the splitting problem deals with covers of finite structures. The most interesting questions in this area ask for splitting the edge covers of (hyper)graphs, and almost optimal solutions of the relevant problems have already been found long ago. But none of the available results concern infinite graphs or infinite-fold covers. In Section 4 we give a complete solution to the splitting problem of infinite-fold edge covers of graphs; we will recall the related finite combinatorial results there.

The situation turns out to be less clear if we are interested in the splitting of finite-fold covers of infinite sets, even in the seemingly simple case of covers of the plane by such familiar objects as circles, triangles or rectangles. To start with positive results, D. Pálvölgyi and G. Tóth [15] showed that for every open convex polygon R in the plane there are constants $c(R)$ and $n(R)$ such that every $c(R)k^{n(R)}$-fold cover of the plane with translates of R can be decomposed into k disjoint covers (see also [18] for the special case of open triangles). The analogous decomposition result in the special case of centrally symmetric open convex polygons was obtained by J. Pach and G. Tóth [13], and in [2] it was shown that for such regions $n(R) = 1$ can be chosen. In contrast with these results, J. Pach, G. Tardos and G. Tóth [12] constructed, for every $1 < k < \omega$, a k-fold cover of the plane (1) by open strips, (2) by axis-parallel open rectangles, (3) by homothets of an arbitrary open concave quadrilateral which cannot be decomposed into two disjoint covers. In fact, D. Pálvölgyi [14] obtained a characterization of the open polygons for which a positive decomposition result holds.

However, the problem whether for a given convex subset R of the plane there is a k such that any k-fold cover of the plane with translates/homothets of R can be decomposed into two disjoint subcovers is far from being solved. E.g. we do not know the answer when R is an open or closed disk; but we remark that a positive answer may be hidden in a more than 100 page-long manuscript of J. Mani-Levitska and J. Pach. We also note that the situation in $\mathbb{R}^3$ can turn out to be completely different: in another unpublished work of J. Mani-Levitska and J. Pach, for every $k < \omega$ a k-fold cover of $\mathbb{R}^3$ with open unit balls is constructed which cannot be decomposed into two disjoint covers. D. Pálvölgyi [14] obtained the analogous negative answer for polyhedra in $\mathbb{R}^3$.

Our investigations were initiated by the question of J. Pach whether any infinite-fold cover of the plane by axis-parallel rectangles can be decomposed into two disjoint subcovers (see also [1, Concluding remarks p. 12]). After answering this question in the negative for ω-fold covers, we started a systematic study of splitting infinite-fold covers in the spirit of J. Pach et al.; in the present paper we would like to publish our first results and state numerous open problems.

We have organized the paper to add structure as we go along. In Section 3, for any pair of cardinals κ and λ, we study the splitting of covers of κ by sets in $[\kappa]^{\leq\lambda}$. In Section 4, we discuss the splitting of edge-covers of finite or infinite graphs. In the remaining sections of the paper we study covers by convex sets. In Section 5, we show that a cover of a linearly ordered set by convex sets is "maximally" decomposable. After completing our work, it turned out that R. Aharoni, A. Hajnal and E. C. Milner [1] obtained results earlier which are similar to our results in Section 5. Since our proofs are significantly simpler and yield slightly stronger results we decided not to leave them out.

In Section 6, as a preliminary study on covers by convex sets on the plane, we show that the splitting problem for covers by closed sets is independent of ZFC. Roughly speaking, under Martin's Axiom an indecomposable cover of $\mathbb{R}$ can be obtained even by the translates of one compact set; while in a Cohen extension of a model with GCH, every uncountable-fold cover by closed sets is "maximally" decomposable. From these results, in Section 7 we easily deduce that the splitting problem for covers of $\mathbb{R}^n$ by convex sets is independent of ZFC. This independence is accompanied by two ZFC results. We show that for very general classes of sets, including e.g. polyhedra, balls or arbitrary affine varieties, an uncountable-fold cover by such sets is "maximally" decomposable. On the other hand, we construct an ω-fold cover of the plane by closed axis-parallel rectangles which cannot be decomposed into two disjoint subcovers. We close the paper with a collection of open problems.

2. Terminology. In this section we fix the notation which will be used in all of the forthcoming sections. We denote by On and Card the class of ordinals and the class of cardinals, respectively. For any set X and cardinal λ, $\mathcal{P}(X)$ denotes the power set of X, and $[X]^{\lambda}$, $[X]^{\leq\lambda}$ and $[X]^{<\lambda}$ stand for the families of those subsets of X which have cardinalities λ, $\leq\lambda$ and $<\lambda$, respectively. If κ is an ordinal, $\mathrm{Lim}(\kappa)$ denotes the set of limit ordinals $<\kappa$. For $\alpha,\beta\in\mathrm{On}$ with $\alpha\leq\beta$, let $[\alpha,\beta]=\{\gamma\in\mathrm{On}\colon \alpha\leq\gamma\leq\beta\}$. The ordinal interval $[\alpha,\beta)$ is defined analogously. If $f\colon X\to Y$ and $A\subseteq X$ are given, $f[A]=\{f(x)\colon x\in A\}$.

When we consider covers of a set X, we do not want to exclude using a set $H\subseteq X$ multiple times. This approach is motivated both by theoretical and by practical reasons. First, the classical results for splitting finite-fold covers of finite graphs allow graphs with multiple edges, so it is reasonable to keep this generality while extending these results to infinite graphs and infinite-fold covers. Second, the natural operation of restricting a cover of X to a subset of X can easily result in a cover where some of the covering sets are used multiple times. Moreover, this generality does not cause any additional complication. The following definition makes our notion of *cover* precise.

Definition 2.1. Let X be an arbitrary set, let $\mathcal{H} \subseteq \mathcal{P}(X)$ be an arbitrary family of subsets of X and let $m\colon \mathcal{H} \to \mathrm{On} \setminus \{0\}$ be an arbitrary function. Then the *cover on X by $\mathcal{H}$ with multiplicity m* is $\mathbf{H} = \{\{H\} \times m(H)\colon H \in \mathcal{H}\}$. For $x \in X$, let $\mathbf{H}(x) = \{\langle H, \alpha\rangle\colon x \in H \in \mathcal{H}, \alpha < m(H)\}$. A cover is *simple* if $m(H) = 1$ $(H \in \mathcal{H})$; for simple covers we identify $\mathbf{H}$ with $\mathcal{H}$.

Let $Y \subseteq X$ and let κ be a cardinal number. Then $\mathbf{H}$ is a *cover of Y* if $|\mathbf{H}(x)| \geq 1$ for every $x \in Y$, and $\mathbf{H}$ is a *κ-fold cover of Y* if $|\mathbf{H}(x)| \geq \kappa$ for every $x \in Y$. We say $\mathbf{H}$ is a *cover* (*κ-fold cover*, resp.) if it is a cover (κ-fold cover, resp.) of $\bigcup \mathcal{H}$.

In the following, $\mathcal{H}$, m and $\mathbf{H}$ will always be as in Definition 2.1. To ease notation, the decomposition of a cover will be realized by coloring the covering sets.

Definition 2.2. Let $\mathbf{H}$ be a cover on X, let $Y \subseteq X$ and let $\kappa \in \mathrm{Card}$. A partial function $c\colon \mathbf{H} \to \mathrm{On}$ is a *κ-good coloring of $\mathbf{H}$ over Y* if for every $x \in Y$ if $|\mathbf{H}(x)| \geq \kappa$ then $\kappa \subseteq c[\mathbf{H}(x)]$. Similarly, $c\colon \mathbf{H} \to \kappa$ is a *κ-good coloring of $\mathbf{H}$*, or simply a *κ-good coloring*, if it is a κ-good coloring of $\mathbf{H}$ over $\bigcup \mathcal{H}$.

The coloring c is a *$[\kappa, \infty)$-good coloring of $\mathbf{H}$ over Y* if it is λ-good for each cardinal $\lambda \geq \kappa$. We say that c is a *maximally good coloring of $\mathbf{H}$ over Y* if it is 1-good and $[\omega, \infty)$-good. The notions of *$[\kappa, \infty)$-good coloring of $\mathbf{H}$* and of *maximally good coloring of $\mathbf{H}$* are defined analogously.

Clearly, a κ-fold cover of Y has a κ-good coloring over Y if and only if it can be partitioned into κ many covers of Y.

Next we prove an easy reduction theorem.

Proposition 2.3. *Let X be a set, $Y \subseteq X$, $\kappa \in \mathrm{Card} \setminus \omega$, and let $\mathcal{H}$, m and $\mathbf{H}$ be as in Definition 2.1. If there is a κ-good coloring of $\mathcal{H}$ over Y then there is a κ-good coloring of $\mathbf{H}$ over Y. The analogous result holds for $[\kappa, \infty)$-good and maximally good colorings, as well.*

Proposition 2.3 allows us to consider only simple covers. We will frequently use this reduction steps in inductive proofs: it allows us to use the inductive assumption for multicovers while we prove our statement in the special case of simple covers. We will always state explicitly when this reduction is used. Note that the assumption of κ being infinite cannot be left out: e.g. the complete graph G on three vertices has an n-good edge coloring for every $n \geq 3$, while if we take each edge of G with multiplicity two, then the resulting graph has no 4-good edge coloring.

Proof of Proposition 2.3. First suppose $c_0 \colon \mathcal{H} \to \mathrm{On}$ is a κ-good coloring of $\mathcal{H}$ over Y. Let $\chi \colon \mathrm{On} \setminus \{0\} \to \mathrm{On}$ satisfy $\chi[\lambda \setminus \{0\}] = \lambda$ ($\lambda \in \mathrm{Card} \setminus \omega$). For every $H \in \mathrm{dom}(c_0)$ we define

$$c(\langle H, \alpha \rangle) = \begin{cases} c_0(H) & \text{if } \alpha = 0, \\ \chi(\alpha) & \text{if } 0 < \alpha < m(H). \end{cases}$$

Then for every $x \in Y$, $c_0[\mathcal{H}(x)] \subseteq c[\mathbf{H}(x)]$; and for every $H \in \mathcal{H}(x)$, $\omega \leq m(H)$ implies $m(H) \subseteq c[\mathbf{H}(x)]$. Hence c is a κ-good coloring of $\mathbf{H}$ over Y. For $[\kappa, \infty)$-good and maximally good colorings the proof is identical. ∎

In some inductive proofs we will also use the following lemma.

Lemma 2.4. *Let $\kappa \in \mathrm{Card}$ and $\alpha \in \mathrm{On}$ satisfy $\omega_1 \leq \kappa = |\alpha|$. Then there is a function $h_\alpha \colon \kappa \to \alpha$ such that $h_\alpha[\kappa] = \alpha$ and $\kappa' \subseteq h_\alpha[\kappa']$ for each cardinal $\omega_1 \leq \kappa' < \kappa$.*

Proof. Let h_α be such that $h_\alpha(\nu) = \mathrm{tp}(\mathrm{Lim}(\kappa) \cap \nu)$ for $\nu \in \mathrm{Lim}(\kappa)$ and $h_\alpha[\kappa \setminus \mathrm{Lim}(\kappa)] = \alpha$. It is obvious that $h_\alpha[\kappa] = \alpha$. Let $\kappa' \in \mathrm{Card}$ satisfy $\omega_1 \leq \kappa' < \kappa$. Since for every $\beta \in [0, \kappa')$ there is a $\nu \in [0, \kappa')$ such that $\mathrm{tp}(\mathrm{Lim}(\kappa') \cap \nu) = \beta$, we get $\kappa' \subseteq h_\alpha[\kappa']$. So h_α satisfies the requirements. ∎

Later on, we will apply Lemma 2.4 the following way. Suppose we have an $[\omega_1, \infty)$-good coloring $c \colon \mathbf{H} \to \kappa$ and an ordinal $\kappa \leq \alpha < \kappa^+$. Then the coloring $h_\alpha \circ c$ is an $[\omega_1, \infty)$-good coloring with the additional property

(m) if $|\mathbf{H}(x)| = \kappa$ then $\alpha \subseteq (h_\alpha \circ c)[\mathbf{H}(x)]$.

2.1. Hunting for the strongest possible decomposition result. In this subsection we investigate how one can obtain colorings with even stronger decomposition properties from maximally good or $[\omega_1, \infty)$-good colorings. The notions and results of this subsection are not used later in the paper. The strongest possible decomposition notion is formulated in the following terminology.

Definition 2.5. Let $\mathbf{H}$ be a cover of X and let $Y \subseteq X$. Let $\mathbf{h} \colon \mathrm{Card} \to \mathrm{On} \setminus \{0\}$ be a partial function satisfying $\mathbf{h}(\kappa) < \kappa^+$ ($\kappa \in \mathrm{dom}(\mathbf{h})$). A partial function $c \colon \mathbf{H} \to \mathrm{On}$ is an $\mathbf{h}$-*good coloring of* $\mathbf{H}$ *over* Y if for every $x \in Y$,

(m^+) if $|\mathbf{H}(x)| \in \mathrm{dom}(\mathbf{h})$ then $\mathbf{h}(|\mathbf{H}(x)|) \subseteq c[\mathbf{H}(x)]$.

Proposition 2.6. *Let X, Y, $\mathbf{H}$ and $\mathbf{h}$ be as in Definition 2.5. Suppose that*

(h1) *if $\mathrm{dom}(\mathbf{h}) \cap \omega$ is unbounded in ω then $\mathbf{h}(\omega) = \omega$;*
(h2) *for each $\mu \in \mathrm{Card}$ the set $\{\nu \in \mathrm{dom}(\mathbf{h}) \cap \mu \colon \nu < \mathbf{h}(\nu)\}$ is not stationary in μ.*

Set $\mathbf{i}\colon \mathrm{dom}(\mathbf{h}) \to \mathrm{On}$, $\mathbf{i}(\kappa) = \kappa$ $(\kappa \in \mathrm{dom}(\mathbf{h}))$. *If there exists an* $\mathbf{i}$*-good coloring of* $\mathbf{H}$ *over* Y *then there exists an* $\mathbf{h}$*-good coloring of* $\mathbf{H}$ *over* Y*, as well.*

Observe that a maximally good coloring is $\mathbf{i}$-good for $\mathrm{dom}(\mathbf{i}) = \{1\} \cup (\mathrm{Card} \setminus \omega)$. Before proving Proposition 2.6, we need a lemma in advance. This lemma is a far-reaching generalization of Lemma 2.4.

LEMMA 2.7. *Let* λ *be an infinite cardinal. Let the partial function* $\mathbf{h}\colon \mathrm{Card} \cap \lambda^+ \to \mathrm{On}$ *satisfy* (h1), (h2) *and* $\kappa \le \mathbf{h}(\kappa) < \kappa^+$ $(\kappa \in \mathrm{dom}(\mathbf{h}))$. *Then there exists a function* $\chi\colon \lambda \to \mathrm{On}$ *such that*

1. $\chi(n) = n$ *for* $n < 1 + \sup(\mathrm{dom}(\mathbf{h}) \cap \omega)$;
2. $[0, \mathbf{h}(\kappa)) \subseteq \chi[[0,\kappa)]$ *for* $\kappa \in \mathrm{dom}(\mathbf{h}) \cap [\omega, \lambda]$.

Proof. By defining $\mathbf{h}$ on $\mathrm{Card} \cap [\omega, \lambda] \setminus \mathrm{dom}(\mathbf{h})$ to be $\mathbf{i}$, we can assume $\mathrm{Card} \cap [\omega, \lambda] \subseteq \mathrm{dom}(\mathbf{h})$. We prove the statement by induction on λ. For $\lambda = \omega$ a bijection $\chi\colon \omega \to \mathbf{h}(\omega)$ with $\chi(n) = n$ $(n < 1 + \sup(\mathrm{dom}(\mathbf{h}) \cap \omega))$ does the job.

Let now $\lambda > \omega$ and suppose that the statement holds for every cardinal $\kappa < \lambda$. If λ is a successor, say $\lambda = \kappa^+$, set $\mathbf{h}_\kappa = \mathbf{h}|_{\kappa^+}$ and let χ_κ satisfy 1 and $[0, \mathbf{h}_\kappa(\kappa')) \subseteq \chi_\kappa[[0,\kappa')]$ for every $\kappa' \in \mathrm{Card} \cap [\omega, \kappa]$. Define $\chi\colon \lambda \to \mathrm{On}$ by $\chi|_\kappa = \chi_\kappa$, and $\chi|_{[\kappa,\kappa^+)}\colon [\kappa, \kappa^+) \to [\kappa, \mathbf{h}(\kappa^+))$ being any bijection. Then $\mathbf{h}$ clearly fulfills the requirements.

If λ is a limit cardinal, take a strictly increasing continuous cofinal sequence $\langle \lambda_\alpha < \lambda\colon \alpha < \mathrm{cf}(\lambda)\rangle$ of infinite cardinals such that $\mathbf{h}(\lambda_\alpha) = \lambda_\alpha$ for every $\alpha < \mathrm{cf}(\lambda)$. Let $\mathbf{h}(\lambda) \setminus \lambda = \bigcup\{K_\alpha\colon \alpha < \mathrm{cf}(\lambda)\}$ be such that for every $\alpha \le \alpha' < \mathrm{cf}(\lambda)$ we have $K_\alpha \subseteq K_{\alpha'}$ and $|K_\alpha| \le \lambda_\alpha$. For every $\alpha < \mathrm{cf}(\lambda)$ define $\mathbf{h}_\alpha\colon \mathrm{Card} \cap \lambda_\alpha^+ \to \mathrm{On}$, $\mathbf{h}_\alpha = \mathbf{h}|_{\mathrm{Card} \cap \lambda_\alpha^+}$. By the inductive hypothesis, for every $\alpha < \mathrm{cf}(\lambda)$ we have $\chi_\alpha\colon \lambda_\alpha \to \mathrm{On}$ satisfying 1 and $[0, \mathbf{h}_\alpha(\kappa)) \subseteq \chi_\alpha[[0,\kappa)]$ $(\kappa \in \mathrm{Card} \cap [\omega, \lambda_\alpha])$.

Set $\chi|_{[0,\lambda_0)} = \chi_0$; then 1 holds, and 2 holds for $\kappa \in \mathrm{Card} \cap [\omega, \lambda_0]$. For every $\alpha < \mathrm{cf}(\lambda)$ fixed, we define $\chi\colon [\lambda_\alpha, \lambda_{\alpha+1}) \to \mathrm{On}$ as follows. Let $\vartheta\colon [\lambda_\alpha, \lambda_\alpha \dot{+} |K_\alpha|) \to K_\alpha$ be a bijection and let $\varepsilon\colon [\lambda_\alpha \dot{+} |K_\alpha|, \lambda_{\alpha+1}) \to \lambda_{\alpha+1}$ be the enumeration. Observe that for every $\kappa \in \mathrm{Card} \cap (\lambda_\alpha, \lambda_{\alpha+1})$ we have $\varepsilon(\kappa) = \kappa$. For $\eta \in [\lambda_\alpha, \lambda_\alpha \dot{+} |K_\alpha|)$ set $\chi(\eta) = \vartheta(\eta)$ while for $\eta \in [\lambda_\alpha \dot{+} |K_\alpha|, \lambda_{\alpha+1})$ set $\chi(\eta) = \chi_{\alpha+1}(\varepsilon(\eta))$. Then for every $\kappa \in \mathrm{Card} \cap (\lambda_\alpha, \lambda_{\alpha+1}]$ we have $\chi[[0,\kappa)] \supseteq K_\alpha$ and

$$\chi[[0,\kappa)] \supseteq \chi_{\alpha+1}[\varepsilon[[\lambda_\alpha \dot{+} |K_\alpha|, \kappa)]] \supseteq \chi_{\alpha+1}[[0,\kappa)] \supseteq [0, \mathbf{h}_{\alpha+1}(\kappa)) = [0, \mathbf{h}(\kappa)).$$

For every limit ordinal $\alpha < \mathrm{cf}(\lambda)$ we have $\lambda_\alpha = \sup_{\beta<\alpha} \lambda_\beta$, hence $\lambda_\alpha \subseteq \chi[[0,\lambda_\alpha)]$ and we have $\chi[[0,\lambda)] \supseteq [0, \mathbf{h}(\lambda))$ as well. This completes the proof. ∎

In Lemma 2.7, the technical assumptions (h1) and (h2) cannot be left out. This is obvious for (h1). For (h2), observe that if λ is a Mahlo cardinal then for the function $\mathbf{h}(\nu) = \nu \dot{+} 1$ ($\nu \in \mathrm{Card} \cap \lambda$), where $\dot{+}$ denotes ordinal addition, a function χ satisfying conclusion 2 of Lemma 2.7 would induce a regressive function on the set $\mathrm{Card} \cap \lambda$, which is stationary in λ. This explains the assumptions on $\mathbf{h}$ in Proposition 2.6.

Proof of Proposition 2.6. Let $c_{\mathbf{i}}\colon \mathbf{H} \to \mathrm{On}$ be an $\mathbf{i}$-good coloring of $\mathbf{H}$ over Y. Set $\lambda = \sup \mathrm{dom}(\mathbf{h})$,

$$\mathbf{h}^+(\kappa) = \begin{cases} \max\{\mathbf{h}(\kappa), \kappa\}, & \kappa \in \mathrm{dom}(\mathbf{h}), \\ \kappa, & \kappa \in \lambda^+ \setminus \mathrm{dom}(\mathbf{h}); \end{cases}$$

and let $\chi\colon \lambda \to \mathrm{On}$ be the function of Lemma 2.7 for $\mathbf{h}^+$. Define $c\colon \mathbf{H} \to \mathrm{On}$ by $c = \chi \circ c_{\mathbf{i}}$. Then c is clearly an $\mathbf{h}$-good coloring of $\mathbf{H}$ over Y. ∎

3. Arbitrary sets. In this section we briefly recall some easy coloring results which hold for covers by arbitrary sets. Let us recall that Axiom Stick is the statement

- there is a family $\mathcal{S} \subseteq [\omega_1]^\omega$ such that $|\mathcal{S}| = \omega_1$ and $\forall X \in [\omega_1]^{\omega_1}\ \exists S \in \mathcal{S}$ $(S \subseteq X)$.

We refer to [8] for the definition of Martin's Axiom for various posets.

Theorem 3.1.

1. *Let μ be an arbitrary cardinal and κ be an infinite cardinal. Then every κ-fold cover $\mathbf{H}$ with $\mathcal{H} \subseteq [\mu]^{\le\kappa}$ has a κ-good coloring.*
2. *For each infinite cardinal μ there is a simple μ-fold cover $\mathbf{H} \subseteq [2^\mu]^{2^\mu}$ of 2^μ with $|\mathbf{H}| = \mu$ which does not have a 2-good coloring.*
3. $\mathrm{MA}_\mu(\mathrm{Fn}(\mu, 2; \omega))$ *implies that every cover $\mathbf{H}$ with $\mathcal{H} \subseteq [\mu]^{\le\mu}$ and $|\mathbf{H}| \le \mu$ has an ω-good coloring.*
4. $\mathrm{MA}_\mu(\mathrm{Fn}(\omega, 2; \omega))$ *implies that every cover $\mathbf{H}$ with $\mathcal{H} \subseteq [\mu]^{\le\mu}$ and $|\mathbf{H}| \le \omega$ has an ω-good coloring.*
5. *Axiom Stick implies that there is a simple cover $\mathbf{H} \subseteq [\omega_1]^{\omega_1}$ with $|\mathbf{H}| = \omega_1$ and $|\mathbf{H}(\xi)| = \omega$ $(\xi \in \omega_1)$ which does not have a 2-good coloring.*
6. *It is consistent that 2^ω is arbitrarily large and the following two statements hold:*
   - (i) *there is a simple cover $\mathbf{H} \subseteq [\omega_1]^{\omega_1}$ with $|\mathbf{H}| = \omega_1$ and $|\mathbf{H}(\xi)| = \omega$ $(\xi \in \omega_1)$ which does not have a 2-good coloring;*
   - (ii) *for each $\mu < 2^\omega$ every cover $\mathbf{H}$ with $\mathcal{H} \subseteq [\mu]^{\le\mu}$ and $|\mathbf{H}| = \omega$ has an ω-good coloring.*

Proof. 1. Set $X = \bigcup \mathcal{H}$. If $\mu \le \kappa$, a straightforward transfinite induction yields the required coloring as follows. Let $\varphi\colon X \times \kappa \to \kappa$ be a bijection.

By induction, for every $\alpha < \kappa$ we define partial colorings $c_\alpha\colon \mathbf{H} \to \kappa$ satisfying $|c_\alpha| = 1$, as follows. If $\alpha < \kappa$ and c_β is defined for $\beta < \alpha$, let $x \in X$ and $\chi < \kappa$ satisfy $\varphi(x, \chi) = \alpha$. Since $|\mathbf{H}(x)| \geq \kappa$, we can pick an $\langle H, \gamma\rangle \in \mathbf{H} \setminus \bigcup_{\beta<\alpha} \mathrm{dom}(c_\beta)$ satisfying $x \in H$ and set $c_\alpha(\langle H, \gamma\rangle) = \chi$. This completes the αth step of the construction. Then $c = \bigcup_{\alpha<\kappa} c_\alpha$ is a κ-good coloring of $\mathbf{H}$.

Suppose now $\mu > \kappa$. For each $\xi \in \mu$ let $\mathbf{H}'_\xi \in [\mathbf{H}(\xi)]^\kappa$. Let $\equiv$ be the equivalence relation on μ generated by the relation $\{(\xi, \zeta)\colon \xi, \zeta \in \mu, \mathbf{H}'_\xi \cap \mathbf{H}'_\zeta \neq \emptyset\}$. Since $\mathcal{H} \subseteq [\mu]^{\leq\kappa}$ and $|\mathbf{H}'_\xi| = \kappa$ for each $\xi \in \mu$, the equivalence classes $\{X_i\colon i \in I\}$ of $\equiv$ have cardinalities $\leq \kappa$. Set $\mathbf{H}_i = \bigcup\{\mathbf{H}'_\xi\colon \xi \in X_i\}$ $(i \in I)$. Then $|X_i| \leq |\mathbf{H}_i| = \kappa$ and $|\mathbf{H}_i(x)| = \kappa$ for each $x \in X_i$. So by the inductive hypothesis, there is a κ-good coloring $c_i\colon \mathbf{H}_i \to \kappa$ over X_i $(i \in I)$. Then $c = \bigcup_{i\in I} c_i$ is a good κ-coloring over X.

2. Set $X = \{f \in 2^\mu\colon |f^{-1}\{1\}| = \mu\}$, and for every $\alpha < \mu$ let $H_\alpha = \{x \in X\colon x(\alpha) = 1\}$. Then $|X| = 2^\mu$ and $\mathbf{H} = \{H_\alpha\colon \alpha < \mu\}$ is a simple cover of X with the required properties.

3. Consider the poset $\mathcal{P} = \{(\mathcal{K}, c)\colon \mathcal{K} \in [\mathbf{H}]^{<\omega},\ c\colon \mathcal{K} \to \omega\}$ with partial order $(\mathcal{K}', c') \leq (\mathcal{K}, c)$ if and only if $\mathcal{K} \subseteq \mathcal{K}'$ and $c \subseteq c'$. Observe that for every $\alpha \in \mu$ with $|\mathbf{H}(\alpha)| \geq \omega$ and $\chi < \omega$, the set

$$D_{\alpha,\chi} = \{(\mathcal{K}, c) \in \mathcal{P}\colon \exists\langle H, \gamma\rangle \in \mathbf{H}\ (\alpha \in H \text{ and } c(\langle H, \gamma\rangle) = \chi)\}$$

is dense in $\mathcal{P}$. So by applying $\mathrm{MA}_\mu(\mathrm{Fn}(\mu, 2; \omega))$ to the poset $\mathcal{P}$ and the family $\mathcal{D} = \{D_{\alpha,\lambda}\colon \alpha \in \mu,\ \chi < \omega\}$ of dense sets of cardinality μ, the statement follows. The argument for 4 is similar.

5. Let $\mathcal{S} \subseteq [\omega_1]^\omega$ be a stick family. Set $\mathcal{X} = \{S \cup \{\alpha\}\colon S \in \mathcal{S},\ \alpha < \omega_1\}$; then $\mathcal{X}$ is also a stick family with the additional properties that

- (i) $|\mathcal{X}(\alpha)| = \omega_1$ $(\alpha < \omega_1)$;
- (ii) for every $\alpha, \beta < \omega_1$ with $\alpha \neq \beta$ there is an $X \in \mathcal{X}$ with $\alpha \in X$ and $\beta \notin X$.

For every $\alpha < \omega_1$ let $H_\alpha = \{X \in \mathcal{X}\colon \alpha \in X\}$. Since $|\mathcal{X}| = \omega_1$, it is enough to show that $\mathbf{H} = \{H_\alpha\colon \alpha < \omega_1\}$ is a cover of $\mathcal{X}$ with the required properties. By (i), $|H_\alpha| = \omega_1$ $(\alpha < \omega_1)$. By (ii), $H_\alpha \neq H_\beta$ for $\alpha \neq \beta$., i.e. $\mathbf{H}$ is a simple cover. For every $X \in \mathcal{X}$ we have $|\mathbf{H}(X)| = |\{H_\alpha\colon \alpha \in X\}| = |X| = \omega$. Finally assume that for some $I \subseteq \omega_1$, $\mathbf{H}_0 = \{H_\alpha\colon \alpha \in I\}$ covers $\mathcal{X}$. This means $I \cap X \neq \emptyset$ for each $X \in \mathcal{X}$. Then $|\omega_1 \setminus I| \leq \omega$ because every $J \in [\omega_1]^{\omega_1}$ contains some element of $\mathcal{X}$. Thus $X \subseteq I$ for some $X \in \mathcal{X}$ so this X is not covered by $\mathbf{H} \setminus \mathbf{H}_0$.

6. In [4] it was proved that it is consistent that 2^ω is arbitrarily large, Axiom Stick holds and $\mathrm{MA}(\mathrm{Fn}(\omega, 2; \omega))$ also holds. So by 4 and 5, that model satisfies the requirements of 6. ■

Theorem 3.1 is to be compared with the results of Section 6 on splitting closed covers.

4. Graphs. Now we investigate the interesting special case of graphs, that is, when each covering set has two elements. In this section "graph" means an undirected possibly infinite graph where multiple edges are allowed, but we exclude loops. We follow the standard notation, i.e. $G = (V, E)$ denotes the graph with vertex set V and edge set E. For every $V' \subseteq V$, $G[V']$ denotes the subgraph of G spanned by V'. According to our convention, for $v, w \in V$, the set of edges containing v is denoted by $E(v)$, and $E(v, w)$ is the edges connecting v to w. For every $v \in V$, $d_G(v)$ is the degree of v in G, i.e. $d_G(v) = |E(v)|$ where multiple edges are counted with multiplicity. Set $\Delta(G) = \sup\{d_G(v) \colon v \in V\}$; the supremum of the edge multiplicities is denoted by $\mu(G)$. For every $E' \subseteq E$, $V[E']$ is the set of vertices of the edge set E'. A graph G is *n-regular* if $d_G(v) = n$ $(v \in V)$. A *complete matching* in G is a subgraph of G' of G satisfying $d_{G'}(v) = 1$ $(v \in V)$.

As we mentioned in the introduction, the splitting problem for finite graphs is much studied (see e.g. [16, Chapter 28]), and the following result, originally due to R. P. Gupta [6, Theorem 2.2 p. 500], solves our problem for finite graphs.

THEOREM 4.1. *Let $1 \le n < \omega$. Let $G = (V, E)$ be a finite graph and let $X \subseteq V$ be such that for every $x \in X$ we have $d_G(x) \ge n + \mu(G)$. Then E has an n-good coloring over X.*

The main result of this section is the extension of Theorem 4.1 to infinite graphs, which, in addition, gives a necessary and sufficient condition for the existence of 2-good colorings. First we show that even for simple graphs, in order to ensure the existence of an n-good coloring, the condition on the degree of vertices cannot be weakened to $d_G(x) \ge n$. To see this, we will use the following constructions proposed by Gyula Pap.

For every $n < \omega$ let K_n denote the complete graph on the vertex set $\{v_0, v_1, \ldots, v_{n-1}\}$. For odd n let K_n^- denote the graph obtained from K_n by *deleting* the edges $\{v_0, v_{n-1}\}$ and $\{v_{2k}, v_{2k+1}\}$ $(k < (n-1)/2)$. Take two disjoint copies of K_{n+2}^-, say on the vertex sets $\{v_0, v_1, \ldots, v_{n+1}\}$ and $\{v'_0, v'_1, \ldots, v'_{n+1}\}$ and let D_n denote the graph obtained as the union of the two copies of K_{n+2}^- and the edge $\{v_0, v'_0\}$.

PROPOSITION 4.2. *Let $2 \le n < \omega$.*

1. *If n is even, then K_{n+1} is an n-regular graph with no n-good coloring.*
2. *If n is odd, then D_n is an n-regular graph with no n-good coloring.*

Proof. We prove the statements simultaneously. It is obvious that K_{n+1} and D_n are n-regular graphs. Hence an n-good coloring of any of them is a partition of the edge set into n disjoint complete matchings.

Now for n even, K_{n+1} has no complete matchings at all since its vertex set has odd cardinality. Also for cardinality reasons, if n is odd any complete matching of D_n must contain the edge $\{v_0, v_0'\}$. Hence D_n has no two disjoint complete matchings; in particular, as $n \geq 2$, the n-regular graph D_n has no n-good coloring. ∎

We also note that Theorem 3.1.1 completely solves the splitting problem for infinite-fold edge-covers.

Theorem 4.3. *Let $G = (V, E)$ be a graph, let $X \subseteq V$ be arbitrary and suppose that $d_G(x) \geq \omega$ for every $x \in X$. Then E has an ω-good coloring over X.*

Proof. The statement follows from Theorem 3.1.1 with $\kappa = \omega$, $\mu = |X|$, and $\mathbf{H} = E \cap [X]^{\leq 2}$ counted with multiplicity. ∎

From now on we work with n-good colorings with $n < \omega$. The case $n = 1$ is trivial, so we start with $n = 2$. We have the following characterization of the existence of 2-good colorings.

Theorem 4.4. *Let $G = (V, E)$ be a graph. Then E has a 2-good coloring if and only if no connected component of G is an odd cycle.*

Proof. It is easy to see that the condition is necessary, as an odd cycle has no 2-good coloring.

To prove sufficiency, observe that we can assume G is connected since it is sufficient to color the connected components separately.

Lemma 4.5. *There exists a subgraph $G' = (V', E')$ of G and a coloring $c' \colon E' \to 2$ such that $V' \neq \emptyset$ and for every $v \in V'$, $d_G(v) \geq 2$ implies $c'[E'(v)] = 2$.*

Proof. If there is $x \in V$ with $d_G(x) = 1$ then $V' = \{x\}$ and $E' = \emptyset$ works. So we can assume that $d_G(x) \geq 2$ for each $x \in V$. Depending on the subgraphs of G we distinguish several cases.

Case I: *G contains an even cycle, or a path which is infinite in both directions.* Since even cycles and paths infinite in both directions have 2-good colorings we can choose G' to be one of these subgraphs. Note that a pair of multiple edges is an even cycle.

From now on we assume that G contains no such subgraphs. Pick an arbitrary vertex $v \in V$ and start a path from v until it first fails to be vertex-disjoint.

Case II: *We get an infinite path in this direction.* Let us start another path from v until it is vertex-disjoint from itself and from the previous infinite path. As we have no doubly infinite paths, the second path has to terminate, say at $w \in V$. We obtained a cycle on w and an infinite path starting from w. Let us call such a configuration an *infinite lasso*. It is easy to check that we get a 2-good coloring of our infinite lasso if we color its edges using alternating colors in such a way that we start the coloring at w and we first color the edges of the cycle on w. Thus in this case we can set G' to be an infinite lasso.

Case III: *Our (first) path from v reaches a vertex visited before.* Since G contains no even cycles we get an odd cycle C. Since G is not an odd cycle, there is a vertex w of C with $d_G(w) \geq 3$. Let us start a path from w disjoint from C. If we get an infinite lasso then we are done by Case II. Otherwise the path reaches either a vertex of C or a vertex of the path itself. If it reaches C then it has to reach it at w: else G would contain an even cycle since for two odd cycles intersecting each other in a finite path, removing the intersection results in an even cycle.

Hence we obtain two disjoint cycles connected by a path, possibly of length 0. Let G' be this graph. As for the infinite lasso, color the edges of this graph alternately, starting from w and coloring first a cycle containing w. It is easy to check that this is a 2-good coloring of G'. ∎

Now we go back to the proof of Theorem 4.4. For an ordinal ξ to be specified later, we define a sequence of partial colorings $c_\alpha \colon E \to 2$ $(\alpha < \xi)$ such that

(i) $\mathrm{dom}(c_\alpha) \subsetneq \mathrm{dom}(c_{\alpha'})$ and $c_{\alpha'}|_{\mathrm{dom}(c_\alpha)} = c_\alpha$ $(\alpha < \alpha' < \xi)$,
(ii) if $v \in V[\mathrm{dom}(c_\alpha)]$ and $d_G(v) \geq 2$ then $c_\alpha[E(v)] = 2$ $(\alpha < \xi)$,
(iii) $V = V[\bigcup_{\alpha<\xi} \mathrm{dom}(c_\alpha)]$.

Once this done the function $c\colon E \to 2$, $c = \bigcup_{\alpha<\xi} c_\alpha$, is a 2-coloring of E by (i) which is 2-good by (ii) and (iii).

To start the construction, by Lemma 4.5 we have a partial coloring c' which works as c_0. Let α be an ordinal and suppose that c_β is defined for every $\beta < \alpha$. If $V = V[\bigcup_{\beta<\alpha} \mathrm{dom}(c_\alpha)]$ set $\xi = \alpha$ and the construction is finished. Else set $c_\alpha^- = \bigcup_{\beta<\alpha} c_\beta$. We have $V \setminus V[\mathrm{dom}(c_\alpha^-)] \neq \emptyset$. As G is connected, there exists an edge $\{u, v\} \in E \setminus \mathrm{dom}(c_\alpha^-)$ such that $u \in V[\mathrm{dom}(c_\alpha^-)]$ and $v \in V \setminus V[\mathrm{dom}(c_\alpha^-)]$. Start a path P from u whose first edge is $\{u, v\}$, and keep extending P as long as it is edge-disjoint from $\mathrm{dom}(c_\alpha^-)$. This P can be infinite, or it can end either in $V[\mathrm{dom}(c_\alpha^-)]$, in a vertex of P, or in a vertex $x \in V$ with $d_G(x) = 1$. Let c_α be the partial coloring extending c_α^- where we color the edges of P alternately starting with the edge $\{u, v\}$. Then c_α clearly satisfies (i); we have to show that it also satisfies (ii). To this end, let $w \in V[\mathrm{dom}(c_\alpha)]$ with $d_G(w) \geq 2$. By the definition of P, either $w \in V[\mathrm{dom}(c_\alpha^-)]$ or $d_P(w) \geq 2$. If $w \in V[\mathrm{dom}(c_\alpha^-)]$ then $c_\alpha^-[E(w)] = 2$ by the inductive assumption (ii); while if $d_P(w) \geq 2$ then $c_\alpha[E(w)] = 2$ because we color the edges of P alternately.

Since $\mathrm{dom}(c_\alpha)$ $(\alpha < \xi)$ are strictly increasing, this transfinite procedure terminates at some ordinal ξ. The resulting sequence $(c_\alpha)_{\alpha<\xi}$ satisfies (i)–(iii), so the proof is complete. ∎

Clearly, an n-regular graph has an n-good coloring if and only if its edge chromatic number is n. It is a well-known theorem of Vizing that the edge chromatic number of a simple finite graph is either $\Delta(G)$ or $\Delta(G)+1$ (see e.g. [16, Theorem 28.2 p. 467]). But to decide e.g. whether a 3-regular graph is 3-chromatic or not is an NP-complete problem (see e.g. [16, Theorem 28.3 p. 468]). Hence we cannot hope for a very simple analogue of Theorem 4.4 for $n \geq 3$.

It remains to extend the Theorem of R. P. Gupta to infinite graphs.

THEOREM 4.6. *Let $1 \leq n < \omega$. Let $G = (V, E)$ be a graph and let $X \subseteq V$ be such that $d_G(x) \geq n + \mu(G)$ for every $x \in X$. Then E has an n-good coloring over X.*

Proof. For finite graphs this is Theorem 4.1 due to Gupta. If G is locally finite, i.e. all degrees are finite, then an easy compactness argument yields the result. If $\mu(G) \geq \omega$ we are done by Theorem 4.3.

If G is arbitrary with $\mu(G) < \omega$, we construct a locally finite graph $G' = (V', E')$, as follows. Set $V_0 = \{v \in V : d_G(v) < \omega\}$ and $V_1 = \{v \in V : d_G(v) \geq \omega\}$. Let $V' = V_0 \cup \{\langle v, \alpha\rangle : v \in V_1,\ \alpha < d_G(v)\}$. We define E' by setting $G'[V_0] = G[V_0]$ and by distributing for $v \in V_1$, using $d_G(v) = d_G(v) \times (n + \mu(G))$, the edges $E(v)$ onto $(\langle v, \alpha\rangle)_{\alpha<d_G(v)}$ "uniformly": that is, E' is constructed in such a way that

(i) for every $v \in V_0$, $d_{G'}(v) = d_G(v)$,
(ii) for every $v \in V_1$ and $\alpha < d_G(v)$, $d_{G'}(\langle v, \alpha\rangle) = n + \mu(G)$,
(iii) for every $v \in V_1$ and $\alpha < d_G(v)$, $E'(\langle v, \alpha\rangle)$ contains no multiple edges.

This is clearly possible, and we have $\mu(G') \leq \mu(G)$. Set $X' = (X \cap V_0) \cup \{\langle v, 0\rangle : v \in X \cap V_1\}$. We see that G' is locally finite and $d_{G'}(x) \geq n + \mu(G)$, hence $d_{G'}(x) \geq n + \mu(G')$ $(x \in X')$. So there is an n-good coloring of G' over X'. By merging, for every $v \in V_1$, the vertices $\langle v, \alpha\rangle \in V'$ $(\alpha < d_G(v))$ to one vertex we get a graph isomorphic to G and X' is mapped onto X. Thus the n-good coloring of G' over X' yields an n-good coloring of G over X. ∎

5. Intervals in linearly ordered sets. Let $\mathcal{L} = (L, \leq)$ be a linearly ordered set and let $\mathrm{conv}(\mathcal{L})$ denote the family of convex subsets of L. In this section we prove the following two results. The first establishes the existence of a maximally good coloring for convex covers.

THEOREM 5.1. *Let $(L, \leq)$ be an ordered set and let $\mathbf{H}$ be a cover of L with $\mathcal{H} \subseteq \mathrm{conv}(L)$. Then $\mathbf{H}$ has a maximally good coloring.*

The second gives the splitting of k-covers.

THEOREM 5.2. *Let* $(L, \leq)$ *be an ordered set and, for some (finite or infinite) cardinal* k, *let* $\mathbf{H}$ *be a* k*-fold cover of* L *with* $\mathcal{H} \subseteq \mathrm{conv}(L)$. *Then* $\mathbf{H}$ *has a* k*-good coloring.*

As we noted in the introduction, Theorem 5.2 was obtained in [1] much earlier than our investigations. In spite of [1], we decided to treat the splitting of convex covers of linearly ordered sets in the present paper because Theorem 5.1 is new, and we found a significantly simpler proof of Theorem 5.2 than the one in [1].

The heart of the proof of Theorem 5.1 is the following general statement on maximally good colorings.

THEOREM 5.3. *Let* X *be a set and let* $\mathbf{H}$ *be a simple cover on* X. *If for each* $\mathbf{K} \subseteq \mathbf{H}$ *there is* $\mathbf{J} \subseteq \mathbf{K}$ *such that*

(e1) $\bigcup \mathbf{J} = \bigcup \mathbf{K}$,
(e2) $\mathbf{J}$ *has a maximally good coloring,*

then $\mathbf{H}$ *has a maximally good coloring.*

Proof. Let $\{H_\alpha \colon \alpha < |\mathbf{H}|\}$ be an enumeration of $\mathbf{H}$. By transfinite recursion on $\alpha < |\mathbf{H}|$ we define families $\mathbf{J}_\alpha \subseteq \mathbf{H}$ satisfying $H_\alpha \in \bigcup_{\beta \leq \alpha} \mathbf{J}_\beta$, and maximally good colorings $c_\alpha \colon \mathbf{J}_\alpha \to \mathrm{On}$, as follows. Let $\alpha < |\mathbf{H}|$ be arbitrary and suppose $\mathbf{J}_\nu$ and c_ν are constructed for $\nu < \alpha$. Set $\mathbf{K}_\alpha = \mathbf{H} \setminus \bigcup \{\mathbf{J}_\nu \colon \nu < \alpha\}$. Since $\mathbf{K}_\alpha \subseteq \mathbf{H}$, by assumption we have a family $\mathbf{J}_\alpha \subseteq \mathbf{K}_\alpha$ with $\bigcup \mathbf{J}_\alpha = \bigcup \mathbf{K}_\alpha$ and a maximally good coloring c_α of $\mathbf{J}_\alpha$. If $H_\alpha \notin \bigcup_{\beta \leq \alpha} \mathbf{J}_\beta$, we put H_α into $\mathbf{J}_\alpha$ and we set $c_\alpha(H_\alpha) = 0$. So we can assume $H_\alpha \in \bigcup_{\beta \leq \alpha} \mathbf{J}_\beta$. This completes the αth step of the construction. Then we have $\mathbf{H} = \bigcup_{\alpha < |\mathbf{H}|} \mathbf{J}_\alpha$.

Let $c \colon \mathbf{H} \to \mathrm{On}$, $c(H) = \alpha + c_\alpha(H)$ for $H \in \mathbf{J}_\alpha$ $(\alpha < |\mathbf{H}|)$. As $\mathbf{K}_0 = \mathbf{H}$ we have $\bigcup \mathbf{J}_0 = \bigcup \mathbf{H}$ and $c|_{\mathbf{J}_0} = c_0$, so c is 1-good.

Before proving that c is $[\omega, \infty)$-good, let us observe that if $x \in X$ and α is an ordinal such that $\mathbf{J}_\beta(x) \neq \emptyset$ for each $\beta < \alpha$, then $0 \in c_\beta[\mathbf{J}_\beta(x)]$ and so $\beta \in c[\mathbf{J}_\beta(x)]$ $(\beta < \alpha)$. Hence $\alpha \subseteq c[\mathbf{H}(x)]$.

To see that c is $[\omega, \infty)$-good, pick $x \in L$ and suppose $\kappa = |\mathbf{H}(x)| \geq \omega$. We distinguish several cases. If $\mathbf{J}_\beta(x) \neq \emptyset$ for each $\beta < \kappa$, then by the previous observation $\kappa \subseteq c[\mathbf{H}(x)]$, as required.

So suppose $\mathbf{J}_\beta(x) = \emptyset$ for some $\beta < \kappa$; fix a minimal such β. Then $\mathbf{H}(x) = \bigcup \{\mathbf{J}_\alpha(x) \colon \alpha < \beta\}$. Thus for each cardinal $\lambda < \kappa$ there is an $\alpha(\lambda) < \beta$ such that $|\mathbf{J}_{\alpha(\lambda)}(x)| \geq \max\{\omega, \lambda^+\}$. Then $\mathbf{J}_\gamma(x) \neq \emptyset$ $(\gamma < \alpha(\lambda))$ and so $\alpha(\lambda) \subseteq c[\mathbf{H}(x)]$ by our observation. Moreover, $\max\{\omega, \lambda^+\} \subseteq c_{\alpha(\lambda)}[\mathbf{J}_{\alpha(\lambda)}]$ and so $[\alpha(\lambda), \alpha(\lambda) + \lambda^+) \subseteq c[\mathbf{H}(x)]$, as well. By putting these together we obtain $\alpha(\lambda) + \lambda^+ \subseteq c[\mathbf{H}(x)]$, so since $\kappa = \sup\{\lambda^+ \colon \lambda < \kappa\}$ we conclude $\kappa \subseteq c[\mathbf{H}(x)]$, as required. ∎

In the following two lemmas, for any linearly ordered set $\mathcal{L}$, we establish the existence of a maximally good coloring for special subfamilies of $\mathrm{conv}(L)$. For $x \in L$ set $(-\infty, x] = \{y \in L \colon y \leq x\}$ and $[x, +\infty) = \{y \in L \colon x \leq y\}$. We define

$$\mathrm{tail}(\mathcal{L}) = \{I \subseteq L \colon [x, +\infty) \subseteq I \text{ for each } x \in I\}.$$

Clearly, $\mathrm{tail}(\mathcal{L}) \subsetneq \mathrm{conv}(\mathcal{L})$ provided $|L| \geq 2$.

LEMMA 5.4. *Every simple cover* $\mathbf{H}$ *with* $\mathcal{H} \subseteq \mathrm{tail}(\mathcal{L})$ *has a maximally good coloring.*

Proof. We intend to apply Theorem 5.3. To this end, it is enough to show that for every $\mathbf{K} \subseteq \mathrm{tail}(\mathcal{L})$ there is a $\mathbf{J} \subseteq \mathbf{K}$ satisfying $\bigcup \mathbf{K} = \bigcup \mathbf{J}$ such that $\mathbf{J}$ has a maximally good coloring.

Let $\mathbf{K} \subseteq \mathrm{tail}(\mathcal{L})$ be a cover. For some regular cardinal κ there is a strictly increasing chain $\mathbf{J} = \{J_\nu \colon \nu < \kappa\}$ of elements of $\mathbf{K}$ such that $\bigcup \mathbf{J} = \bigcup \mathbf{K}$. Note that we may have $\kappa = 1$.

Let $f \colon \kappa \to \kappa$ be a κ-abundant map, i.e. for every $\lambda < \kappa$ we have $|f^{-1}(\lambda)| = \kappa$. Define $c \colon \mathbf{J} \to \mathrm{On}$ by $c(J_\lambda) = f(\lambda)$ $(\lambda < \kappa)$. Clearly, c is a maximally good coloring of $\mathbf{J}$, which completes the proof. ∎

LEMMA 5.5. *Fix* $a \in L$. *Let* $\mathbf{H}$ *be a simple cover with* $\mathcal{H} \subseteq \mathrm{conv}(\mathcal{L})(a)$. *Then* $\mathbf{H}$ *has a maximally good coloring.*

Proof. Again, we intend to apply Theorem 5.3, thus it is enough to show that for every $\mathbf{K} \subseteq \mathrm{conv}(\mathcal{L})(a)$ there is a $\mathbf{J} \subseteq \mathbf{K}$ satisfying $\bigcup \mathbf{K} = \bigcup \mathbf{J}$ such that $\mathbf{J}$ has a maximally good coloring.

So let $\mathbf{K} \subseteq \mathrm{conv}(\mathcal{L})(a)$ be a cover. By the definition of maximally good coloring we can assume $\bigcup \mathbf{K} = L$. For some regular cardinal κ there is a family $\mathbf{J}^+ = \{J_\nu \colon \nu < \kappa\}$ of elements of $\mathbf{K}$ such that $\{J_\nu \cap [a, +\infty) \colon \nu < \kappa\}$ is strictly increasing and $\mathbf{J}^+$ covers $[a, +\infty)$. Note that we may have $\kappa = 1$. We can apply Lemma 5.4 for $\mathbf{J}^+$ as a cover over $(-\infty, a]$ to obtain a maximally good coloring $c \colon \mathbf{J}^+ \to \mathrm{On}$ of $\mathbf{J}^+$ over $(-\infty, a]$.

Let $f \colon \kappa \to \kappa$ be a κ-abundant map. Define $h \colon \kappa \to \kappa$ by

$$h(\beta) = \begin{cases} f(\xi + n) & \text{if } \beta = \xi + 2n + 1 \text{ for some } \xi \in \mathrm{Lim}(\kappa), \\ \xi + n & \text{if } \beta = \xi + 2n \text{ for some } \xi \in \mathrm{Lim}(\kappa), \end{cases}$$

and let $d^+ \colon \mathbf{J}^+ \to \mathrm{On}$, $d^+(J_\nu) = h(c(J_\nu))$ $(\nu < \kappa)$. Then d^+ is a maximally good coloring of $\mathbf{J}^+$.

If $\mathbf{J}^+$ covers L, then $\mathbf{J} = \mathbf{J}^+$ satisfies the requirements. If not, take $\mathbf{K}' = \mathbf{K} \setminus \mathbf{J}^+$. Then $\mathbf{K}'$ covers $(-\infty, a]$, so by repeating the previous argument for $(-\infty, a]$ instead of $[a, +\infty)$ we can find a family $\mathbf{J}^- \subseteq \mathbf{K}'$ covering $(-\infty, a]$ with a maximally good coloring d^-.

Put $\mathbf{J} = \mathbf{J}^+ \cup \mathbf{J}^-$ and $d = d^- \cup d^+$. Then $\bigcup \mathbf{J} = L$ and d is a maximally good coloring of $\mathbf{J}$, which completes the proof. ∎

Proof of Theorem 5.1. By Proposition 2.3 we can assume $\mathbf{H}$ is simple. By Theorem 5.3 it is enough to prove that for every cover $\mathbf{K} \subseteq \operatorname{conv}(\mathcal{L})$ there is a subfamily $\mathbf{J} \subseteq \mathbf{K}$ such that

(a) $\bigcup \mathbf{J} = \bigcup \mathbf{K}$,
(b) $\mathbf{J}$ has a maximally good coloring $c_{\mathbf{J}}$.

Consider the equivalence relation R on L generated by the relation $\bigcup\{I \times I : I \in \mathbf{K}\}$. The equivalence classes of R give a partition of L and every $I \in \mathbf{K}$ is contained in some equivalence class. Hence we can construct $\mathbf{J}$ and $c_{\mathbf{J}}$ for each equivalence class separately. Therefore we can assume we have only one equivalence class. Hence for every $z^-, z^+ \in L$, $[z^-, z^+]$ can be covered by finitely many members of $\mathbf{K}$.

Let $z \in L$ be arbitrary.

Proposition 5.6. *If*

($\circ$) *for each $x \in [z, +\infty)$ there is $y \in [z, +\infty)$ such that $\bigcup \mathbf{K}(x) \subseteq (-\infty, y]$*

then there is $\mathbf{J}^+ \in [\mathbf{K}]^\omega$ such that

($\circ\circ$) *$\mathbf{J}^+$ covers $[z, +\infty)$ and $|\mathbf{J}^+(x)| < \omega$ for each $x \in L$.*

Proof. We define recursively a partition $\{L(n) : n \in \omega\}$ of $[z, +\infty)$ by setting $L(0) = \{z\}$, and for $0 < n < \omega$,

$$L(n) = \{y \in [z, +\infty) : I \cap L(n-1) \neq \emptyset \text{ for some } I \in \mathbf{K}(y)\} \setminus \bigcup_{k<n} L(k).$$

Since L is one equivalence class of R, $[z, +\infty) = \bigcup_{n<\omega} L(n)$ indeed. Note that some $L(n)$ can be empty, e.g. if L has a maximal element.

We show that for each $n < \omega$ there is an $\mathbf{I}_n \in [\mathbf{K}]^{\leq 2}$ such that $\mathbf{I}_n$ covers $L(n)$ and $I \cap L(n) \neq \emptyset$ ($I \in \mathbf{I}_n$). This is obvious if $n = 0$ or $L(n) = \emptyset$. If $n \neq 0$, $L(n) \neq \emptyset$ but $L(n+1) = \emptyset$ then by ($\circ$), L has a maximal element m and $m \in L(n)$. By definition, there is an $I \in \mathbf{K}(m)$ with $I \cap L(n-1) \neq \emptyset$, so $\mathbf{I}_n = \{I\}$ fulfills the requirements. Finally if $n \neq 0$, $L(n) \neq \emptyset$ and $L(n+1) \neq \emptyset$ then pick a $y \in L(n+1)$. By definition, there is an $I \in \mathbf{K}$ with $y \in I$ and $I \cap L(n) \neq \emptyset$. Let $y' \in I \cap L(n)$ and let $I' \in \mathbf{K}$ with $y' \in I'$ and $I' \cap L(n-1) \neq \emptyset$. Then $\mathbf{I}_n = \{I, I'\}$ fulfills the requirements.

Let $\mathbf{J}^+ = \bigcup\{\mathbf{I}_n : n < \omega\}$. Since $[z, +\infty) = \bigcup_{n<\omega} L(n)$, $\mathbf{J}^+$ covers $[z, +\infty)$. Observe that for each $n < \omega$, if $x \in L(n)$ then $I \in \mathbf{I}_{n+2}$ implies $x \notin I$. Hence $|\mathbf{J}^+(x)| < \omega$ ($x \in L$), as required. ∎

Let us return to the proof of Theorem 5.1. If ($\circ$) holds then let $z^+ = z$ and fix a family $\mathbf{J}^+ \in [\mathcal{I}]^\omega$ satisfying ($\circ\circ$). Otherwise pick $z^+ \in [z, +\infty)$ such that $\mathbf{K}(z^+)$ covers $[z^+, +\infty)$ and let $\mathbf{J}^+ = \mathbf{K}(z^+)$.

By applying Proposition 5.6 to L with reversed order, we can show that if

($\diamond$) for each $x \in (-\infty, z]$ there is $y \in (-\infty, z]$ such that $\bigcup \mathbf{K}(x) \subseteq [y, +\infty)$

then there is a family $\mathbf{J}^- \in [\mathbf{K}]^\omega$ such that

($\diamond\diamond$) $\mathbf{J}^-$ covers $(-\infty, z]$ and $|\mathbf{J}^-(x)| < \omega$ for each $x \in L$.

If ($\diamond$) holds let $z^- = z$ and fix a family $\mathbf{J}^-$ satisfying ($\diamond\diamond$). Otherwise pick $z^- \in (-\infty, z]$ such that $\mathbf{K}(z^-)$ covers $(-\infty, z^-]$ and let $\mathbf{J}^- = \mathbf{K}(z^-)$. Finally pick $\mathbf{J}^0 \in [\mathbf{K}]^{<\omega}$ which covers $[z^-, z^+]$. Let $\mathbf{J} = \mathbf{J}^- \cup \mathbf{J}^0 \cup \mathbf{J}^+$. Then $\mathbf{J}$ covers L.

The families $\mathbf{J}^+$, $\mathbf{J}^- \setminus \mathbf{J}^+$ and $\mathbf{J}^0 \setminus (\mathbf{J}^+ \cup \mathbf{J}^-)$ have maximally good colorings c^+, c^- and c^0 respectively, because they are either "locally finite" or Lemma 5.5 can be applied. Thus $c_{\mathbf{J}} = c^+ \cup c^- \cup c^0$ is a maximally good coloring of $\mathbf{J}$. ∎

We close this section with the proof of Theorem 5.2.

Proof of Theorem 5.2. If k is an infinite cardinal the statement follows immediately from Theorem 5.1. So let $k < \omega$; we prove the statement by induction on k. For $k = 1$ the statement is trivial.

Let $k \geq 2$ and suppose the statement is true for $k - 1$. As in the proof of Theorem 5.1, consider the equivalence relation R on L generated by the relation $\bigcup\{H \times H : H \in \mathcal{H}\}$. The equivalence classes of R give a partition of L and every $H \in \mathcal{H}$ is contained in some equivalence class, hence we can construct the k-good coloring of $\mathbf{H}$ for each equivalence class separately. Therefore we can assume that we have only one equivalence class.

PROPOSITION 5.7. *Let $I \subseteq L$ be a convex set and $y \in I$. If $\mathbf{H}$ has a k-good coloring over $I \cap (-\infty, y]$ and another k-good coloring over $I \cap [y, +\infty)$ then it has a k-good coloring over I as well.*

Proof. Fix two k-good colorings $c_- : \mathbf{H} \to k$ and $c_+ : \mathbf{H} \to k$ over $I \cap (-\infty, y]$ and $I \cap [y, +\infty)$, respectively. By thinning out the domain of c_- we can assume that for each $i < k$ the family $[c_-^{-1}(i)](y)$ has an enumeration $\{J_-^i(\gamma) : \gamma < \kappa_i\}$ for some regular cardinal κ_i such that $\{J_-^i(\gamma) \cap (-\infty, y] : \gamma < \kappa_i\}$ is strictly increasing and so for each cofinal subset $\Gamma \subseteq \kappa_i$ the family $(c_-^{-1}(i) \setminus [c_-^{-1}(i)](y)) \cup \{J_-^i(\gamma) : \gamma \in \Gamma\}$ covers $I \cap (-\infty, y]$. Let us remark that κ_i can be finite, namely 1.

Similarly, we can thin out the domain of c_+ so that for each $i < k$ the family $[c_+^{-1}(i)](y)$ has an enumeration $\{J_+^i(\gamma) : \gamma < \lambda_i\}$ for some regular cardinal λ_i such that for each cofinal subset $\Gamma \subseteq \lambda_i$ the family $(c_+^{-1}(i) \setminus [c_+^{-1}(i)](y)) \cup \{J_+^i(\gamma) : \gamma \in \Gamma\}$ covers $I \cap [y, +\infty)$.

Then by passing to cofinal subsets of $[c_-^{-1}(i)](y)$ and $[c_+^{-1}(i)](y)$ we can assume that for each $i, j < k$ if $[c_-^{-1}(i)](y) \cap [c_+^{-1}(j)](y) \neq \emptyset$ then $\kappa_i = \lambda_j = 1$ and so $[c_-^{-1}(i)](y) = [c_+^{-1}(j)](y)$. So there is a bijection $f : k \to k$ such that if $[c_-^{-1}(i)](y) \cap [c_+^{-1}(j)](y) \neq \emptyset$ then $j = f(i)$.

Define $c\colon \mathbf{H} \to k$ by $c(\langle H, \alpha\rangle) = i$ if $c_-(\langle H, \alpha\rangle) = i$ or $c_+(\langle H, \alpha\rangle) = f(i)$ ($\langle H, \alpha\rangle \in \mathbf{H}$). The definition of c is valid and c is a k-good coloring of $\mathbf{H}$ over I. This completes the proof. ∎

Define the relation $\equiv$ on L by $x \equiv y$ if and only if there exists a k-good coloring of $\mathbf{H}$ over $[x, y]$. By Proposition 5.7, $\equiv$ is an equivalence relation on L. Moreover, we have the following.

Proposition 5.8. *For every $H \in \mathcal{H}$, H is contained in one equivalence class of $\equiv$.*

Proof. Let $H \in \mathcal{H}$ and $\{x, y\} \in [H]^2$. Then $\mathbf{H} \setminus \{\langle H, 0\rangle\}$ is a $(k-1)$-fold cover of L. Hence by the inductive hypothesis, $\mathbf{H}\setminus\{\langle H, 0\rangle\}$ has a $(k-1)$-good coloring $c\colon \mathbf{H}\setminus\{\langle H, 0\rangle\} \to k-1$ over $[x, y]$. Extend c by setting $c(\langle H, 0\rangle) = k$; then c is a k-good coloring over $[x, y]$. ∎

Proposition 5.9. *Let E be an equivalence class of $\equiv$. Then there is a k-good coloring of $\mathbf{H}$ over E.*

Proof. Take an arbitrary $y \in E$. Since E is convex, by Proposition 5.7 it is enough to prove that $\mathbf{H}$ has a k-good coloring over $E \cap [y, +\infty)$ and over $E \cap (-\infty, y]$. We only prove that $\mathbf{H}$ has a k-good coloring over $E \cap [y, +\infty)$, the proof of the other statement is similar. We distinguish several cases.

Suppose first that there is $H \in \mathcal{H}$ such that H is cofinal in E. Fix $z \in H \cap E \cap [y, +\infty)$; then $[z, +\infty) \subseteq [y, +\infty)$. So $\mathbf{H}\setminus\{\langle H, 0\rangle\}$ is a $k-1$-fold cover of $E \cap [z, +\infty)$. Hence by the inductive hypothesis, there is $c\colon \mathbf{H}\setminus\{\langle H, 0\rangle\} \to k-1$ that is a $k-1$-good coloring of $\mathbf{H} \setminus \{\langle H, 0\rangle\}$ over $E \cap [z, +\infty)$. Then extending c by setting $c(\langle H, 0\rangle) = k$ yields a k-good coloring of $\mathbf{H}$ over $E \cap [z, +\infty)$. Since $y \equiv z$, $\mathbf{H}$ has a k-good coloring over $[y, z]$, so by Proposition 5.7 we find that $\mathbf{H}$ has a k-good coloring over $E \cap [y, +\infty)$ as well.

From now on assume that there is no $H \in \mathcal{H}$ such that H is cofinal in $E \cap [y, +\infty)$. If there is $z \in [y, +\infty)$ such that $\bigcup \mathcal{H}(z)$ is cofinal in E, then since for $H \in \mathcal{H}$, H is not cofinal in E, $\mathbf{H}$ has a k-good coloring over $E \cap [z, +\infty)$. Since $y \equiv z$, $\mathbf{H}$ has a k-good coloring over $[y, z]$. So by Proposition 5.7, $\mathbf{H}$ has a k-good coloring over $E \cap [y, +\infty)$ as well.

In what follows we assume in addition that for every $z \in E \cap [y, +\infty)$, $\bigcup \mathcal{H}(z)$ has an upper bound in E. We define recursively a strictly increasing sequence $(x_n)_{n<\omega} \subseteq E \cap [y, +\infty)$ as follows. Let $x_0 = y$. If $0 < n < \omega$ and x_{n-1} is already defined, let b_{n-1} be an upper bound of $\bigcup \mathbf{H}(x_{n-1})$, and let x_n be an upper bound of $\bigcup \mathbf{H}(b_{n-1})$. Then $\bigcup \mathbf{H}(x_{n-1}) \subseteq (-\infty, b_{n-1}]$ and $\bigcup \mathbf{H}(x_n) \subseteq (b_{n-1}, +\infty)$ imply

$$\mathbf{H}(x_n) \cap \mathbf{H}(x_{n'}) = \emptyset \qquad (n < n' < \omega),$$

and by our assumption that L is one equivalence class of R, $\{x_n : n < \omega\}$ is cofinal in E.

For every $n < \omega$ we have $x_n \equiv x_{n+1}$ so there is $c_n \colon \mathbf{H} \to k$ that is a k-good coloring of $\mathbf{H}$ over $[x_n, x_{n+1}]$. Fix $n < \omega$; by thinning out the domain of c_n we can assume that for each $i < k$ the family $[c_n^{-1}(i)](x_n)$ has an enumeration $\{J_n^i(\gamma) \colon \gamma < \kappa_n^i\}$ for some regular cardinal κ_n^i such that $\{J_n^i(\gamma) \cap [x_n, x_{n+1}] \colon \gamma < \kappa_n^i\}$ is strictly increasing, and so for each cofinal subset $\Gamma \subseteq \kappa_n^i$ the family $(c_n^{-1}(i) \setminus [c_n^{-1}(i)](x_n)) \cup \{J_n^i(\gamma) \colon \gamma \in \Gamma\}$ covers $[x_n, x_{n+1}]$. Similarly, we can assume that for each $i < k$ the family $[c_n^{-1}(i)](x_{n+1})$ has an enumeration $\{B_n^i(\gamma) \colon \gamma < \lambda_n^i\}$ for some regular cardinal λ_n^i such that $\{B_n^i(\gamma) \cap [x_n, x_{n+1}] \colon \gamma < \lambda_n^i\}$ is strictly increasing, therefore for each cofinal subset $\Gamma \subseteq \lambda_n^i$ the family $(c_n^{-1}(i) \setminus [c_n^{-1}(i)](x_{n+1})) \cup \{B_n^i(\gamma) \colon \gamma \in \Gamma\}$ covers $[x_n, x_{n+1}]$.

Then for every $n < \omega$ and $i < k$, we can pass to cofinal subsets of $[c_n^{-1}(i)](x_{n+1})$ and $[c_{n+1}^{-1}(i)](x_{n+1})$ in such a way that for each $i, j < k$ if $[c_n^{-1}(i)](x_{n+1}) \cap [c_{n+1}^{-1}(j)](x_{n+1}) \neq \emptyset$ then $\lambda_n^i = \kappa_{n+1}^j = 1$ and so $[c_n^{-1}(i)](x_{n+1}) = [c_{n+1}^{-1}(j)](x_{n+1})$. So there is a bijection $f_n \colon k \to k$ such that if $[c_n^{-1}(i)](x_{n+1}) \cap [c_{n+1}^{-1}(j)](x_{n+1}) \neq \emptyset$ then $j = f_n(i)$. Write $g_0 = \mathrm{Id}$ and $g_n = f_{n-1} \circ f_{n-2} \circ \cdots \circ f_0$ $(0 < n < \omega)$.

For every $\langle H, \alpha \rangle \in \mathbf{H}$ and $i < k$ define $c(\langle H, \alpha \rangle) = i$ if and only if for some $n < \omega$, $\langle H, \alpha \rangle \in \mathrm{dom}(c_n)$ and $c_n(\langle H, \alpha \rangle) = g_n(i)$. This definition makes sense and c is a k-good coloring of $\mathbf{H}$ over $E \cap [y, +\infty)$, which completes the proof. ∎

We are ready to complete the proof of Theorem 5.2. By assumption, L is one equivalence class of the relation R. So by Proposition 5.8, L is one equivalence class of $\equiv$. Therefore by Proposition 5.9 there is a k-good coloring of $\mathbf{H}$, which finishes the proof. ∎

6. Closed sets. Towards the investigation of splitting of covers with special geometric properties let us tackle closed covers, i.e. the variant of the problem where the sets in the cover are closed. The study of this special case is motivated by the facts that, apart from considering open covers, this is the simplest topological constraint one can impose; even for closed covers we get independence from ZFC by very strong means; and these results will be very useful for treating the problem of covers by compact convex sets.

Obviously, we have to specify the topological spaces where closed covers are considered. Observe that similarly to the proof of Theorem 6.1 below, the construction of Theorem 3.1.2 can be carried out in such a way that the covering sets H_α are closed in 2^κ endowed with the product topology. Since our purpose is not to find suitable topologies for general constructions but to establish independence from ZFC for natural topological spaces, in this section we restrict our attention to covers of $\mathbb{R}$, or equivalently to covers of ω^ω and 2^ω. We refer to [3] and [7] for basic notions in descriptive set theory.

As we shall see in Proposition 6.9, if $\mathbf{H}$ is a closed cover of $\mathbb{R}$ and $|\mathbf{H}| < \operatorname{cov}(\mathcal{M})$ then $\mathbf{H}$ has a countable subcover. In particular, for $\omega < \kappa < \operatorname{cov}(\mathcal{M})$, a κ-fold closed cover *of cardinality* κ has a κ-good coloring. There are models of ZFC where even Borel covers of special cardinalities of the real line satisfy a similar Lindelöf like property. In [9], A. Miller showed that in a model obtained from a model of CH by adding ω_3 Cohen reals, every cover of $\mathbb{R}$ by ω_2 Borel sets has an ω_1-fold subcover. Here the corresponding splitting result says that if $\mathbf{H}$ is an ω_2-fold Borel cover of $\mathbb{R}$ and $|\mathcal{H}| = \omega_2$ then $\mathbf{H}$ has an ω_2-good coloring. However, these are very special settings as far as splitting is concerned, so we do not pursue our investigations in this direction. For more background on covering numbers related to closed sets see [10].

In this section our main results are the following. In Theorem 6.1 we prowe that if $\mathrm{MA}_\kappa(\sigma\text{-centered})$ holds, then there exists a κ-fold closed cover of $\mathbb{R}$, consisting of translates of *one* compact set, which cannot be partitioned into two subcovers. In particular, we prove in ZFC that there exists an ω-fold closed cover of $\mathbb{R}$, consisting of translates of one compact set, which cannot be partitioned into two subcovers. Finally in Theorem 6.5 we establish that in the Cohen real model every closed cover of $\mathbb{R}$ has an $[\omega_1, \infty)$-good coloring. In this section X denotes any of $\mathbb{R}$, ω^ω or 2^ω; and 2^ω is identified with $\mathcal{P}(\omega)$ the usual way.

6.1. Martin's Axiom. This section is devoted to the following theorem.

Theorem 6.1. *Let κ be a cardinal satisfying $\omega \leq \kappa < 2^\omega$ and assume* MA_κ*(σ-centered). Then there exists a κ-fold simple closed cover of X which cannot be decomposed into two disjoint subcovers. Moreover, in $\mathbb{R}$ the cover can consist of translates of one compact set.*

Since $\mathrm{MA}_\omega(\sigma\text{-centered})$ holds in ZFC we obtain the following corollary.

Corollary 6.2. *There exists an ω-fold closed cover of X which cannot be partitioned into two subcovers. If $X = \mathbb{R}$ the cover can consist of translates of one compact set.*

We prove Theorem 6.1 first in $\mathbb{R}$ since there we need to construct the cover using translates of one compact set. We fix some notation in advance. For a set $F \subseteq \mathbb{R}$ let $\langle F \rangle_{\mathbb{Q}}$ denote the linear span of F in $\mathbb{R}$ considered as a vector space over the rationals $\mathbb{Q}$. We set $\Sigma = 4^\omega$.

In order to construct a cover of $\mathbb{R}$ using translates of one compact set, we need the following auxiliary construction.

Lemma 6.3. *There exist perfect sets $F, W \subseteq [0,1]$ and a sequence $(v_n)_{n<\omega} \subseteq [0,1]$ with $\lim_{n\to\infty} v_n = 0$ such that*

(i) $F = \{\sum_{i<\omega} \sigma(i)/4^{k_i} : \sigma \in \Sigma\}$ *for some sequence* $(k_i)_{i<\omega} \subseteq \omega \setminus \{0\}$;
(ii) $W \cup \{v_n : n < \omega\}$ *is linearly independent over* $\mathbb{Q}$;
(iii) $\langle F\rangle_{\mathbb{Q}} \cap \langle W \cup \{v_n : n < \omega\}\rangle_{\mathbb{Q}} = \{0\}$.

Proof. Let $(j_i)_{i<\omega} \subseteq \omega \setminus \{0\}$ satisfy $j_{i+1} - j_i > i$ $(i < \omega)$. Write $k_i = j_{2i}$ and $\ell_i = j_{2i+1}$ $(i < \omega)$, and set $F = \{\sum_{i<\omega} \sigma(i)/4^{k_i} : \sigma \in \Sigma\}$ and $U = \{\sum_{i<\omega} \sigma(i)/4^{\ell_i} : \sigma \in \Sigma\}$. Then (i) holds.

By [11, Theorem 1 p. 141], there is a nonempty perfect set $U' \subseteq U$ such that $U' \cup \{1\}$ is linearly independent over $\mathbb{Q}$; in particular, $\langle U'\rangle_{\mathbb{Q}} \cap \mathbb{Q} = \{0\}$. Let $(w_n)_{n<\omega} \subseteq U'$ be a strictly decreasing sequence. Set $v_n = w_n/(n+1)$ $(n < \omega)$ and let $W \subseteq U' \setminus \{w_n : n < \omega\}$ be a nonempty perfect set. Then $\lim_{n\to\infty} v_n = 0$, and (ii) holds.

It remains to verify (iii). First we show that $\langle F\rangle_{\mathbb{Q}} \cap \langle U\rangle_{\mathbb{Q}} \subseteq \mathbb{Q}$. To see this, for some $m, n < \omega$, let $f_a \in F$, $p_a \in \mathbb{Q} \setminus \{0\}$ $(a < m)$ and $u_b \in U$, $q_b \in \mathbb{Q} \setminus \{0\}$ $(b < n)$ satisfy

$$\sum_{a<m} p_a f_a = \sum_{b<n} q_b u_b. \tag{1}$$

For every $a < m$ and $b < n$, let $\sigma_a, \tau_b \in \Sigma$ be such that $f_a = \sum_{i<\omega} \sigma_a(i)/4^{k_i}$ and $u_b = \sum_{i<\omega} \tau_b(i)/4^{l_i}$. By multiplying both sides of (1) with an appropriate integer, we can assume $p_a, q_b \in \mathbb{Z}$ $(a < m,\ b < n)$. Let $j < \omega$ satisfy

$$3\max\Big\{\sum_{a<m} |p_a|, \sum_{b<n} |q_b|\Big\} < j. \tag{2}$$

It is enough to show that $\sum_{a<m} p_a\sigma_a(h) = 0$ for every $j < h < \omega$; then the sums in (1) have a rational value.

So suppose that $\sum_{a<m} p_a\sigma_a(h) \neq 0$ for some $j < h < \omega$. We consider every real in its base 4 decimal expansion, and for every $c < \omega$, the cth digit of $r \in \mathbb{R}$ is the coefficient of 4^{-c} in this expansion of r.

By (2) and since $j < h$, we have $3\sum_{a<m} |p_a| < 4^{k_h - l_{h-1}}$, so there are $l_{h-1} < c_0 < c_1 \le k_h$ such that the c_0th and c_1th digits of $\sum_{a<m} p_a f_a = \sum_{i<\omega} (\sum_{a<m} p_a\sigma_a(i))/4^{k_i}$ in its base 4 expansion differ. Also since $j < h$, we have $3\sum_{b<n} |q_b| < 4^{l_h - k_h}$. So the dth digits of

$$\sum_{j<n} q_b u_b = \sum_{i<\omega} \Big(\sum_{b<n} q_b\tau_b(i)\Big)/4^{l_i}$$

for $l_{h-1} < d \le k_h$ are either all 0 or all 3. This contradicts (1), so the proof of $\langle F\rangle_{\mathbb{Q}} \cap \langle U\rangle_{\mathbb{Q}} \subseteq \mathbb{Q}$ is complete.

Now $W \cup \{w_n : n < \omega\} \subseteq U' \subseteq U$ so $\langle F\rangle_{\mathbb{Q}} \cap \langle W \cup \{v_n : n < \omega\}\rangle_{\mathbb{Q}} \subseteq \mathbb{Q}$. Also as $W \cup \{w_n : n < \omega\} \subseteq U'$ we have $\langle W \cup \{v_n : n < \omega\}\rangle_{\mathbb{Q}} \cap \mathbb{Q} = \{0\}$. To summarize, $\langle F\rangle_{\mathbb{Q}} \cap \langle W \cup \{v_n : n < \omega\}\rangle_{\mathbb{Q}} = \{0\}$, as stated. ∎

Once we have the compact set, its translates will be coded by the members of an almost disjoint family in $[\omega]^\omega$ of size κ. In the end we will need the following amended version of Solovay's Lemma.

LEMMA 6.4. (MA_κ(σ-*centered*)) *Let* $\mathcal{A} \subseteq [\omega]^\omega$ *be an almost disjoint family of size* κ. *Let* $\mathcal{B} \subseteq \mathcal{A}$ *and suppose that for every* $A \in \mathcal{B}$ *a set* $C_A \in [A]^\omega$ *is given. Then there exists* $X \in [\omega]^\omega$ *such that*

1. $\max(X \cap A) \in C_A$ *for* $A \in \mathcal{B}$;
2. $|X \cap A| = \omega$ *for* $A \in \mathcal{A} \setminus \mathcal{B}$.

Proof. Let

$$P = \{\langle x, b\rangle : x \in [\omega]^{<\omega},\ b \in [\mathcal{B}]^{<\omega},\ \max(x \cap B) \in C_B \text{ for } B \in b\},$$

and put $\langle x, b\rangle \leq_P \langle x', b'\rangle$ if and only if $x' \subseteq x$, $b' \subseteq b$ and $x \cap B' = x' \cap B'$ for each $B' \in b'$. Since the conditions $\langle x, b_0\rangle, \langle x, b_1\rangle, \ldots, \langle x, b_{n-1}\rangle$ have the joint extension $\langle x, b_0 \cup b_1 \cup \cdots \cup b_{n-1}\rangle$, $P = \bigcup\{\{\langle x, b\rangle : b \in [\mathcal{B}]^{<\omega}\} : x \in [\omega]^{<\omega}\}$ shows that $\langle P, \leq_P\rangle$ is σ-centered.

For every $B \in \mathcal{B}$ the set $D_B = \{\langle x, b\rangle : B \in b\}$ is dense in P since if $B \notin b$ then we have $n \in C_B \setminus \max(B \cap \bigcup b)$ and $\langle x \cup \{n\}, b \cup \{B\}\rangle \leq \langle x, b\rangle$ is in D_B.

For every $A \in \mathcal{A} \setminus \mathcal{B}$ and $m < \omega$ the set $D_{A,m} = \{\langle x, b\rangle : \max(x \cap A) \geq m\}$ is dense in P since for $n \in (A \setminus m) \setminus \bigcup b$, $\langle x \cup \{n\}, b\rangle \leq \langle x, b\rangle$ is in $D_{A,m}$.

If G is a $\{D_B : B \in \mathcal{B}\} \cup \{D_{A,m} : A \in \mathcal{A} \setminus \mathcal{B}, m < \omega\}$-generic filter then $X = \bigcup\{x : \langle x, b\rangle \in G\}$ satisfies the requirements. ∎

Proof of Theorem 6.1. We can apply Lemma 6.3 to get $(k_i)_{i<\omega} \subseteq \omega \setminus \{0\}$, $(v_n)_{n<\omega} \subseteq [0,1]$, $W \subseteq [0,1]$ and $F = \{\sum_{i<\omega} \sigma(i)/4^{k_i} : \sigma \in \Sigma\}$ such that (ii) and (iii) of Lemma 6.3 hold. Let $\mathcal{A} \subseteq [\omega]^\omega$ be an almost disjoint family of size κ and for every $A \in \mathcal{A}$ set $x(A) = \sum_{i<\omega} \chi_A(i)/4^{k_i}$. Recall $\Sigma = 4^\omega$, and for every $n < \omega$ let $\Sigma_n = \{\sigma \in \Sigma : n = \max\{i < \omega : \sigma(i) = 2\}\}$.

For every $n < \omega$ let $F_n = \{\sum_{i<\omega} \sigma(i)/4^{k_i} : \sigma \in \Sigma_n\}$ and set

$$K = W \cup F \cup \bigcup\{F_n + v_n : n < \omega\}.$$

Note that F_n $(n < \omega)$ are closed,

$$F \cup \bigcup\{F_n + v_n : n < \omega\} \subseteq [0,2], \qquad 0 \in K \subseteq [0,2], \tag{3}$$

and $\lim_{n\to\infty} v_n = 0$ implies $\lim_{n\to\infty}(F_n + v_n) = F$, hence K is a compact set. We define

$$K_{n,A} = K + x(A) - v_n \quad (A \in \mathcal{A},\ n < \omega)$$

and $\mathbf{H}_0 = \{K_{n,A} : A \in \mathcal{A},\ n < \omega\}$.

Set $Z = \{z \in F : |\mathbf{H}_0(z)| = \kappa\}$. Let $\mathbf{H}_1$ consist of all translates of K which avoid Z and do not show up in $\mathbf{H}_0$, i.e.

$$\mathbf{H}_1 = \{K + d : (K + d) \cap Z = \emptyset,\ d \neq x(A) - v_n\ (A \in \mathcal{A},\ n < \omega)\}.$$

We show that the simple closed cover $\mathbf{H} = \mathbf{H}_0 \cup \mathbf{H}_1$ of $\mathbb{R}$ is κ-fold and has no two disjoint subcovers over F.

Pick an arbitrary $x \in \mathbb{R}$. If $|\mathbf{H}_0(x)| \geq \kappa$ we are done; so suppose $|\mathbf{H}_0(x)| < \kappa$. If $(K+x-w)\cap Z = \emptyset$ for every $w \in W$, then $|\mathbf{H}_1(x)| \geq \kappa$ by the definition of $\mathbf{H}_1$. Similarly, if $(K+x-f)\cap Z = \emptyset$ for every $f \in F$, then again $|\mathbf{H}_1(x)| \geq \kappa$. If these cases fail to happen, then there are $w \in W$, $f \in F$, $y_1, y_2 \in K$ and $z_1, z_2 \in Z$ such that $z_1 = y_1 + x - w$ and $z_2 = y_2 + x - f$. Thus $x = z_1 + w - y_1 = z_2 + f - y_2$.

Let D be a Hamel basis of $\mathbb{R}$ extending $W \cup \{v_n : n < \omega\}$. By Lemma 6.3(iii), for every $y \in K$, if $y \in W$ then the expression of y in the Hamel basis D is y, while if $y \notin W$ then in the expression of y in the Hamel basis D no member of W appears.

Consider the expression of x in the Hamel basis D. We have $y_1, y_2 \in K$ and $z_1, z_2, f \in F$, so in particular, by Lemma 6.3(iii) we have $z_1, z_2, f \in K \setminus W$. Thus in the expression of $z_2 + f - y_2$ in the Hamel basis D no member of W appears with positive coefficient. This implies $y_1 = w$, hence $x = z_1$. Thus $x \in Z$ and so $|\mathbf{H}_0(x)| = \kappa$.

It remains to see that $\mathbf{H}$ has no two disjoint subcovers over F. Let $c\colon \mathbf{H}_0 \to 2$. We will find an $\varepsilon \in \{0,1\}$ and an $x \in F$ such that $|\mathbf{H}_0(x)| = \kappa$, $\mathbf{H}_1(x) = \emptyset$ and for every $A \in \mathcal{A}$ and $n < \omega$, $x \in K_{n,A}$ implies $c(K_{n,A}) = \varepsilon$. This will complete the proof.

For each $A \in \mathcal{A}$ there exists an $\varepsilon_A \in \{0,1\}$ and a $C_A \in [A]^\omega$ such that $c(K_{n,A}) = \varepsilon_A$ for $n \in C_A$. Then there are $\varepsilon \in \{0,1\}$ and $\mathcal{B} \in [\mathcal{A}]^\kappa$ such that $\varepsilon_B = \varepsilon$ for $B \in \mathcal{B}$.

By applying Lemma 6.4 we obtain $X \in [\omega]^\omega$ satisfying $\max(X \cap A) \in C_A$ $(A \in \mathcal{B})$ and $|X \cap A| = \omega$ $(A \in \mathcal{A} \setminus \mathcal{B})$. Let $x = \sum_{i<\omega}(1 + 2\chi_X(i))/4^{k_i}$, i.e. $x \in F$ and x has digits 1 and 3 only. We show that this x fulfills the requirements.

First we show $|\mathbf{H}_0(x)| = \kappa$. For every $z \in \mathbb{R}$ and $j < \omega$, let $z[j]$ denote the coefficient of 4^{-j} in the base 4 expansion of z. For every $A \in \mathcal{A}$ and $j < \omega$ we have

$$[x - x(A)][j] = \begin{cases} 3 & \text{if } j = k_i \text{ with } i \in X \setminus A, \\ 2 & \text{if } j = k_i \text{ with } i \in X \cap A, \\ 1 & \text{if } j = k_i \text{ with } i \in \omega \setminus (X \cup A), \\ 0 & \text{if } j = k_i \text{ with } i \in A \setminus X. \end{cases} \tag{4}$$

Thus for each $A \in \mathcal{B}$, $x - x(A) \in F_{\max(X\cap A)}$, hence

$$x \in F_{\max(X\cap A)} + v_{\max(X\cap A)} + x(A) - v_{\max(X\cap A)} \subseteq K_{\max(X\cap A),A}$$

and so $|\mathbf{H}_0(x)| = \kappa$. In particular, $x \in Z$ and so $\mathbf{H}_1(x) = \emptyset$.

It remains to show that for every $A \in \mathcal{A}$ and $n < \omega$, $x \in K_{n,A}$ implies $c(K_{n,A}) = \varepsilon$. Suppose $x \in K_{n,A}$ for some $A \in \mathcal{A}$ and $n < \omega$, i.e.

$$x \in K + x(A) - v_n = (W + x(A) - v_n) \\ \cup (F + x(A) - v_n) \cup \bigcup\{F_m + v_m + x(A) - v_n \colon m < \omega\}.$$

From $x, x(A) \in F$ and Lemma 6.3(iii),

$$x \notin W + x(A) - v_n,\ x \notin F + x(A) - v_n,\ x \notin F_m + v_m + x(A) - v_n\ (m \neq n),$$

hence $x \in F_n + x(A)$. By (4), for $A \in \mathcal{A} \setminus \mathcal{B}$ we have $x - x(A) \notin \bigcup_{n<\omega} F_n$. Thus $x \in K_{n,A}$ implies $A \in \mathcal{B}$. Again by (4) we have $n = \max(X \cap A) \in C_A$, so $c(K_{n,A}) = \varepsilon$. This completes the proof in $\mathbb{R}$.

If $X = \omega^\omega$ or $X = 2^\omega$ take a continuous surjective map $\varphi \colon X \to [0,2]$ and set $\mathbf{H}_X = \{\varphi^{-1}(H) \colon H \in \mathbf{H}\}$. Then $\mathbf{H}_X$ is clearly a κ-fold closed cover of X. It is a simple cover since $K \subseteq [0,2]$ implies that if $d_1, d_2 \in \mathbb{R}$ satisfy $d_1 \neq d_2$ and $(K+d_1) \cap [0,2] \neq \emptyset$, $(K+d_2) \cap [0,2] \neq \emptyset$ then $(K+d_1) \cap [0,2] \neq (K+d_2) \cap [0,2]$. Since $\mathbf{H}$ has no two disjoint subcovers over F and $F \subseteq [0,2]$, $\mathbf{H}_X$ has no two disjoint subcovers. ∎

Corollary 6.2 implies in particular that in a positive partition result for closed covers the points covered by ω many sets only must be ignored.

6.2. The Cohen real model. In this section we will prove that in the Cohen real model every closed cover of the reals has an $[\omega_1, \infty)$-good coloring. Note that by Corollary 6.2 it is impossible to get an ω-good coloring. Thus we have, in a sense, a best possible decomposition result. The proof is based on the weak Freese–Nation property (see Proposition 6.6 below), for which we need standard additional assumptions, such as GCH and $\square_\lambda$ for cardinals λ with $\mathrm{cf}(\lambda) = \omega$.

Following [8], we recall some notation. Let V be our ground model and let κ be a cardinal. We denote by V^{C_κ} the model obtained from V by adding κ many Cohen reals the usual way.

We will prove the following theorem.

THEOREM 6.5. *Suppose that GCH holds in V and let κ be a cardinal. Suppose also that in V we have $\square_\lambda$ for every cardinal λ satisfying $\omega < \lambda \leq |\kappa|$, $\mathrm{cf}(\lambda) = \omega$. In V^{C_κ}, let (X, τ) be a topological space which has a countable base, and let $\mathbf{H}$ be a cover of X by closed sets. Then in V^{C_κ} there exists an $[\omega_1, \infty)$-good coloring of $\mathbf{H}$.*

The proof of Theorem 6.5 is based on the fact that in V^{C_κ} the poset $(\mathcal{P}(\omega), \subseteq)$ has the *weak Freese–Nation property*. We recall it in the following proposition and we introduce the corresponding notion of good coloring on $\mathcal{P}(\omega)$.

Proposition 6.6 ([5, Theorem 15]). *Under the assumptions of Theorem 6.5, in V^{C_κ} the poset $(\mathcal{P}(\omega),\subseteq)$ has the weak Freese–Nation property, i.e. there is a function $f\colon \mathcal{P}(\omega)\to[\mathcal{P}(\omega)]^{\le\omega}$ such that for every $A,B\in\mathcal{P}(\omega)$ with $A\subseteq B$ there exists $C\in f(A)\cap f(B)$ satisfying $A\subseteq C\subseteq B$.*

Definition 6.7. Let $\mathcal{A},\mathcal{B}\subseteq\mathcal{P}(\omega)$ be arbitrary. A *$\mathcal{B}$-good coloring* of $\mathcal{A}$ is a function $c\colon \mathcal{A}\to\mathrm{On}$ such that for every $B\in\mathcal{B}$, $|\mathcal{P}(B)\cap\mathcal{A}|\ge\omega_1$ implies $|\mathcal{P}(B)\cap\mathcal{A}|\subseteq c[\mathcal{P}(B)\cap\mathcal{A}]$.

To get Theorem 6.5, it is enough to prove the following theorem.

Theorem 6.8. *Let $\mathcal{A},\mathcal{B}\subseteq\mathcal{P}(\omega)$ be arbitrary. Assume $(\mathcal{P}(\omega),\subseteq)$ has the weak Freese–Nation property. Then $\mathcal{A}$ has a $\mathcal{B}$-good coloring.*

Proof of Theorem 6.5. By Proposition 2.3 we can assume that $\mathbf{H}$ is simple, i.e. $\mathbf{H}=\mathcal{H}$. Let $\{U_n\colon n<\omega\}$ be a base of X. For every closed set $Z\subseteq X$, define $B(Z)=\{n<\omega\colon U_n\cap Z=\emptyset\}$. Since $Z\subseteq Z'$ if and only if $B(Z)\supseteq B(Z')$, B is injective.

Let $\mathcal{A}=\{B(H)\colon H\in\mathcal{H}\}$, $\mathcal{B}=\{B(\{x\})\colon x\in X\}$. By Theorem 6.8 we have a $\mathcal{B}$-good coloring $c^\star\colon \mathcal{A}\to\mathrm{On}$. We show that $c\colon \mathcal{H}\to\mathrm{On}$, $c=c^\star\circ B$, is an $[\omega_1,\infty)$-good coloring of $\mathcal{H}$.

To see this, let $x\in X$ satisfy $|\mathcal{H}(x)|\ge\omega_1$. Clearly, B is a bijection between $\mathcal{H}(x)$ and $\mathcal{P}(B(\{x\}))\cap\mathcal{A}$. Hence $|\mathcal{P}(B(\{x\}))\cap\mathcal{A}|\ge\omega_1$ and so $c[\mathcal{H}(x)]=c^\star[\mathcal{P}(B(\{x\}))\cap\mathcal{A}]\supseteq|\mathcal{P}(B(\{x\}))\cap\mathcal{A}|=|\mathcal{H}(\{x\})|$, as required. ∎

It remains to show Theorem 6.8.

Proof of Theorem 6.8. We prove the statement by induction on $\lambda=|\mathcal{A}\cup\mathcal{B}|$.

If $\lambda\le\omega$, an arbitrary coloring $c\colon \mathcal{A}\to\mathrm{On}$ works. Consider now $\lambda=\omega_1$. Enumerate $\mathcal{B}$ as $\{B_\alpha\colon \alpha<\omega_1\}$ so that each $B\in\mathcal{B}$ occurs ω_1 many times. We define $c\colon \mathcal{A}\to\omega_1$ by transfinite induction of length ω_1, extending c to at most one further member of $\mathcal{A}$ at each step as follows. For every $B\in\mathcal{B}$ let $I_B=\{\alpha<\omega_1\colon B_\alpha=B\}$. In the αth step of the coloring if $\alpha\in I_B$ and $|\mathcal{P}(B)\cap\mathcal{A}|=\omega_1$ pick one $A\in\mathcal{A}$ such that $c(A)$ is not defined yet and $A\in\mathcal{P}(B)$. Define $c(A)=\mathrm{tp}(\alpha\cap I_B)$. This coloring clearly fulfills the requirements.

Assume now that $\lambda>\omega_1$ and the statement holds for every $\omega_1\le\lambda'<\lambda$. Let $\mathcal{A},\mathcal{B}\subseteq\mathcal{P}(\omega)$ with $|\mathcal{A}\cup\mathcal{B}|=\lambda$. Let $f\colon \mathcal{P}(\omega)\to[\mathcal{P}(\omega)]^{\le\omega}$ be a function witnessing the weak Freese–Notion property of $\mathcal{P}(\omega)$. By closing $\mathcal{B}$ under f we can assume that $\mathcal{B}$ is f-closed.

Let $\langle M_\alpha : \omega_1\le\alpha<\lambda\rangle$ be a continuous, increasing sequence of models of a large enough fragment of ZFC such that $\mathcal{A},\mathcal{B},f\in M_{\omega_1}$, $M_\alpha\in M_{\alpha+1}$, $\alpha\subseteq M_\alpha$ and $|M_\alpha|=|\alpha|$ $(\omega_1\le\alpha<\lambda)$. Let $\mathcal{A}_\alpha=\mathcal{A}\cap(M_{\alpha+1}\setminus M_\alpha)$, $\mathcal{B}_\alpha=\mathcal{B}\cap M_{\alpha+1}$.

For every $\omega_1 \le \alpha < \lambda$ we have $|\mathcal{A}_\alpha \cup \mathcal{B}_\alpha| = |\alpha|$. So by the inductive hypothesis there is a coloring $c'_\alpha \colon \mathcal{A}_\alpha \to |\alpha|$ which is $\mathcal{B}_\alpha$-good for $\mathcal{A}_\alpha$. By Lemma 2.4, there is a function $h_\alpha \colon |\alpha| \to \alpha$ such that $h_\alpha[|\alpha|] = \alpha$ and $\kappa \subseteq h_\alpha[\kappa]$ for every cardinal $\omega \le \kappa < |\alpha|$. Let $c_\alpha = h_\alpha \circ c'_\alpha$. Then for every $B \in \mathcal{B}_\alpha$,

(i) $|\mathcal{P}(B) \cap \mathcal{A}_\alpha| \ge \omega_1$ implies $|\mathcal{P}(B) \cap \mathcal{A}_\alpha| \subseteq c_\alpha[\mathcal{P}(B) \cap \mathcal{A}_\alpha]$,
(ii) $|\mathcal{P}(B) \cap \mathcal{A}_\alpha| = |\alpha|$ implies $\alpha \subseteq c_\alpha[\mathcal{P}(B) \cap \mathcal{A}_\alpha]$.

Let $c = \bigcup\{c_\alpha \colon \alpha < \lambda\}$; the definition makes sense since for $\alpha \ne \beta$ we have $\mathcal{A}_\alpha \cap \mathcal{A}_\beta = \emptyset$. We show that c is a $\mathcal{B}$-good coloring.

Assume on the contrary that there is $B \in \mathcal{B}$ such that $|\mathcal{P}(B) \cap \mathcal{A}| \ge \omega_1$ but $|\mathcal{P}(B) \cap \mathcal{A}| \not\subseteq c[\mathcal{P}(B) \cap \mathcal{A}]$. Let $\alpha < \lambda$ be minimal such that we can have such a B in $M_{\alpha+1}$ and let $\mu \le \lambda$ be an *uncountable regular* cardinal such that $|\mathcal{P}(B) \cap \mathcal{A}| \ge \mu$ but $\mu \not\subseteq c[\mathcal{P}(B) \cap \mathcal{A}]$. We distinguish three cases.

Suppose first $|(\mathcal{P}(B) \cap \mathcal{A}) \setminus M_\alpha| \ge \mu$ and $\alpha \ge \mu$. Then as $\mu \subseteq M_{\alpha+1}$ we have $|((\mathcal{P}(B) \cap \mathcal{A}) \setminus M_\alpha) \cap M_{\alpha+1}| \ge \mu$. Hence $|\mathcal{A}_\alpha \cap \mathcal{P}(B)| \ge \mu$ and so $c[\mathcal{A} \cap \mathcal{P}(B)] \supseteq c_\alpha[\mathcal{A}_\alpha \cap \mathcal{P}(B)] \supseteq \mu$, a contradiction.

Suppose next $|(\mathcal{P}(B) \cap \mathcal{A}) \setminus M_\alpha| \ge \mu$ but $\alpha < \mu$. Let $\sigma \in \mu \setminus c[\mathcal{P}(B) \cap \mathcal{A}]$ and let $\beta = \max(\alpha, \sigma + 1) < \mu$. Then $|(\mathcal{P}(B) \cap \mathcal{A}) \setminus M_\beta| \ge \mu$ and so $\omega_1 \le \beta \subseteq M_\beta$ implies $|\mathcal{P}(B) \cap \mathcal{A}_\beta| = |\beta|$. Thus $\beta \subseteq c_\beta[\mathcal{P}(B) \cap \mathcal{A}_\beta]$ by (ii) and so $\sigma \in c_\beta[\mathcal{P}(B) \cap \mathcal{A}_\beta] \subseteq c[\mathcal{P}(B) \cap \mathcal{A}]$, a contradiction.

Finally suppose $|(\mathcal{P}(B) \cap \mathcal{A}) \setminus M_\alpha| < \mu$. As $\nu = |M_\alpha \cap f(B) \cap \mathcal{P}(B)| \le \omega$, enumerate $M_\alpha \cap f(B) \cap \mathcal{P}(B)$ as $\{B_i \colon i < \nu\}$. For each $A \in \mathcal{A} \cap \mathcal{P}(B) \cap M_\alpha$ there is $B' \in f(B) \cap f(A)$ with $A \subseteq B' \subseteq B$. Since M_α is f-closed, $A \in M_\alpha$ implies $f(A) \subseteq M_\alpha$. Thus we have our $B' \in M_\alpha$, i.e. $B' = B_{n(A)}$ for some $n(A) < \nu$. Therefore

$$\mathcal{A} \cap \mathcal{P}(B) \cap M_\alpha = \bigcup_{n<\nu}\{A \in \mathcal{A} \colon A \in M_\alpha,\ A \subseteq B_n\}.$$

Since $|\mathcal{A} \cap \mathcal{P}(B) \cap M_\alpha| \ge \mu$ there is $n < \nu$ such that $|\{A \in \mathcal{A} \colon A \in M_\alpha,\ A \subseteq B_n\}| \ge \mu$. Since $B_n \in M_{\alpha^*+1}$ for some $\alpha^* < \alpha$, by the minimality of α we see that $|\mathcal{A} \cap \mathcal{P}(B_n)| \ge \mu$ implies $\mu \subseteq c[\mathcal{A} \cap \mathcal{P}(B_n)]$. But $c[\mathcal{A} \cap \mathcal{P}(B_n)] \subseteq c[\mathcal{A} \cap \mathcal{P}(B)]$, a contradiction. This completes the proof. ∎

To close this section, we prove the following Lindelöf-like property mentioned in the introduction.

PROPOSITION 6.9. *Let* $\mathbf{H}$ *be a closed cover of* $\mathbb{R}$ *such that* $|\mathbf{H}| < \mathrm{cov}(\mathcal{M})$. *Then* $\mathbf{H}$ *has a countable subcover of* $\mathbb{R}$. *In particular, for* $\omega < \kappa < \mathrm{cov}(\mathcal{M})$, *every* κ*-fold closed cover of* $\mathbb{R}$ *of cardinality* κ *has a* κ*-good coloring.*

Proof. Let $\mathcal{U}$ be the collection of those open sets $U \subseteq \mathbb{R}$ for which $\mathbf{H}$ has a countable subcover of U; i.e.

$$\mathcal{U} = \{U \subseteq \mathbb{R} \colon U \text{ is open},\ \exists \mathcal{C} \in [\mathcal{H}]^{\le\omega}\ (U \subseteq \bigcup \mathcal{C})\}.$$

Then $V = \bigcup \mathcal{U}$ is open. Since $\mathbb{R}$ is hereditarily Lindelöf, there is a $\mathcal{V} \in [\mathcal{U}]^{\le\omega}$ such that $V = \bigcup \mathcal{V}$; in particular, $V \in \mathcal{U}$.

To complete the proof of the first statement, it is enough to show that $V = \mathbb{R}$. Suppose $V \neq \mathbb{R}$ and set $F = \mathbb{R} \setminus V$. Then F is a nonempty closed set and $\mathbf{H}$ is a cover of F. Since $|\mathbf{H}| < \mathrm{cov}(\mathcal{M})$, there is an $H \in \mathcal{H}$ such that $H \cap F$ is non-meager in F in the relative topology on F. Thus there is an open set $U \subseteq \mathbb{R}$ such that $U \cap F \neq \emptyset$ and $U \cap F \subseteq H \cap F$. To summarize, we have shown that $\mathbf{H}$ has a countable subcover of $V \cup U$. This contradicts the definition of $\mathcal{U}$.

The second statement immediately follows from the first, so the proof is complete. ∎

7. Convex sets in $\mathbb{R}^n$

7.1. Arbitrary convex sets. In this section we observe that Theorems 6.1 and 6.5 imply that it is independent of ZFC whether an uncountable-fold cover of $\mathbb{R}^n$ $(1 < n < \omega)$ by isometric copies of one compact convex set can be split into two disjoint subcovers.

THEOREM 7.1. *Let $1 < n < \omega$. Under the assumptions of Theorem 6.1, there exists a κ-fold simple closed cover of $\mathbb{R}^n$ by isometric copies of one compact convex set which cannot be decomposed into two disjoint subcovers.*

Proof. By rescaling the construction for Theorem 6.1, there is a compact set $K \subseteq [\pi/4, \pi/2]$ and a set of translations $T \subseteq [-\pi/4, \pi/4]$ such that $\mathbf{K} = \{K + t : t \in T\}$ is a κ-fold simple cover over $[\pi/4, \pi/2]$ which cannot be split into two subcovers over $[\pi/4, \pi/2]$.

Let $\mathbb{O} \in \mathbb{R}^{n-2}$ denote the origin. For every $t \in \mathbb{R}$ set

$$H(t) = \mathrm{conv}\{(\cos(\vartheta + t), \sin(\vartheta + t)) : \vartheta \in K\} \times \{\mathbb{O}\}$$

and let $\mathbf{H}_0 = \{H(t) : t \in T\}$. Set $Y = \{(\cos(\vartheta), \sin(\vartheta)) : \vartheta \in [\pi/4, \pi/2]\} \times \{\mathbb{O}\}$ and let $\mathbf{H}_1$ be a κ-fold simple cover of $\mathbb{R}^n \setminus Y$ by isometric copies of $H(0)$ which do not intersect Y. Such an $\mathbf{H}_1$ clearly exists. Then $\mathbf{H} = \mathbf{H}_0 \cup \mathbf{H}_1$ fulfills the requirements. ∎

The consistency of the existence of $[\omega_1, \infty)$-good colorings for compact covers follows from Theorem 6.5.

7.2. Axis-parallel closed rectangles

THEOREM 7.2. *There exists a countable family $\mathcal{R}$ of axis-parallel closed rectangles in $\mathbb{R}^2$ such that $\mathcal{R}$ is an ω-fold cover of $\mathbb{R}^2$ without two disjoint subcovers.*

We prove Theorem 7.2 in two steps: first we find an ω-fold cover of an abstract space without two disjoint subcovers; then we show how this cover can be realized using axis-parallel closed rectangles in $\mathbb{R}^2$.

Let $X = (\omega+1)^\omega \cup (\omega+1)^{<\omega}$. For each $\sigma \in (\omega+1)^{<\omega}$ and $n \le \omega$, set

$$C_{\sigma^\frown n} = \{\sigma\} \cup \{f \in (\omega+1)^\omega : \sigma^\frown n \subseteq f\},$$

and let $\mathcal{C} = \{C_{\sigma^\frown n} : \sigma \in (\omega+1)^{<\omega},\ n \le \omega\}$.

Lemma 7.3. *$\mathcal{C}$ is an ω-fold cover of X which cannot be split into two disjoint subcovers.*

Proof. Pick an arbitrary $x \in X$. If $x \in (\omega+1)^\omega$ then $x \in C_{x|_k}$ for each $k > 0$. If $x \in (\omega+1)^{<\omega}$ then $x \in C_{x^\frown n}$ $(n < \omega)$ so $\mathcal{C}$ is an ω-fold cover of X indeed.

Split $\mathcal{C} = \mathcal{C}_0 \cup \mathcal{C}_1$ where $\mathcal{C}_0 \cap \mathcal{C}_1 = \emptyset$. We show that if $\mathcal{C}_0$ is a cover of X then $\mathcal{C}_1$ is not a cover of X. So suppose $X = \bigcup \mathcal{C}_0$. We define inductively a sequence $s \in (\omega+1)^\omega$ such that $C_{s|_{n+1}} \in \mathcal{C}_0$ $(n < \omega)$; then we get $\mathcal{C}_1(s) = \emptyset$, which shows that $\mathcal{C}_1$ is not a cover of X. Let $n < \omega$ and suppose that $s|_n$ is defined. Since $s|_n \in X$ and $\mathcal{C}_0$ is a cover of X, we have $\mathcal{C}_0(s|_n) \ne \emptyset$. So there is an $m \le \omega$ for which $C_{(s|_n)^\frown m} \in \mathcal{C}_0$. Defining $s(n) = m$ completes the inductive step and the proof. ∎

Proof of Theorem 7.2. First we show that it is enough to construct a bijection φ between X and a closed subset F of $\mathbb{R}^2$ such that for each $C \in \mathcal{C}$ there is an axis-parallel rectangle $\Phi(C)$ such that $\varphi[C] = F \cap \Phi(C)$. Indeed, since F is closed, $\mathbb{R}^2 \setminus F$ has a countable, ω-fold cover $\mathcal{D}$ by axis-parallel closed rectangles which are all disjoint from F. Then $\mathcal{D} \cup \{\Phi(C) : C \in \mathcal{C}\}$ is a countable ω-fold cover of $\mathbb{R}^2$ by axis-parallel closed rectangles which by Lemma 7.3 cannot be split into two disjoint subcovers.

We construct F as $F_1 \cup F_2$ such that $\varphi[(\omega+1)^{<\omega}] = F_1$ and $\varphi[(\omega+1)^\omega] = F_2$. Let F_1 be a countable closed subset of the closed line segment connecting the points $(-1,1), (0,2) \in \mathbb{R}^2$, and let $\varphi|_{(\omega+1)^{<\omega}}$ be an arbitrary bijection between $(\omega+1)^{<\omega}$ and F_1.

We need some preparation to construct F_2. For each $\sigma \in (\omega+1)^{<\omega}$, by induction on $|\sigma|$, we construct a closed interval $I_\sigma \subseteq [0,1]$ as follows (see Figure 1). We set $I_\emptyset = [0,1]$. If I_σ is constructed then we choose $I_{\sigma^\frown n} \subseteq I_\sigma$ for $n \le \omega$ such that

(I1) $\max I_{\sigma^\frown n} < \min I_{\sigma^\frown n'}$ and $(\max I_{\sigma^\frown n} - \min I_{\sigma^\frown n}) < 2^{-|\sigma|}$ $(n < n' \le \omega)$;
(I2) $\lim_{n<\omega} \max I_{\sigma^\frown n} = \min I_{\sigma^\frown\omega}$, $\min I_{\sigma^\frown 0} = \min I_\sigma$ and $\max I_{\sigma^\frown\omega} = \max I_\sigma$.

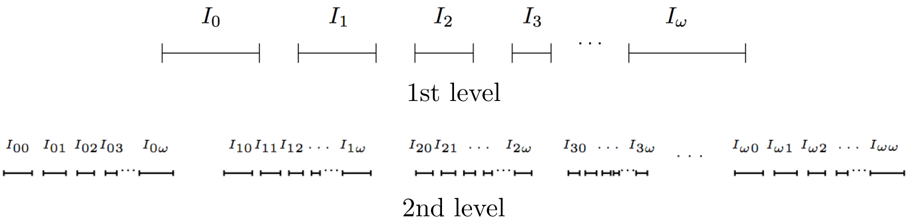

Fig. 1

For every $x \in (\omega+1)^\omega$ set $h(x) = \bigcap_{l<\omega} I_{x|_l}$. By (I2), for every $l < \omega$ the set $H_l = \bigcup\{I_\sigma \colon \sigma \in (\omega+1)^l\}$ is closed. So $H = \bigcap_{l<\omega} H_l$ is also closed.

We define $\varphi(x) = (h(x), h(x))$ for $x \in (\omega+1)^\omega$. We have $H = h[(\omega+1)^\omega]$, so $F_2 = \varphi[(\omega+1)^\omega] = \{(z,z) \colon z \in H\}$ is closed in $\mathbb{R}^2$.

It remains to define Φ. For every $\sigma \in (\omega+1)^{<\omega}$ and $n \le \omega$ let $\Phi(C_{\sigma^\frown n})$ be the unique axis-parallel closed rectangle on $\mathbb{R}^2$ whose upper left corner is $\varphi(\sigma)$ and whose lower right corner is the point $(\max I_{\sigma^\frown n}, \min I_{\sigma^\frown n})$. Then for every $\sigma \in (\omega+1)^{<\omega}$ and $n \le \omega$, $\Phi(C_{\sigma^\frown n}) \cap F_1 = \{\varphi(\sigma)\}$ and

$$\Phi(C_{\sigma^\frown n}) \cap F_2 = \{(z,z) \colon z \in I_{\sigma^\frown n}\} = \varphi[\{x \colon \sigma^\frown n \subseteq x\}] = \varphi[C_{\sigma^\frown n} \cap (\omega+1)^\omega].$$

Thus $\varphi[C_{\sigma^\frown n}] = \Phi(C_{\sigma^\frown n}) \cap F$. So the functions φ and Φ satisfy the requirements. ∎

7.3. Polyhedra. The purpose of this section is to show that an uncountable-fold cover of $\mathbb{R}^n$ by polyhedra has an $[\omega_1, \infty)$-good coloring. We managed to obtain the following general result in this direction, which allows us to treat covers by sets with very different geometric constraints in a unified way (similar ideas appeared in [17]).

We introduce some notation in advance. Let (X, τ) be a topological space, where τ stands for the family of all open subsets of X. For $\mathfrak{B} \subseteq \mathcal{P}(X)$ let

$$\coprod(\mathfrak{B} \sqcap \tau) = \Big\{ \bigcup \mathcal{X} \colon \mathcal{X} \in [B \cap G \colon B \in \mathfrak{B},\ G \in \tau]^\omega \Big\}$$

Theorem 7.4. *Let (X, τ) be a hereditarily Lindelöf space and let $\mathfrak{B} \subseteq \mathcal{P}(X)$ be an intersection-closed family which is well-founded under $\subseteq$. Then every cover $\mathbf{H}$ with $\mathcal{H} \subseteq \coprod(\mathfrak{B} \sqcap \tau)$ has an $[\omega_1, \infty)$-good coloring.*

From Theorem 7.4 we have the following immediate corollaries.

Corollary 7.5. *Let κ be an uncountable cardinal. Any κ-fold cover of $\mathbb{R}^n$*

1. *by sets which can be obtained as countable unions of relatively open subsets of real affine varieties,*
2. *by open or closed polyhedra,*
3. *by open or closed balls,*

can be split into κ many disjoint subcovers.

Proof. Since the polynomial ring of n variables over the reals is Noetherian, the family of real affine varieties in $\mathbb{R}^n$ is intersection-closed and well-founded under $\subseteq$. So for 1, we can apply Theorem 7.4 with $\mathfrak{B}$ standing for the real affine varieties in $\mathbb{R}^n$. Statements 2 and 3 are special cases of 1. ∎

We proceed to the proof of Theorem 7.4.

Proof of Theorem 7.4. By Proposition 2.3, we can assume that $\mathbf{H}$ is a simple cover. For each cardinal λ, let $(\circ_\lambda)$ denote the following statement:

($\circ_\lambda$) If $\mathfrak{A} = \{\mathcal{A}_\alpha \colon \alpha \in I\} \subseteq [B \cap G \colon B \in \mathfrak{B},\, G \in \tau]^\omega$, $|I| \le \lambda$ then there is a function $c \colon I \to \lambda$ such that for each $x \in X$, $|\{\alpha \colon x \in \bigcup \mathcal{A}_\alpha\}| \ge \omega_1$ implies $|\{\alpha \colon x \in \bigcup \mathcal{A}_\alpha\}| \subseteq \{c(\alpha) \colon x \in \bigcup \mathcal{A}_\alpha\}$.

Thus it is enough to prove that ($\circ_\lambda$) holds for each λ. We do it by induction on λ. Clearly, we can assume $I = \lambda$. For $\lambda \le \omega$ any coloring fulfills the requirements. So let first $\lambda = \omega_1$.

Take an arbitrary $\mathfrak{A} = \{\mathcal{A}_\alpha \colon \alpha < \omega_1\}$ and let

$$\mathcal{B} = \{B \in \mathfrak{B} \colon \exists \alpha < \omega_1,\ \exists G \in \tau\ (B \cap G \in \mathcal{A}_\alpha)\}.$$

We have $|\mathcal{B}| = \omega_1$, so since $\mathfrak{B}$ is well-founded, the intersection-closed hull $\mathcal{B}^\cap$ of $\mathcal{B}$ satisfies $|\mathcal{B}^\cap| = \omega_1$. Hence we can take an enumeration $\mathcal{B}^\cap = \{B_\alpha \colon \alpha < \omega_1\}$. We also fix a bijection $\varphi \colon \omega_1 \times \omega_1 \to \omega_1$.

For every $\alpha < \omega_1$ we construct inductively a countable partial coloring $c_\alpha \colon \omega_1 \to \mathrm{On}$ as follows. Let $\alpha < \omega_1$ and suppose that c_η is defined for every $\eta < \alpha$. Set $\mathfrak{R}_\alpha = \omega_1 \setminus \bigcup_{\beta<\alpha} \mathrm{dom}(c_\beta)$. Let $\beta, \chi < \omega_1$ be such that $\varphi(\beta, \chi) = \alpha$. For $\gamma \in \mathfrak{R}_\alpha$ let

$$G(\gamma) = \bigcup\{G \in \tau \colon \exists B \in \mathfrak{B}\ (B_\beta \subseteq B,\ B \cap G \in \mathcal{A}_\gamma)\}, \tag{5}$$

and

$$C_\alpha = \{x \in B_\beta \colon |\{\gamma \in \mathfrak{R}_\alpha \colon x \in G(\gamma)\}| = \omega_1\}.$$

Since X is hereditarily Lindelöf, C_α is Lindelöf. Hence there is an $I_\alpha \in [\mathfrak{R}_\alpha]^\omega$ such that $C_\alpha \subseteq \bigcup_{\gamma \in I_\alpha} G(\gamma)$; thus by (5), $C_\alpha \subseteq \bigcup\{\bigcup \mathcal{A}_\gamma \colon \gamma \in I_\alpha\}$ as well. We define c_α by $\mathrm{dom}(c_\alpha) = I_\alpha$ and $c_\alpha(\gamma) = \chi$ ($\gamma \in I_\alpha$). This completes the αth step of the construction. We set $c = \bigcup_{\alpha<\omega_1} c_\alpha$.

We show that c witnesses condition ($\circ_{\omega_1}$). Suppose that $x \in X$ satisfies $|\{\alpha \colon x \in \bigcup \mathcal{A}_\alpha\}| = \omega_1$, i.e. there are $I \in [\omega_1]^{\omega_1}$ and $B_\alpha \in \mathfrak{B}$, $G_\alpha \in \tau$ ($\alpha \in I$) such that $x \in B_\alpha \cap G_\alpha \in \mathcal{A}_\alpha$ ($\alpha \in I$). Let $B = \bigcap_{\alpha \in I} B_\alpha$. Then $B \in \mathcal{B}^\cap$, that is, $B = B_\beta$ for some $\beta < \omega_1$. Pick an arbitrary $\chi < \omega_1$ and let $\alpha = \varphi(\beta, \chi)$.

Recall the construction of c_α: since c_η ($\eta < \alpha$) are countable, we have $|I \cap \mathfrak{R}_\alpha| = \omega_1$. Hence $x \in C_\alpha$ and so $x \in \bigcup \mathcal{A}_\gamma$ for some $\gamma \in I_\alpha$. Since $c_\alpha(\gamma) = \chi$ and $\chi < \omega_1$ was arbitrary, the proof of the $\lambda = \omega_1$ case is complete.

Let now $\lambda > \omega_1$ and suppose the statement holds for every $\omega_1 \le \kappa < \lambda$. Let $\mathcal{M} = \langle M, \in \rangle$ be a large enough model of a large enough fragment of ZFC. Let $\langle M_\alpha \colon \omega_1 \le \alpha < \lambda \rangle$ be a continuous, increasing chain of elementary submodels of $\mathcal{M}$ such that $|M_\alpha| = |\alpha|$, $M_\alpha \in M_{\alpha+1}$, and $(X, \tau), \tau, \mathfrak{B}, \mathfrak{A} \in M_{\omega_1}$. For every set $y \in M_\lambda$, let $\mathrm{rank}(y) = \min\{\alpha \colon y \in M_{\alpha+1}\}$.

For every $\omega_1 \le \alpha < \lambda$, let $J_\alpha = \lambda \cap (M_{\alpha+1} \setminus M_\alpha) = \{\eta < \lambda \colon \mathrm{rank}(\eta) = \alpha\}$. Then $|J_\alpha| = |\alpha|$. By the inductive hypothesis, there is a coloring $c'_\alpha \colon J_\alpha \to |\alpha|$ witnessing $(\circ)_{|\alpha|}$ for $\{\mathcal{A}_\xi \colon \xi \in J_\alpha\}$. By Lemma 2.4, there is a function $h_\alpha \colon |\alpha| \to \alpha$ such that $h_\alpha[|\alpha|] = \alpha$ and $\kappa \subseteq h_\alpha[\kappa]$ for every cardinal $\omega_1 \leq \kappa < |\alpha|$. Let $c_\alpha = h_\alpha \circ c'_\alpha$. Then for every $x \in X$, with $\kappa = |\{\eta \in J_\alpha : x \in \bigcup \mathcal{A}_\eta\}|$,

(c1) $\omega_1 \leq \kappa \leq |\alpha|$ implies $\kappa \subseteq \{c_\alpha(\eta) : x \in \bigcup \mathcal{A}_\eta\}$;
(c2) $\kappa = |\alpha|$ implies $\alpha = \{c_\alpha(\eta) : x \in \bigcup \mathcal{A}_\eta\}$.

Set $c = \bigcup_{\alpha<\lambda} c_\alpha$; we show c witnesses $(\circ_\lambda)$.

To this end, let $x \in X$ be such that $\kappa = |\{\alpha : x \in \bigcup \mathcal{A}_\alpha\}| \geq \omega_1$. Take $\nu_\xi \in \mathrm{On}$, $B_\xi \in \mathfrak{B}$ and $G_\xi \in \tau$ $(\xi < \kappa)$ such that $(\nu_\xi)_{\xi<\kappa}$ are pairwise different and $x \in B_\xi \cap G_\xi \in \mathcal{A}_{\nu_\xi}$ $(\xi < \kappa)$. Let $\rho_\xi = \mathrm{rank}(\nu_\xi)$ $(\xi < \kappa)$; we can assume $(\rho_\xi)_{\xi<\kappa}$ is an increasing sequence. Let $\rho = \sup\{\rho_\xi \dot{+} 1 : \xi < \kappa\}$. We can also assume that if ρ is a successor ordinal, $\rho = \rho' + 1$, then $\rho_\xi = \rho'$ $(\xi < \kappa)$.

If ρ is successor then $|\{\eta \in J_{\rho'} : x \in \bigcup \mathcal{A}_\eta\}| = \kappa$ so we are done because $c_{\rho'}$ satisfies (c1). From now on assume ρ is a limit ordinal. By the well-foundedness of $\mathfrak{B}$ there are $B \in \mathfrak{B}$ and $F \in [\kappa]^{<\omega}$ such that

$$B = \bigcap_{\xi<\kappa} B_\xi = \bigcap_{\xi\in F} B_\xi. \tag{6}$$

Let $\sigma = \mathrm{rank}(F) \geq \mathrm{rank}(B)$. Since ρ is a limit ordinal we have $\sigma < \rho$. Thus $|\{\xi < \kappa : \sigma < \rho_\xi\}| = \kappa$ and so

$$\left|\left\{\eta \in \lambda \setminus M_\sigma : x \in \bigcup \mathcal{A}_\eta\right\}\right| = \kappa. \tag{7}$$

For every $\alpha < \lambda$ let $G_\alpha = \bigcup\{G \in \tau : \exists B' \in \mathfrak{B}\ (B \subseteq B',\ B' \cap G \in \mathcal{A}_\alpha\}$.

We distinguish two cases. First suppose $\kappa \leq \sigma$. Set

$$C = \{y \in B : |\{\eta \in \lambda \setminus M_\sigma : y \in G_\eta\}| \geq \kappa\}. \tag{8}$$

Since C is Lindelöf, by (8)

(9) there is a sequence $(K^\star(\zeta))_{\zeta<\kappa}$ of pairwise disjoint countable subsets of $\lambda \setminus M_\sigma$ such that $C \subseteq \bigcup_{\eta\in K^\star(\zeta)} G_\eta$ for each $\zeta < \kappa$.

But $\kappa \leq \sigma$ implies $B, \kappa \in M_{\sigma+1}$, therefore $C \in M_{\sigma+1}$ as well. So by elementarity (9) holds in $M_{\sigma+1}$, i.e. there is a sequence $(K(\zeta))_{\zeta<\kappa}$ of pairwise disjoint countable subsets of $\lambda \cap (M_{\sigma+1} \setminus M_\sigma) = J_\sigma$ in $M_{\sigma+1}$ such that $C \subseteq \bigcup_{\eta\in K(\zeta)} G_\eta$ $(\zeta < \kappa)$. For every $\zeta < \kappa$ we have $K(\zeta) \subseteq M_{\sigma+1}$ because $K(\zeta) \in M_{\sigma+1}$ and $K(\zeta)$ is countable. So $K(\zeta) \subseteq J_\sigma$ $(\zeta < \kappa)$. Since $G_\eta \cap B \subseteq \bigcup \mathcal{A}_\eta$ $(\eta < \lambda)$, we have $C \subseteq \bigcup_{\eta\in K(\zeta)} \bigcup \mathcal{A}_\eta$ $(\zeta < \kappa)$. Thus $\{\bigcup \mathcal{A}_\eta : \eta \in J_\sigma\}$ is a κ-fold cover of C. Since $x \in C$ and c_σ satisfies (c1), $\kappa \subseteq \{c_\sigma(\eta) : x \in \bigcup \mathcal{A}_\eta\}$, as required.

Finally suppose $\sigma < \kappa$. Fix an arbitrary $\beta \in \mathrm{On}$ satisfying $\sigma < \beta < \kappa$. Set

$$C = \{y \in B : |\{\eta \in \lambda \setminus M_\beta : y \in G_\eta\}| \geq \beta\}. \tag{10}$$

Since C is Lindelöf, by (10)

(11) there is a sequence $(K^\star(\zeta))_{\zeta<\beta}$ of pairwise disjoint countable subsets of $\lambda \setminus M_\beta$ such that $C \subseteq \bigcup_{\eta\in K^\star(\zeta)} G_\eta$ for each $\zeta < \beta$.

But $\sigma \leq \beta$ implies $B, \beta \in M_{\beta+1}$, therefore $C \in M_{\beta+1}$ as well. So by elementarity (11) holds in $M_{\beta+1}$, i.e. there is a sequence $(K(\zeta))_{\zeta<\beta}$ of pairwise disjoint countable subsets of $\lambda \cap (M_{\beta+1} \setminus M_\beta) = J_\beta$ in $M_{\beta+1}$ such that $C \subseteq \bigcup_{\eta\in K(\zeta)} G_\eta$ $(\zeta < \beta)$. For every $\zeta < \beta$ we have $K(\zeta) \subseteq M_{\beta+1}$ because $K(\zeta) \in M_{\beta+1}$ and $K(\zeta)$ is countable. So $K(\zeta) \subseteq J_\beta$ $(\zeta < \beta)$. Since $G_\eta \cap B \subseteq \bigcup \mathcal{A}_\eta$ $(\eta < \lambda)$, we have $C \subseteq \bigcup_{\eta\in K(\zeta)} \bigcup \mathcal{A}_\eta$ $(\zeta < \beta)$. Thus $\{\bigcup \mathcal{A}_\eta : \eta \in J_\beta\}$ is a $|\beta|$-fold cover of C.

By (7) and by elementarity we have $x \in C$, hence $|\{\eta \in J_\beta : x \in \bigcup \mathcal{A}_\eta\}| = |\beta|$. Since c_β satisfies (c2), $\beta \subseteq \{c_\beta(\eta) : x \in \bigcup \mathcal{A}_\eta\} \subseteq \{c(\eta) : x \in \bigcup \mathcal{A}_\eta\}$. Since $\beta < \kappa$ was arbitrary, the proof is complete. ∎

We remark that in the proof of Theorem 7.4, formally we used only the assumption that X is a hereditarily ω_1-Lindelöf space. However, a space is hereditarily ω_1-Lindelöf if and only if it is hereditarily Lindelöf.

8. Open problems. Whenever we considered the splitting problem of κ-fold covers for infinite κ, either we could establish the existence of a κ-good coloring or we could construct a κ-fold cover which cannot be split into two disjoint subcovers. However, we have not been able to prove that for κ-fold covers the existence of a 2-good coloring is equivalent to the existence of a κ-good coloring.

PROBLEM 8.1. *Let X be a set, κ be an infinite cardinal, and let $\mathcal{F} \subseteq \mathcal{P}(X)$ be arbitrary. Suppose every κ-fold cover $\mathbf{H}$ of X satisfying $\mathcal{H} \subseteq \mathcal{F}$ has a 2-good coloring. Is it then true that every κ-fold cover $\mathbf{H}$ of X satisfying $\mathcal{H} \subseteq \mathcal{F}$ has a κ-good coloring as well?*

In Section 4 we did not consider the splitting problem for hypergraphs.

PROBLEM 8.2. *Examine the splitting problem of finite-fold and infinite-fold edge covers of hypergraphs.*

It would be interesting to know more on the consistency strength of the splitting of closed covers. In particular, one could examine whether maximal coloring of closed covers is possible in well-known extensions other than the Cohen model. A special case is the following.

PROBLEM 8.3. *Let κ be an uncountable cardinal. Is it true in a random real extension of a model with GCH that every κ-fold closed cover of $\mathbb{R}$ can be split into two disjoint subcovers?*

We have seen that both under CH and under $\omega_1 < \mathrm{cov}(\mathcal{M})$, an ω_1-fold closed cover $\mathbf{H}$ of $\mathbb{R}$ with $|\mathbf{H}| = \omega_1$ has an ω_1-good coloring. However, we have not been able to obtain it as a ZFC result.

PROBLEM 8.4. *Is it consistent with ZFC that there exists an ω_1-fold closed cover $\mathbf{H}$ of $\mathbb{R}$ such that $|\mathbf{H}| = \omega_1$ but $\mathbf{H}$ cannot be split into two disjoint subcovers?*

As we mentioned in the introduction, there are numerous open problems concerning the splitting of finite-fold covers of $\mathbb{R}^n$ by sets with special geometric properties. The interested reader is referred to [13] for more details. Here we propose problems for ω-fold covers only.

PROBLEM 8.5. *Is it true that every ω-fold cover of $\mathbb{R}^2$ by* translates *of one compact convex set can always be decomposed into two disjoint subcovers?*

PROBLEM 8.6. *Is it true that every ω-fold cover of $\mathbb{R}^n$*

1. *by translates or homothets of the unit cube,*
2. *by translates of the unit ball*

can be decomposed into two disjoint subcovers?

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