FUNDAMENTA
MATHEMATICAE
212 (2011)

Does a billiard orbit determine its (polygonal) table?

by

Jozef Bobok (Praha) and **Serge Troubetzkoy** (Marseille)

Abstract. We introduce a new equivalence relation on the set of all polygonal billiards. We say that two billiards (or polygons) are order equivalent if each of the billiards has an orbit whose footpoints are dense in the boundary and the two sequences of footpoints of these orbits have the same combinatorial order. We study this equivalence relation under additional regularity conditions on the orbit.

1. Introduction. In mathematics one often wants to know if one can reconstruct an object (often a geometric object) from certain discrete data. A famous example of this is the celebrated problem posed by Mark Kac, "Can one hear the shape of a drum", i.e., whether one can reconstruct a drum head from knowing the frequencies at which it vibrates [K]. This problem was resolved negatively by Milnor in dimension 16 [M] and then by Gordon, Webb and Wolpert in dimension 2 [GWW]. Another well known example is a question posed by Burns and Katok whether a negatively curved surface is determined by its marked length spectrum [BK]. The *marked length spectrum* of a surface S is the function that associates to each conjugacy class in $\pi_1(S)$ the length of the geodesic in the associated free homotopy class. This question was resolved positively by Otal [O].

In this article we ask if a polygonal billiard table is determined by the combinatorial data of the footpoints of a billiard orbit. For this purpose we introduce a new equivalence relation on the set of all polygonal billiards. Namely we say that two polygonal billiards (polygons) are order equivalent if each of the billiards has an orbit whose footpoints are dense in the boundary and the two sequences of footpoints of these orbits have the same combinatorial order. We study this equivalence relation under additional regularity conditions on the orbit. Our main results are that, under a weak regularity condition on the orbits, an irrational polygon cannot be order equivalent to a rational polygon, and two order equivalent rational

2010 *Mathematics Subject Classification*: Primary 37C15, 37E35.
Key words and phrases: polygon, billiard map, order equivalence.

DOI: 10.4064/fm212-2-2

polygons must have the same number of sides with corresponding corners having the same angle. In the case of triangles, only similar triangles can be order equivalent. In general, in the rational case, one cannot say more since any two rectangles are order equivalent. If we furthermore assume that the greatest common denominator of the rational angles is at least 3, then under a slightly stronger regularity condition we show two order equivalent rational polygons must be similar. In the case where the greatest common denominator is 2, the two order equivalent rational polygons must be affinely similar.

In Section 2 we start by recalling basic facts about polygonal billiards, while in Section 3 we prove various preparatory lemmas. We prove our main results in Sections 4 (rational versus irrational) and 5 (rational versus rational). Finally in Section 6 we summarize our results and ask some open questions.

2. Polygonal billiard. A *polygonal billiard* table is a planar simply connected compact polygon P. The billiard flow $\{T_t\}_{t\in\mathbb{R}}$ in P is generated by the free motion of a mass-point subject to elastic reflection in the boundary. This means that the point moves along a straight line in P with a constant speed until it hits the boundary. At a smooth boundary point the billiard ball reflects according to the well known law of geometrical optics: the angle of incidence equals the angle of reflection. If the billiard ball hits a corner (a non-smooth boundary point), its further motion is not defined. Moreover, the billiard trajectory is not defined for a direction tangent to a side.

We denote by D the group generated by the reflections in the lines through the origin, parallel to the sides of the polygon P. The group D is either

- finite, when all the angles of P are of the form $\pi m_i/n_i$ with distinct coprime integers m_i, n_i; in this case $D = D_N$, the dihedral group generated by the reflections in lines through the origin that meet at angles π/N, where N is the least common multiple of n_i's, or
- countably infinite, when at least one angle between sides of P is an irrational multiple of π.

In the two cases we will refer to the polygon as *rational*, respectively *irrational.*

Consider the phase space $P \times S^1$ of the billiard flow T_t, and for $\theta \in S^1$, let $R_\theta \subset P\times S^1$ consist of the points whose second coordinate belongs to the orbit of θ under D. Since a trajectory changes its direction by an element of D under each reflection, R_θ is an invariant set of the billiard flow T_t in P.

The *billiard map* $T : V_P = \bigcup e \times \theta \subset \partial P \times S^1 \to V_P$ associated with the flow T_t is the first return map to the boundary ∂P of P. Here the union

$\bigcup e \times \theta$ is taken over all sides of P and for each side e over the inner pointing directions θ from S^1. We will denote points of V_P by $u = (x, \theta)$.

The set $P \times \theta$, resp. $\partial P \times \theta$ will be called a *floor* of the phase space of the flow T_t, resp. of the billiard map T.

Let ∂P be oriented counterclockwise. For $x, x' \in \partial P$, we denote by $[x, x']$ (resp. (x, x')) the closed (resp. open) arc of ∂P with outgoing endpoint x and incoming endpoint x'.

If P, Q are simply connected polygons with counterclockwise oriented boundaries, two sequences $\{x_n\}_{n\geq 0} \subset \partial P$ and $\{y_n\}_{n\geq 0} \subset \partial Q$ *have the same combinatorial order* if for all non-negative integers k, l, m,

$$x_k \in [x_l, x_m] \Leftrightarrow y_k \in [y_l, y_m].$$

The next definition introduces a new relation on the set of all simply connected polygons. The reader can verify that it is reflexive, symmetric and transitive, i.e., it is an equivalence relation. As usual, π_1 denotes the natural projection to the first coordinate (the footpoint).

Definition 2.1. We say that two polygons (or polygonal billiards) P, Q are *order equivalent* if there are points $u_0 \in V_P$, $v_0 \in V_Q$ such that

(i) $\overline{\{\pi_1(T^n(u_0))\}_{n\geq 0}} = \partial P$, $\overline{\{\pi_1(S^n(v_0))\}_{n\geq 0}} = \partial Q$,
(ii) the sequences $\{\pi_1(T^n(u_0))\}_{n\geq 0}$, $\{\pi_1(S^n(v_0))\}_{n\geq 0}$ have the same combinatorial order.

In that case we will write $(P; u_0) \approx (Q; v_0)$; the points u_0, v_0 will sometimes be called the *leaders*.

Let $t = \{x_n\}_{n\geq 0}$ be a sequence which is dense in ∂P. The *t-address* $a_t(x)$ of a point $x \in \partial P$ is the set of all increasing sequences $\{n(k)\}_k$ of non-negative integers satisfying $\lim_k x_{n(k)} = x$. It is clear that any $x \in \partial P$ has a non-empty t-address, and the t-addresses of two distinct points from ∂P are disjoint.

For order equivalent polygons P, Q with leaders u_0, v_0, we will consider addresses with respect to the sequences given by Definition 2.1(ii):

$$t = \{\pi_1(T^n(u_0))\}_{n\geq 0}, \qquad s = \{\pi_1(S^n(v_0))\}_{n\geq 0}.$$

It is an easy exercise to prove that the map $\phi\colon \partial P \to \partial Q$ defined by

$$\phi(x) = y \quad \text{if } a_t(x) = a_s(y) \tag{1}$$

is a homeomorphism.

One can pose many questions about the equivalence classes. In particular, can rational and irrational polygons lie in the same class? Can order equivalent polygons have different (number of) angles, or at least different lengths of sides?

It follows immediately from the well known results that order equivalent polygons need not be similar (see for example [MT]).

Example 2.2. Any two rectangles are order equivalent.

We finish this section by recalling several well known and useful (for our purpose) results about polygonal billiards (see for example [MT]). Recall that a *flat strip* $\mathcal{T}$ is an invariant subset of the phase space of the billiard flow/map such that

- $\mathcal{T}$ is contained in a finite number of floors,
- the billiard flow/map dynamics on $\mathcal{T}$ is minimal in the sense that any orbit which does not hit a corner is dense in $\mathcal{T}$,
- the boundary of $\mathcal{T}$ is non-empty and consists of a finite union of generalized diagonals (a *generalized diagonal* is a billiard trajectory that goes from a corner to a corner).

We denote by π_2 the second natural projection (to the direction). A direction, resp. a point u from the phase space is *exceptional* if it is the direction of a generalized diagonal, resp. $\pi_2(u)$ is such a direction. Obviously there are countably many generalized diagonals, hence also exceptional directions. A direction, resp. a point u from the phase space, which is not exceptional will be called *non-exceptional*.

The set of the corners of P is denoted by C_P. As usual, the ω-limit set of a point u is denoted by $\omega(u)$.

Proposition 2.3 ([MT]). *Let P be rational and $u_0 \in V_P$. Then exactly one of the following three possibilities holds:*

(i) *u_0 is periodic.*
(ii) *$\overline{\mathrm{orb}(u_0)}$ is a flat strip.*
(iii) *$\omega(u_0) = R_{\pi_2(u_0)}$; then*

$$\#(\{\pi_2(T^n(u_0)) : n \geq 0\}) = 2N,$$

and for every $x \in \partial P \setminus C_P$,

$$\#\{u \in \omega(u_0) : \pi_1(u) = x\} = N,$$

where N is the least common multiple of the denominators of the angles of P. Moreover, in this case

$$\pi_2(\{u \in \omega(u_0) : \pi_1(u) = x\}) = \pi_2(\{u \in \omega(u_0) : \pi_1(u) = x'\})$$

whenever $x' \notin C_P$ belongs to the same side as x. The possibility (iii) *holds whenever u_0 is non-exceptional.*

The billiard map T has a natural invariant measure on its phase space given by the formula $\mu = \sin\theta\, dx\, d\theta$, where $\theta \in (0,\pi)$ is measured with respect to the direction of an oriented side e. In a rational polygon we say that a point u is *generic* if it is non-exceptional, has bi-infinite orbit and the billiard map restricted to the invariant surface $R_{\pi_2(u)}$ has a unique invariant measure (this measure is then automatically the measure μ).

Finally we will use the following remarkable results.

THEOREM 2.4 ([V1]). *If P is a regular n-gon ($n \neq 3$) then any non-exceptional u with infinite orbit is generic. Any exceptional u whose trajectory does not hit a corner is periodic.*

The details concerning the next definition can be found in [Vo].

DEFINITION 2.5. A rational polygon P will be called *Veech* if the corresponding directional surface $R = R_{\pi_2(u)}$, u non-exceptional, has a stabilizer $S(R)$ which is a lattice in $\mathrm{SL}(2,\mathbb{R})$.

THEOREM 2.6 ([V2]). *If P is Veech then each non-exceptional direction is generic.*

3. Preparatory lemmas. To define some important objects in dynamical systems, for example an ω-limit set, the corresponding phase space has to be equipped with a topology providing suitable convergence. In the context of polygonal billiards this often means the simultaneous convergence of footpoints and directions (briefly, pointwise convergence). Since it simplifies several of our proofs, throughout the paper we will use the following modified definition of convergence.

DEFINITION 3.1. Let $\{T^n(u_0)\}_{n\geq 0}$ be an infinite trajectory. The sequence $\{T^{n(k)}(u_0)\}_{k\geq 0}$ *converges* if and only if

(i) $\pi_1(T^{n(k)}(u_0)) \to_k x \in \partial P$,
(ii) $\pi_1(T^{n(k)+1}(u_0)) \to_k x' \in \partial P$,
(iii) x and x' do not belong to the same side of P.

In particular, the above definition excludes the case when a sequence $\{T^{n(k)}(u_0)\}_{k\geq 0}$ satisfies (i), (ii) but $x = x'$. Instead of rewriting known results in slightly modified versions we will use the following

FACT 3.2. Despite the fact that this notion is stronger than the usual convergence we will use classical results, such as Proposition 2.3(ii), (iii). All the results we use remain true under our notion of convergence.

LEMMA 3.3. *Let $(P; u_0) \approx (Q; v_0)$, and assume that for any point $x' \in C_P \cup \phi^{-1}(C_Q)$ there exists a sequence $\{n(k)\}_{k\geq 1}$ such that the items* (i) *and* (ii) *of Definition* 3.1 *are satisfied, $x \neq x'$ and x' is a both-sided limit point of $\{\pi_1(T^{n(k)+1}(u_0))\}_{k\geq 0}$. Then $\phi(C_P) = C_Q$.*

Proof. Choose $x' \in C_P \setminus \phi^{-1}(C_Q)$. Let $\{T^{n(k)}(u_0)\}_{k\geq 0}$ be a sequence satisfying the assumptions of the lemma. Consider its "counterpart" in Q, $\{S^{n(k)}(v_0)\}_{k\geq 0}$. Let

$$y = \phi(x) = \lim_k \pi_1(S^{n(k)}(v_0)), \qquad y' = \phi(x') = \lim_k \pi_1(S^{n(k)+1}(v_0)).$$

Then $\lim_k \pi_1(T^{n(k)+2}(u_0))$ does not exist since x' is a corner and a both-sided limit. However, $\lim_k \pi_1(S^{n(k)+2}(v_0))$ does exist; this is impossible for order equivalent polygons $(P; u_0) \approx (Q; v_0)$.

The case when $x' \in \phi^{-1}(C_Q) \setminus C_P$ can be excluded analogously. Thus, $C_P = \phi^{-1}(C_Q)$, i.e., $\phi(C_P) = C_Q$. ∎

Let us define the map $\Phi : \{T^n(u_0)\}_{n\ge 0} \to \{S^n(v_0)\}_{n\ge 0}$ by

$$\Phi(T^n(u_0)) = S^n(v_0). \tag{2}$$

LEMMA 3.4. *Let $(P; u_0) \approx (Q; v_0)$ and $\phi(C_P) = C_Q$. Then:*

(i) $T^{n(k)}(u_0) \to_k u \in V_P$ *if and only if* $S^{n(k)}(v_0) \to_k v \in V_Q$.

(ii) *The map Φ from* (2) *can be extended homeomorphically to the map $\Phi : \omega(u_0) \to \omega(v_0)$ satisfying (for all $n \in \mathbb{Z}$ for which the image is defined)*

$$\Phi(T^n(u)) = S^n(\Phi(u)), \qquad u \in \omega(u_0);$$

in particular, for every $x \in \partial P$ we have

$$\Phi(\{u \in \omega(u_0) : \pi_1(u) = x\}) = \{v \in \omega(v_0) : \pi_1(v) = \phi(x)\}. \tag{3}$$

(iii) *For every $u, u' \in \omega(u_0)$ with $\pi_1(u) = \pi_1(u') \notin C_P$,*

$$\pi_2(u) < \pi_2(u') \iff \pi_2(\Phi(u)) < \pi_2(\Phi(u')), \tag{4}$$

(iv) u_0 *is recurrent if and only if* v_0 *is recurrent.*

Proof. (i) Let $T^{n(k)}(u_0) \to_k u$ and x, x' be points from Definition 3.1. Then both the limits $y = \phi(x) = \lim_k \pi_1(S^{n(k)}(v_0))$ and $y' = \phi(x') = \lim_k \pi_1(S^{n(k)+1}(v_0))$ exist, and since $\phi(C_P) = C_Q$, the points y, y' cannot lie on the same side of Q. Thus $S^{n(k)}(v_0) \to_k v$ where v is the unit vector with footpoint y and pointing towards y'. To prove the "only if" part, interchange the roles of u and v.

(ii) Denote $u_n = T^n(u_0)$. For $u \in \omega(u_0)$ for which $u = \lim_k u_{n(k)}$ put

$$\Phi(u) = \lim_k \Phi(u_{n(k)});$$

using (2) and part (i) of the lemma one can easily show that Φ is well defined and a homeomorphism. Thus (3) follows.

(iii) Since the homeomorphism ϕ preserves the corners, we have $\pi_1(\Phi(u)) = \pi_1(\Phi(u')) \notin C_Q$. From (ii) we infer that $\pi_2(u) \neq \pi_2(u')$ if and only if $\pi_2(\Phi(u)) \neq \pi_2(\Phi(u'))$. If (4) does not hold then there are positive integers n, n' such that u_n, resp. $u_{n'}$ is close to u, resp. u' (with the footpoints on the same side, in particular) and either

$$\pi_2(u_n) < \pi_2(u_{n'}) \ \&\ \pi_2(\Phi(u_n)) > \pi_2(\Phi(u_{n'})),$$

or the analogous possibility with both inequality signs reversed, is true. Without loss of generality consider the first possibility. Then for the corresponding arcs we have

$$[\pi_1(u_{n'+1}), \pi_1(u)) \cap (\pi_1(u), \pi_1(u_{n+1})] = \emptyset$$

and

$$\begin{aligned}[\pi_1(\Phi(u_{n'+1})), \pi_1(\Phi(u))) \cap (\pi_1(\Phi(u)), \pi_1(\Phi(u_{n+1}))] \\ = [\pi_1(\Phi(u_{n'+1})), \pi_1(\Phi(u_{n+1}))] \neq \emptyset,\end{aligned}$$

which is impossible for order equivalent polygons $(P; u_0) \approx (Q; v_0)$.

Part (iv) easily follows from (i). ∎

Let g be a function defined on a neighborhood of y. The derived numbers $D^+g(y)$, $D_+g(y)$ of g at y are given by

$$D^+g(y) = \limsup_{h \to 0_+} \frac{g(y+h) - g(y)}{h}, \qquad D_+g(y) = \liminf_{h \to 0_+} \frac{g(y+h) - g(y)}{h}$$

and the analogous limits from the left are denoted by $D^-g(y)$, $D_-g(y)$.

Let (z, y) be the coordinates of $\mathbb{R}^2$ and let $p_{a,b} \subset \mathbb{R}^2$ be the line with equation $y = a + z \tan b$. For short we denote $p_{y_0, g(y_0)}$ by $p_{g(y_0)}$.

Lemma 3.5. *Let $g : (c, d) \to (-\pi/2, \pi/2)$ be a continuous function, and fix $C \subset (c, d)$ countable. Assume that for some y_0 one of the four possibilities*

$$D^+g(y_0) > 0, \quad D_+g(y_0) < 0, \quad D^-g(y_0) > 0, \quad D_-g(y_0) < 0 \tag{5}$$

holds. Then there exists a sequence $\{y_n\}_{n \geq 1} \subset (c, d) \setminus C$ such that $\lim_n y_n = y_0$ and the set of crossing points $\{p_{g(y_0)} \cap p_{g(y_n)} : n \geq 1\}$ is bounded in $\mathbb{R}^2$.

Proof. We will prove the conclusion when $\epsilon = D^+g(y_0) > 0$. The remaining three possibilities can be shown analogously.

By our assumption there exists a decreasing sequence $\{y_n\}_{n \geq 1} \subset (c, d) \setminus C$ such that $\lim_n y_n = y_0$ and for each n,

$$g(y_n) > g_n = g(y_0) + \frac{\epsilon}{2}(y_n - y_0) > g(y_0).$$

This means that the crossing point $p_{g(y_0)} \cap p_{g(y_n)}$ is closer to $(0, y_0)$ than the crossing point $A_n = p_{g(y_0)} \cap p_{y_n, g_n}$. Let $A_n = (\tilde{z}_n, \tilde{y}_n)$. Then

$$y_0 + \tilde{z}_n \tan g(y_0) = y_n + \tilde{z}_n \tan g_n,$$

hence

$$\tilde{z}_n = \frac{y_0 - y_n}{\tan[g(y_0) + \frac{\epsilon}{2}(y_n - y_0)] - \tan g(y_0)}.$$

Since

$$\lim_n \tilde{z}_n = \frac{-2}{\epsilon} \cos^2 g(y_0),$$

both the sets $\{A_n : n \geq 1\}$, $\{p_{g(y_0)} \cap p_{g(y_n)} : n \geq 1\}$ are bounded in $\mathbb{R}^2$. ∎

Remark 3.6. Note that if $g'(y_0) \neq 0$ at some point y_0 then at least one of the four possibilities in (5) has to hold.

To apply this lemma to billiards, recall the notion of an unfolded billiard trajectory. Namely, instead of reflecting it in a side of P one may reflect P in this side and unfold the trajectory to a straight line.

Lemma 3.7. *Let $(P; u_0) \approx (Q; v_0)$ with P rational. Then the set of directions*

$$\{\pi_2(S^n(v_0)) : n \geq 0\}$$

along the trajectory of v_0 is finite.

Remark 3.8. In Lemma 3.7 we do not assume u_0 is non-exceptional.

Proof. We know that $\phi(\partial P) = \partial Q$, where ϕ is the homeomorphism defined in (1). Under our assumption we cannot exclude the possibility $\phi(C_P) \neq C_Q$. So, it will be convenient to consider extended sets of corners, C_P^*, C_Q^*, defined as

$$C_P^* = \{x \in \partial P : x \in C_P \text{ or } \phi(x) \in C_Q\}, \qquad C_Q^* = \phi(C_P^*),$$

and the sides of P, resp. Q will be taken with respect to C_P^*, resp. C_Q^*.

By Definition 2.1(i) the first projection of the forward trajectory of u_0 is dense in ∂P, so in particular u_0 is not periodic. In such a case the trajectory of u_0 is minimal either in a flat strip $\mathcal{T}$ or in an invariant surface $R_{\pi_2(u_0)}$ (see Proposition 2.3). Since the conclusion can be verified for both possibilities in a similar way, we will only treat the flat strip case.

For an interval $(a, b) \subset \partial P$ and an $\alpha \in S^1$ denote

$$I := \{n \in \mathbb{N}_0 : \pi_1(T^n(u_0)) \in (a, b) \;\&\; \pi_2(T^n(u_0)) = \alpha\}. \tag{6}$$

By Proposition 2.3 there are finitely many directions along the trajectory of u_0 and, by our assumption, the footpoints of it are dense in ∂P; passing to a subinterval and choosing a suitable α we can assume that (a, b), resp. $\phi((a, b)) = (c, d)$ is a subinterval of a "side" e of P, resp. a "side" f of Q and the sequence $\{\pi_1(T^n(u_0))\}_{n\in I}$ is dense in (a, b). Obviously,

$$\tau = (a, b) \times \{\alpha\} \subset \omega(u_0), \qquad \sigma = \Phi(\tau) \subset \omega(v_0);$$

there is a countable subset τ_0 of τ such that each point of $\tau \setminus \tau_0$ has a bi-infinite trajectory (either the forward or backward trajectory starting from any point of τ_0 finishes in a corner from C_P^*).

Clearly, the sequence $\{\pi_1(S^n(v_0))\}_{n\in I}$ is dense in $(c, d) = \phi((a, b))$. Define a continuous function $g : (c, d) \to S^1$ by

$$g(y) = \pi_2(\Phi(\phi^{-1}(y) \times \{\alpha\})). \tag{7}$$

In what follows we will show that since the polygons P, Q are order equivalent, the continuous function g has to be constant. It is sufficient to show that $g'(y_0) = 0$ whenever $y_0 \in (c,d) \setminus C$, where $C = \pi_1(\Phi(\tau_0))$ is countable. Suppose that the direction perpendicular to the side of Q containing (c,d) is the origin $0 \in S^1$. Fix $y_0 \in (c,d) \setminus C$; then for a sufficiently small neighborhood $U(y_0)$ of y_0, $g(U(y_0)) \subset (-\pi/2, \pi/2)$.

For $y' \in U(y_0) \setminus C$ consider the unfolded (bi-infinite) billiard trajectory of $(y', g(y'))$ under the billiard flow $\{S_t\}_{t\in\mathbb{R}}$ in Q. Via unfolding, this trajectory corresponds to the line $p_{g(y')}$ with equation $y = y' + z \tan g(y')$. Denote by $S^m(y)$ the copy of $S^m((y, g(y)))$ on $p_{g(y)}$. A *link* $(S^{m-1}(y_0), S^m(y_0)) \subset p_{g(y_0)}$ is a segment of $p_{g(y_0)}$ with endpoints corresponding to copies of $\pi_1(S^{m-1}(y_0))$ and $\pi_1(S^m(y_0))$.

Claim 3.9. *There is no sequence* $\{y_n\}_{n\ge1} \subset (c,d) \setminus C$ *such that* $\lim_n y_n = y_0$ *and the set of crossing points* $\{p_{g(y_0)} \cap p_{g(y_n)} : n \ge 1\}$ *is bounded.*

Proof. On the contrary, suppose that $\{y_n\}_{n\ge1} \subset (c,d) \setminus C$, $\lim_n y_n = y_0$ and the set of crossing points $\{p_{g(y_0)} \cap p_{g(y_n)} : n \ge 1\}$ is bounded. Without loss of generality we can assume that the crossing points lie in the part $\{y_0 + z \tan g(y_0) : z \ge 0\}$ of $p_{g(y_0)}$ and that there exists a value m for which all crossing points from $\{p_{g(y_0)} \cap p_{g(y_n)} : n \ge 1\}$ lie on the link $(S^{m-1}(y_0), S^m(y_0)) \subset p_{g(y_0)}$. Choose a point y_n such that for each $i \in \{0, \dots, m\}$ both points $S^i(y_0)$, $S^i(y_n)$ lie on the same side of Q and the crossing point $p_{g(y_0)} \cap p_{g(y_n)}$ lies on the link $(S^{m-1}(y_n), S^m(y_n)) \subset p_{g(y_n)}$. Such a y_n does exist since the function g is continuous and $p_{g(y_0)}$ does not contain any copy of a corner from C_Q^*. Thus,

$$(S^{m-1}(y_0), S^m(y_0)) \cap (S^{m-1}(y_n), S^m(y_n)) \neq \emptyset. \tag{8}$$

At the same time, for the points $x_0 = \phi^{-1}(y_0)$, $x_n = \phi^{-1}(y_n)$, $\lim_n x_n = x_0$, for $i = 0, \dots, m$ the iterates $T^i((x_0, \alpha))$, $T^i((x_n, \alpha))$ lie on the same side of P, and the unfolded bi-infinite billiard trajectories of (x_0, α), (x_n, α) under the billiard flow $\{T_t\}_{t\in\mathbb{R}}$ in P are parallel lines. In particular, for the links,

$$(T^{m-1}(x_0), T^m(x_0)) \cap (T^{m-1}(x_n), T^m(x_n)) = \emptyset,$$

which contradicts (8) for order equivalent polygons $(P; u_0) \approx (Q; v_0)$ and the claim is proved. ∎

Applying Lemma 3.5 and Claim 3.9 we deduce that the function g defined in (7) satisfies $g'(y_0) = 0$ for every $y_0 \in (c,d) \setminus C$, hence being a continuous function, it has to be constant. This means that

$$S^n(v_0) \in \sigma \quad \text{and} \quad \pi_2(S^n(v_0)) = g(y_0) \quad \text{whenever } n \in I \tag{9}$$

and I is given by (6). We assumed that the trajectory of u_0 is minimal in a flat strip $\mathcal{T}$. It is known that in that case the gaps in I are bounded. This fact together with (9) implies that the set of directions $\{\pi_2(S^n(v_0)) : n \ge 0\}$ along the trajectory of v_0 is finite. This proves the lemma. ∎

4. Rational versus irrational. As in Section 2, we denote by D the group generated by the reflections in the lines through the origin, parallel to the sides of the polygon P. The group D is (countably) infinite if and only if the polygon P is irrational [MT]. In our case when P is simply connected, this is equivalent to the condition that some angle of P is not π-rational.

THEOREM 4.1. *Let P be irrational, and u a point of the phase space.*

(i) *If the orbit of u does not hit a corner and $\theta = \pi_2(u)$ is non-exceptional then $\{\pi_2(T^n(u)) : n \geq 0\}$ is infinite.*
(ii) *If u is not periodic, but visits only a finite number of floors then (u is recurrent and) $\overline{\mathrm{orb}}(u)$ is a flat strip.*

Proof. (i) Fix P and θ as in the statement of the lemma. Enumerate the infinite orbit $D\theta$ as $\{\theta_1, \theta_2, \ldots\}$. Consider $I := \bigcup e_j \times \theta_i$. This union is taken over all sides and for each side over the inner pointing (at some interior point of the side) θ_i from S^1. Equip each $e_j \times \theta_i$ with the measure $\sin\theta_i\, dx$ where dx is the arc length along e_j and $\theta_i \in (0, \pi)$ is measured with respect to the oriented side e_j. We will refer to this measure as *length*. The billiard map T leaves I invariant and acts as an infinite interval exchange map. The boundary point of each interval of continuity corresponds to a (first) preimage of a corner of P.

Suppose that the result is not true; then there is a point $x \in \partial P$ such that the orbit of (x, θ) is infinite but takes on only a finite number of angles $\{\theta_{i_1}, \ldots, \theta_{i_n}\}$. Let $I(x) := \bigcup e_j \times \theta_{i_k} \subset I$ be the (finite) set of intervals in the directions visited by the orbit of (x, θ).

First note that the set $I(x)$ cannot be T-invariant: this would contradict the fact that the D-orbit of θ is infinite.

Let $J^{\pm} := J^{\pm}(x) := \{(x', \theta') \in I(x) : T^{\pm}(x', \theta') \in I(x)\}$. Each of these sets is a finite union of intervals and the total length of J^+ is equal to the total length of J^-. Thus the total length of the intervals of $I(x)$ where the forward map is not defined is equal to the total length of the intervals where the backward map is not defined. Thus we can complete (in an arbitrary manner) the definition of the partially defined map to an interval exchange transformation G (IET). The IET G agrees with the partially defined first return billiard map whenever the latter was defined.

Since G is an IET, the well known topological decomposition holds: the interval of definition is decomposed into periodic and minimal components, with the boundary of each component consisting of saddle connections, i.e., orbits starting and ending at a point of discontinuity of the IET (see for example [MT]). Saddle connections correspond to generalized diagonals of the billiard.

Since the original billiard direction is non-exceptional, the G-orbit of (x, θ) cannot be periodic. By the topological decomposition theorem, if the orbit of (x, θ) is not all of $I(x)$ it must accumulate on a saddle connection of G. This saddle connection must be in the closure of the set J^+, thus it corresponds to a generalized diagonal of the billiard map. This contradicts the fact that θ is non-exceptional.

(ii) The proof is similar to that for (i). Define the ghost map and its extension in the same way as above. The conclusion is just a consequence of the topological decomposition theorem for IETs. ∎

The main result of this section follows.

THEOREM 4.2. *Let $(P; u_0) \approx (Q; v_0)$, P rational, u_0 non-exceptional. Then Q is rational with v_0 non-exceptional.*

Proof. All the assumptions of Lemma 3.3 are fulfilled, thus $\phi(C_P) = C_Q$. By Lemma 3.7, v_0 visits a finite number of floors.

First let us assume that v_0 is exceptional, i.e., parallel to a generalized diagonal d. Let y, resp. y' be an outgoing, resp. incoming corner of d with $y' = \pi_1(S^m((y, \beta)))$ for some $m \in \mathbb{N}$ and a direction β with respect to a side f. Since by Lemma 3.4(iv), v_0 is also recurrent and visits a finite number of floors, there is a sequence $\{n(k)\}_{k \geq 0}$ such that for each k, $(y_k, \beta) = S^{n(k)}(v_0) \in \omega(v_0)$ with $y \neq y_k \in f$ and $\lim_k y_k = y$. Put $e = \phi^{-1}(f)$, $x_k = \phi^{-1}(y_k)$, $x = \phi^{-1}(y)$ and $x' = \phi^{-1}(y')$. Then $x, x' \in C_P$ and since u_0 also visits a finite number of floors, the sequence $\{\pi_2(\Phi^{-1}((y_k, \beta)))\}_{k \geq 0}$ has to be eventually constant, i.e., $\Phi^{-1}((y_k, \beta)) = (x_k, \alpha) = T^{n(k)}(u_0) \in \omega(u_0)$ for each sufficiently large k. The reader can verify that u_0 is parallel to a generalized diagonal outgoing from x and incoming to x'. This is impossible for u_0 non-exceptional; consequently, v_0 has to be non-exceptional.

If the polygon Q were irrational, Theorem 4.1(i) would imply, since the orbit of v_0 is infinite, that v_0 is exceptional. This is impossible by Theorem 4.1. ∎

5. Rational versus rational. In a rational polygon a billiard trajectory may have only finitely many different directions. In Section 2 we introduced the invariant subset R_θ of the phase space consisting of the points whose second projection belongs to the orbit of θ under the dihedral group D_N; R_θ has the structure of a surface. For non-exceptional θ's the faces of R_θ can be glued according to the action of D_N to obtain a flat surface depending only on the polygon P but not on the choice of θ; we will denote it R_P.

Let us recall the construction of R_P. Consider $2N$ disjoint parallel copies $P_1, \ldots, P_{2N}$ of P in the plane. Orient the even ones clockwise and the odd ones counterclockwise. We will glue their sides together pairwise, according to the action of the group D_N. Let $0 < \theta = \theta_1 < \pi/N$ be some angle, and let θ_i be its ith image under the action of D_N. Consider P_i and reflect the direction θ_i in one of its sides. The reflected direction is θ_j for some j. Glue the chosen side of P_i to the identical side of P_j. After these gluings are done for all the sides of all the polygons, one obtains an oriented compact surface R_P.

Let p_i be the ith vertex of P with the angle $\pi m_i/n_i$ and denote by G_i the subgroup of D_N generated by the reflections in the sides of P, adjacent to p_i. Then G_i consists of $2n_i$ elements. According to the construction of R_P the number of copies of P that are glued together at p_i equals the cardinality of the orbit of the test angle θ under the group G_i, that is, equals $2n_i$.

Each corner $p \in C_P$ of P corresponds to an angle $A(p) \in (0, 2\pi) \setminus \{\pi\}$.

Proposition 5.1. *Let $(P; u_0) \approx (Q; v_0)$, P rational with u_0 non-exceptional. Then $\phi(C_P) = C_Q$ and $A(p) = A(\phi(p))$ for each $p \in C_P$.*

A triangle is determined (up to similarity) by its angles, thus Proposition 5.1 implies

Corollary 5.2. *Let $(P; u_0) \approx (Q; v_0)$, P a rational triangle with u_0 non-exceptional. Then Q is similar to P.*

Proof. Theorem 4.2 implies that Q is also rational with a non-exceptional leader v_0. From Lemma 3.3 it follows that $\phi(C_P) = C_Q$; let $k = \#C_P = \#C_Q$, and number the corners p_i of P, resp. q_i of Q so that $\phi(p_i) = q_i$, $i = 1, \dots, k$. Since P, Q are rational and simply connected, $A(p_i) = \pi m_i^P/n_i^P$ and $A(q_i) = \pi m_i^Q/n_i^Q$, where m_i^P, n_i^P, resp. m_i^Q, n_i^Q are coprime integers. In what follows, we will show that $n_i^P = n_i^Q$ and $m_i^P = m_i^Q$.

First of all, it is known that, since u_0 is non-exceptional, the least common multiple N_P of n_i^P's is equal to

$$\#\{u \in \omega(u_0) : \pi_1(u) = x\} = \#\{u \in \omega(u_0) : \pi_1(u) = x'\} \tag{10}$$

whenever $x, x' \in \partial P \setminus C_P$; moreover, if in addition x and x' are from the same side then

$$\pi_2(\{u \in \omega(u_0) : \pi_1(u) = x\}) = \pi_2(\{u \in \omega(u_0) : \pi_1(u) = x'\}). \tag{11}$$

The analogous equalities are true for N_Q.

Using (10) and Lemma 3.4(ii)(3) we get $N_P = N_Q = N$. Thus, both rational billiards correspond to the same dihedral group D_N.

Second, consider the local picture around the ith vertex p_i. Denote the two sides which meet at p_i by e and e'. Suppose there are $2n_i^P$ copies of P which are glued at p_i. Label them $1, 2, \dots, 2n_i^P$ in a cyclic counterclockwise fashion. Since u_0 is non-exceptional, its orbit is minimal, so it visits each of the copies of P glued at p_i. In particular the orbit crosses each of the gluings (copy j glued to copy $j+1$).

Now consider the orbit of v_0. We need to show that there are the same number of copies of Q glued at $q_i = \phi(p_i)$. Suppose that j is an element of the cyclic group $\{1, \dots, 2n_i^P\}$. Since u_0 is non-exceptional, the orbit of u_0 must pass from copy j to copy $j+1$ of P or vice versa. Suppose that we are at the instant that the orbit of u_0 passes from copy j to copy $j+1$ of P. At this same instant the orbit of v_0 passes through a side. We label the two copies of Q by j and $j+1$ respectively. This labeling is consistent for each crossing from j to $j+1$.

Since this is true for each j, the combinatorial data of the orbit u_0 glue the corresponding $2n_i^P$ copies of Q together in the same cyclic manner as the corresponding copies of P. Note that the common point of the copies of Q is a common point of $\phi(e)$ and $\phi(e')$, thus it is necessarily the point $q_i = \phi(p_i)$. In particular, since Lemma 3.4(iii) applies, we have $2n_i^P$ copies of Q glued around q_i to obtain an angle which is a multiple of 2π. The total angle at this corner is by definition $2\pi m_i^Q$. Thus $2n_i^Q$ must divide $2n_i^P$. The argument is symmetric, thus $2n_i^P$ divides $2n_i^Q$. We conclude that $n_i^P = n_i^Q$.

Third, let us show that $m_i^Q = m_i^P$. Realizing the gluing of $2n_i^P$ copies of P together at p_i we get a point $p \in R_P$ with total angle of $2\pi m_i^P$. If $m_i^P > 1$, the point p is a cone angle $2\pi m_i^P$ singularity. In any case, for the direction θ and the corresponding constant flow on R_P, there are m_i^P incoming trajectories that enter p on the surface R_P, hence also m_i^P points in V_P that finish their trajectory after the first iterate at the corner p_i. Repeating all arguments for Q and $\vartheta = \pi_2(v_0)$, one obtains m_i^Q points in V_Q that finish their trajectory after the first iterate at the corner $q_i = \phi(p_i)$. Since such a number has to be preserved by the homeomorphism $\varPhi$, the inequality $m_i^P \neq m_i^Q$ contradicts our assumption $(P; u_0) \approx (Q; v_0)$. Thus, $m_i^Q = m_i^P$. ∎

We have recalled in Section 2 that the billiard map T has a natural invariant measure on its phase space given by the formula $\mu = \sin\theta\, dx\, d\theta$ and the notion of a generic point u. In the case when P is rational and the corresponding billiard flow is dense in the surface R_P, the measure μ sits on the skeleton K_P (union of edges) of R_P. In particular, an edge e of K_P associated with θ has μ-length $|e|\sin\theta$. As before, the number N is defined as the least common multiple of n_i's, where the angles of a simply connected rational polygon P are $\pi m_i/n_i$.

We have seen in Example 2.2 that for $N = 2$ order equivalent non-similar polygons exist even if they are uniquely ergodic (note that rectangular billiards are uniquely ergodic in every non-exceptional direction). For any rational polygon with $N = 2$ we can speak (up to rotation) about horizontal, resp. vertical sides. Two such polygons, P and Q with sides e_i resp. f_i, are *affinely similar* if they have the same number of corners/sides, the corresponding angles are equal, and there are positive numbers $a, b \in \mathbb{R}$ such that $|e_i|/|f_i| = a$, resp. $|e_i|/|f_i| = b$ for any pair of corresponding horizontal, resp. vertical sides. Recall the map $\varPhi$ defined in Lemma 3.4.

THEOREM 5.3. *Let $(P; u) \approx (Q; v)$, P rational with u non-exceptional. Denote by μ and ν the natural measures sitting on the skeletons K_P and K_Q*

respectively. If $\nu = \Phi^*\mu$ *then*

(i) Q *is similar to* P *whenever* $N = N_P \geq 3$*;*
(ii) Q *is affinely similar to* P *whenever* $N = N_P = 2$*.*

Proof. We know from Theorem 4.2 that under our assumptions, Q is also rational with v non-exceptional. By Proposition 5.1, the polygons P, Q are quasisimilar, i.e., they have the same angles including their counterclockwise order. In particular, $N_P = N_Q$. Moreover, Proposition 2.3 and Lemma 3.4 imply $\Phi(K_P) = K_Q$.

1) For a side e of P and a $\theta \in [0, \pi]$ denote by $[e, \theta]$ the edge of K_P associated with e and θ. Let $[f, \vartheta] = \Phi([e, \theta])$. Since $\nu = \Phi^*\mu$ and μ, ν are natural,

$$|e| \sin\theta = |f| \sin\vartheta. \tag{12}$$

Assume that the least common multiple N of the denominators of the angles of P is greater than or equal to 3. The polygons P, Q correspond to the same dihedral group D_N generated by the reflections in lines through the origin that meet at angles π/N. The orbit of $\theta_0^+ = \pi_2(u_0)$, resp. $\vartheta_0^+ = \pi_2(u_0)$ under D_N consists of the $2N$ angles

$$\theta_j^+ = \theta_0^+ + 2j\pi/N, \qquad \theta_j^- = \theta_0^- + 2j\pi/N,$$

resp.

$$\vartheta_j^+ = \vartheta_0^+ + 2j\pi/N, \qquad \vartheta_j^- = \vartheta_0^- + 2j\pi/N.$$

Since $N \geq 3$, for each side e, resp. f one can find the angles

$$\theta, \theta + 2\pi/N, \quad \text{resp.} \quad \vartheta, \vartheta + 2\pi/N$$

such that by Lemma 3.4, $\Phi([e, \theta]) = [f, \vartheta]$ and $\Phi([e, \theta + 2\pi/N]) = [f, \vartheta + 2\pi/N]$. Then as in (12),

$$|e| \sin\theta = |f| \sin\vartheta, \qquad |e| \sin(\theta + 2\pi/N) = |f| \sin(\vartheta + 2\pi/N),$$

hence after some routine computation we get $|e| = |f|$.

2) Since the polygons P and Q are quasisimilar we can speak about corresponding horizontal, resp. vertical sides. As above, for a side e of P, some $\theta \in [0, \pi]$ and $[f, \vartheta] = \Phi([e, \theta])$,

$$|e| \sin\theta = |f| \sin\vartheta,$$

where θ, resp. ϑ can be taken the same for any pair of corresponding horizontal, resp. vertical sides. Thus, the number $a = |e|/|f|$, resp. $b = |e|/|f|$ does not depend on the concrete choice of a pair of corresponding horizontal, resp. vertical sides. This finishes the proof of our theorem. ∎

In a rational polygon we say that a point u is *generic* if it is non-exceptional, has bi-infinite orbit and the billiard map restricted to the skeleton K_P of an invariant surface $R_P \sim R_{\pi_2(u)}$ has a unique invariant measure (this measure is then automatically the measure μ).

COROLLARY 5.4. *Let P, Q be order equivalent with leaders u, v, P rational, u generic. Then*

(i) *Q is similar to P whenever $N = N_P \geq 3$;*
(ii) *Q is affinely similar to P whenever $N = N_P = 2$.*

Proof. Obviously the dynamical systems (K_P, T), (K_Q, S) are conjugate via the conjugacy Φ, hence by our assumption on the element u, both are uniquely ergodic. This means that $\nu = \Phi^*\mu$, where μ, ν are natural, and Theorem 5.3 applies. ■

6. Conclusions and open questions. Let us summarize our main results in terms of order equivalence classes. We call an order equivalence class non-exceptional (resp. generic, resp. Veech) if the point u_0 in P is additionally supposed to be non-exceptional (resp. generic with respect to μ, resp. if P is Veech due to Definition 2.5). Two order equivalent polygons P, Q are said to be *quasisimilar* if $\phi(C_P) = C_Q$ and $A(p) = A(\phi(p))$ for each $p \in C_P$.

THEOREM 6.1. *Let us consider order equivalence classes defined in Definition* 2.1. *Then:*

- *An irrational non-exceptional order equivalence class does not contain rational polygons* (Th. 4.2).
- *A rational non-exceptional order equivalence class contains quasisimilar polygons* (Prop. 5.1).
- *A rational non-exceptional triangle order equivalence class is (up to similarity) one point* (Cor. 5.2).
- *A rational generic order equivalence class with $N \geq 3$ is (up to similarity) one point* (Th. 5.3(1)).
- *A rational generic order-equivalence class with $N = 2$ is (up to affine similarity) one point* (Th. 5.3(2)).
- *For each $n \geq 3$ ($n \neq 4$), the order equivalence class of the regular n-gon is (up to similarity) one point* (Th. 2.4, Cor. 5.4).
- *A Veech order equivalence class with $N \geq 3$ is (up to similarity) one point* (Th. 2.6, Cor. 5.4).

The following questions remain open:

(1) Can one replace the assumption that both orbits are dense (Definition 2.1(i)) by only one orbit being dense?
(2) Can one replace the assumption that the orbit is generic by the orbit being non-exceptional in Corollary 5.4?
(3) Can two irrational (non-similar) polygons be order equivalent?

(4) For a rational polygon P, does the density of the (foot) orbit in ∂P imply that the direction is non-exceptional?

Acknowledgements. We gratefully acknowledge the support of the "poste tcheque" of the University of Toulon. The first author was also supported by MYES of the Czech Republic via contract MSM 6840770010 and by the Grant Agency of the Czech Republic contract number 201/09/0854.

References

[BK] K. Burns and A. Katok, *Manifolds with nonpositive curvature*, Ergodic Theory Dynam. Systems 5 (1985), 307–317.

[GWW] C. Gordon, D. Webb and S. Wolpert, *Isospectral plane domains and surfaces via Riemannian orbifolds*, Invent. Math. 110 (1992), 1–22.

[K] M. Kac, *Can one hear the shape of a drum?*, Amer. Math. Monthly 73 (1966), 1–23.

[MT] H. Masur and S. Tabachnikov, *Rational billiards and flat surfaces*, in: Handbook of Dynamical Systems, Vol. 1A, B. Hasselblatt and A. Katok (eds.), Elsevier, 2002, 1015–1089.

[M] J. Milnor, *Eigenvalues of the Laplace operator on certain manifolds*, Proc. Nat. Acad. Sci. U.S.A. 51 (1964), 542.

[O] J.-P. Otal, *Le spectre marqué des longueurs des surfaces à courbure négative*, Ann. of Math. 131 (1990), 151–162.

[V1] W. Veech, *The billiard in a regular polygon*, Geom. Funct. Anal. 2 (1992), 341–379.

[V2] —, *Teichmüller curves in moduli space, Eisenstein series and an application to triangular billiards*, Invent. Math. 97 (1989), 553–583; Erratum, ibid. 103 (1991), 447.

[Vo] Ya. Vorobets, *Plane structures and billiards in rational polygons: the Veech alternative*, Russian Math. Surveys 51 (1996), 779–817.

Jozef Bobok
KM FSv ČVUT
Thákurova 7
166 29 Praha 6, Czech Republic
E-mail: bobok@mat.fsv.cvut.cz

Serge Troubetzkoy
Centre de physique théorique
Fédération de Recherches des Unités
de Mathématique de Marseille
Institut de mathématiques de Luminy
and
Université de la Méditerranée
Luminy, Case 907
F-13288 Marseille Cedex 9, France
E-mail: troubetz@iml.univ-mrs.fr
http://iml.univ-mrs.fr/~troubetz/

Received 6 November 2008;
in revised form 2 February 2011