Connected economically metrizable spaces

by

Taras Banakh (Kielce and Lviv), Myroslava Vovk (Lviv) and Michał Ryszard Wójcik (Louisville, KY)

Abstract. A topological space is *non-separably connected* if it is connected but all of its connected separable subspaces are singletons. We show that each connected sequential topological space X is the image of a non-separably connected complete metric space $\mathsf{E}X$ under a monotone quotient map. The metric $d_{\mathsf{E}X}$ of the space $\mathsf{E}X$ is *economical* in the sense that for each infinite subspace $A \subset X$ the cardinality of the set $\{d_{\mathsf{E}X}(a,b) : a, b \in A\}$ does not exceed the density of A, $|d_{\mathsf{E}X}(A \times A)| \leq \mathrm{dens}(A)$.

The construction of the space $\mathsf{E}X$ determines a functor $\mathsf{E} : \mathbf{Top} \to \mathbf{Metr}$ from the category $\mathbf{Top}$ of topological spaces and their continuous maps into the category $\mathbf{Metr}$ of metric spaces and their non-expanding maps.

1. Introduction. This paper was motivated by the problem of constructing a non-separably connected complete metric space, posed in [10] and [11]. A topological space X is called *separably connected* if any two points of X lie in a connected separable subspace. On the other hand, a topological space X is *non-separably connected* if it is connected but all connected separable subspaces of X are singletons.

The first example of a non-separably connected metric space was constructed by R. Pol [14] in 1975. Another example was given by P. Simon [17] in 2001. In 2008, M. Morayne and M. R. Wójcik obtained a non-separably connected metric group as the graph of an additive function from the real line to a non-separable Banach space (see [18] or [11]). Recently E. Pol and R. Pol [13] (answering a question posed in the preliminary version of this paper) constructed a non-separably connected metric space which is connected and locally connected. However, none of these non-separably connected spaces is completely metrizable.

[145]

In this paper we shall suggest a general (functorial) construction of non-separably connected complete metric spaces. Our approach is based on the notion of an *economical* metric space. This is a metric space (X, d) such that for each infinite subspace $A \subset X$ the set $d(A \times A) = \{d(a, b) : a, b \in A\}$ has cardinality $|d(A \times A)| \leq \mathrm{dens}(A)$ where $\mathrm{dens}(A) = \min\{|D| : D \subset A \subset \overline{D}\}$ stands for the density of A.

It is easy to see that each separable subspace of an economical metric space is zero-dimensional and hence each connected economically metrizable space is non-separably connected. The following theorem, which is the main result of the paper, yields many examples of connected economical complete metric spaces, thus resolving Problem 2 of [11]. This theorem is proved in Section 9 (see Theorem 9.1).

THEOREM 1.1. *Each* (*connected*) *sequential topological space* X *is the image of a* (*connected*) *economical complete metric space* $\mathsf{E}X$ *under a monotone quotient map* $\xi_X : \mathsf{E}X \to X$.

As we shall see, the construction of the space $\mathsf{E}X$ determines a functor

$$\mathsf{E} : \mathbf{Top} \to \mathbf{Metr}$$

from the category $\mathbf{Top}$ of topological spaces and their continuous maps to the category $\mathbf{Metr}$ of metric spaces and their non-expanding maps. The functor E will be defined as the composition $\mathsf{E} = \circledast^\omega \circ \mathsf{D}$ of the functor of sequential decomposition $\mathsf{D} : \mathbf{Top} \to \mathbf{PMetr}$ and the functor of infinite iterated cobweb $\circledast^\omega : \mathbf{PMetr} \to \mathbf{Metr}$. Here $\mathbf{PMetr}$ is the category of premetric spaces and their non-expanding maps. A *premetric space* is a pair (X, d) consisting of a set X and a function $d : X \times X \to [0, \infty)$ such that $d(x, x) = 0$ for all $x \in X$. Premetric spaces will be considered in Section 3; the functors D and $\circledast^\omega$ are defined in Sections 4 and 6, respectively.

2. Economical metric spaces. We recall that the metric d of a metric space (X, d) is *economical* if for each infinite subset $A \subset X$ the set $d(A \times A) = \{d(a, b) : a, b \in A\}$ has cardinality $|d(A \times A)| \leq \mathrm{dens}(A)$.

Typical examples of economical metric spaces are ultrametric spaces. We recall that a metric d on a set X is called an *ultrametric* if it satisfies the strong triangle inequality

$$d(x, z) \leq \max\{d(x, y), d(y, z)\} \quad \text{for all } x, y, z \in X.$$

PROPOSITION 2.1. *Each ultrametric space is economical.*

Proof. We should check that $|d(A \times A)| \leq \mathrm{dens}(A)$ for any infinite subset A of an ultrametric space (X, d). Assuming that $|d(A \times A)| > \mathrm{dens}(A)$, we conclude that the set $D = d(A \times A) \setminus \{0\}$ has cardinality $|D| > \mathrm{dens}(A)$. For every $t \in D$ select a pair of points $x_t, y_t \in A$ with $d(x_t, y_t) = t$. Since the

subspace $B = \{(x_y, y_t)\}_{t \in D} \subset A \times A$ has cardinality $|B| = |D| > \mathrm{dens}(A) = \mathrm{dens}(A \times A) \geq \mathrm{dens}(B)$, it is not discrete and hence has a non-isolated point $(x_t, y_t) \in B$. Then we can find $s \in D \setminus \{t\}$ such that the point (x_s, y_s) is so near to (x_t, y_t) that

$$\max\{d(x_s, x_t), d(y_s, y_t)\} < \tfrac{1}{3}d(x_t, t_t) = t/3.$$

The triangle inequality for d guarantees that

$$s = d(x_s, y_s) \geq d(x_t, y_t) - d(x_t, x_s) - d(y_t, y_s) > \tfrac{1}{3}d(x_t, y_t) = t/3.$$

On the other hand, the strong triangle inequality implies that

$$s = d(x_s, y_s) \leq \max\{d(x_s, x_t), d(x_t, y_t), d(y_t, y_s)\} = d(x_t, y_t) = t.$$

For the same reason,

$$t = d(x_t, y_t) \leq \max\{d(x_t, x_s), d(x_s, y_s), d(y_s, y_t)\} = d(x_s, y_s) = s.$$

Unifying those inequalities, we get $t = s$, which contradicts the choice of s. ∎

Proposition 2.1 implies that the Cantor cube $2^\omega = \{0, 1\}^\omega$ endowed with the ultrametric

$$d(\{x_n\}, \{y_n\}) = \max_{n \in \omega} |x_n - y_n|/2^n$$

is an economical metric space. Yet, the Cantor cube 2^ω is homeomorphic to the Cantor set $C \subset \mathbb{R}$ which, when endowed with the Euclidean metric $d(x, y) = |x - y|$, is not economical. This justifies the following definition.

A topological space X is defined to be *economically metrizable* if the topology of X is generated by an economical metric.

PROPOSITION 2.2. *If X is an economically metrizable space, then each subspace $A \subset X$ of density $\mathrm{dens}(A) < \mathfrak{c}$ is zero-dimensional. Consequently, each connected economically metrizable space is non-separably connected.*

Proof. Let d be an economical metric generating the topology of X. Given any subspace $A \subset X$ of density $\mathrm{dens}(A) < \mathfrak{c}$, we see that $|d(A \times A)| \leq \mathrm{dens}(A) < \mathfrak{c}$ and hence the set $R = [0, \infty) \setminus d(A \times A)$ is dense in $(0, \infty)$. Since each ball $B_A(a, r)$, $a \in A$, $r \in R$, is open-and-closed in A, the space A has a base of the topology consisting of open-and-closed sets, which means that A is zero-dimensional. ∎

PROBLEM 2.3. *Let X be an economical metric space and $A \subset X$ be a subspace of density $\mathrm{dens}(A) < \mathfrak{c}$. Is A strongly zero-dimensional?*

We recall that a metric space X is *strongly zero-dimensional* if for any disjoint closed subsets $A, B \subset X$ there is an open-and-closed subset $U \subset X$ such that $A \subset U \subset X \setminus B$ (see [7, 6.2.4]).

The following example constructed by E. Pol and R. Pol [13] shows that Proposition 2.2 cannot be reversed.

EXAMPLE 2.4. There is a metrizable space X of weight $\aleph_1$ such that each separable subspace of X is zero-dimensional but X is not economically metrizable.

In light of Proposition 2.2 and Example 2.4 the following problem arises naturally:

PROBLEM 2.5. *Characterize topological spaces whose topology is generated by an economical metric.*

We can also ask about the characterization of metrizable spaces X such that any metric generating the topology of X is economical. This question has the following answer:

THEOREM 2.6. *For a metrizable topological space X the following conditions are equivalent:*

(1) *each metric generating the topology of X is economical;*
(2) $|f(A)| \leq \mathrm{dens}(A)$ *for any subspace $A \subset X$ and a continuous map $f : X \to \mathbb{R}$;*
(3) $\min\{|A|, \mathfrak{c}\} \leq \mathrm{dens}(A)$ *for any subspace $A \subset X$.*

Proof. (1)$\Rightarrow$(2). Assume that each metric generating the topology of X is economical and fix any such metric d. Assume that $|f(A)| > \mathrm{dens}(A)$ for some continuous function $f : X \to \mathbb{R}$ and some subspace $A \subset X$. In this case the set A is infinite. Without loss of generality, $f(a) = 0$ for some $a \in A$ and hence

$$|f(A)| = |\{|f(x)| : x \in A\}| = |\{|f(x) - f(a)| : x \in A\}|.$$

It is easy to see that the metric ρ on X defined by

$$\rho(x, y) = d(x, y) + |f(x) - f(y)|$$

generates the topology of X.

By our assumption, both metrics d and ρ are economical. Consequently the sets $\{d(x, a) : x \in A\}$, $\{\rho(x, a) : x \in A\}$ have cardinality $\leq \mathrm{dens}(A)$. Then

$$|f(A)| = |\{|f(x) - f(a)| : x \in A\}| = |\{\rho(x, a) - d(x, a) : x \in A\}| \leq \mathrm{dens}(A),$$

which is the desired contradiction.

(2)$\Rightarrow$(3). Assume that $\min\{|A|, \mathfrak{c}\} > \mathrm{dens}(A)$ for some subset $A \subset X$ but $|f(A)| \leq \mathrm{dens}(A)$ for any continuous function $f : X \to \mathbb{R}$. Without loss of generality, A is a closed subspace of X. We claim that the space A is strongly zero-dimensional. Given two disjoint closed subsets $E, F \subset A$ we should find an open-and-closed subset $U \subset A$ such that $E \subset U \subset A \setminus F$.

By the normality of X there is a continuous function $f : X \to [0, 1]$ such that $f(E) \subset \{0\}$ and $f(F) \subset \{1\}$. By our hypothesis, the set $f(A)$ has cardinality $|f(A)| \leq \mathrm{dens}(A) < \mathfrak{c}$. Consequently, we can find a number

$t \in (0,1) \setminus f(A)$. Then $U = A \cap f^{-1}([0,t))$ is the required open-and-closed set in A separating E from F.

By [7, 6.2.4, 7.3.15], the space A, being metrizable and strongly zero-dimensional, embeds into the countable power D^ω of the discrete space D of cardinality $|D| = \operatorname{dens}(A) < \mathfrak{c}$. Since D admits a continuous injective map into the Cantor cube 2^ω, the countable power D^ω also admits such a map. Taking into account that the Cantor cube embeds into the real line, we conclude that the space A admits a continuous injective map $g : A \to \mathbb{R}$. By the Tietze–Urysohn Theorem [7, 2.1.8], the map g has a continuous extension $f : X \to \mathbb{R}$. Then $|f(A)| = |g(A)| = |A| > \operatorname{dens}(A)$ and this is a contradiction.

The implication (3)$\Rightarrow$(1) is trivial. ∎

3. Premetric spaces. The definition of a metric on a set X is well-known: this is a function $d : X \times X \to [0, \infty)$ satisfying four axioms:

(1) $d(x, x) = 0$,
(2) $d(x, y) = 0$ implies $x = y$,
(3) $d(x, y) = d(y, x)$,
(4) $d(x, z) \leq d(x, y) + d(y, z)$,

for any points $x, y, z \in X$.

Deleting some of these axioms we obtain various generalizations of metrics: pseudometrics (they obey axioms (1), (3), (4)), quasimetrics (1), (2), (4), symmetrics (1), (2), (3). The most radical generalization of a metric is that of a premetric (see [3, §2.4]).

DEFINITION 3.1. A *premetric* on a set X is any function $d : X \times X \to [0, \infty)$ such that $d(x, x) = 0$ for all $x \in X$. A *premetric space* is a pair (X, d) consisting of a set X and a premetric d on X. In what follows, the premetric of a premetric space X will be denoted by d_X or just d if the set X is clear from the context.

A map $f : X \to Y$ between two premetric spaces is called *non-expanding* if $d_Y(f(x), f(y)) \leq d_X(x, y)$ for all $x, y \in X$.

Premetric spaces and their non-expanding maps form a category **PMetr** that contains the category **Metr** of metric spaces as a full subcategory.

Many notions related to metric spaces can still be defined for premetric spaces. In particular, for any point x of a premetric space X we can define the ball of radius r centered at x by the familiar formula

$$B_X(x, r) = \{y \in X : d_X(x, y) < r\}.$$

Also we can define a subset U of a premetric space X to be *open* if for each point $x \in U$ there is $r > 0$ with $B_X(x, r) \subset U$. Open subsets of a premetric space X form a topology called the *premetric topology* (see [3,

§2.4]). Speaking about topological properties of premetric spaces we shall always refer to this topology.

The following proposition can be immediately derived from the definition of the premetric topology.

PROPOSITION 3.2. *Each non-expanding map between premetric spaces is continuous.*

Each subset A of a premetric space X carries the induced premetric $d_A = d_X|A \times A$. Then the identity inclusion id $: A \to X$ is non-expanding and hence is continuous with respect to the premetric topologies. However this inclusion is not necessarily a topological embedding.

EXAMPLE 3.3 (Arens' space). Consider the set

$$S_2 = \{(0,0)\} \cup \{(1/n, 0) : n \in \mathbb{N}\} \cup \{(1/n, 1/(nm)) : n, m \in \mathbb{N}\} \subset \mathbb{R}^2$$

endowed with the premetric

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 1/n & \text{if } x = (0,0) \text{ and } y = (1/n, 0), \\ 1/(nm) & \text{if } x = (1/n, 0) \text{ and } y = (1/n, 1/(nm)), \\ 1 & \text{otherwise.} \end{cases}$$

The topology on S_2 generated by this premetric coincides with the largest topology that induces the Euclidean topology on the convergent sequences

$$\{(0,0)\} \cup \{(1/n, 0) : n \in \mathbb{N}\}$$

and

$$\{(1/n, 0)\} \cup \{(1/n, 1/(nm)) : m \in \mathbb{N}\}, \quad n \in \mathbb{N}.$$

It follows that $(0,0)$ is a non-isolated point of the subset $A = \{(0,0)\} \cup \{(1/n, 1/(nm)) : n, m \in \mathbb{N}\} \subset S_2$. On the other hand, the induced premetric $d_A = d|A \times A$ on A is $\{0,1\}$-valued and generates the discrete topology on A. This means that the indentity inclusion id $: A \to X$ is not a topological embedding.

This example suggests the following

DEFINITION 3.4. A premetric space X is called *hereditary* if for any subset $A \subset X$ the subspace topology on A coincides with the topology generated by the premetric $d_A = d_X|A^2$ induced from X.

In order to characterize hereditary premetric spaces let us introduce another

DEFINITION 3.5. A premetric d on X is called *basic at a point* $x \in X$ if the family of balls $\{B_X(x,r)\}_{r>0}$ is a neighborhood basis at x. This is equivalent to saying that x is an interior point of each ball $B_X(x,r)$, $r > 0$.

A premetric space X is called *basic* if its premetric d_X is basic at each point $x \in X$. In this case the premetric d_X is also called *basic*.

THEOREM 3.6. *A premetric space is hereditary if and only if it is basic.*

Proof. Assume that a premetric space X is not basic. This means that some point $x \in X$ fails to be an interior point of some ball $B_X(x, r)$, $r > 0$. It follows that x is a non-isolated point of the subspace $A = \{x\} \cup (X \setminus B_X(x, r))$ of X. On the other hand, this point is isolated in the topology generated by the restriction $d_A = d_X | A \times A$ of the premetric d_X on A. This means that the premetric space X is not hereditary.

Now assume that a premetric space X is basic. Given any subset $A \subset X$, endow it with the induced premetric $d_A = d_X | A^2$ and consider the identity inclusion $\mathrm{id}_A : A \to X$. The heredity property of X will follow as soon as we check that the map id_A is a topological embedding. The continuity of id_A follows from the non-expanding property of id_A. To show the continuity of the inverse map $\mathrm{id}_A^{-1} : \mathrm{id}_A(A) \to A$, take any point $a \in \mathrm{id}_A(A)$ and a neighborhood $U_a \subset A$ of a. Next, find a positive $r > 0$ such that $B_A(a, r) \subset U_a$. Since the space X is basic, the ball $B_X(a, r) \subset X$ contains some open neighborhood $V \subset X$ of x. Since $\mathrm{id}_A^{-1}(V) \subset \mathrm{id}_A^{-1}(B_X(a, r)) = B_A(a, r) \subset U_a$, we see that the map $\mathrm{id}_A^{-1} : \mathrm{id}_A(A) \to X$ is continuous. ∎

Premetric spaces are tightly connected with weakly first-countable spaces introduced by A. V. Arhangel'skii in [2]. We recall that a topological space X is *weakly first-countable* if to each point $x \in X$ one can assign a decreasing sequence $\{B_n(x)\}_{n \in \omega}$ of subsets of X that contain x so that a subset $U \subset X$ is open if and only if for each $x \in U$ there is $n \in \omega$ with $B_n(x) \subset U$.

It is clear that each first-countable space is weakly first-countable. Arens' space defined in Example 3.3 is weakly first-countable but not first countable.

The following proposition shows that premetric spaces relate to weakly first-countable spaces in the same way as metric spaces relate to metrizable topological spaces.

PROPOSITION 3.7. *A topological space X is weakly first-countable* (*resp. first-countable*) *if and only if the topology of X is generated by a premetric* (*resp. by a basic premetric*).

Proof. If the topology of X is generated by a (basic) premetric $d : X \times X \to [0, \infty)$, then the family of balls $\{B_X(x, 1/n)\}_{n \in \mathbb{N}}$, $x \in X$, witnesses that X is weakly first-countable (resp. first-countable).

Now assume conversely that X is weakly first-countable and for every $x \in X$ fix a decreasing sequence of sets $\{B_n(x)\}_{n \in \omega}$ witnessing that X is weakly first-countable. If X is first-countable, then we can additionally require that each set $B_n(x)$ is a neighborhood of x.

Those sequences determine the premetric $d : X \times X \to [0, \infty)$ defined by

$$d(x, y) = \inf\{2^{-n} : y \in B_n(x)\}$$

for $(x, y) \in X \times X$. This premetric is basic if and only if each $B_n(x)$ is a neighborhood of x in X.

Finally, observe that the topology of X is generated by the premetric d. ∎

Our next aim is to characterize premetric spaces whose topology is first-countable. It is clear that each basic premetric space is first-countable. However the converse is not true.

EXAMPLE 3.8. On the set of integers $\mathbb{Z}$ consider the premetric

$$d(n, m) = \begin{cases} 2^{-m} & \text{if } n = 0 \text{ and } m > 0, \\ 0 & \text{otherwise.} \end{cases}$$

The topology on $\mathbb{Z}$ generated by this premetric is anti-discrete and hence first-countable. However, the premetric space $(\mathbb{Z}, d)$ is not basic.

On the other hand, we shall show that for sequentially Hausdorff premetric spaces the first-countability of the premetric topology is equivalent to the basic property of the premetric.

A topological space X is called *sequentially Hausdorff* if any convergent sequence $\{x_n\}_{n \in \omega} \subset X$ has a unique limit in X. Each Hausdorff space is sequentially Hausdorff. The converse is not true: any uncountable space X endowed the cocountable topology

$$\tau = \{\emptyset\} \cup \{U \subset X : |X \setminus U| \leq \aleph_0\}$$

is sequentially Hausdorff but not Hausdorff.

We shall characterize sequentially Hausdorff premetric spaces as 2-separating premetric spaces.

DEFINITION 3.9. A premetric space X is called

- 1-*separating* if for any distinct points $x, y \in X$ there is $r > 0$ such that $y \notin B_X(x, r)$;
- 2-*separating* if for any distinct points $x, y \in X$ there is $r > 0$ such that $B_X(x, r) \cap B_X(y, r) = \emptyset$.

The following characterization of 1-separating premetric spaces is immediate.

PROPOSITION 3.10. *A premetric space X is 1-separating if and only if it is a topological T_1-space.*

It is clear that a premetric space is 2-separating if its topology is Hausdorff. The converse is not true:

EXAMPLE 3.11. Take any non-metrizable first-countable compact Hausdorff space K. By the Katětov Theorem [9] (see also [7, 4.5.15]), the cube

K^3 is not hereditarily normal and hence contains a non-normal subspace $Y \subset K^3$. Consequently, Y contains two closed disjoint sets $A, B \subset Y$ that have no disjoint neighborhoods in Y. It follows that the closures $\bar{A}$ and $\bar{B}$ of those sets in K^3 contain a common point $c \in \bar{A} \cap \bar{B}$. Fix a decreasing neighborhood base $(B_n(x))_{n \in \mathbb{N}}$ at each point $x \in K^3$ such that $B_1(x) = K^3$. It is easy to see that the space $X = Y \cup \{A, B\}$ endowed with the premetric d defined by

$$d(x, y) = \begin{cases} \inf\{1/n : y \in B_n(x)\} & \text{if } x, y \in Y, \\ \inf\{1/n : y \in B_n(c)\} & \text{if } x = A \text{ and } y \in A, \\ \inf\{1/n : y \in B_n(c)\} & \text{if } x = B \text{ and } y \in B, \\ 1 & \text{otherwise} \end{cases}$$

is 2-separating but not Hausdorff.

On the other hand, we have the following characterization:

THEOREM 3.12. *A premetric space is 2-separating if and only if it is sequentially Hausdorff.*

This theorem can be easily derived from the following useful characterization of convergence in 2-separating premetric spaces.

PROPOSITION 3.13. *A sequence $\{x_n\}_{n \in \omega}$ in a 2-separating premetric space X converges to a point $x \in X$ if and only if $\lim_{n \to \infty} d_X(x, x_n) = 0$.*

Proof. The "if" part is trivial. To prove the "only if" part, assume that a sequence $\{x_n\}_{n \in \omega}$ converges to x but $\lim_{n \to \infty} d_X(x, x_n) \neq 0$. This means that for some $r > 0$ the set $N = \{n \in \omega : x_n \notin B_X(x, r)\}$ is infinite. The convergence of the sequence $\{x_n\}$ to x implies that the set $U = X \setminus \{x_n\}_{n \in N}$ is not open in X.

We shall derive a contradiction by showing that for every $y \in U$ there is $\varepsilon > 0$ with $B_X(y, \varepsilon) \subset U$. Assuming that no such ε exists, we conclude that for some infinite subset $M \subset N$ we get $d_X(y, x_m) \to 0$ as $M \ni m \to \infty$. In this case we show that the set $V = X \setminus (\{y\} \cup \{x_m\}_{m \in M})$ is open in X. Indeed given any point $z \in V$, we can use the 2-separating property of X to find $\varepsilon > 0$ so small that $B_X(z, \varepsilon) \cap B(y, \varepsilon) = \emptyset$. Since the set $F = \{m \in M : x_m \notin B_X(y, \varepsilon)\}$ is finite, we can find a positive $\delta \leq \varepsilon$ such that $B_X(z, \delta) \cap F = \emptyset$. It follows that $B(z, \delta) \subset V$, witnessing that the set V is open.

Taking into account that

$$\{x_m\}_{m \in M} \cap B_X(x, r) = \emptyset \quad \text{and} \quad \lim_{M \ni m \to \infty} d_X(y, x_m) = 0,$$

we conclude that $x \neq y$ and hence V is an open neighborhood of x. Since $M \subset \{n \in \omega : x_n \notin V\}$ is infinite, we conclude that the sequence $\{x_n\}_{n \in \omega}$ does not converge to x, which is the desired contradiction. ∎

Now we are ready to characterize first-countable spaces among 2-separating premetric spaces. This will be done with the help of the Fréchet–Urysohn property.

We recall that a topological space X is

- *Fréchet–Urysohn* if for each $A \subset X$ and $x \in \mathrm{cl}_X(A)$ there is a sequence $\{a_n\}_{n \in \omega} \subset A$ that converges to x;
- *sequential* if for each non-closed subset $A \subset X$ there is a sequence $\{a_n\}_{n \in \omega} \subset A$ that converges to a point $x \in X \setminus A$.

By [2], each weakly first-countable space is sequential. Combining this result with Proposition 3.7 we get

PROPOSITION 3.14. *Each premetric space is sequential.*

Proof. We shall give a direct proof of this important fact for the convenience of the reader. We need to show that a subset $A \subset X$ of a premetric space is closed if for each convergent sequence $\{a_n\}_{n \in \omega} \subset A$ we have $\lim_{n \to \infty} x_n \in A$. Assuming that such a set A is not closed in the premetric topology of X, we can find a point $x \in X \setminus A$ such that $B_X(x, 1/n) \not\subset X \setminus A$ for any $n \in \mathbb{N}$. This allows us to choose a point $a_n \in B_X(x, 1/n) \cap A$. The definition of the premetric topology on X guarantees that the sequence $\{a_n\}_{n \in \mathbb{N}} \subset A$ converges to the points $x \in X \setminus A$, which contradicts the choice of the set A. ■

The following theorem characterizes first-countable premetric spaces among 2-separating premetric spaces.

PROPOSITION 3.15. *For a 2-separating premetric space X the following conditions are equivalent:*

(1) X *is basic;*
(2) X *is hereditary;*
(3) X *is first-countable;*
(4) X *is Fréchet–Urysohn.*

If the equivalent conditions (1)–(4) *hold, then X is Hausdorff.*

Proof. The equivalence (1)⇔(2) has been proved in Theorem 3.6. The implications (1)⇒(3)⇒(4) are trivial.

(4)⇒(1). Assume that a 2-separated premetric space X is Fréchet–Urysohn. Assuming that X is not basic, we would find a point $x \in X$ and a radius $r > 0$ such that x is not an interior point of the ball $B_X(x, r)$. Since X is Fréchet–Urysohn, there is a sequence $\{x_n\}_{n \in \omega} \subset X \setminus B_X(x, r)$ that converges to x. But this contradicts Proposition 3.13. ■

4. The sequential decomposition of a topological space. In this section we shall describe a functor $\mathsf{D} : \mathbf{Top} \to \mathbf{PMetr}$ assigning to each topological space X a premetric space $\mathsf{D}X$ called the *sequential decomposition* of X. This space is defined as follows.

Let $S_0 = \{0\} \cup \{1/n : n \in \mathbb{N}\}$ be a standard convergent sequence on the real line. Then each convergent sequence $\{x_n\}_{n \in \mathbb{N}}$ in a topological space X can be identified with the continuous map $f : S_0 \to X$ that maps $1/n$ onto x_n and 0 onto the limit $\lim_{n \to \infty} x_n$.

Therefore, the set $C(S_0, X)$ of all continuous functions from S_0 to X can be identified with the family of all convergent sequences in X.

The premetric space $\mathsf{D}X$ is defined as the set

$$\mathsf{D}X = C(S_0, X) \times S_0$$

endowed with the premetric

$$d_{\mathsf{D}X}((f, t), (g, s)) = \begin{cases} 0 & \text{if } f(t) = g(s), \\ |t - s| & \text{if } f(t) \neq g(s) \text{ but } f = g, \\ 1 & \text{otherwise.} \end{cases}$$

Any continuous map $f : X \to Y$ between topological spaces induces a non-expanding map

$$\mathsf{D}f : \mathsf{D}X \to \mathsf{D}Y, \quad \mathsf{D}f : (\xi, s) \mapsto (f \circ \xi, s).$$

In this way we have defined a functor

$$\mathsf{D} : \mathbf{Top} \to \mathbf{PMetr}$$

from the category $\mathbf{Top}$ of topological spaces and their continuous maps to the category $\mathbf{PMetr}$ of premetric spaces and their non-expanding maps.

For every topological space X the spaces $\mathsf{D}X$ and X are linked by the *evaluation map*

$$c_X : \mathsf{D}X \to X, \quad c_X : (f, x) \mapsto f(x).$$

This map is natural in the sense that for any continuous map $f : X \to Y$ between topological spaces the following diagram is commutative:

$$
\begin{array}{ccc}
\mathsf{D} & \xrightarrow{\;\mathsf{D}f\;} & \mathsf{D}Y \\
{\scriptstyle c_X}\downarrow & & \downarrow{\scriptstyle c_Y} \\
X & \xrightarrow{\;f\;} & Y
\end{array}
$$

The evaluation map will be used to describe the topology of the space $\mathsf{D}X$. Namely, we shall show that a subset $U \subset \mathsf{D}X$ is open if and only if $U = c_X^{-1}(V)$ for some sequentially open subset $V \subset X$.

A subset $V \subset X$ is called *sequentially open* if for each sequence $\{x_n\}_{n \in \omega}$ in X that converges to a point $x \in V$ there is $n \in \omega$ such that $x_m \in V$ for all $m \geq n$.

LEMMA 4.1. *A subset $U \subset \mathsf{D}X$ is open if and only if $U = c_X^{-1}(V)$ for some sequentially open subset $V \subset X$.*

Proof. First assume that $U = c_X^{-1}(V)$ for some sequentially open subset $V \subset X$. In order to show that U is open in $\mathsf{D}X$ we should find for any point $a \in U \subset \mathsf{D}X$ a radius $r > 0$ such that $B_{\mathsf{D}X}(a, r) \subset U$.

Being an element of $\mathsf{D}X$, the point a is of the form $a = (f, t)$ for some continuous map $f : S_0 \to X$ and some $t \in S_0$. If $t \neq 0$, then we can find $r > 0$ such that $(t - r, t + r) \cap S_0 = \{t\}$. For such r the ball $B_{\mathsf{D}X}(a, r)$ coincides with the set $\{(g, s) \in \mathsf{D}X : g(s) = f(t)\}$ and then $c_X(B_{\mathsf{D}X}(a, r)) = \{f(t)\} = \{c_X(a)\} \subset V$ and hence $B_{\mathsf{D}X}(a, r) \subset U$. If $t = 0$, then the sequence $\{f(1/n)\}_{n=1}^{\infty}$ converges to $f(0) = c_X(a) \in V$. As V is sequentially open, there is $m \geq 2$ such that $f(1/n) \in V$ for all $n \geq m$. Then for the radius $r = 1/m$ we get $B_{\mathsf{D}X}(a, r) \subset c_X^{-1}(V) = U$.

Now assume that a set $U \subset \mathsf{D}X$ is open. First we show that $U = c_X^{-1}(c_X(U))$. Given any $a = (f, t) \in U$ find $r > 0$ with $B_{\mathsf{D}X}(a, r) \subset U$ and observe that

$$c_X^{-1}(c_X(a)) = \{(g, s) \in \mathsf{D}X : g(s) = f(t)\}$$
$$\subset \{b \in \mathsf{D}X : d_{\mathsf{D}X}(a, b) = 0\} \subset B_{\mathsf{D}X}(a, r) \subset U,$$

witnessing that $U = c_X^{-1}(c_X(U))$. It remains to check that the set $c_X(U)$ is sequentially open in X. Fix a sequence $\{x_n\}_{n \in \mathbb{N}}$ in X that converges to a point $x_0 \in c_X(U)$. Then the map $f : S_0 \to X$ defined by $f(0) = x_0$ and $f(1/n) = x_n$ for $n \in \mathbb{N}$ is continuous. Consider the point $a = (f, 0) \in \mathsf{D}X$ and observe that $c_X(a) = x_0 \in c_X(U)$. Since U is open, for the point $a \in c_X^{-1}(x_0) \subset U$ there is a radius $r > 0$ such that $B_{\mathsf{D}X}(a, r) \subset U$. Then for every $n > 1/r$ we get $(f, 1/n) \in B_{\mathsf{D}X}(a, r) \subset U$ and hence $x_n = c_X(f, 1/n) \in c_X(U)$, witnessing that the set $c_X(U)$ is sequentially open in X. ∎

Sequentially open subsets of X form a topology on X. The set X endowed with this topology is called the *sequential coreflexion* of X and is denoted by $\mathsf{s}X$.

Since each open subset of X is sequentially open, the identity map id : $\mathsf{s}X \to X$ is continuous. This map is a homeomorphism if and only if the space X is sequential.

Lemma 4.1 implies the following important result:

COROLLARY 4.2. *For every topological space X the map $c_X : \mathsf{D}X \to \mathsf{s}X$ is surjective, continuous, monotone, pseudo-open, and hereditarily quotient.*

We recall that a surjective map $f : X \to Y$ between topological spaces is

- *monotone* if $f^{-1}(y)$ is connected for every $y \in Y$;
- *quotient* if a set $U \subset Y$ is open if and only if its preimage $f^{-1}(U)$ is open in X;

- *hereditarily quotient* if for every $A \subset Y$ the restriction $f|f^{-1}(A) : f^{-1}(A) \to A$ is a quotient map;
- *pseudo-open* if for every $y \in Y$ and an open set $U \subset X$ containing $f^{-1}(y)$ the point y lies in the interior of $f(U)$ in Y.

We shall often use the following characterization of hereditarily quotient maps, due to Arhangel'skii [1] (see also [7, 2.4.F(a)]).

THEOREM 4.3. *A map between topological spaces is hereditarily quotient if and only if it is continuous and pseudo-open.*

It is known that each quotient map onto a sequentially Hausdorff Fréchet–Urysohn space is hereditarily quotient and pseudo-open (see [1] or [7, 2.4.F(c)]).

Quotient maps are important for us because of the following known fact (see [7, 6.1.28]).

LEMMA 4.4. *Let $f : X \to Y$ be a monotone quotient map. The space X is connected if and only if Y is connected.*

This lemma combined with Corollary 4.2 implies:

COROLLARY 4.5. *The sequential decomposition $\mathsf{D}X$ of a topological space X is connected if and only if the sequential coreflexion $\mathsf{s}X$ of X is connected. In particular, $\mathsf{D}X$ is connected for every connected sequential topological space X.*

5. The complete oriented graph over a set. The *complete oriented graph ΓX* over a set X is the set

$$\Gamma X = X \cup \{(x, y, t) \in X \times X \times (0, 1) : x \neq y\}$$

endowed with a special (path) metric. The points $x \in X \subset \Gamma X$ are called *vertices* of ΓX, and the sets

$$(x, y) = \{(x, y, t) : t \in (0, 1)\} \quad \text{and} \quad [x, y] = \{x, y\} \cup (x, y)$$

are called *oriented edges* connecting distinct vertices $x, y \in X$. It will be convenient to consider also degenerate edges $(x, x) = \emptyset$ and $[x, x] = \{x\}$ for $x \in X$.

Therefore,

$$\Gamma X = \bigcup_{x,y \in X} [x, y] = X \cup \bigcup_{x \neq y} (x, y).$$

Observe that the oriented edges $[x, y]$ and $[y, x]$ meet only in their endpoints: $[x, y] \cap [y, x] = \{x, y\}$.

The graph ΓX is the image of the product $X \times X \times [0,1]$ under the map $\langle \cdot \rangle : X \times X \times [0,1] \to \Gamma X$ defined by

$$\langle \cdot \rangle : (x,y,t) \mapsto \langle x,y,t \rangle = \begin{cases} (x,y,t) & \text{if } x \neq y \text{ and } t \in (0,1), \\ x & \text{if } x = y \text{ or } t = 0, \\ y & \text{if } x = y \text{ or } t = 1. \end{cases}$$

In particular, $\langle x,y,0 \rangle = \langle x,x,t \rangle = \langle y,x,1 \rangle = x$ for every $x,y \in X$ and $t \in [0,1]$.

Now, we define a metric $d_{\Gamma X}$ on ΓX such that each edge $[x,y] \subset \Gamma X$, $x \neq y$, is isometric to the unit interval $[0,1]$. For this we put

$$\tilde{d}_{\Gamma X}(\langle x,y,t \rangle, \langle x,y,s \rangle) = |t - s|$$

for any distinct vertices $x,y \in X$ and $t,s \in [0,1]$. Next, we extend $\tilde{d}_{\Gamma X}$ to a metric $d_{\Gamma X}$ on ΓX by letting $d_{\Gamma X}(a,b)$ be the minimal value of the sum $\sum_{i=1}^{n} \tilde{d}_{\Gamma X}(a_{i-1}, a_i)$ where $a = a_0, a_1, \dots, a_n = b$ and for any $i \leq n$ the points a_{i-1}, a_i lie on some common edge $[x,y] \subset \Gamma X$.

It is easy to check that the metric $d_{\Gamma X}$ has the following properties:

PROPOSITION 5.1.

(1) *The metric $d_{\Gamma X}$ is complete and* $\operatorname{diam} \Gamma X \leq 2$.
(2) *Each edge $[x,y]$, $x \neq y$, is isometric to $[0,1]$.*
(3) *Each edge (x,y), $x \neq y$, is open in ΓX.*
(4) *Any two points $a,b \in \Gamma X$ lying on closed disjoint edges are at distance $d_{\Gamma X}(a,b) \geq 1$.*
(5) *If $d_{\Gamma X}(a,b) < 1$ for some $a,b \in \Gamma X$, then either a,b lie on the same edge or else there is a vertex $x \in X$ such that $d_{\Gamma X}(a,b) = d_{\Gamma X}(a,x) + d_{\Gamma X}(x,b)$.*

The construction of the complete oriented graph determines a functor

$$\Gamma : \mathbf{Set} \to \mathbf{Metr}$$

from the category $\mathbf{Set}$ of sets and their maps to the category $\mathbf{Metr}$ of metric spaces and their non-expanding maps.

In fact, each map $f : X \to Y$ between sets determines a map $\Gamma f : \Gamma X \to \Gamma Y$ defined by

$$\Gamma f : \langle x,z,t \rangle \mapsto \langle f(x), f(z), t \rangle.$$

It follows from the definition of the metrics on the graphs ΓX and ΓY that the map $\Gamma f : \Gamma X \to \Gamma Y$ is non-expanding in the sense that $d_{\Gamma Y}(\Gamma f(a), \Gamma f(b)) \leq d_{\Gamma X}(a,b)$ for any $a,b \in \Gamma X$. Also it is clear that the identity map $\operatorname{id} : X \to X$ induces the identity map $\Gamma \operatorname{id} : \Gamma X \to \Gamma X$ and for any maps $f : X \to Y$ and $g : Y \to Z$ the composition $\Gamma g \circ \Gamma f$ equals the map $\Gamma(g \circ f)$. This means that $\Gamma : \mathbf{Set} \to \mathbf{Metr}$ is a functor.

PROPOSITION 5.2. *For every injective map $f : X \to Y$ between sets the induced map $\Gamma f : \Gamma X \to \Gamma Y$ is an isometric embedding.*

Proof. The map f, being injective, admits a left inverse map $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$. By functoriality of Γ, the maps $\Gamma f : \Gamma X \to \Gamma Y$ and $\Gamma g : \Gamma Y \to \Gamma X$ are non-expanding and their composition satisfies $\Gamma g \circ \Gamma f = \Gamma(g \circ f) = \Gamma \, \mathrm{id}_X = \mathrm{id}_{\Gamma X}$. Then for every $a, b \in \Gamma X$ the non-expanding property of the maps Γf and Γg yields

$$d_{\Gamma X}(a, b) = d_{\Gamma X}(\Gamma g(\Gamma f(a)), \Gamma g(\Gamma f(b))) \leq d_{\Gamma Y}(\Gamma f(a), \Gamma f(b)) \leq d_{\Gamma X}(a, b),$$

which implies that $d_{\Gamma Y}(\Gamma f(a), \Gamma f(b)) = d_{\Gamma X}(a, b)$. This witnesses that $\Gamma f : \Gamma X \to \Gamma Y$ is an isometric embedding. ∎

6. The cobweb construction over a premetric space. In this section we define the cobweb functor $\circledast : \mathbf{PMetr} \to \mathbf{Metr}$.

By the *cobweb* of a premetric space X we understand the following closed subspace $\circledast X$ of the complete oriented graph ΓX:

$$\circledast X = \{\langle x, y, t \rangle \in \Gamma X : t \leq 1 - \bar{d}_X(y, x)\}\} \subset \Gamma X,$$

where $\bar{d}_X = \min\{1, d_X\}$ and d_X is the premetric of X.

Let us explain this construction in more detail. For any distinct points $x, y \in X$ consider the unique point x_y on the edge $[x, y] \subset \Gamma X$ that lies at distance $d_{\Gamma X}(x_y, y) = \bar{d}_X(y, x) = \min\{1, d_X(y, x)\}$ from the vertex y. It follows that the set

$$[x, x_y] = \{\langle x, y, t \rangle \in \Gamma X : t \leq d_{\Gamma X}(x, x_y) = 1 - \bar{d}_X(y, x)\}$$

is a subarc of length $1 - \bar{d}_X(y, x)$ in $[x, y] \subset \Gamma X$. If $d_X(y, x) \geq 1$, then the subarc $[x, x_y]$ degenerates to the singleton $\{x\}$.

The union

$$S_x = \bigcup_{y \neq x} [x, x_y] \setminus \{y\}$$

will be called the *spider* centered at a point $x \in X \subset \circledast X \subset \Gamma X$. Each spider S_x is an arcwise connected subspace of ΓX. Therefore

$$\circledast X = \bigcup_{x \neq y} [x, x_y] = \bigcup_{x \in X} S_x$$

is the union of all such spiders.

The cobweb $\circledast X$ contains X as a discrete set of vertices. Consequently,

$$|X| \leq \mathrm{dens}(\circledast X) \leq |\circledast X| \leq \mathfrak{c} \cdot |X|.$$

The reader should not be confused by the fact that the space X that has the premetric topology will also be considered as the discrete subspace of the cobweb $\circledast X$ over X.

Since $\circledast X$ can be obtained from ΓX by deleting selected open arcs from the edges of ΓX, the subspace $\circledast X$ is closed in ΓX and hence is complete with respect to the metric $d_{\circledast X} = d_{\Gamma X}|\circledast X \times \circledast X$ induced from the graph ΓX.

Next, we show that the construction of the cobweb space determines a functor

$$\circledast : \mathbf{PMetr} \to \mathbf{Metr}$$

from the category $\mathbf{PMetr}$ of premetric spaces and their non-expanding maps to the category $\mathbf{Metr}$ of metric spaces.

Given any non-expanding map $f : X \to Y$ between premetric spaces, consider the induced non-expanding map $\Gamma f : \Gamma X \to \Gamma Y$ between the complete oriented graphs.

LEMMA 6.1. $\Gamma f(\circledast X) \subset \circledast Y$.

Proof. Given any point $a \in \circledast X \subset \Gamma X$, find vertices $x, y \in X$ such that $a \in [x, x_y]$. It follows that $d_{\Gamma X}(x, a) \leq d_{\Gamma X}(x, x_y) = 1 - \bar{d}_X(y, x)$ where $\bar{d}_X = \min\{1, d_X\}$. Now consider the image $\Gamma f(a) \subset \Gamma Y$. If $f(a) \in Y$, then we are done because $Y \subset \circledast Y \subset \Gamma Y$. So, we assume that $f(a) \in \Gamma Y \setminus Y$. In this case $f(a) = (f(x), f(y), t)$ where $a = (x, y, t)$ and $t \leq 1 - \bar{d}_X(y, x)$. Since the map f is non-expanding, $\bar{d}_Y(f(y), f(x)) \leq \bar{d}_X(y, x)$, and consequently $t \leq 1 - \bar{d}_X(y, x) \leq 1 - \bar{d}_Y(f(y), f(x))$. Now the definition of $\circledast Y$ guarantees that $f(a) = (f(x), f(y), t) \in [f(x), f(x)_{f(y)}] \subset \circledast Y$. ∎

Lemma 6.1 allows us to define a map $\circledast f : \circledast X \to \circledast Y$ as the restriction $\circledast f = \Gamma f|\circledast X$. In this way we have proved

THEOREM 6.2. $\circledast : \mathbf{PMetr} \to \mathbf{Metr}$ *is a functor.*

Proposition 5.2 implies that this functor preserves isometric embeddings.

PROPOSITION 6.3. *For any injective non-expanding map $f : X \to Y$ between premetric spaces the induced map $\circledast f : \circledast X \to \circledast Y$ is an isometric embedding.*

7. The compression map. In this section, given a premetric space X we construct an important map $\pi_X : \circledast X \to X$ called the *compression map.* It is defined by the formula

$$\pi_X(a) = \begin{cases} a & \text{if } a \in X \subset \circledast X, \\ x & \text{if } a = (x, y, t) \in \circledast X \setminus X. \end{cases}$$

Observe that for every $x \in X$ the preimage

$$\pi_X^{-1}(x) = S_x = \bigcup_{X \ni y \neq x} [x, x_y] \setminus \{y\}$$

coincides with the spider S_x centered at x.

The compression map π_X is natural in the sense that for any non-expanding map $f : X \to Y$ between premetric spaces the following diagram commutes:

$$
\begin{CD}
\circledast X @>{\pi_X}>> X \\
@V{\circledast f}VV @VV{f}V \\
\circledast Y @>{\pi_Y}>> Y
\end{CD}
$$

In the following proposition we collect some simple properties of the compression map.

PROPOSITION 7.1. *Let X be a premetric space. Then:*

(1) $\pi_X(x) = x$ *for all $x \in X$;*
(2) $\pi_X : \circledast X \to X$ *is a monotone surjection;*
(3) π_X *is locally constant on the open subset $\circledast X \setminus X$ of $\circledast X$;*
(4) *for any subset $A \subset \circledast X$ we get $|\pi_X(A)| \leq \mathrm{dens}(A)$.*

Proof. (1), (3) These items follow immediately from the definition of π_X.

(2) Since each spider $S_x = \pi_X^{-1}(x)$, $x \in X$, is non-empty and (arcwise) connected, the map π_X is a monotone surjection.

(4) Let $A \subset \circledast X$. If A is finite, then $|\pi_X(A)| \leq |A| = \mathrm{dens}(A)$. So, we assume that A is infinite.

Since X is discrete in $\circledast X$, we have $|A \cap X| \leq \mathrm{dens}(A)$. Since π_X is locally constant on $\circledast X \setminus X$, it follows that $|\pi_X(A \setminus X)| \leq \mathrm{dens}(A)$. Combining those two facts, we get

$$|\pi_X(A)| \leq |\pi_X(A \cap X)| + |\pi_X(A \setminus X)| \leq \mathrm{dens}(A) + \mathrm{dens}(A) = \mathrm{dens}(A). \quad \blacksquare$$

Next, we establish some metric properties of the compression map.

PROPOSITION 7.2. *Let X be a premetric space, d_X be the premetric of X and $\bar{d}_X = \min\{1, d_X\}$. Then*

   (1) $\bar{d}_X(x, \pi_X(a)) \leq d_{\circledast X}(x, a)$ *for every $x \in X$ and $a \in \circledast X$;*
   (2) $\pi_X(B_{\circledast X}(x, r)) = B_X(x, r)$ *for any $x \in X$ and $r \in (0, 1]$;*
   (3) $d_{\circledast X}(a, b) \geq \inf_{x \in X}[\bar{d}_X(x, \pi_X(a)) + \bar{d}_X(x, \pi_X(b))]$ *for any $a, b \in \circledast X$;*
   (4) *the compression map $\pi_X : \circledast X \to X$ is non-expanding if and only if $\bar{d}_X$ is a pseudometric.*

Proof. (1) Fix any $x \in X$ and $a \in \circledast X$. We need to check that $\bar{d}_X(x, y) \leq d_{\circledast X}(x, a)$ where $y = \pi_X(a)$. This inequality is trivial if $d_{\circledast X}(x, a) \geq 1$. So, we assume that $d_{\circledast X}(x, a) < 1$.

Since $a \in \pi_X^{-1}(y) = S_y = \bigcup_{z \neq y}[y, y_z] \setminus \{z\}$, there is a point $z \in X \setminus \{y\}$ such that $a \in [y, y_z]$. By Proposition 5.1(4), the inequality $d_{\circledast X}(x, a) < 1$ implies that $x \in \{y, z\}$. If $x = y$, then we are done because $\bar{d}_X(x, y) = 0 \leq d_{\circledast X}(x, a)$. If $x = z$, then $\bar{d}_X(x, y) = \bar{d}_X(z, y) = d_{\circledast X}(z, y_z) \leq d_{\circledast X}(z, a) = d_{\circledast X}(x, a)$.

(2) Fix any $x \in X$ and a positive $r \leq 1$. The preceding item implies that $\pi_X(B_{\circledast X}(x, r)) \subset B_X(x, r)$. The reverse inclusion will follow as soon as, given any point $z \in B_X(x, r)$, we find a point $y \in \pi_X^{-1}(z) = S_z$ with $d_{\circledast X}(x, y) < r$. If $z = x$, then we can put $y = x$. So, assume that $z \neq x$.

Consider the point $z_x \in [z, x]$ and observe that $d_{\circledast X}(x, z_x) = \bar{d}_X(x, z) \leq d_X(x, z) < r$. If $z_x \neq x$, then we can put $y = z_x \in S_z$. If $z_x = x$, then $[z, z_x] = [z, x]$ and we can take any point $y \in [z, z_x) \subset S_z$ with $d_{\circledast X}(x, y) < r$.

(3) Take any two points $a, b \in \circledast X$ and consider their images $y = \pi_X(a)$ and $z = \pi_X(b)$. If $y = z$, then

$$\inf_{x \in X} [\bar{d}_X(x, y) + \bar{d}_X(x, z)] = 0 \leq d_{\circledast X}(a, b).$$

So, we can assume that $y \neq z$.

If $d_{\circledast X}(a, b) \geq 1$, then

$$d_{\circledast X}(a, b) \geq 1 \geq \bar{d}_X(y, z) \geq \inf_{x \in X} [\bar{d}_X(x, y) + \bar{d}_X(x, z)]$$

and we are done. So, we can assume that $d_{\circledast X}(a, b) < 1$. In this case Proposition 5.1(5) guarantees that $d_{\circledast X}(a, b) = d_{\circledast X}(a, v) + d_{\circledast X}(v, b)$ for some vertex $v \in X \subset \Gamma X$. Applying (1) we conclude that

$$d_{\circledast X}(a, b) = d_{\circledast X}(v, a) + d_{\circledast X}(v, b) \geq \bar{d}_X(v, \pi_X(a)) + \bar{d}_X(v, \pi_X(b))$$
$$\geq \inf_{x \in X} [\bar{d}_X(x, \pi_X(a)) + \bar{d}_X(x, \pi_X(b))].$$

(4) If $\bar{d}_X$ is a pseudometric, then for any $a, b \in \circledast X$ the triangle inequality for $\bar{d}_X$ combined with the preceding item implies

$$d_{\circledast X}(a, b) \geq \inf_{x \in X} [\bar{d}_X(x, \pi_X(a)) + \bar{d}_X(x, \pi_X(b))] \geq d_X(\pi_X(a), \pi_X(b)),$$

witnessing the non-expanding property of the compression map π_X.

Now assume that the compression map $\pi_X : \circledast X \to X$ is non-expanding. Given any three points $x, y, z \in X$ we shall prove that

$$(7.1) \qquad \bar{d}_X(x, z) \leq \bar{d}_X(y, x) + \bar{d}_X(y, z).$$

This inequality is trivial if $x = z$ or $x = y$. So, we assume that $x \neq z$ and $x \neq y$. Consider the points $x_y, z_y \in \circledast X$. If $x_y \neq y \neq z_y$, then $\pi_X(x_y) = x$, $\pi_X(z_y) = z$ and then

$$\bar{d}_X(x, z) = \bar{d}_X(\pi_X(x_y), \pi_X(z_y)) \leq d_{\circledast X}(x_y, z_y)$$
$$\leq d_{\circledast X}(x_y, y) + d_{\circledast X}(z_y, y) = \bar{d}_X(y, x) + \bar{d}_X(y, z).$$

If $x_y = y$ and $z_y \neq y$, then for every $\varepsilon > 0$ we can find a point $c \in [x, x_y) = [x, y)$ such that $d_{\circledast X}(c, y) < \varepsilon$ and conclude that

$$\bar{d}_X(x, z) = \bar{d}_X(\pi_X(c), \pi_X(z_y)) \leq d_{\circledast X}(c, z_y) \leq d_{\circledast X}(c, y) + d_{\circledast X}(y, z_y)$$
$$< \varepsilon + \bar{d}_X(y, z) \leq \bar{d}_X(y, x) + \bar{d}_X(y, z) + \varepsilon.$$

Letting $\varepsilon \to 0$, we get the desired inequality (7.1). By a similar argument we can treat the cases when $z_y = y$.

For points x and $y = z$, inequality (7.1) yields $\bar{d}_X(x, y) \leq \bar{d}_X(y, x)$. This implies that the premetric $\bar{d}_X$ is symmetric and satisfies the triangle inequality, so is a pseudometric. ∎

Next, we establish some topological properties of the compression map. In the following theorem we endow the premetric space X with the premetric topology.

THEOREM 7.3. *For any premetric space X,*

(1) *the compression map $\pi_X : \circledast X \to X$ is a monotone quotient surjection;*

(2) *$\pi_X : \circledast X \to X$ is hereditarily quotient if and only if the premetric space X is basic;*

(3) *the cobweb space $\circledast X$ is connected if and only if so is the space X.*

Proof. (1) By Proposition 7.1(2), π_X is a monotone surjection. To show that π_X is continuous, take any open subset $U \subset X$. To check that $\pi_X^{-1}(U)$ is open in $\circledast X$, fix any point $a \in \pi_X^{-1}(U)$. If $a \in \circledast X \setminus X$, then the map π_X is locally constant and hence continuous at a. So, we assume that $a \in X$. In this case the definition of the premetric topology on X yields a radius $r \in (0, 1)$ such that $B_X(\pi_X(a), r) \subset U$. By Proposition 7.2(2), $\pi_X(B_{\circledast X}(a, r)) \subset B_X(\pi_X(a), r) \subset U$ and hence $B_{\circledast X}(a, r) \subset \pi_X^{-1}(U)$, witnessing that $\pi_X^{-1}(U)$ is open in $\circledast X$.

To show that π_X is quotient, take any subset $A \subset X$ with $\pi_X^{-1}(A)$ open in $\circledast X$. For every $x \in A$ we get $x \in \pi_X^{-1}(A)$ and hence $B_{\circledast X}(x, r) \subset \pi_X^{-1}(A)$ for some $r \in (0, 1)$. It follows from Proposition 7.2(2) that

$$B_X(x, r) = \pi_X(B_{\circledast X}(x, r)) \subset \pi_X(\pi_X^{-1}(A)) = A.$$

Consequently, the set A is open in the premetric topology of X.

(2) Assuming that the premetric space X is basic, we check that the compression map $\pi_X : \circledast X \to X$ is pseudo-open and hence hereditarily quotient. Fix any point $x \in X$ and any open set $U \subset \circledast X$ containing the fiber $\pi_X^{-1}(x)$. Since $x \in U$, there is a positive $r < 1$ such that $B_{\circledast X}(x, r) \subset U$. Now Proposition 7.2(2) implies that $B_X(x, r) = \pi_X(B_{\circledast X}(x, r)) \subset \pi_X(U)$ and hence $\pi_X(U)$ is a neighborhood of x in X (because the ball $B_X(x, r)$ is a neighborhood of x).

Now assume conversely that the compression map π_X is hereditarily quotient and hence pseudo-open. We need to check that the premetric d_X of X is basic at each point $x \in X$. This will follow as soon as we check that for any positive $r < 1$ the ball $B_X(x, r)$ is a neighborhood of x in X. Observe that the set $U = S_x \cup B_{\circledast X}(x, r)$ is an open neighborhood of the fiber $S_x = \pi_X^{-1}(x)$

in $\circledast X$. Since π_X is pseudo-open, the image $\pi_X(U) = \{x\} \cup \pi_X(B_{\circledast X}(x,r)) = B_X(x,r)$ is a neighborhood of x in X.

(3) This item follows from (1) and Lemma 4.4. ∎

Finally, we apply the compression map to study separablewise components of the cobweb space $\circledast X$.

By the *separablewise component* of a point x of a topological space X we understand the union $C(x)$ of all separable connected subspaces of X that contain the point x. It is standard to show that two separablewise components are either disjoint or coincide. In a countably tight topological space all separablewise components are closed. We recall that a topological space X is *countably tight* if for each subset $A \subset X$ and a point $x \in \bar{A}$ in its closure there is a countable subset $B \subset A$ such that $x \in \bar{B}$.

We shall say that a topological space X *contains no countable connected subspaces* if each non-empty at most countable connected subset of X is a singleton. For example, each regular space contains no non-trivial countable connected subspace.

PROPOSITION 7.4. *If a premetric space X contains no countable connected subspace, then the fibers of the compression map $\pi_X : \circledast X \to X$ coincide with the separablewise components of $\circledast X$ and also with the arcwise components of $\circledast X$. Consequently, X can be identified with the space of separablewise (or arcwise) connected components of $\circledast X$.*

Proof. Assuming that X contains no countable connected subspace, we shall show that each spider $S_x = \pi_X^{-1}(x)$, $x \in X$, coincides with the separablewise component $C(x)$ of the point x in the cobweb $\circledast X$. Taking into account the arcwise connectedness of the spider S_x, we conclude that $S_x \subset C(x)$. To show that $C(x) \subset S_x = \pi_X^{-1}(x)$, fix any point $y \in C(x)$ and find a connected separable subspace $A \subset \circledast X$ containing the points x and y. By Proposition 7.1(4), the image $\pi_X(A)$ is at most countable. Being a connected subspace of X, the set $\pi_X(A)$ coincides with the singleton $\{x\}$. Consequently, $y \in A \subset \pi_X^{-1}(x) = S_x$. ∎

COROLLARY 7.5. *Two premetric spaces X, Y containing no countable connected subspaces are homeomorphic provided their cobwebs $\circledast X$ and $\circledast Y$ are homeomorphic.*

REMARK 7.6. It is well-known that a connected locally connected complete metric space is arcwise connected [7, 6.3.11]. The cobweb over a connected metric space is not locally connected, although it is locally connected off a metrically discrete subset. This illustrates how important it is to assume that the space is locally connected at each point if we want to conclude that it is arcwise connected.

8. The iterated cobweb construction. In this section we shall iterate the cobweb functors and at the limit stage obtain the functor $\circledast^\omega : \mathbf{PMetr} \to \mathbf{Metr}$ assigning to each premetric space X an economical complete metric space $\circledast^\omega X$.

Given a premetric space X put $\circledast^1 X = \circledast X$ and inductively $\circledast^{n+1} X = \circledast(\circledast^n X)$ for $n \in \mathbb{N}$. In such a way we define the functors $\circledast^n : \mathbf{PMetr} \to \mathbf{Metr}$ for all $n \in \mathbb{N}$. For every $n \in \mathbb{N}$ the spaces $\circledast^{n+1} X$ and $\circledast^n X$ are linked by the compression map $\pi_{\circledast^n X} : \circledast^{n+1} X \to \circledast^n X$. This map is non-expanding, surjective, monotone, and hereditarily quotient according to Proposition 7.2(4) and Theorem 7.3.

The iterated cobweb spaces and their compression maps form the inverse sequence

$$(8.1) \qquad \cdots \to \circledast^{n+1} X \to \circledast^n X \to \cdots \to \circledast^1 X.$$

Let

$$\circledast^\omega X = \left\{ (x_n)_{n\in\omega} \in \prod_{n\in\mathbb{N}} \circledast^n X : \forall n \in \mathbb{N} \ \ \pi_{\circledast^n X}(x_{n+1}) = x_n \right\}$$

be the limit of this inverse sequence, and for every $n \in \mathbb{N}$ let

$$\pi_n^\omega : \circledast^\omega X \to \circledast^n X, \qquad \pi_n^\omega : (x_k)_{k\in\mathbb{N}} \mapsto x_n,$$

denote the limit projection.

The space $\circledast^\omega X$ is endowed with the metric

$$d_{\circledast^\omega X}(a,b) = \max_{n\in\mathbb{N}} \frac{1}{n} d_{\circledast^n X}(\pi_n^\omega(a), \pi_n^\omega(b))$$

that generates the topology of $\circledast^\omega X$ inherited from the product $\prod_{n\in\mathbb{N}} \circledast^n X$.

LEMMA 8.1. *For every $n \in \mathbb{N}$ the limit projection $\pi_n^\omega : \circledast^\omega X \to \circledast^n X$ is a monotone hereditarily quotient surjection.*

Proof. By Theorem 7.3(2), the bonding projections $\pi_{\circledast^n X} : \circledast^{n+1} X \to \circledast^n X$ of the inverse sequence (8.1) are monotone hereditarily quotient surjections. Applying Theorem 9 and Corollary to Theorem 11 of [15], we conclude that each limit projection $\pi_n^\omega : \circledast^\omega X \to \circledast^n X$ is monotone and hereditarily quotient. ∎

Taking the composition of the projection $\pi_1^\omega : \circledast^\omega X \to \circledast X$ with the compression map $\pi_X : \circledast X \to X$ we get an important map

$$\pi_X^\omega = \pi_X \circ \pi_1^\omega : \circledast^\omega X \to X.$$

Some properties of this map are collected in:

THEOREM 8.2. *For a premetric space X,*

(1) *$\circledast^\omega X$ is an economical complete metric space of cardinality $|\circledast^\omega X| \le |X|^\omega$;*
(2) *the map $\pi_X^\omega : \circledast^\omega X \to X$ is a monotone quotient surjection;*

(3) *the map π_X^ω is hereditarily quotient if and only if the premetric space X is basic;*

(4) *the map $\pi_X^\omega : \circledast^\omega X \to X$ is non-expanding provided the premetric $\bar{d}_X = \min\{1, d_X\}$ is a pseudometric;*

(5) *the space $\circledast^\omega X$ is connected if and only if $\circledast^\omega X$ is non-separably connected if and only if the space X is connected.*

Proof. (1) The completeness of the metric $d_{\circledast^\omega X}$ on $\circledast^\omega X$ follows from the completeness of the iterated cobwebs $\circledast^n X$ and the closedness of $\circledast^\omega X$ in the Tikhonov product $\prod_{n \in \mathbb{N}} \circledast^n X$.

To show that $\circledast^\omega X$ is economical, take any infinite subset $A \subset \circledast^\omega X$ and observe that for every $n \in \mathbb{N}$,

$$(8.2) \qquad |\pi_n^\omega(A)| = |\pi_{\circledast^n X}(\pi_{n+1}^\omega(A))| \le \mathrm{dens}(\pi_{n+1}^\omega(A)) \le \mathrm{dens}(A)$$

according to Proposition 7.1(4).

Observe that for any $a, b \in A$ we have

$$d_{\circledast^\omega X}(a, b) \in \{n^{-1} d_{\circledast^n X}(\pi_n^\omega(a), \pi_n^\omega(b)) : n \in \mathbb{N}\}$$

and thus

$$d_{\circledast^\omega X}(A \times A) \subset \{n^{-1} d_{\circledast^n X}(x, y) : x, y \in \pi_n^\omega(A), \ n \in \mathbb{N}\}.$$

Combining this with (8.2) we get the desired inequality

$$|d_{\circledast^\omega X}(A \times A)| \le \sum_{n \in \mathbb{N}} |\pi_n^\omega(A) \times \pi_n^\omega(A)| \le \aleph_0 \cdot \mathrm{dens}(A)^2 = \mathrm{dens}(A),$$

confirming the economical property of the metric $d_{\circledast^\omega X}$.

Finally, we show that $|\circledast^\omega X| \le |X|^\omega$. This is clear if X is a singleton. So, we assume that $|X| \ge 2$. It follows from the definition of cobweb space that $|\circledast X| \le \mathfrak{c} \cdot |X|$. By induction, $|\circledast^n X| \le \mathfrak{c} \cdot |X|$ and then $|\circledast^\omega X| \le (\mathfrak{c} \cdot |X|)^\omega = |X|^\omega$.

(2) By Lemma 8.1 and Theorem 7.3(1) the maps $\pi_1^\omega : \circledast^\omega X \to \circledast X$ and $\pi_X : \circledast X \to X$ are quotient surjections. Then so is their composition $\pi_X^\omega = \pi_X \circ \pi_1^\omega$.

Next, we check that the map π_X^ω is monotone. Given any point $x \in X$, let $C = (\pi_X^\omega)^{-1}(x) = (\pi_1^\omega)^{-1}(S_x)$. Since the map $\pi_1^\omega : \circledast^\omega X \to \circledast X$ is hereditarily quotient, the restriction $\pi_X^\omega | C : C \to S_x$ is quotient. Now the connectedness of the spider S_x and Lemma 4.4 guarantee that C is connected.

(3) If the map $\pi_X^\omega = \pi_X \circ \pi_1^\omega$ is hereditarily quotient, then so is the map $\pi_X : \circledast X \to X$ and then the premetric space X is basic according to Theorem 7.3(2). On the other hand, if the premetric space X is basic, then by Theorem 7.3(2), the compression map $\pi_X : \circledast X \to X$ is hereditarily

quotient, and then the map $\pi_X^\omega = \pi_X \circ \pi_1^\omega$ is hereditarily quotient as the composition of two hereditarily quotient maps.

(4) If the premetric $\bar{d}_X = \{1, d_X\}$ is a pseudometric, then Proposition 7.2(4) guarantees that the compression map $\pi_X : \circledast X \to X$ is non-expanding. Then $\pi_X^\omega = \pi_X \circ \pi_1^\omega$ is non-expanding as the composition of two non-expanding maps.

(5) This follows from (2), Lemma 4.4 and Proposition 2.2. ∎

Now we show that the construction of $\circledast^\omega X$ can be completed to a functor $\circledast^\omega : \mathbf{PMetr} \to \mathbf{Metr}$. Given any non-expanding map $f : X \to Y$ between premetric spaces, consider the non-expanding maps $\circledast^n f : \circledast^n X \to \circledast^n Y$ for all $n \in \mathbb{N}$. The naturality of the compression maps implies the commutativity of the following diagrams for all n:

$$
\begin{array}{ccc}
\circledast^{n+1} X & \xrightarrow{\circledast^{n+1} f} & \circledast^{n+1} Y \\
\downarrow{\scriptstyle \pi_{\circledast^n X}} & & \downarrow{\scriptstyle \pi_{\circledast^n Y}} \\
\circledast^n (X) & \xrightarrow{\circledast^n f} & \circledast^n Y
\end{array}
$$

The commutativity of those diagrams ensures that the map

$$\circledast^\omega f : \circledast^\omega X \to \circledast^\omega Y, \qquad \circledast^\omega f : (x_n)_{n \in \mathbb{N}} \mapsto (\circledast^n f(x_n))_{n \in \mathbb{N}},$$

is well-defined and that also the following diagram is commutative:

$$
\begin{array}{ccc}
\circledast^\omega X & \xrightarrow{\circledast^\omega f} & \circledast^\omega Y \\
\downarrow{\scriptstyle \pi_X^\omega} & & \downarrow{\scriptstyle \pi_Y^\omega} \\
X & \xrightarrow{f} & Y
\end{array}
$$

The commutativity of this diagram means that the maps $\pi_X^\omega : \circledast^\omega X \to X$ compose components of the natural transformation $\pi^\omega : \circledast^\omega \to \mathrm{Id}$ of the functor $\circledast^\omega$ into the identity functor.

9. The functor of economical resolution. In this section we define and study the functor $\mathsf{E} : \mathbf{Top} \to \mathbf{Metr}$ of economical resolution. This functor is defined as the composition

$$\mathsf{E} = \circledast^\omega \circ \mathsf{D}$$

of the functors $\mathsf{D} : \mathbf{Top} \to \mathbf{PMetr}$ and $\circledast^\omega : \mathbf{PMetr} \to \mathbf{Metr}$.

Thus $\mathsf{E} X = \circledast^\omega(\mathsf{D} X)$ for every topological space X. Next, define the resolution map $\xi_X : \mathsf{E} X \to \mathsf{s} X$ as the composition $\xi_X = c_X \circ \pi_{\mathsf{D} X}^\omega$ of two maps: $\pi_{\mathsf{D} X}^\omega : \circledast^\omega(\mathsf{D} X) \to \mathsf{D} X$ and $c_X : \mathsf{D} X \to \mathsf{s} X$. We recall that $\mathsf{s} X$ stands for the sequential coreflexion of X (which is X endowed with the topology consisting of all sequentially open subsets).

The maps ξ_X can be seen as the components of a natural transformation $\xi : \mathsf{E} \to \mathsf{s}$ from the functor E to the functor s of sequential coreflexion.

The following theorem describes some properties of the spaces $\mathsf{E}X$ and maps $\xi_X : \mathsf{E}X \to X$. Below by a *convergent sequence* in a topological space X we understand the image $f(S_0)$ of the standard convergent sequence $S_0 = \{0\} \cup \{1/n : n \in \mathbb{N}\} \subset \mathbb{R}$ under a continuous map $f : S_0 \to X$.

THEOREM 9.1. *For any topological space X,*

(1) $\mathsf{E}X = \circledast^\omega \mathsf{D}X$ *is an economical complete metric space of cardinality* $|\mathsf{E}X| \le \mathfrak{c} \cdot |X|^\omega$;

(2) *the map* $\xi_X : \mathsf{E}X \to \mathsf{s}X$ *is a monotone quotient surjection*;

(3) *the space* $\mathsf{E}X$ *is connected if and only if the sequential coreflexion* $\mathsf{s}X$ *of X is connected*;

(4) *the map* $\xi_X : \mathsf{E}X \to X$ *is quotient if and only if the space X is sequential*;

(5) *the map* $\xi_X : \mathsf{E}X \to X$ *is hereditarily quotient if and only if X is Fréchet–Urysohn*;

(6) *each point $a \in \mathsf{E}X$ has a neighborhood $U_a \subset \mathsf{E}X$ whose image $\xi_X(U_a)$ lies in a convergent sequence in X*.

Proof. (1) This follows from Theorem 8.2(1) and the fact that $|\mathsf{D}X| \le \aleph_0 \cdot |X|^\omega$.

(2) The map $\xi_X = \pi_X^\omega \circ c_X : \mathsf{E}X \to \mathsf{s}X$ is quotient, being the composition of two maps which are quotient by Theorem 8.2(2) and Corollary 4.2. To show that ξ_X is monotone, take any point $x \in X$ and consider the preimage $c_X^{-1}(x) \subset \mathsf{D}X$. For any points $y, z \in c_X^{-1}(x)$ we get $d_{\mathsf{D}X}(y, z) = 0$ by the definition of the premetric d_D and $[y, z] \subset \bar{S}_y$ by the definition of $\circledast(\mathsf{D}X)$. Now we see that $\bigcup_{y \in c_X^{-1}(x)} \bar{S}_y = (c_X \circ \pi_X)^{-1}(x)$ is a connected subspace of $\circledast(\mathsf{D}X)$. By Lemma 8.1, the map $\pi_1^\omega : \circledast^\omega X \to \circledast X$ is monotone and hereditarily quotient. Consequently, the preimage

$$\xi_X^{-1}(x) = (\pi_1^\omega)^{-1}(c_X \circ \pi_X)^{-1}(x))$$

is connected, witnessing that the map ξ_X is monotone.

(3) This follows from (2) and Lemma 4.4.

(4) If the space X is sequential, then $\mathsf{s}X = X$ and hence the map $\xi_X : \mathsf{E}X \to X$ is quotient by (2). If the map $\xi_X : \mathsf{E}X \to X$ is quotient, then the space X is sequential, being the image of a metrizable space under a quotient map (see [7, 2.4.G]).

(5) If the map $\xi_X : \mathsf{E}(X) \to X$ is hereditarily quotient, then the space X is Fréchet–Urysohn, being the image of a metrizable space under a hereditarily quotient map (see [7, 2.4.G]).

Now assume conversely that the space X is Fréchet–Urysohn. In this case $\mathsf{s}X = X$. First we show that the composition $c_X \circ \pi_{\mathsf{D}X} : \circledast \circ \mathsf{D}X \to X$ is pseudo-open. Given any point $x \in X$ and an open set $U \subset \circledast(\mathsf{D}X)$ containing the preimage $(c_X \circ \pi_{\mathsf{D}X})^{-1}(x)$, we need to check that the image $V = c_X \circ \pi_{\mathsf{D}X}(U)$ is a neighborhood of x in X. Assuming the converse, we could find a sequence $\{x_n\}_{n\in\mathbb{N}} \subset X \setminus V$ that converges to the point x. Consider the continuous map $f : S_0 \to X$ such that $f(0) = x_0$ and $f(1/n) = x_n$. Consider the point $a = (f, 0) \in \mathsf{D}X$. This point also belongs to the cobweb $\circledast(\mathsf{D}X)$ and lies in the set $(c_X \circ \pi_{\mathsf{D}X})^{-1}(x) \subset U$. Since U is open, there is $r \in (0, 1]$ such that $B_{\circledast\mathsf{D}X}(a, r) \subset U$. Choose any integer $n > 1/r$. By Proposition 7.2(2),

$$(f, 1/n) \in B_{\mathsf{D}X}(a, r) = \pi_{\mathsf{D}X}(B_{\circledast\mathsf{D}X}(a, r)) \subset \pi_{\mathsf{D}X}(U)$$

and hence $x_n = f(1/n) = c_X(f, 1/n) \subset c_X \circ \pi_{\mathsf{D}X}(U) = V$, which contradicts the choice of x_n. This contradiction shows that the map $c_X \circ \pi_{\mathsf{D}X}$ is pseudo-open and hence hereditarily quotient.

By Lemma 8.1, the map $\pi_1^\omega : \circledast^\omega(\mathsf{D}X) \to \circledast(\mathsf{D}X)$ is hereditarily quotient. Then $\xi_X = c_X \circ \pi_{\mathsf{D}X} \circ \pi_1^\omega$ is hereditarily quotient as the composition of hereditarily quotient maps.

(6) Since $\xi_X = c_X \circ \pi_{\mathsf{D}X} \circ \pi_1^\omega$ and the map $\pi_1^\omega : \circledast^\omega\mathsf{D}X \to \circledast\mathsf{D}X$ is continuous, it suffices to check that each point $a \in \circledast\mathsf{D}X$ has a neighborhood U_a whose image $c_X \circ \pi_{\mathsf{D}X}(U_a)$ lies in a convergent sequence in X. If $a \in \circledast\mathsf{D}X \setminus \mathsf{D}X$, then the map $\pi_{\mathsf{D}X}$ is locally constant at a. Consequently, a has a neighborhood U_a whose image $c_X(\pi_{\mathsf{D}X}(U_a))$ is a singleton.

If $a \in \mathsf{D}X \subset \circledast\mathsf{D}X$, then $a = (f, t)$ for some continuous map $f \in C(S_0, X)$ and some $t \in S_0$. By the definition of the premetric $d_{\mathsf{D}X}$, we get $c_X(B_{\mathsf{D}X}(a, 1)) \subset f(S_0)$. Letting $U_a = B_{\circledast\mathsf{D}X}(a, 1)$ and applying Proposition 7.2(2), we conclude that the image

$$c_X \circ \pi_{\mathsf{D}X}(U_a) = c_X(B_{\mathsf{D}X}(a, 1)) \subset f(S_0)$$

lies in the convergent sequence $f(S_0)$ in X. ∎

10. Locally extremal functions on connected metric spaces. In this section we shall apply the cobweb construction to exhibit a non-constant locally extremal function defined on a connected complete metric space.

A function $f : X \to \mathbb{R}$ is called *locally extremal* if each $x \in X$ is a point of local maximum or local minimum of f. In 1912 W. Sierpiński [16] proved that each continuous locally extremal function $f : \mathbb{R} \to \mathbb{R}$ is constant. In [6] this result was generalized to continuous locally extremal functions $f : X \to \mathbb{R}$

defined on a connected topological space X of weight $w(X) < \mathfrak{c}$. Another generalization was proved by Fedeli and Le Donne [8]:

THEOREM 10.1 (Fedeli, Le Donne). *A continuous locally extremal function $f : X \to \mathbb{R}$ defined on a connected topological space X with cellularity $c(X) < \mathfrak{c}$ is constant.*

The cellularity requirement is essential in this theorem as shown by the projection $\mathrm{pr} : [0, 1] \times [0, 1] \to [0, 1]$ of the lexicographic square onto the interval (see [6]). This projection is locally extremal but not constant. On the other hand, the lexicographic square $[0, 1] \times [0, 1]$ is a first-countable linearly ordered connected compact Hausdorff space.

Having in mind this example, Morayne and Wójcik asked in [10] and [11] if there is a non-constant locally extremal function defined on a connected (complete) metric space.

An example of such a function was first constructed by Fedeli and Le Donne [8] and independently by the authors in [4]. In this paper we construct such a non-constant locally extremal function with the help of the cobweb construction.

Consider the set $\mathrm{II} = \{-1, +1\} \times [0, 1]$ endowed with the premetric

$$d((i, x), (j, y)) = \begin{cases} x - y & \text{if } i = -1 \text{ and } x \geq y, \\ y - x & \text{if } i = +1 \text{ and } y \geq x, \\ 1 & \text{otherwise.} \end{cases}$$

By Theorem 7.3(1) the map $\pi_{\mathrm{II}} : \circledast \mathrm{II} \to \mathrm{II}$ is a monotone quotient surjection. Consider its composition with the projection

$$\mathrm{pr} : \mathrm{II} \to [0, 1], \quad \mathrm{pr} : (i, x) \mapsto x.$$

THEOREM 10.2. *The composition $f = \mathrm{pr} \circ \pi_{\mathrm{II}} : \circledast \mathrm{II} \to [0, 1]$ is a surjective continuous locally extremal function defined on the connected complete metric space $\circledast \mathrm{II}$.*

Proof. It is easy to check that the premetric d on II generates the smallest topology turning the projection $\mathrm{pr} : \mathrm{II} \to [0, 1]$ into a continuous map. Endowed with this topology, the space II is connected. Applying Theorem 7.3(3) we conclude that $\circledast \mathrm{II}$ is a connected complete metric space.

We claim that the map $\mathrm{pr} \circ \pi_{\mathrm{II}} : \circledast \mathrm{II} \to [0, 1]$ is locally extremal. Take any point $a \in \circledast \mathrm{II}$. If $a \notin \mathrm{II}$, then f is locally constant at a by Proposition 7.1(3). If $a \in \mathrm{II}$, then consider the open unit ball $U_a = B_{\circledast \mathrm{II}}(a, 1)$ and apply Proposition 7.2(2) to conclude that

$$f(U_a) = \mathrm{pr}(\pi_{\mathrm{II}}(B_{\circledast \mathrm{II}}(a, 1))) = \mathrm{pr}(B_{\mathrm{II}}(a, 1)).$$

Let $a = (i, x)$ where $i \in \{-1, +1\}$ and $x \in [0, 1]$. If $i = -1$, then $f(U_a) = \mathrm{pr}(B_{\mathrm{II}}(a, r)) \subset (x - r, x]$ and thus a is a point of local maximum of f. If $i = +1$, then $f(U_a) = \mathrm{pr}(B_{\mathrm{II}}(a, r)) \subset [x, x + r)$ and hence a if a point of local minimum of f. ∎

11. Some open problems. By Corollary 7.5, two metric spaces X, Y are homeomorphic provided their cobwebs $\circledast X$, $\circledast Y$ are homeomorphic.

PROBLEM 11.1. *Are metric spaces X, Y homeomorphic if so are the spaces $\circledast^\omega X$ and $\circledast^\omega Y$? In particular, are the spaces $\circledast^\omega(\mathbb{R}^n)$ and $\circledast^\omega(\mathbb{R}^m)$ homeomorphic for some $n \neq m$?*

In [12] E. Pol constructed for every $n \in \mathbb{N}$ a metric space of covering dimension n having no separable subspaces of positive dimension.

PROBLEM 11.2. *Let $n \in \mathbb{N}$. Is there an economical complete metric space X_n of covering dimension $\dim X_n = n$?*

A natural candidate for such a space would be $(\circledast^\omega X)^n$ for a connected metric space X.

PROBLEM 11.3. *Is $\dim(\circledast^\omega X)^n = n$ for every connected metric space X and every $n \in \mathbb{N}$?*

In [11], Morayne and Wójcik constructed a non-separably connected metric group which is an example of a (topologically) homogeneous non-separably connected metric space.

A metric space X is called (*topologically*) *homogeneous* if for any two points $x, y \in X$ there is a bijective isometry (a homeomorphism) $f : X \to X$ with $f(x) = y$.

PROBLEM 11.4. *Can a non-separably connected complete metric space be (topologically) homogeneous? In particular, is there an economical connected complete metric group?*

PROBLEM 11.5. *Is the space $\circledast^\omega X$ (topologically) homogeneous for some metric space X that contains more than one point?*

This paper was developed over a long period of time gaining in generality and complexity of the mechanisms that we first discovered in considerably simpler settings. Therefore, an earlier draft [5], which is less advanced but

more compact, may still be of interest to those who want to see the short-est solution of the original problem of constructing a non-separably connected complete metric space, without studying all the mechanisms in full generality.

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Taras Banakh
Instytut Matematyki
Uniwersytet Humanistyczno-Przyrodniczy
Jana Kochanowskiego
Kielce, Poland
and
Department of Mathematics
Ivan Franko National University of Lviv
Universytetska 1
79000, Lviv, Ukraine
E-mail: t.o.banakh@gmail.com

Michał Ryszard Wójcik
Department of Mathematics
University of Louisville
Louisville, KY, U.S.A.
E-mail: michal.ryszard.wojcik@gmail.com
Current address:
Institute of Geography and Regional Development
University of Wrocław
50-137 Wrocław, Poland

Myroslava Vovk
National University
Lvivska Politechnika
Lviv, Ukraine
E-mail: mira.vovk1@gmail.com