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A two-dimensional univoque set

by

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Abstract. Let $\mathbf{J} \subset \mathbb{R}^2$ be the set of couples (x, q) with $q > 1$ such that x has at least one representation of the form $x = \sum_{i=1}^{\infty} c_i q^{-i}$ with integer coefficients c_i satisfying $0 \le c_i < q$, $i \ge 1$. In this case we say that $(c_i) = c_1c_2\ldots$ is an expansion of x in base q. Let $\mathbf{U}$ be the set of couples $(x, q) \in \mathbf{J}$ such that x has exactly one expansion in base q. In this paper we deduce some topological and combinatorial properties of the set $\mathbf{U}$. We characterize the closure of $\mathbf{U}$, and we determine its Hausdorff dimension. For $(x, q) \in \mathbf{J}$, we also prove new properties of the lexicographically largest expansion of x in base q.

1. Introduction. Let $\mathbf{J}$ be the set consisting of all elements $(x, q) \in \mathbb{R} \times (1, \infty)$ such that there exists at least one sequence $(c_i) = c_1c_2\ldots$ of integers satisfying $0 \le c_i < q$ for all i, and

$$x = \frac{c_1}{q} + \frac{c_2}{q^2} + \cdots . \tag{1.1}$$

If (1.1) holds, we say that (c_i) is an *expansion of x in base q*, and if the base q is understood from the context, we sometimes simply say that (c_i) is an expansion of x. The numbers c_i of an expansion (c_i) are usually referred to as *digits*. We denote by $\lceil q \rceil$ the smallest integer larger than or equal to q. The *alphabet* A_q is the set of "admissible" digits in base q, i.e., $A_q = \{0, \ldots, \lceil q \rceil - 1\}$.

If $q > 1$ and $0 \le x \le (\lceil q \rceil - 1)/(q - 1)$, then a particular expansion of x in base q, the so-called *quasi-greedy expansion* $(a_i(x, q))$, may be defined recursively as follows. For $x = 0$ we set $(a_i(x, q)) := 0^\infty$. If $x > 0$ and $a_i(x, q)$ has already been defined for $1 \le i < n$ (no condition if $n = 1$), then $a_n(x, q)$ is the largest element of A_q satisfying

$$\frac{a_1(x, q)}{q} + \cdots + \frac{a_n(x, q)}{q^n} < x.$$

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One easily verifies that $(a_i(x,q))$ is indeed an expansion of x in base q. Therefore

$$(x,q) \in \mathbf{J} \;\Leftrightarrow\; q > 1 \text{ and } x \in J_q := \left[0, \frac{\lceil q \rceil - 1}{q-1}\right].$$

Let us denote by $\mathbf{U}$ the set of couples $(x,q) \in \mathbf{J}$ such that x has *exactly one* expansion in base q. For example, $(0,q) \in \mathbf{U}$ for every $q > 1$, but $\mathbf{U}$ has many more elements. The main purpose of this paper is to describe the topological and combinatorial nature of $\mathbf{U}$. We will prove the following theorem:

THEOREM 1.1.

(i) *The set* $\mathbf{U}$ *is not closed. Its closure* $\overline{\mathbf{U}}$ *is a Cantor set* ([1]).
(ii) *Both* $\mathbf{U}$ *and* $\overline{\mathbf{U}}$ *are two-dimensional Lebesgue null sets.*
(iii) *Both* $\mathbf{U}$ *and* $\overline{\mathbf{U}}$ *have Hausdorff dimension two.*

As far as we know, this two-dimensional *univoque* set has not yet been investigated. There exist, however, a number of papers devoted to the study of its one-dimensional sections

$$\mathcal{U} := \{q > 1 : (1,q) \in \mathbf{U}\}$$

and

$$\mathcal{U}_q := \{x \in J_q : (x,q) \in \mathbf{U}\}, \qquad q > 1.$$

The study of $\mathcal{U}$ started with the paper of Erdős, Horváth and Joó [6] and continued in [4], [5], [7], [8], [15], [16], [17]. We recall in particular that $\mathcal{U}$ and its closure $\overline{\mathcal{U}}$ have Lebesgue measure zero and Hausdorff dimension one.

The sets $\mathcal{U}_q$ have been investigated in [3], [4], [5], [11], [13], [14]. It is known (see [5]) that $\mathcal{U}_q$ is closed if and only if q does not belong to the null set $\overline{\mathcal{U}}$, and that the closure $\overline{\mathcal{U}_q}$ has Lebesgue measure zero for all noninteger bases $q > 1$. Moreover, the set of numbers $x \in J_q$ having a continuum of expansions in base q has full Lebesgue measure for each noninteger $q > 1$ (see [2], [20], [21]).

The key to the proof of Theorem 1.1 is an algebraic characterization of $\overline{\mathbf{U}}$ by using the quasi-greedy expansions $(a_i(x,q))$. We write for brevity $\alpha_i(q) := a_i(1,q)$, $i \in \mathbb{N} := \{1,2,\ldots\}$, $q > 1$. Note that $\alpha_1(q) = \lceil q \rceil - 1$, the largest admissible digit in base q. In the statement of the following theorem we use the lexicographic order between sequences and we define the *conjugate* (in base q) of a digit $c \in A_q$ by $\overline{c} := \alpha_1(q) - c$. If $c_i \in A_q$, $i \geq 1$, we shall also write $\overline{c_1 \ldots c_n}$ instead of $\overline{c_1} \ldots \overline{c_n}$ and $\overline{c_1 c_2 \ldots}$ instead of $\overline{c_1}\,\overline{c_2} \ldots$.

([1]) We recall that a *Cantor set* is a nonempty closed set having neither interior nor isolated points.

Theorem 1.2. *A point* $(x,q) \in \mathbf{J}$ *belongs to* $\overline{\mathbf{U}}$ *if and only if*

$$\overline{a_{n+1}(x,q)a_{n+2}(x,q)\dots} \le \alpha_1(q)\alpha_2(q)\dots \qquad \textit{whenever } a_n(x,q) > 0.$$

Along with the quasi-greedy expansion, we also need the notion of the *greedy expansion* $(b_i(x,q))$ for $x \in J_q$, introduced by Rényi [19]. It can be defined by a slight modification of the above recursion: if $b_i(x,q)$ has already been defined for all $1 \le i < n$ (no condition if $n = 1$), then $b_n(x,q)$ is the largest element of A_q satisfying

$$\frac{b_1(x,q)}{q} + \dots + \frac{b_n(x,q)}{q^n} \le x.$$

Note that the greedy expansion $(b_i(x,q))$ of a number $x \in J_q$ is the lexicographically largest expansion of x in base q. We denote the greedy expansion of 1 in base q by $(\beta_i(q)) := (b_i(1,q))$.

The rest of this paper is organized as follows. In the next section we give a short overview of some basic results on greedy and quasi-greedy expansions, and we prove some new results concerning the coordinatewise convergence of sequences of these expansions. We shall prove (see Theorem 2.7) that the set of numbers $x \in J_q$ for which the greedy expansion of x in base q is not the greedy expansion of a number belonging to J_p in any smaller base $p \in (1,q)$ is of full Lebesgue measure and its complement in J_q is a set of first category and Hausdorff dimension one. We shall also prove (see Theorem 2.8) that for each word $v := b_{\ell+1}(x,q)\dots b_{\ell+m}(x,q)$ ($\ell \ge 0$, $m \ge 1$, $x \in [0,1)$) there exists a set $Y_v \subset J_q$ of first category and Hausdorff dimension less than one such that the word v occurs in the greedy expansion in base q of every number belonging to $J_q \setminus Y_v$. Using (some of) the results of Section 2 we prove Theorem 1.2 in Section 3 and Theorem 1.1 in Section 4.

2. Greedy and quasi-greedy expansions. In this paper we consider only one-sided sequences of nonnegative integers. We equip this set of sequences $\{0,1,\dots\}^{\mathbb{N}}$ with the topology of coordinatewise convergence. We say that an expansion is *infinite* if it has infinitely many nonzero elements; otherwise it is called *finite*. Using this terminology, the quasi-greedy expansion $(a_i(x,q))$ of a number $x \in J_q \setminus \{0\}$ is the lexicographically largest *infinite* expansion of x in base q. Moreover, if the greedy expansion of $x \in J_q$ is infinite, then $(a_i(x,q)) = (b_i(x,q))$.

The family of all quasi-greedy expansions is characterized by the following propositions (see [1] or [5] for a proof):

Proposition 2.1. *The map* $q \mapsto (\alpha_i(q))$ *is an increasing bijection from the open interval* $(1,\infty)$ *onto the set of all* infinite *sequences* (α_i) *satisfying*

$$\alpha_{k+1}\alpha_{k+2}\dots \le \alpha_1\alpha_2\dots \qquad \textit{for all } k \ge 1.$$

Proposition 2.2. *For each $q > 1$, the map $x \mapsto (a_i(x,q))$ is an increasing bijection from $J_q \setminus \{0\}$ onto the set of all* infinite *sequences (a_i) satisfying*

$$a_n \in A_q \quad \text{for all } n \geq 1$$

and

$$a_{n+1}a_{n+2}\ldots \leq \alpha_1(q)\alpha_2(q)\ldots \quad \text{whenever } a_n < \alpha_1(q).$$

The quasi-greedy expansions have a lower semicontinuity property for the order topology induced by the lexicographic order. More precisely, we have the following result.

Lemma 2.3. *Let $(x,q) \in \mathbf{J}$ and $(y_n, r_n) \in \mathbf{J}$, $n \in \mathbb{N}$. Then*

(i) *for each positive integer m there exists a neighborhood $\mathbf{W} \subset \mathbb{R}^2$ of (x,q) such that*

$$a_1(y,r)\ldots a_m(y,r) \geq a_1(x,q)\ldots a_m(x,q) \quad \text{for all } (y,r) \in \mathbf{W}\cap\mathbf{J}; \tag{2.1}$$

(ii) *if $y_n \uparrow x$ and $r_n \uparrow q$, then $(a_i(y_n,r_n))$ converges to $(a_i(x,q))$.*

Proof. (i) We may assume that $x \neq 0$. By definition of the quasi-greedy expansion we have

$$\sum_{i=1}^{n} \frac{a_i(x,q)}{q^i} < x \quad \text{for all } n = 1,2,\ldots.$$

For any fixed positive integer m, if $(y,r) \in \mathbf{J}$ is sufficiently close to (x,q), then $r > \lceil q \rceil - 1$, i.e., $A_q \subset A_r$, and

$$\sum_{i=1}^{n} \frac{a_i(x,q)}{r^i} < y, \quad n = 1,\ldots,m.$$

These inequalities imply (2.1).

(ii) If $y_n \leq x$ and $r_n \leq q$, we deduce from the definition of the quasi-greedy expansion that

$$(a_i(x,q)) \geq (a_i(y_n,r_n))$$

for every n. Equivalently, we have

$$a_1(x,q)\ldots a_m(x,q) \geq a_1(y_n,r_n)\ldots a_m(y_n,r_n)$$

for all positive integers m and n. It remains to notice that by the previous part the reverse inequality also holds for each fixed m if n is large enough. ∎

The family of greedy expansions has already been characterized by Parry [18]:

Proposition 2.4. *For a given base $q > 1$, the map $x \mapsto (b_i(x,q))$ is an increasing bijection from J_q onto the set of all sequences (b_i) satisfying*

$$b_n \in A_q \quad \text{for all } n \geq 1$$

and

$$b_{n+1}b_{n+2}\ldots < \alpha_1(q)\alpha_2(q)\ldots \qquad \text{whenever } b_n < \alpha_1(q).$$

The greedy expansions have the following upper semicontinuity property:

LEMMA 2.5. *Let $(x,q)\in \mathbf{J}$, $(y_n,r_n)\in \mathbf{J}$, $n\in\mathbb{N}$ and suppose $q\notin\mathbb{N}$. Then*

(i) *for each positive integer m there exists a neighborhood $\mathbf{W}\subset\mathbb{R}^2$ of (x,q) such that*

$$b_1(y,r)\ldots b_m(y,r)\le b_1(x,q)\ldots b_m(x,q) \qquad \text{for all } (y,r)\in \mathbf{W}\cap\mathbf{J}; \tag{2.2}$$

(ii) *if $y_n\downarrow x$ and $r_n\downarrow q$, then $(b_i(y_n,r_n))$ converges to $(b_i(x,q))$.*

Proof. (i) By the definition of greedy expansions we have

$$\sum_{i=1}^{n}\frac{b_i(x,q)}{q^i} > x-\frac{1}{q^n} \qquad \text{whenever } b_n(x,q)<\alpha_1(q).$$

If $(y,r)\in\mathbf{J}$ is sufficiently close to (x,q), then $A_r=A_q$, $\alpha_1(r)=\alpha_1(q)$, and

$$\sum_{i=1}^{n}\frac{b_i(x,q)}{r^i} > y-\frac{1}{r^n} \qquad \text{whenever } n\le m \text{ and } b_n(x,q)<\alpha_1(r).$$

These inequalities imply (2.2).

(ii) If $y_n\ge x$ and $r_n\ge q$, we deduce from the definition of the greedy expansion that

$$(b_i(x,q))\le (b_i(y_n,r_n))$$

for every n. Equivalently, we have

$$b_1(x,q)\ldots b_m(x,q)\le b_1(y_n,r_n)\ldots b_m(y_n,r_n)$$

for all positive integers m and n. It remains to notice that by the previous part the reverse inequality also holds for each fixed m if n is large enough. ∎

From Lemmas 2.3 and 2.5 we deduce the following result:

PROPOSITION 2.6. *Consider $(x,q)\in\mathbf{J}$ with a noninteger base q and assume that the greedy expansion $(b_i(x,q))$ is infinite. If (y_n,r_n) converges to (x,q) in $\mathbf{J}$, then both $(a_i(y_n,r_n))$ and $(b_i(y_n,r_n))$ converge to $(b_i(x,q))=(a_i(x,q))$.*

Proof. For each positive integer m there exists a neighborhood $\mathbf{W}\subset\mathbb{R}^2$ of (x,q) such that for all $(y,r)\in\mathbf{W}\cap\mathbf{J}$,

$$\begin{aligned}a_1(x,q)\ldots a_m(x,q)&\le a_1(y,r)\ldots a_m(y,r)\le b_1(y,r)\ldots b_m(y,r)\\&\le b_1(x,q)\ldots b_m(x,q).\end{aligned}$$

The result follows from our assumption that $(a_i(x,q))=(b_i(x,q))$. ∎

THEOREM 2.7. *Let $q > 1$ be a real number. Then*

(i) *for each $r \in (1, q)$, the Hausdorff dimension of the set*

$$G_{r,q} := \left\{ \sum_{i=1}^{\infty} \frac{b_i(x,r)}{q^i} : x \in J_r \right\}$$

equals $\log r/\log q$*;*

(ii) *the set*

$$G_q := \bigcup \{G_{r,q} : r \in (1, q)\}$$

is of first category, has Lebesgue measure zero and Hausdorff dimension one.

Proof. (i) It is well known (see, e.g., [17], [18]) and easy to prove that the set of numbers $r > 1$ for which $(\beta_i(r))$ is finite is dense in $[1, \infty)$. Moreover, if $(\beta_i(r))$ is finite and $\beta_n(r)$ is its last nonzero element, then $(\alpha_i(r)) = (\beta_1(r) \dots \beta_{n-1}(r)\beta_n^-(r))^\infty$ (where $\beta_n^-(r) := \beta_n(r) - 1$). By virtue of Propositions 2.1 and 2.4 we have $G_{s,q} \subset G_{t,q}$ whenever $1 < s < t < q$. Hence it is enough to prove that $\dim_{\mathrm{H}} G_{r,q} = \log r/\log q$ for those values $r \in (1, q)$ for which $(\alpha_i(r))$ is periodic.

Fix $r \in (1, q)$ such that $(\alpha_i) := (\alpha_i(r))$ is periodic and let $n \in \mathbb{N}$ be such that $(\alpha_i) = (\alpha_1 \dots \alpha_n)^\infty$. Let us denote by W_r the set consisting of the finite words

$$w_{ij} := \alpha_1 \dots \alpha_{j-1} i, \qquad 0 \le i < \alpha_j,\ 1 \le j \le n,$$

and

$$w_{\alpha_n n} := \alpha_1 \dots \alpha_{n-1}\alpha_n.$$

Let $\mathcal{F}_r'$ be the set of sequences $(c_i) = c_1c_2$ such that for each $k \ge 0$ the inequality $c_{k+1} \dots c_{k+n} \le \alpha_1 \dots \alpha_n$ holds. Note that the set $\mathcal{F}_r'$ consists of those sequences (c_i) such that each tail of (c_i) (including (c_i) itself) starts with a word belonging to W_r. It follows from Propositions 2.1 and 2.4 that a sequence (b_i) is greedy in base r if and only if $b_m \in A_r$ for all $m \ge 1$ and

$$b_{m+k+1}b_{m+k+2} \dots < \alpha_1\alpha_2 \dots \qquad \text{for all } k \ge 0, \text{ whenever } b_m < \alpha_1.$$

Therefore, any greedy expansion $(b_i) \ne \alpha_1^\infty$ in base r can be written as $\alpha_1^\ell c_1 c_2 \dots$ for some $\ell \ge 0$ (where α_1^0 denotes the empty word) and some sequence (c_i) belonging to $\mathcal{F}_r'$. Conversely, if no tail of a sequence belonging to $\mathcal{F}_r'$ equals (α_i), then it is the greedy expansion in base r of some $x \in J_r$. Hence if we set

$$\mathcal{F}_{r,q} := \left\{ \sum_{i=1}^{\infty} \frac{c_i}{q^i} : (c_i) \in \mathcal{F}_r' \right\},$$

then $\mathcal{F}_{r,q} \setminus G_{r,q}$ is countable and $G_{r,q}$ can be covered by countably many sets similar to $\mathcal{F}_{r,q}$. Since the union of countably many sets of Hausdorff dimension s is still of Hausdorff dimension s, we have $\dim_{\mathrm{H}} G_{r,q} = \dim_{\mathrm{H}} \mathcal{F}_{r,q}$.

We associate with each word $w_{ij} \in W_r$ a similarity $S_{ij} : J_q \to J_q$ defined by the formula

$$S_{ij}(x) := \frac{\alpha_1}{q} + \cdots + \frac{\alpha_{j-1}}{q^{j-1}} + \frac{i}{q^j} + \frac{x}{q^j}, \qquad x \in J_q.$$

It follows from Proposition 2.1 and the definition of $\mathcal{F}_{r,q}$ that

$$\mathcal{F}_{r,q} = \bigcup S_{ij}(\mathcal{F}_{r,q}) \tag{2.3}$$

where the union runs over all i and j for which $w_{ij} \in W_r$. Applying Proposition 2.1 again, it follows that r is the largest element of the set of numbers $t > 1$ for which $\alpha_i(t) = \alpha_i$, $1 \le i \le n$. Hence $\alpha_1 \ldots \alpha_n < \alpha_1(q) \ldots \alpha_n(q)$ and therefore each sequence in $\mathcal{F}'_r$ is the greedy expansion in base q of some $x \in \mathcal{F}_{r,q}$. It follows that the sets $S_{ij}(\mathcal{F}_{r,q})$ on the right side of (2.3) are disjoint. Moreover, the function $x \mapsto (b_i(x,q))$ that maps $\mathcal{F}_{r,q}$ onto $\mathcal{F}'_r$ is increasing. Using the definition of $\mathcal{F}'_r$ it is easily seen that the limit of each monotonic sequence of elements in $\mathcal{F}_{r,q}$ belongs to $\mathcal{F}_{r,q}$. We conclude that the closed set $\mathcal{F}_{r,q}$ is the (nonempty compact) invariant set of this system of similarities. An application of Propositions 9.6 and 9.7 in [9] yields

$$\dim_{\mathrm{H}} \mathcal{F}_{r,q} = \dim_{\mathrm{H}} G_{r,q} = s,$$

where s is the real solution of the equation

$$\frac{\alpha_1}{q^s} + \cdots + \frac{\alpha_{n-1}}{q^{(n-1)s}} + \frac{\alpha_n + 1}{q^{ns}} = 1.$$

Since

$$\frac{\alpha_1}{r} + \cdots + \frac{\alpha_{n-1}}{r^{n-1}} + \frac{\alpha_n + 1}{r^n} = 1$$

we have $s = \log r / \log q$.

(ii) It follows at once from (i) that $\dim_{\mathrm{H}} G_q = 1$. Let $r \in (1,q)$ be such that $(\alpha_i(r))$ is periodic. The proof of (i) shows that

$$G_{r,q} \subset \bigcup_{n=1}^{\infty} (a_n + b_n \mathcal{F}_{r,q})$$

for some constants $a_n, b_n \in \mathbb{R}$ $(n \in \mathbb{N})$. Since $\mathcal{F}_{r,q}$ is a closed set of Hausdorff dimension less than one, it follows in particular that the sets $a_n + b_n\mathcal{F}_{r,q}$ are nowhere dense null sets. Since $G_{s,q} \subset G_{t,q}$ whenever $1 < s < t < q$, the set G_q is a null set of first category. ∎

Theorem 2.8. *Let $q > 1$ be a real number.*

(i) *Let $v := b_{\ell+1}(y,q) \ldots b_{\ell+m}(y,q)$ for some $y \in [0,1)$ and some integers $\ell \ge 0$ and $m \ge 1$. The set Y_v of numbers $x \in J_q$ for which the word v does not occur in the greedy expansion of x in base q has Hausdorff dimension less than one.*

(ii) *The set* Y *of numbers* $x \in J_q$ *for which* at least one *word of the form* $b_{\ell+1}(y,q)\dots b_{\ell+m}(y,q)$ $(\ell \ge 0,\ m \ge 1,\ y \in [0,1))$ *does not occur in the greedy expansion of* x *in base* q *is of first category, has Lebesgue measure zero and Hausdorff dimension one.*

Proof. (i) Using the inequality $(b_i(y,q)) < (\alpha_i(q))$, it follows from Proposition 2.4 that for some $k \in \mathbb{N}$, there exist positive integers $m_1,\dots,m_k$ and nonnegative integers $\ell_1,\dots,\ell_k$ satisfying $\alpha_{m_j}(q) > 0$ and $\ell_j < \alpha_{m_j}(q)$ for each $1 \le j \le k$, such that v is a subword of

$$w := \alpha_1(q)\dots\alpha_{m_1-1}(q)\ell_1\dots\alpha_1(q)\dots\alpha_{m_k-1}(q)\ell_k.$$

Let W_q and $\mathcal{F}'_q$ be the same as the sets W_r and $\mathcal{F}'_r$ defined in the proof of the previous theorem, but now with $(\alpha_i) := (\alpha_i(q))$ and $n \ge \max\{m_1,\dots,m_k\}$ large enough that

$$\left(1+\frac{1}{q^n}\right)^k < 1+\frac{1}{q^{m_1+\dots+m_k}}. \tag{2.4}$$

If $w_{i_1j_1},\dots,w_{i_kj_k}$ are k words belonging to W_q such that

$$i_1j_1\dots i_kj_k \ne \ell_1m_1\dots\ell_km_k,$$

we associate with them a similarity $S_{i_1j_1\dots i_kj_k}: J_q \to J_q$ defined by

$$\begin{aligned}S_{i_1j_1\dots i_kj_k}(x) = {}&\frac{\alpha_1}{q}+\dots+\frac{\alpha_{j_1-1}}{q^{j_1-1}}+\frac{i_1}{q^{j_1}}\\&+\frac{\alpha_1}{q^{j_1+1}}+\dots+\frac{\alpha_{j_2-1}}{q^{j_1+j_2-1}}+\frac{i_2}{q^{j_1+j_2}}+\cdots\\&+\frac{\alpha_1}{q^{j_1+\dots+j_{k-1}+1}}+\dots+\frac{\alpha_{j_k-1}}{q^{j_1+\dots+j_k-1}}+\frac{i_k}{q^{j_1+\dots+j_k}}\\&+\frac{x}{q^{j_1+\dots+j_k}},\qquad x \in J_q.\end{aligned}$$

Let $\mathcal{G}'_q$ denote the set of those sequences belonging to $\mathcal{F}'_q$ which do not contain the word w, and let

$$\mathcal{G}_q := \left\{\sum_{i=1}^{\infty}\frac{c_i}{q^i} : (c_i) \in \mathcal{G}'_q\right\}.$$

Since $(\alpha_i) = (\alpha_i(q))$, a sequence belonging to $\mathcal{F}'_q$ is not necessarily the greedy expansion in base q of a number $x \in J_q$, but this does not affect our proof. It is important, however, that any greedy expansion $(b_i) \ne \alpha_1^\infty$ in base q can be written as $\alpha_1^\ell c_1c_2\dots$ for some $\ell \ge 0$ and some sequence (c_i) belonging to $\mathcal{F}'_q$. If Y_w denotes the set of numbers $x \in J_q$ for which the word w does not occur in $(b_i(x,q))$ then the latter fact implies that the set $Y_w \setminus \{\alpha_1/(q-1)\}$ can be covered by countably many sets similar to $\mathcal{G}_q$.

It follows from the definition of $\mathcal{G}_q$ that

$$\mathcal{G}_q \subset \bigcup S_{i_1 j_1 \ldots i_k j_k}(\mathcal{G}_q)$$

where the union runs over all $i_1 j_1 \ldots i_k j_k$ for which the similarity $S_{i_1 j_1 \ldots i_k j_k}$ is defined above. Hence

$$\overline{\mathcal{G}_q} \subset \bigcup S_{i_1 j_1 \ldots i_k j_k}(\overline{\mathcal{G}_q})$$

and thus $\mathcal{G}_q \subset \mathcal{H}_q$ where $\mathcal{H}_q$ is the (nonempty compact) invariant set of this system of similarities. Let $\tilde{\alpha}_i := \alpha_i$ for $1 \le i < n$ and $\tilde{\alpha}_n := \alpha_n + 1$. From Proposition 9.6 in [9] we know that $\dim_{\mathrm{H}} \mathcal{H}_q \le s$ where s is the real solution of the equation

$$\sum_{j_1=1}^{n} \sum_{j_2=1}^{n} \cdots \sum_{j_k=1}^{n} \left(\frac{\prod_{i=1}^{k} \tilde{\alpha}_{j_i}}{q^{(j_1+\cdots+j_k)s}} \right) - \frac{1}{q^{(m_1+\cdots+m_k)s}} = 1. \tag{2.5}$$

Denoting the left side of (2.5) by $C(s)$, we have

$$C(1) + \frac{1}{q^{m_1+\cdots+m_k}} = \left(\sum_{i=1}^{n} \frac{\tilde{\alpha}_i}{q^i} \right)^k < \left(1 + \frac{1}{q^n} \right)^k.$$

By (2.4) we have $C(1) < 1$, and thus $\dim_{\mathrm{H}} Y_v \le \dim_{\mathrm{H}} Y_w \le \dim_{\mathrm{H}} \mathcal{H}_q < 1$.

(ii) The proof of (i) shows that

$$Y_v \subset Y_w \subset \bigcup_{n=1}^{\infty} (c_n + d_n \mathcal{H}_q)$$

for some constants $c_n, d_n \in \mathbb{R}$ ($n \in \mathbb{N}$). Arguing as in the proof of Theorem 2.7(ii) we may conclude that Y_v is a null set of first category. Since Y is a countable union of sets of the form Y_v the same properties hold for Y. Let $r \in (1, q)$ and let $G_{r,q}$ be the set defined in Theorem 2.7. Due to Theorem 2.7(i) it is now sufficient to show that $G_{r,q} \subset Y$. By Proposition 2.1 there exists an integer $n \in \mathbb{N}$ such that $\alpha_1(r) \ldots \alpha_n(r) < \alpha_1(q) \ldots \alpha_n(q)$. Note that the greedy expansion in base q of a number $x \in G_{r,q}$ equals $(b_i(x', r))$ for some $x' \in J_r$ by Proposition 2.4. Applying Propositions 2.1 and 2.4 once more we conclude that the sequence $0\alpha_1(q) \ldots \alpha_n(q) 0^\infty$ equals $(b_i(y, q))$ for some $y \in [0, 1)$ while the word $b_1(y, q) \ldots b_{n+1}(y, q) = 0\alpha_1(q) \ldots \alpha_n(q)$ does not occur in the greedy expansion in base r of any number belonging to J_r. ■

Remark. In this remark we will briefly sketch a proof of Theorems 2.7(i) and 2.8(i) that was pointed out to us by the anonymous referee. For $q > 1$, let $B_n(q)$ be the number of possible blocks of length n that may occur in $(b_i(x, q))$ for some $x \in J_q$. Since $b_{n+1}(x, q) b_{n+2}(x, q) \ldots$ is the greedy expansion of $\sum_{i=1}^{\infty} b_{n+i} q^{-i}$ for each $n \in \mathbb{N}$ and $x \in J_q$, we have

$$B_n(q) = |\{(b_1(x, q), \ldots, b_n(x, q)) : x \in J_q\}|.$$

Let σ_q be the one-sided left shift on the set $\{(b_i(x,q)) : x \in J_q\}$. It is well known (see [12]) that its topological entropy $h_{\mathrm{top}}(\sigma_q)$, defined by

$$h_{\mathrm{top}}(\sigma_q) := \lim_{n\to\infty} \frac{\log(B_n(q))}{n}, \tag{2.6}$$

equals $\log q$. By some modifications of the proof of Proposition III.1 in [10], one shows that $\dim_{\mathrm{H}} G_{r,q} = h_{\mathrm{top}}(\sigma_r)/\log q = \log r/\log q$. Theorem 2.8(i) may also be deduced from (2.6) and Proposition III.1 in [10]. On the other hand, our proof of these results enables us to show that the sets G_q and Y in Theorem 2.7(ii) and 2.8(ii) are of first category. Moreover, Theorem 2.7(i) combined with the formula $\dim_{\mathrm{H}} G_{r,q} = h_{\mathrm{top}}(\sigma_r)/\log q$ gives an alternative proof of the fact that $h_{\mathrm{top}}(\sigma_q) = \log q$ for each $q > 1$.

3. Proof of Theorem 1.2. The following characterization of unique expansions readily follows from Proposition 2.4.

PROPOSITION 3.1. *Fix $q > 1$. A sequence (c_i) of integers $c_i \in A_q$ is the unique expansion of some $x \in J_q$ if and only if*

$$c_{n+1}c_{n+2}\ldots < \alpha_1(q)\alpha_2(q)\ldots \qquad \textit{whenever } c_n < \alpha_1(q)$$

and

$$\overline{c_{n+1}c_{n+2}\ldots} < \alpha_1(q)\alpha_2(q)\ldots \qquad \textit{whenever } c_n > 0.$$

In what follows we use the notation $(a_i(x,q))$, $(b_i(x,q))$, $(\alpha_i(q))$ and $(\beta_i(q))$ as introduced in Section 1. If x and q are clear from the context, then we omit these arguments and we simply write a_i, b_i, α_i and β_i. If two couples (x,q) and (x',q') are considered simultaneously, then we also write a_i', b_i', α_i' and β_i' in place of $a_i(x',q')$, $b_i(x',q')$, $\alpha_i(q')$ and $\beta_i(q')$.

LEMMA 3.2. *Given $(x,q) \in \mathbf{J}$, the following two conditions are equivalent:*

$$\overline{a_{n+1}a_{n+2}\ldots} \le \alpha_1\alpha_2\ldots \qquad \textit{whenever } a_n > 0;$$
$$\overline{a_{n+1}a_{n+2}\ldots} \le \beta_1\beta_2\ldots \qquad \textit{whenever } a_n > 0.$$

Proof. Since $(\alpha_i) \le (\beta_i)$, it suffices to show that if there exists a positive integer n such that

$$a_n > 0 \quad \text{and} \quad \overline{a_{n+1}a_{n+2}\ldots} > \alpha_1\alpha_2\ldots,$$

then there also exists a positive integer m such that

$$a_m > 0 \quad \text{and} \quad \overline{a_{m+1}a_{m+2}\ldots} > \beta_1\beta_2\ldots.$$

If the greedy expansion (β_i) is infinite, then $(\beta_i) = (\alpha_i)$ and we may choose $m = n$. If (β_i) has a last nonzero digit β_ℓ, then $(\alpha_i) = (\alpha_1\ldots\alpha_\ell)^\infty$ with $\alpha_1\ldots\alpha_{\ell-1}\alpha_\ell = \beta_1\ldots\beta_{\ell-1}\beta_\ell^-$ (where $\beta_\ell^- := \beta_\ell - 1$), and thus $\alpha_\ell < \alpha_1$. Since we have

$$\overline{a_{n+1}a_{n+2}\ldots} > (\alpha_1\ldots\alpha_\ell)^\infty$$

by assumption, there exists a nonnegative integer j satisfying

$$\overline{a_{n+1}\dots a_{n+j\ell}} = (\alpha_1\dots\alpha_\ell)^j \quad \text{and} \quad \overline{a_{n+j\ell+1}\dots a_{n+(j+1)\ell}} > \alpha_1\dots\alpha_\ell.$$

Putting $m := n + j\ell$ it follows that

$$a_m > 0 \quad \text{and} \quad \overline{a_{m+1}\dots a_{m+\ell}} \ge \beta_1\dots\beta_\ell.$$

Our assumption $\overline{a_{n+1}a_{n+2}\dots} > \alpha_1\alpha_2\dots$ implies $(\alpha_i) < \alpha_1^\infty$ and $(a_i) \ne \alpha_1^\infty$. It follows from Proposition 2.2 that (a_i) has no tail equal to α_1^∞, so that $\overline{a_{m+\ell+1}a_{m+\ell+2}\dots} > 0^\infty$. We conclude that

$$\overline{a_{m+1}a_{m+2}\dots} > \beta_1\beta_2\dots. \ \blacksquare$$

Definition. We say that $(x, q) \in \mathbf{J}$ belongs to the set $\mathbf{V}$ if the conditions of the preceding lemma are satisfied. Moreover, we define

$$\mathcal{V}_q := \{x \in J_q : (x, q) \in \mathbf{V}\}, \quad q > 1.$$

It follows from Proposition 3.1 that $\mathbf{U} \subset \mathbf{V} \subset \mathbf{J}$.

Proof of Theorem 1.2. We need to prove that $\overline{\mathbf{U}} \cap \mathbf{J} = \mathbf{V}$.

First we show that $\mathbf{V} \subset \overline{\mathbf{U}}$. To this end we introduce for each fixed $q > 1$ the sets $\mathcal{U}_q'$ and $\mathcal{V}_q'$, defined by

$$\mathcal{U}_q' := \{(a_i(x, q)) : x \in \mathcal{U}_q\} \quad \text{and} \quad \mathcal{V}_q' := \{(a_i(x, q)) : x \in \mathcal{V}_q\}.$$

Observe that $\mathcal{U}_q'$ is simply the set of unique expansions in base q. It follows easily from Propositions 2.1, 2.2 and 3.1 that $\mathcal{U}_q' \subset \mathcal{V}_q'$ for each $q > 1$, and that $\mathcal{V}_q' \subset \mathcal{U}_r'$ for each $r > q$ such that $\lceil q\rceil = \lceil r\rceil$. Since we also have $\overline{\mathcal{U}_q} = \mathcal{V}_q = [0, 1]$ if $q > 1$ is an integer, the result follows.

Next we show that $\overline{\mathbf{U}} \cap \mathbf{J} \subset \mathbf{V}$. Since $\mathbf{U} \subset \mathbf{V}$ it is sufficient to prove that if $(x, q) \in \mathbf{J} \setminus \mathbf{V}$, then $(x', q') \notin \mathbf{V}$ for all $(x', q') \in \mathbf{J}$ close enough to (x, q). By Lemma 3.2 there exist two positive integers n and m such that

$$a_n > 0 \quad \text{and} \quad \overline{a_{n+1}\dots a_{n+m}} > \beta_1\dots\beta_m. \tag{3.1}$$

This implies in particular that q is not an integer, because otherwise $(\alpha_i) = (\beta_i) = \beta_1^\infty$. Hence, if q' is sufficiently close to q, then

$$\beta_1'\dots\beta_m' \le \beta_1\dots\beta_m \tag{3.2}$$

by Lemma 2.5. It follows from the definition of quasi-greedy expansions that

$$\frac{a_1}{q} + \dots + \frac{a_{j-1}}{q^{j-1}} + \frac{a_j^+}{q^j} + \frac{1}{q^{j+m}} > x \quad \text{whenever } a_j < \alpha_1,$$

where $a_j^+ := a_j + 1$. If $(x', q') \in \mathbf{J}$ is sufficiently close to (x, q), then $\alpha_1 = \alpha_1'$, the inequality (3.2) is satisfied, $a_1'\dots a_{n+m}' \ge a_1\dots a_{n+m}$ by Lemma 2.3, and

$$\frac{a_1}{q'} + \cdots + \frac{a_{j-1}}{(q')^{j-1}} + \frac{a_j^+}{(q')^j} + \frac{1}{(q')^{j+m}} > x' \tag{3.3}$$

$$\text{whenever } j \leq n+m \text{ and } a_j < \alpha_1.$$

Now we distinguish two cases.

If $a'_1 \ldots a'_{n+m} = a_1 \ldots a_{n+m}$, then we have

$$a'_n > 0 \quad \text{and} \quad \overline{a'_{n+1} \ldots a'_{n+m}} > \beta_1 \ldots \beta_m \geq \beta'_1 \ldots \beta'_m$$

by (3.1) and (3.2). This proves that $(x', q') \notin \mathbf{V}$.

If $a'_1 \ldots a'_{n+m} > a_1 \ldots a_{n+m}$, then let us consider the smallest j for which $a'_j > a_j$. It follows from (3.2) and (3.3) that

$$a'_j = a_j^+ > 0 \quad \text{and} \quad \overline{a'_{j+1} \ldots a'_{j+m}} = \beta_1^m > \beta_1 \ldots \beta_m \geq \beta'_1 \ldots \beta'_m.$$

Hence $(x', q') \notin \mathbf{V}$ again. ∎

Remark. It is the purpose of this remark to describe the set $\overline{\mathbf{U}} \setminus \mathbf{J}$. For each $m \in \mathbb{N}$, we define the number $q_m \in (m, m+1)$ by the equation

$$1 = \frac{m}{q_m} + \frac{1}{q_m^2}.$$

Fix $q \in (m, q_m]$. Since $\alpha_1(q) = m$ and $\alpha_2(q) = 0$, Proposition 3.1 implies that a sequence $(c_i) \in \{0, \ldots, m\}^{\mathbb{N}}$ belongs to $\mathcal{U}'_q$ if and only if for each $n \in \mathbb{N}$, we have

$$c_n < m \ \Rightarrow \ c_{n+1} < m \quad \text{and} \quad c_n > 0 \ \Rightarrow \ c_{n+1} > 0.$$

Denoting the set of all such sequences by D'_m and putting, for $m > 1$ (note that $D'_1 = \{0^\infty, 1^\infty\}$),

$$D_m := \left\{ \sum_{i=1}^{\infty} \frac{c_i}{m^i} : (c_i) \in D'_m \right\},$$

one may verify that

$$\overline{\mathbf{U}} \setminus \mathbf{J} = \{(0,1)\} \cup \bigcup_{m=2}^{\infty} (D_m \setminus [0,1]) \times \{m\}.$$

For $x \geq 0$, let $\mathcal{U}(x) = \{q > 1 : (x, q) \in \mathbf{U}\}$, and denote its closure by $\overline{\mathcal{U}(x)}$. With this notation, the set $\mathcal{U}$ introduced in Section 1 equals $\mathcal{U}(1)$. The following corollary implies in particular that the sets $\overline{\mathcal{U}(x)} \setminus \mathcal{U}(x)$ are (at most) countable.

Corollary 3.3. *Each element $q \in \overline{\mathcal{U}(x)} \setminus \mathcal{U}(x)$ is algebraic over the field $\mathbb{Q}(x)$.*

Proof. If $q \in \overline{\mathcal{U}(x)} \setminus \mathcal{U}(x)$ and $q \notin \mathbb{N}$, then $(x, q) \in \mathbf{J}$ and thus $(x, q) \in \mathbf{V}$ by Theorem 1.2. If the sequence $(b_i(x, q))$ is infinite, then it ends with

$\overline{\alpha_1(q)\alpha_2(q)\ldots}$, as follows from the definition of $\mathbf{V}$ and Propositions 2.4 and 3.1. Hence x has a finite expansion in base q or x can be written as

$$x = \frac{b_1(x,q)}{q} + \cdots + \frac{b_n(x,q)}{q^n} + \frac{1}{q^n}\left(\frac{\alpha_1}{q-1} - 1\right)$$

for some $n \geq 0$, whence q is algebraic over $\mathbb{Q}(x)$. ∎

4. Proof of Theorem 1.1. We need some results on the Hausdorff dimension of the sets $\mathcal{U}_q$ and $\mathcal{V}_q$ for $q > 1$. It follows from Theorem 1.2 that $\mathcal{U}_q \subset \overline{\mathcal{U}_q} \subset \mathcal{V}_q$. Moreover, if an element $x \in \mathcal{V}_q \setminus \mathcal{U}_q$ has an infinite greedy expansion in base q, then $(b_i(x,q))$ must end with $\overline{\alpha_1(q)\alpha_2(q)\ldots}$, as follows from Propositions 2.4 and 3.1; hence $\mathcal{V}_q \setminus \mathcal{U}_q$ is (at most) countable and the sets $\mathcal{U}_q$, $\overline{\mathcal{U}_q}$ and $\mathcal{V}_q$ have the same Hausdorff dimension for each $q > 1$. Proposition 4.1 below is contained in the works of Daróczy and Kátai [4], Kallós [13], [14], Glendinning and Sidorov [11], and Sidorov [21]; for the reader's convenience we provide here an elementary proof.

PROPOSITION 4.1. *We have*

(i) $\lim_{q\uparrow 2} \dim_{\mathrm{H}} \mathcal{U}_q = 1$;
(ii) $\dim_{\mathrm{H}} \mathcal{U}_q < 1$ *for all noninteger* $q > 1$.

Proof. (i) Assume that $q \in (1,2)$ is larger than the tribonacci number, i.e.,

$$\frac{1}{q} + \frac{1}{q^2} + \frac{1}{q^3} < 1,$$

and let $N = N(q) \geq 2$ be the largest integer satisfying

$$\frac{1}{q} + \cdots + \frac{1}{q^{2N-1}} < 1.$$

Hence $\alpha_1(q) = \cdots = \alpha_{2N-1}(q) = 1$. Let us denote by $\mathcal{I}_q$ the set of numbers $x \in J_q$ which have an expansion (c_i) in base q satisfying $0 < c_{kN+1} + \cdots + c_{(k+1)N} < N$ for every nonnegative integer k. Since in such expansions (c_i), a zero (resp. one) is followed by at most $2N-2$ consecutive ones (resp. zeros), it follows from Proposition 3.1 that $\mathcal{I}_q \subset \mathcal{U}_q$. It now suffices to prove that

$$\dim_{\mathrm{H}} \mathcal{I}_q = \frac{\log(2^N - 2)}{N \log q}; \tag{4.1}$$

indeed, $q \uparrow 2$ implies $N \to \infty$, hence $\dim_{\mathrm{H}} \mathcal{I}_q \to 1$ and consequently $\dim_{\mathrm{H}} \mathcal{U}_q \to 1$.

Observe that

$$\mathcal{I}_q = \bigcup S_{c_1\ldots c_N}(\mathcal{I}_q) \tag{4.2}$$

where the union is over the words $c_1 \dots c_N$ of length N consisting of zeros and ones satisfying $0 < c_1 + \dots + c_N < N$, and $S_{c_1 \dots c_N} : J_q \to J_q$ is given by

$$S_{c_1 \dots c_N}(x) := \left(\frac{c_1}{q} + \dots + \frac{c_N}{q^N} \right) + \frac{x}{q^N}, \qquad x \in J_q.$$

Moreover, the set $\mathcal{I}_q$ is closed (and thus compact) because the limit of a monotonic sequence in I_q converges to an element of I_q. In other words, $\mathcal{I}_q$ is the (nonempty compact) invariant set of the iterated function system formed by these $2^N - 2$ similarities. The sets $S_{c_1 \dots c_N}(\mathcal{I}_q)$ on the right side of (4.2) are disjoint because $S_{c_1 \dots c_N}(\mathcal{I}_q) \subset \mathcal{I}_q \subset \mathcal{U}_q$, and since all similarity ratios are equal to q^{-N}, it follows from Propositions 9.6 and 9.7 in [9] that the Hausdorff dimension s of $\mathcal{I}_q$ is the real solution of the equation

$$(2^N - 2) q^{-Ns} = 1,$$

which is equivalent to (4.1).

(ii) Let $q > 1$ be a noninteger and let $n \in \mathbb{N}$ be such that $\alpha_n(q) < \alpha_1(q)$. It follows from Proposition 3.1 that the word $1(0)^n$ does not occur in $(b_i(x,q))$ if x belongs to $\mathcal{U}_q$. Applying Theorem 2.8(i) with $y = q^{-1}$, $\ell = 0$ and $m = n + 1$, we conclude that $\dim_{\mathrm{H}} \mathcal{U}_q < 1$. ∎

Proof of Theorem 1.1. (ii) Let $q > 1$ be a noninteger. Since $\mathcal{V}_q \setminus \mathcal{U}_q$ is countable, Proposition 4.1(ii) yields that $\dim_{\mathrm{H}} \mathcal{V}_q < 1$. This implies in particular that the set $\mathcal{V}_q$ is a one-dimensional null set. Applying Theorem 1.2 (and the Remark following its proof) and Fubini's theorem we conclude that $\overline{\mathbf{U}}$ is a two-dimensional null set.

(i) Since $\mathcal{U}_q$ is not closed for all $q > 1$, $\mathbf{U}$ cannot be closed. Since $\overline{\mathbf{U}}$ is a two-dimensional null set, it has no interior points. It remains to show that $\mathbf{U}$ (and thus $\overline{\mathbf{U}}$) has no isolated points. If $q > 1$ is an integer, then, as is well known, $\mathcal{U}_q$ is dense in $J_q = [0,1]$. If $q > 1$ is a noninteger, then each $(x,q) \in \mathbf{U}$ is not isolated because $\mathcal{U}_q' \subset \mathcal{U}_r'$ whenever $q < r$ and $\lceil q \rceil = \lceil r \rceil$.

(iii) From Corollary 7.10 in [9] we may conclude that for almost all $q > 1$,

$$\dim_{\mathrm{H}} \mathcal{U}_q \le \max\{0, \dim_{\mathrm{H}} \mathbf{U} - 1\},$$

which would contradict Proposition 4.1(i) if we had $\dim_{\mathrm{H}} \mathbf{U} < 2$. ∎

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