FUNDAMENTA
MATHEMATICAE
226 (2014)

A groupoid formulation of the Baire Category Theorem

by

Jonathan Brown (Manhattan, KS) and **Lisa Orloff Clark** (Dunedin)

Abstract. We prove that the Baire Category Theorem is equivalent to the following: Let G be a topological groupoid such that the unit space is a complete metric space, and there is a countable cover of G by neighbourhood bisections. If G is effective, then G is topologically principal.

1. Introduction

THEOREM 1.1 (The Baire Category Theorem; see, for example, [7, Theorem 7.7.2]). *Suppose X is a complete metric space. If $\{C_n\}$ is a countable collection of closed subsets of X, each with empty interior, then $\bigcup_n C_n$ has empty interior.*

The proof of the Baire Category Theorem, originally formulated by Baire in the 1890's [1], requires a variant of the Axiom of Choice [4, Chapter 13]. In fact, [2] and [5] show that the Baire Category Theorem is equivalent to the *Principle of Dependent Choice* which says:

> Suppose X is a set and $R \subseteq X \times X$ is a relation such that for each $x \in X$, there exists $y \in X$ such that $(x, y) \in R$. Then there is a sequence $\{x_n\} \subseteq X$ such that $(x_n, x_{n+1}) \in R$ for all n.

The Principle of Dependent Choice falls strictly between the Countable Axiom of Choice and the Axiom of Choice; see [6] for more details.

In this note, we show under the Zermelo–Fraenkel axioms ZF (no choice) that the Baire Category Theorem is equivalent to Theorem 3.4 below. Theorem 3.4 is a strengthening of [9, Proposition 3.6] and provides hypotheses on a topological groupoid to ensure that *effective* implies *topologically principal* (see Section 2). The most substantial contribution of the present paper comes in Section 3 where the precise formulation of Theorem 3.4 is developed.

2010 *Mathematics Subject Classification*: 03E25, 54E52, 22A22.
Key words and phrases: Baire Category Theorem, topologically principal groupoid, effective groupoid, Principle of Dependent Choice.

DOI: 10.4064/fm226-2-2

© Instytut Matematyczny PAN, 2014

That Theorem 3.4 implies the Baire Category Theorem is entirely new; but even our proof that the Baire Category Theorem implies Theorem 3.4 is different from the one in [9].

2. Preliminaries. A groupoid is a generalisation of a group in which multiplication is only partially defined. Equivalently, a groupoid G is a small category in which every morphism is invertible. We identify the set of objects of G with the set of identity morphisms and denote this set $G^{(0)}$. For $\gamma \in G$, we denote the range and source (domain) of γ by $r(\gamma)$ and $s(\gamma)$ respectively. Thus $r, s : G \to G^{(0)} \subseteq G$. We define

$$G^{(2)} := \{(\gamma, \eta) \in G \times G : r(\eta) = s(\gamma)\};$$

$G^{(2)}$ consists of precisely those pairs of morphisms that can be composed in G. See [8] for more details on groupoids. For any $x \in G^{(0)}$, the *isotropy group* at x is the group

$$xGx := \{\gamma \in G : r(\gamma) = s(\gamma) = x\}.$$

The *isotropy subgroupoid* of G is

$$\operatorname{Iso}(G) := \bigcup_{x \in G^{(0)}} xGx$$

which is itself a groupoid. If $B \subseteq G$, then we also write $\operatorname{Iso}(B) := \operatorname{Iso}(G) \cap B$. We say G is a *group bundle* if $\operatorname{Iso}(G) = G$.

A groupoid G is a *topological groupoid* if G is equipped with a topology so that composition and inversion are continuous. In this case, r and s are continuous maps. If $G^{(0)}$ is Hausdorff, then the continuity of r and s implies that $\operatorname{Iso}(G)$ is a closed subset of G.

An open set $A \subseteq G$ is called an *open bisection* if $r(A)$ and $s(A)$ are open in G, and r and s restricted to A are homeomorphisms onto their image; in particular r and s are injective on A.

We say a topological groupoid G is *topologically principal* if the subset $\{x \in G^{(0)} : xGx \neq \{x\}\}$ has empty interior in $G^{(0)}$. A topological groupoid G is *effective* if $\operatorname{Iso}(G) - G^{(0)}$ has empty interior.

3. When does effective imply topologically principal? In [9, Proposition 3.6(ii)] Renault considers effective groupoids whose unit spaces are 'Baire'. We can interpret Renault's result as saying that Theorem 1.1 implies the following:

THEOREM 3.1 ([9, Proposition 3.6(ii)]). *Suppose G is a topological groupoid such that:*

(1) *the unit space is a complete metric space;*
(2) *G has a countable cover consisting of open bisections.*

If G is effective, then G is topologically principal.

Our original intention was to show that Theorem 3.1 is equivalent to Theorem 1.1. However, we eventually realised that such a result will only hold if we weaken the hypotheses of Theorem 3.1. To see why, consider the class of effective groupoids constructed in Example 3.2 below; each groupoid has the property that Theorem 1.1 implies it is topologically principal. At the same time, groupoids in this class may not satisfy the hypotheses of Theorem 3.1. (We will also use this class later in the proof of our main result.)

Example 3.2. Let X be a complete metric space, $\{C_n\}$ be a countable collection of closed subsets of X, each with empty interior, and define

$$C := \bigcup_n C_n.$$

Let G be the group bundle with unit space X and isotropy groups

$$xGx := \begin{cases} \mathbb{Z}_2 & \text{if } x \in C, \\ \{0\} & \text{otherwise.} \end{cases}$$

We identify the identity element $0 \in xGx$ with x. For each $x \in C$, we write γ_x for the nontrivial element of xGx. Notice that

$$\begin{aligned} G^{(2)} = \{(x,x) : x \in X\} &\cup \{(\gamma_x, \gamma_x) : x \in C\} \\ &\cup \{(x, \gamma_x) : x \in C\} \cup \{(\gamma_x, x) : x \in C\}. \end{aligned}$$

To make G into a topological groupoid, first let $\mathcal{T}$ be the topology for X. Define the collection

$$\mathcal{B} := \mathcal{T} \cup \big\{V \subseteq G : V = (W - \{x\}) \cup \{\gamma_x\} \text{ for some } W \in \mathcal{T} \text{ and } x \in C \cap W\big\}.$$

The collection $\mathcal{B}$ forms a basis for a topology on G. To see this, note that $\mathcal{B}$ covers G, and since $\mathcal{T}$ is the topology for X, it is easy to see that $U, V \in \mathcal{B}$ implies $U \cap V \in \mathcal{B}$. We claim that G endowed with the topology generated by $\mathcal{B}$ is a topological groupoid. Indeed, inversion is given by the identity, and is thus continuous. Now let $m : G^{(2)} \to G$ be the composition map. Fix $V \in \mathcal{B}$. If $V \in \mathcal{T}$, then

$$m^{-1}(V) = \{(x,x) : x \in V\} \cup \{(\gamma_x, \gamma_x) : x \in V\}.$$

If $V = (W - \{y\}) \cup \{\gamma_y\}$ for some $W \in \mathcal{T}$ and $y \in C \cap W$, then

$$\begin{aligned} m^{-1}(V) = \{(x,x) : x \in W - \{y\}\} &\cup \{(\gamma_y, y), (y, \gamma_y)\} \\ &\cup \{(\gamma_x, \gamma_x) : x \in (W - \{y\}) \cap C\}. \end{aligned}$$

In both cases, it is straightforward to show that $m^{-1}(V)$ is open in $G^{(2)}$, hence composition is continuous as claimed.

Since every element of $\mathcal{B}$ intersects the unit space, the set $\mathrm{Iso}(G) - G^{(0)} = G - G^{(0)}$ contains no open sets, so G is effective. By construction, $G^{(0)} = X$ is a complete metric space and

$$C = \{x \in G^{(0)} : xGx \neq \{x\}\},$$

so Theorem 1.1 implies that G is also topologically principal. Notice that G need not satisfy item (2) of Renault's Theorem 3.1. Indeed, if $X = [0, 1]$ and $C_n = C$ is the Cantor set for all n, then there is no countable cover of G consisting of open bisections. To see this, suppose $\{U_i\}$ is any countable open cover of G. Since the Cantor set is uncountable, there exists an i_0 such that $A_{i_0} := \{x \in C : \gamma_x \in U_{i_0}\}$ is uncountable. For each $x \in A_{i_0}$ pick a basis element $(V_x - \{x\}) \cup \{\gamma_x\}$ contained in U_{i_0}. Since the standard basis for $[0, 1]$ is given by connected intervals, we can assume that V_x is connected. For each $n \in \mathbb{Z}^+$ define $D_n := \{x \in A_{i_0} : \text{diameter of } V_x \text{ is greater than } 2/n\}$. Since A_{i_0} is uncountable, there exists n_0 such that D_{n_0} is uncountable. Now consider the partition $\{P_m := [m/(2n_0), (m+1)/(2n_0)]\}$ of $[0, 1]$ where $0 \le m \le 2n_0 - 1$. Since D_{n_0} is uncountable, there exists an m_0 such that $D_{n_0} \cap P_{m_0}$ is uncountable. By the definition of D_{n_0} this implies that for every $x \in D_{n_0} \cap P_{m_0}$ both x and γ_x are in U_{i_0}, and so U_{i_0} is *not* an open bisection.

While the groupoids considered above need not have a countable cover of open bisections, they do have a countable cover consisting of 'well-behaved sets'. We call these sets *neighbourhood bisections*. (We denote the interior of a set D by $\mathrm{Int}(D)$.)

Definition 3.3. A set $B \subseteq G$ is called a *neighbourhood bisection* if the following hold:

(1) $B \subseteq \overline{\mathrm{Int}(B)}$;
(2) $r|_B$ and $s|_B$ are injective;
(3) $r(B)$ and $s(B)$ are open in G;
(4) $\mathrm{Int}(B)$ is an open bisection;
(5) $B - \mathrm{Int}(B) \subseteq \mathrm{Iso}(B) - G^{(0)}$.

In the next section we prove that the following theorem is equivalent to Theorem 1.1. One part of our proof involves showing that the class of groupoids constructed in Example 3.2 do indeed have a countable cover consisting of neighbourhood bisections.

Theorem 3.4. *Suppose G is a topological groupoid such that:*

(1) *the unit space is a complete metric space;*
(2) *G has a countable cover consisting of neighbourhood bisections.*

If G is effective, then G is topologically principal.

Remark 3.5. Suppose G is a groupoid satisfying the hypotheses of Theorem 3.4, then $G^{(0)}$ is open in G. To see this, let $\{B_n\}$ be a countable cover of G by neighbourhood bisections. Then $G^{(0)} = \bigcup_n r(B_n)$, which is open.

Remark 3.6. An *étale groupoid* is a topological groupoid that has a cover consisting of open bisections. When studying C^*-algebras associated to groupoids, one often considers second-countable, locally compact, Hausdorff groupoids that are étale. These groupoids satisfy the hypotheses of Theorem 3.4.

Remark 3.7. Suppose G is a topological groupoid. If r is an open map, then G topologically principal implies that G is effective. See [3, Examples 6.3 and 6.4] for examples of groupoids (that do not satisfy the hypotheses of Theorem 3.4) that are effective but not topologically principal.

4. Main result

Theorem 4.1. *Theorem* 1.1 *is equivalent to Theorem* 3.4 *in* ZF.

Before we prove Theorem 4.1, we establish the following two lemmas. The first lemma is used to prove the second one, which is a key step in our proof of Theorem 4.1. The background theory throughout this section is ZF.

Remark 4.2. The proof that Theorem 1.1 implies [9, Proposition 3.6] given in [9] relies heavily on G being étale. In our situation, we have to work harder because we do not have a basis of open bisections.

Lemma 4.3. *Suppose G is a topological groupoid such that $G^{(0)}$ is open in G, $B \subseteq G$ is a neighbourhood bisection and $D \subseteq B$ is closed in B where B is endowed with the subspace topology. Suppose that $B - \mathrm{Int}(B) \subseteq D$. Then $r(D)$ is closed in $r(B)$ where $r(B)$ is endowed with the subspace topology.*

Proof. Let G, B and D be as stated. Then

$$D = (D \cap \mathrm{Int}(B)) \cup (B - \mathrm{Int}(B)),$$

which means

$$r(D) = r(D \cap \mathrm{Int}(B)) \cup r(B - \mathrm{Int}(B)).$$

Since $r|_B$ is a bijection onto its image, $r(B - \mathrm{Int}(B)) = r(B) - r(\mathrm{Int}(B))$ which is closed in $r(B)$ as $r(\mathrm{Int}(B))$ is open. Further $r(D \cap \mathrm{Int}(B))$ is closed in $r(\mathrm{Int}(B))$ because r restricted to $\mathrm{Int}(B)$ is a homeomorphism. Thus there exists a closed set C such that $r(\mathrm{Int}(B)) \cap C = r(\mathrm{Int}(B) \cap D)$. Therefore

$$\begin{aligned} r(D) &= r(\mathrm{Int}(B) \cap D) \cup r(B - \mathrm{Int}(B)) \\ &= (r(\mathrm{Int}(B)) \cap C) \cup (r(B - \mathrm{Int}(B)) \cap C) \cup r(B - \mathrm{Int}(B)) \\ &= (r(B) \cap C) \cup r(B - \mathrm{Int}(B)) \end{aligned}$$

which is closed in $r(B)$. ∎

Lemma 4.4. *Suppose G is an effective groupoid such that $G^{(0)}$ is open in G, and B is a neighbourhood bisection. Then*

(1) *$r(\mathrm{Iso}(B) - G^{(0)})$ has empty interior, and*
(2) *$\overline{r(\mathrm{Iso}(B) - G^{(0)})}$ has empty interior.*

Proof. For (1), by way of contradiction, suppose there exists a nonempty open set $W \subseteq r(\mathrm{Iso}(B) - G^{(0)})$. Thus $W \cap r(B) \neq \emptyset$, and since $B \subseteq \overline{\mathrm{Int}(B)}$, we have $W \cap r\big(\overline{\mathrm{Int}(B)}\big) \neq \emptyset$. Therefore

$$W \cap \overline{r(\mathrm{Int}(B))} \neq \emptyset \quad \text{because} \quad r\big(\overline{\mathrm{Int}(B)}\big) \subseteq \overline{r(\mathrm{Int}(B))}.$$

Hence $W \cap r(\mathrm{Int}(B))$ is a nonempty open set contained in $G^{(0)}$. Since

$$\phi := r|_{\mathrm{Int}(B)}$$

is a homeomorphism,

$$\phi^{-1}(W \cap r(\mathrm{Int}(B)))$$

is a nonempty open subset of $\mathrm{Int}(B)$ and thus is open in G. Since r is injective on B and $W \subseteq r(\mathrm{Iso}(B) - G^{(0)})$,

$$\phi^{-1}(W \cap r(\mathrm{Int}(B))) \subseteq \mathrm{Iso}(B) - G^{(0)} \subseteq \mathrm{Iso}(G) - G^{(0)}.$$

This is a contradiction because G is effective.

For (2), by way of contradiction, assume there exists a nonempty open subset

$$V \subseteq \overline{r(\mathrm{Iso}(B) - G^{(0)})}.$$

Notice that $V \cap r(B)$ is a nonempty open subset of $G^{(0)}$. Further,

$$V \cap r(B) \subseteq \overline{r(\mathrm{Iso}(B) - G^{(0)})} \cap r(B).$$

We show that $\overline{r(\mathrm{Iso}(B) - G^{(0)})} \cap r(B) = r(\mathrm{Iso}(B) - G^{(0)})$. Since $\mathrm{Iso}(B)$ is closed in B and $G^{(0)}$ is open, $\mathrm{Iso}(B) - G^{(0)}$ is also closed in B. Also, $B - \mathrm{Int}(B) \subseteq \mathrm{Iso}(B) - G^{(0)}$ by assumption. Therefore we apply Lemma 4.3 to see that $r(\mathrm{Iso}(B) - G^{(0)})$ is closed in $r(B)$. Thus

$$r(\mathrm{Iso}(B) - G^{(0)}) = \overline{r(\mathrm{Iso}(B) - G^{(0)})} \cap r(B),$$

and so

$$V \cap r(B) \subseteq r(\mathrm{Iso}(B) - G^{(0)}),$$

which contradicts item (1). ∎

Proof of Theorem 4.1. Suppose Theorem 1.1 holds. Let G be a topological groupoid with a countable cover of neighbourhood bisections $\{B_n\}$ such that $G^{(0)}$ is a complete metric space. Suppose also that G is effective.

By Lemma 4.4(2), the set $\overline{r(\mathrm{Iso}(B_n) - G^{(0)})}$ has empty interior for every n. Define $C_n := \overline{r(\mathrm{Iso}(B_n) - G^{(0)})} \cap G^{(0)}$ for each n. Notice that each C_n is a closed subset of $G^{(0)}$. Because $G^{(0)}$ is open in G, each C_n also has empty interior in $G^{(0)}$. Applying Theorem 1.1 (Baire Category Theorem) to the collection $\{C_n\}$ we see that

$$C := \bigcup_n C_n$$

has empty interior. By construction, C contains the units with nontrivial isotropy. Therefore, G is topologically principal.

Conversely, suppose that Theorem 3.4 holds. Let X be a complete metric space with topology $\mathcal{T}$ and $\{C_n\}$ be a countable collection of closed subsets of X, each with empty interior. Without loss of generality we can assume $C_0 = \emptyset$. Let $C = \bigcup_n C_n$. Define G as in Example 3.2. Since $G^{(0)} = X$ as a topological space, G satisfies (1) of Theorem 3.4.

For each n, define

$$B_n := (X - C_n) \cup \{\gamma_x : x \in C_n\}.$$

We claim that each B_n is a neighbourhood bisection. To prove this, we must check each of the items in Definition 3.3. To see (1), first note that $X - C_n$ is open in G and contained in B_n. Thus $X - C_n \subseteq \mathrm{Int}(B_n)$. We show that $B_n \subseteq \overline{X - C_n} \subseteq \overline{\mathrm{Int}(B_n)}$. Consider γ_x for some $x \in C_n$. For every $V \in \mathcal{B}$ with $\gamma_x \in V$ we have $V = (W - \{x\}) \cup \{\gamma_x\}$, where $W \in \mathcal{T}$ and $x \in W$. Now $V \cap (X - C_n) = W \cap (X - C_n)$ is nonempty because C_n has empty interior. Therefore $\gamma_x \in \overline{X - C_n}$. Since $B_n = X - C_n \cup \{\gamma_x : x \in C_n\}$, we have $B_n \subseteq \overline{X - C_n}$. That B_n satisfies item (2) is clear, thus $r(B_n) = X = s(B_n)$ is open in G, giving us item (3). Since $r(V)$ is in $\mathcal{T}$ for every $V \in \mathcal{B}$, $r = s$ is an open map, and hence $r|_{\mathrm{Int}(B_n)} = s|_{\mathrm{Int}(B_n)}$ is a homeomorphism with open image, giving item (4). Lastly, since $B_n \cap G^{(0)} = X - C_n \subseteq \mathrm{Int}(B_n)$ and $G = \mathrm{Iso}(G)$, we get item (5). Thus $\{B_n\}_n$ is a countable cover of G by neighbourhood bisections and G satisfies (2) of Theorem 3.4.

We showed in Example 3.2 that G is effective. Therefore G is topologically principal by Theorem 3.4. Thus

$$C = \{x \in G^{(0)} : xGx \neq \{x\}\}$$

has empty interior, proving Theorem 1.1. ∎

Acknowledgements. Thanks to François Dorais for the very helpful email correspondence.

References

[1] R. Baire, *Sur les fonctions de variables réelles*, Ann. Mat. Pura Appl. 3 (1899), 1–123.

[2] C. E. Blair, *The Baire category theorem implies the principle of dependent choices*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 25 (1977), 933–934.

[3] J. H. Brown, L. O. Clark, C. Farthing and A. Sims, *Simplicity of algebras associated to étale groupoids*, Semigroup Forum 88 (2014), 433–452.

[4] W. Dunham, *The Calculus Gallery: Masterpieces from Newton to Lebesgue*, Princeton Univ. Press, Princeton, NJ, 2005.

[5] R. Goldblatt, *On the role of the Baire Category Theorem and dependent choice in the foundations of logic*, J. Symbolic Logic 50 (1985), 412–422.

[6] T. J. Jech, *The Axiom of Choice*, Dover Publ., Mineola, NY, 2008.

[7] J. R. Munkres, *Topology*, 2nd ed., Prentice-Hall, Upper Saddle River, NJ, 2000.
[8] A. Paterson, *Groupoids, Inverse Semigroups, and Their Operator Algebras*, Progr. Math. 170, Birkhäuser Boston, Boston, MA, 1999.
[9] J. Renault, *Cartan subalgebras in C^*-algebras*, Irish Math. Soc. Bull. 61 (2008), 29–63.

Jonathan Brown
Mathematics Department
Kansas State University
138 Cardwell Hall
Manhattan, KS 66506-2602, U.S.A.
E-mail: brownjh@math.kansas.edu

Lisa Orloff Clark
Department of Mathematics and Statistics
University of Otago
P.O. Box 56, Dunedin 9054, New Zealand
E-mail: lclark@maths.otago.ac.nz

Received 2 December 2012;
in revised form 8 January 2014