FUNDAMENTA
MATHEMATICAE
226 (2014)

On generalized shift graphs

by

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Abstract. In 1968 Erdős and Hajnal introduced shift graphs as graphs whose vertices are the k-element subsets of $[n] = \{1, \ldots, n\}$ (or of an infinite cardinal κ) and with two k-sets $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_k\}$ joined if $a_1 < a_2 = b_1 < a_3 = b_2 < \cdots < a_k = b_{k-1} < b_k$. They determined the chromatic number of these graphs. In this paper we extend this definition and study the chromatic number of graphs defined similarly for other types of mutual position with respect to the underlying ordering. As a consequence of our result, we show the existence of a graph with interesting disparity of chromatic behavior of finite and infinite subgraphs. For any cardinal κ and integer l, there exists a graph G with $|V(G)| = \chi(G) = \kappa$ but such that, for any finite subgraph $F \subset G$, $\chi(F) \le \log_{(l)} |V(F|$, where $\log_{(l)}$ is the l-iterated logarithm. This answers a question raised by Erdős, Hajnal and Shelah.

1. Introduction. In this paper we study the chromatic number of graphs whose vertices are subsets of $[n] = \{1, \ldots, n\}$ and whose edges are induced by patterns of mutual positions of subsets.

The chromatic number of such graphs has been investigated in a number of papers. For example, Erdős and Rado [7] considered graphs whose vertices are triples such that $\{x_1, x_2, x_3\}$, $\{y_1, y_2, y_3\}$ form an edge if $x_1 < x_2 < y_1 < x_3 < y_2 < y_3$. Other well known graphs considered in [4] are shift graphs whose vertices are pairs of integers, and $\{x_1, x_2\}$, $\{y_1, y_2\}$ are adjacent if $x_1 < x_2 = y_1 < y_2$. Both these graphs provide examples of triangle free graphs with large chromatic number. Here we study the chromatic number of graphs defined in a similar way.

Definition 1.1. For a fixed k and ℓ, $k < \ell$, we call a sequence $\tau = (\tau_i)_{i=1}^{\ell}$, $\tau_i \in \{1, 2, 3\}$, a *type* of *width* k and *length* ℓ if

$$|\{i : \tau_i \in \{1, 2\}\}| = |\{i : \tau_i \in \{1, 3\}\}| = k\,.$$

2010 *Mathematics Subject Classification*: Primary 05C15; Secondary 05C63.
Key words and phrases: shift graphs, chromatic number, infinite graphs, odd girth.

DOI: 10.4064/fm226-2-6

Definition 1.2. Let $k < n$ be integers and $X, Y \in \binom{[n]}{k}$ be subsets of size k of $[n]$. We say that the *type* of a pair X, Y is $\tau = (\tau_i)_{i=1}^{\ell}$, and write $\tau(X,Y) = \tau$, if $|X \cup Y| = \ell$ and, for $X \cup Y = \{z_1, \ldots, z_\ell\}_<$, we have $z_i \in X \setminus Y$ for $\tau_i = 1$, $z_i \in Y \setminus X$ for $\tau_i = 2$, and $z_i \in X \cap Y$ when $\tau_i = 3$.

Definition 1.3. For a type τ of width k and $n > k$, the *type graph* $G(n, \tau)$ is the graph with vertex set $\binom{[n]}{k}$ in which two vertices X, Y are adjacent if $\tau(X, Y) = \tau$.

In this paper we are interested in the chromatic number of type graphs. The definition of $G(n, \tau)$ can be extended by replacing n with any totally ordered set. In fact, the first results in this direction concerned infinite cardinals. For example, Erdős and Rado [7] proved that for any infinite cardinal κ and the type $\delta_3 = 112122$, the chromatic number satisfies $\chi(G(\kappa, \delta_3)) = \kappa$. Erdős and Hajnal [3] showed that for any infinite cardinal κ and $\sigma_2 = 132$, we have $\chi(G(\kappa, \sigma)) = \min\{\alpha : \exp(\alpha) \geq \kappa\}$. They also proved a generalization of this result: for $\sigma_k = 13\ldots32$, of length $l = k+1$, we have $\chi(G(\kappa, \sigma_k)) = \min\{\alpha : \exp_{(k-1)}(\alpha) \geq \kappa\}$, where $\exp_{(k-1)}(\alpha) = \exp\ldots\exp(\alpha)$, taken $k-1$ times. Their proof extends to finite graphs and yields $\chi(G(n, \sigma_k)) = (1+o(1)) \log_{(k-1)} n$, where $\log_{(k-1)}$ is the $(k-1)$-iterated binary logarithm. It is easy to check that $G(n, \sigma_k)$ contains no odd cycles shorter than $2k+1$. Consequently, the $G(n, \sigma_k)$ provide examples of graphs with arbitrarily large odd girth and chromatic number.

While the chromatic number of graphs defined by σ_k behaves similarly in the finite and infinite case (in the sense that the chromatic number is the $k-1$ times iterated logarithm of the size of the graph), the chromatic behavior of the graphs $G(n, \delta_3)$ and $G(\kappa, \delta_3)$ is quite different. Indeed, while the result of Erdős and Rado [7] states that the chromatic number and the cardinality of the graph $G(\kappa, \delta_3)$ are the same, i.e. $\chi(G(\kappa, \delta_3)) = \kappa$, the chromatic number of finite subgraphs of $G(\kappa, \delta_3)$ grows much slower, i.e. $\chi(G(n, \delta_3)) \leq 2 \log n$ (see Lemma 3.8 below). Our result is much more general. Let $\delta_k = 1121\ldots122$ be the type of length $2k$ consisting of 1 followed by $k-1$ copies of 12 and ending with the last 2. We show that while $\chi(G(\kappa, \delta_k)) = \kappa$ for any infinite cardinal κ, the chromatic number of its finite subgraphs grows only as an iterated logarithm. This follows from Theorems 1.5 and 2.1 below.

Theorem 1.4. *If τ is a type of width k which contains no threes, then*

$$\chi(G(n, \tau)) \geq (1 + o(1)) \log_{(\lfloor (k-1)/2 \rfloor)} n.$$

Theorem 1.5. *For the canonical type δ_k of width k we have*

$$\chi(G(n, \delta_k)) \leq b_k (\log_{(\lfloor (k-1)/2 \rfloor)} n)^{\binom{\lfloor (k+1)/2 \rfloor}{2}}$$

for some $b_k > 0$.

We prove Theorem 1.5 in the following form, presenting a slight improvement for even k.

THEOREM 1.6. *Let* $k \geq 3$.

(i) *If* $k = 2l+1$ *is odd, then* $\chi(G(n,\delta_{2l+1})) \leq b_{2l+1}(\log_{(l)} n)^{\binom{l+1}{2}}$ *for some* $b_k > 0$.

(ii) *If* k *is even, then* $\chi(G(n,\delta_k)) \leq \chi(G(n,\delta_{k-1}))$.

In the last section of the paper, we extend both results above to a larger class of types.

REMARK 1. It is easy to see that $G(n,\delta_k)$ contains no odd cycle shorter than $2\lceil k/2 \rceil$. Thus Theorem 1.4 gives an example of graphs with large odd girth and large chromatic number.

REMARK 2. For an infinite graph G with $\chi(G) = \kappa$ and κ infinite, let $f_G(n)$ be the maximum chromatic number of its n-vertex subgraph. The function f_G is clearly non-decreasing, and a result of de Bruijn and Erdős [1] implies that $\lim_n f_G(n) = \infty$. Erdős and Hajnal established in [3] that the graphs $S_k = G(\kappa,\sigma_k)$ for $\kappa \geq \exp_{(k-1)}(\lambda)$ are examples of graphs with $\chi(S_k) \geq \lambda$ and $f_{S_k}(n) \leq \log_{(k-1)} n$. In other words, the chromatic number of infinite graphs as well as of their finite subgraphs is slowly growing with their size.

As a corollary of Theorems 1.5 and 2.1, we observe in Theorem 2.3 that the graphs $D_k = G(\kappa,\delta_k)$ provide examples of graphs with chromatic number largest possible while the chromatic number of their finite subgraphs is very slowly growing with their size. More precisely: $\chi(D_k) = |V(D_k)| = \kappa$ and $f_{D_k}(n) \leq b_k(\log_{(\lfloor (k-1)/2 \rfloor)} n)^{\binom{\lfloor (k+1)/2 \rfloor}{2}}$ for some constant b_k.

Komjáth and Shelah proved in [8] the following related result: given an arbitrarily slowly growing function $\varphi : \mathbb{N} \to \mathbb{N}$, it is consistent that there exists a graph $G_\varphi(\kappa)$ with $|V(G_\varphi(k))| = \chi(G_\varphi(\kappa)) = \kappa$, while $\chi(G') \leq \varphi(n)$ for any subgraph $G' \subset G$ with n vertices. The graphs D_k are examples of graphs with $\chi(D_k) = |V(D_k)|$ and $f_{G_\varphi(\kappa)}(n) \leq \varphi(n)$ for $\varphi(n) = b_k(\log_{(\lfloor (k-1)/2 \rfloor)} n)^{\binom{\lfloor (k+1)/2 \rfloor}{2}}$. It would be interesting to find explicit examples of such graphs for even slower growing functions $\varphi(n)$.

2. Infinite graphs.

A type consisting only of 1's and 2's is called a *disjoint type*. In the infinite case, the behavior of the chromatic number of graphs defined using disjoint types is simple. A similar result was proved in [3, Theorem 7.4].

THEOREM 2.1. *If* τ *is any disjoint type and* κ *is any infinite cardinal, then*

$$\chi(G(\kappa,\tau)) = \kappa.$$

Note that it is sufficient to prove the theorem for κ regular. Indeed, if κ is a singular cardinal then

$$\kappa = \sup\{\mu^+ : \mu < \kappa\},$$

and assuming Theorem 2.1 for regular cardinals we obtain

$$\chi(G(\kappa,\tau)) \geq \sup_{\mu<\kappa} \chi(G(\mu^+,\tau)) = \kappa.$$

For an infinite cardinal κ, a subset $S \subset \kappa$ is said to be *cofinal* if it is unbounded in κ. We extend this notion to families of k-tuples the following way. For $k = 1$, a family S of singletons of κ is *cofinal* if $\bigcup S$ is unbounded in κ. For $k > 1$, we say that a family of k-tuples of κ is *cofinal* if it satisfies the following two conditions:

- the set $S_1 = \{a_1 : \exists\{a_1,\ldots,a_k\} \in S\}$ is cofinal in κ;
- for every $a_1 \in S_1$, the set $\{\{x_2,\ldots,x_k\} : \{a_1,x_2,x_3,\ldots,x_k\} \in S\}$ is a cofinal family of $(k-1)$-tuples.

LEMMA 2.2. *Let κ be an infinite regular cardinal. For any coloring $\chi : [\kappa]^k \to \gamma$ of the k-tuples of κ using $\gamma < \kappa$ colors, there exists a monochromatic cofinal family of k-tuples.*

Proof. We use induction on k. For $k = 1$, this is clear since $\gamma < \kappa$. We assume the statement is true for $k \geq 1$ and show it for $k+1$. For every $\alpha \in \kappa$ define $f_\alpha : [\kappa]^k \to \gamma$ by setting $f_\alpha(\beta_1,\ldots,\beta_k) = f(\alpha,\beta_1,\ldots,\beta_k)$. By the induction hypothesis, there exists a monochromatic cofinal family S_α of k-tuples for the coloring f_α. Let $c_\alpha \in \gamma$ be the color of the elements of S_α. Since $\gamma < \kappa$, there exists $A \subset \kappa$, $|A| = \kappa$, such that c_α is the same for each $\alpha \in A$. Since $|A| = \kappa$, it is cofinal in κ, and the family

$$\{(\alpha,a_2,\ldots,a_k) : \alpha \in A \text{ and } (a_2,\ldots,a_k) \in S_\alpha\}$$

is a cofinal family of $(k+1)$-tuples. ∎

Theorem 2.1 for a regular cardinal κ is an immediate consequence of the above lemma. Rather than giving a somewhat tedious and technical proof of the statement, we illustrate this with an example for δ_3. The general case follows from a similar argument. Let κ be an infinite cardinal and let $\chi : [\kappa]^3 \to \gamma$ be a coloring for some $\gamma < \kappa$. According to Lemma 2.2, there exists a monochromatic cofinal family S of 3-tuples for the coloring χ. For every $a_1 \in S_1$, we define $S_2^{(a_1)} = \{a_2 : \exists x_3 \text{ with } \{a_1,a_2,x_3\} \in S\}$. For any $a_1 \in S_1$ and $a_2 \in S_2^{(a_1)}$ we define $S_3^{(a_1,a_2)} = \{a_3 : \{a_1,a_2,a_3\} \in S\}$.

Note that because S is cofinal, all three sets S_1, $S_2^{(a_1)}$ and $S_3^{(a_1,a_2)}$ are unbounded. Choose $a_1 \in S_1$ and $a_2 \in S_2^{(a_1)}$ such that $a_2 > a_1$. Then select $b_2 \in S_1$ such that $b_2 > a_2$, and then $a_3 \in S_3^{(a_1,a_2)}$ satisfying $a_3 > b_2$. Finally, using again the cofinality of the family S, choose $b_2 \in S_2^{(b_1)}$ and

$b_3 \in S_3^{(b_1,b_2)}$, $b_3 > b_2 > a_3$. The resulting triples $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ satisfy $a_1 < a_2 < b_1 < a_3 < b_2 < b_3$ and since they both belong to S they are of the same color. Consequently, χ is not a proper coloring and $\chi(\delta_3, \kappa) = \kappa$.

The following theorem is a consequence of Theorems 1.4, 1.5 and 2.1. It asserts that while the chromatic number of the graph $G_{\delta_k,\kappa}$ is the same as its vertex size κ, a very different behavior is demonstrated by its finite subgraphs, the chromatic number of which grows only very slowly with their size.

THEOREM 2.3. *For any integer k and infinite cardinal κ, the graph $D_k = G(\kappa, \delta_k)$ has the following properties:*

(i) $\chi(D_k) = |V(D_k)| = \kappa$.
(ii) *For any finite subgraph $G \subset D_k$, there exists a constant a_k such that* $\chi(G) \le a_k(\log_{(\lfloor (k-1)2 \rfloor)} |V(D_k)|)^{3^k}$.

Proof. Each element of $V(G)$ is a k-set so that $n = \bigcup V(G) \le k|V(G)|$. By Theorem 1.5, there exists a constant $b_k > 0$ such that

$$\begin{aligned}\chi(G) &\le b_k(\log_{(\lfloor (k-1)/2 \rfloor)} n)^{\binom{\lfloor (k+1)/2 \rfloor}{2}} \le b_k(\log_{(\lfloor (k-1)/2 \rfloor)} k|V(G)|)^{\binom{\lfloor (k+1)/2 \rfloor}{2}} \\ &\le a_k(\log_{\lfloor (k-1)/2 \rfloor} |V(G)|)^{\binom{\lfloor (k+1)/2 \rfloor}{2}}\end{aligned}$$

for an appropriately chosen $a_k > b_k$. ■

We remark that if S is a totally ordered set with $|S| = \kappa$, then the chromatic number of a naturally defined $G(S, \tau)$ does not have to satisfy $\chi(G(\kappa, \tau)) = \chi(G(S, \tau))$.

Indeed, for an infinite cardinal α let us consider the set S of functions $f : \alpha \to \{0, 1\}$ with the lexicographical order. Let us color a triple $(f_1 < f_2 < f_3)$ by a pair (a, ℓ), where a is the minimum element of α for which the three values $f_1(a), f_2(a), f_3(a)$ are not all the same (i.e. $f_1(a) = 0$ while $f_3(a) = 1$), and $\ell = f_2(a)$. It is easy to see that it is a proper coloring of $G(S, \delta_3)$ and so $\chi(G(S, \delta_3)) \le \alpha$ while Theorem 2.1 states that

$$\chi(G(|S|, \delta_3)) = 2^\alpha > \chi(G(S, \delta_3)).$$

On the other hand, assuming GCH, any totally ordered set of size κ^+ contains a subset of size κ which is either well ordered or its reverse is well ordered. Consequently, $\chi(G(S, \tau)) \ge \kappa$ for any disjoint type τ and any totally ordered set S with $|S| = \kappa^+$.

3. Proof of Theorem 1.4. We say that the disjoint type τ is *simple* if when reading it from left to right it is possible to group it into disjoint blocks of the form $1 \ldots 12 \ldots 2$ or $2 \ldots 21 \ldots 1$ (each with the same number of 1's and 2's). Here is a precise definition of a simple type.

Definition 3.1. A disjoint type $\tau = (\tau_1, \ldots, \tau_l)$ is *simple* if there exists a partition of $[1, l]$ into intervals $I_1, \ldots, I_m$, each of even length, $|I_j| = 2l_j$, $j = 1, \ldots, m$ (with m possibly equal to 1), and so that for each $j = 1, \ldots, m$, $(\tau_i)_{i \in I_j}$ is a sequence of l_j 1's followed by l_j 2's or vice versa, i.e. $(\tau_i)_{i \in I_j}$ is either $(1, \ldots, 1, 2, \ldots, 2)$ or $(2, \ldots, 2, 1, \ldots, 1)$.

Example. The type 112221222111 is simple with $m = 3$ and $l_1 = 2$, $l_2 = 1$ and $l_3 = 3$, since it can be split into the following three groups:

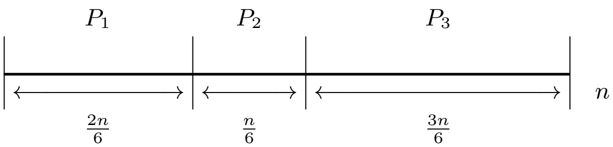
$$1122 \quad 21 \quad 222111.$$

Proposition 3.2. *If τ is a simple type of width k, then $\chi(G(n, \tau)) = \lfloor n/k \rfloor$.*

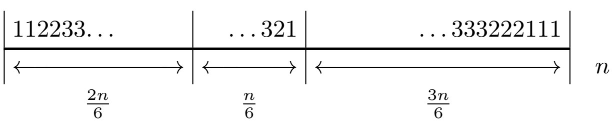
Proof. Fix a simple type τ of width k and let $X, Y \in [n]^k$. First observe that if $\tau(X, Y) = \tau(Y, Z) = \tau$ then also $\tau(X, Z) = \tau$. Consequently, the relation $\prec$ on $\binom{[n]}{k}$, defined by $X \prec Y$ if and only if $\tau(X, Y) = \tau$, is a partial order. Then $G(n, \tau)$ is the comparability graph of the partial order $\prec$, which implies that it is perfect (see Dilworth [2]). Thus, in order to determine $\chi(G(n, \tau))$, it is sufficient to find the largest clique.

We will show that the largest clique in $G(n, \tau)$ is of size $\lfloor n/k \rfloor$. Since the vertices of a clique in $G(n, \tau)$ are disjoint k-tuples, there are no cliques of size larger than $\lfloor n/k \rfloor$. Consequently, we just need to show the existence of a clique of size $\lfloor n/k \rfloor$. Since the size of the largest clique can only decrease in a subgraph, we can assume that k divides n.

To avoid tedious notation, instead of giving a formal proof of a general case we outline it for a special case. The argument can be mimicked for all other simple types. We will work with the simple type $\tau = 1122\ 21\ 222111$ of width $k = 6$. Consider the following partition of n into three parts:



The partition classes correspond respectively to the blocks 1122, 21 and 222111 of the simple type τ and are of sizes proportional to the length of each block. In each block we embed a monotone sequence of numbers as shown in the following picture:



It is easy to see that the 6-tuples of vertices with the same label form a clique of size $n/6$. ∎

Before giving a proof to Theorem 1.4 we introduce some definitions and prove a few auxiliary results.

Definition 3.3. Given two types $\tau = \tau_1 \dots \tau_l$ and $\tau' = \tau'_1 \dots \tau'_{l'}$, the concatenation $\tau_1 \dots \tau_l \tau'_1 \dots \tau'_{l'}$ will be denoted by $\tau\tau'$.

Definition 3.4. A type which cannot be written as a concatenation of two types is called *irreducible.*

In other words, irreducible types starting with 1 are those types $\tau_1 \dots \tau_l$ such that for any $j < l$ the number of 1's exceeds the numbers of 2's in $\{\tau_1 \dots \tau_j\}$. For example, 112122 is irreducible while 11212212 is not since it is the concatenation of 112122 and 12. Note that if in a type τ' we switch the occurrences of 1's and 2's we obtain a type τ'' with the property that $G(n, \tau') \simeq G(n, \tau'')$. Since it may be convenient to have a 1-1 relationship between graphs and types, we will usually assume that each type begins with 1 (or possibly 3). However we cannot make this assumption in the above definition since the graphs $G(n, \tau\tau')$ and $G(n, \tau\tau'')$ do not have to be isomorphic. For example, setting $\tau = \tau' = 12$ and hence $\tau'' = 21$ yields the graphs $G(n, 1212)$ and $G(n, 1221)$ which are not isomorphic (the latter has vertices of higher degree than any vertex in the former).

Definition 3.5. Given a disjoint type $\tau = \tau_1 \dots \tau_{2k}$, let $t_1^{(1)} < \dots < t_k^{(1)}$ be the indices of the 1's, i.e.

$$\tau_{t_i^{(1)}} = 1, \quad 1 \le i \le k.$$

Similarly let $t_1^{(2)} < \dots < t_k^{(2)}$ be the indices of the 2's. We also set $T_1 = T_1(\tau) = \{t_1^{(1)}, \dots, t_k^{(1)}\}$ and $T_2 = T_2(\tau) = \{t_1^{(2)}, \dots, t_k^{(2)}\}$.

Definition 3.6. Let τ be a disjoint type of width k and $S \subseteq \{1, \dots, k\}$. Then τ^S is the type obtained by deleting $\tau_{t_i^{(1)}}$ and $\tau_{t_i^{(2)}}$ from τ for every $i \notin S$. Any type of this kind is called a *subtype* of τ.

In other words, if τ describes the position of the sets $A = \{a_1, \dots, a_k\}_<$ and $B = \{b_1, \dots, b_k\}_<$, then τ^S describes the position of the sets $A' = \{a_i : i \in S\}$ and $B' = \{b_i : i \in S\}$. For example, if $\tau = 12211122$, then $\tau^{\{2,3\}} = 2112$ and $\tau^{\{1,3,4\}} = 121122$.

Lemma 3.7. *If ρ is a subtype of τ, then for every integer n,*

$$\chi(G(n, \tau)) \le \chi(G(n, \rho)).$$

Proof. Fix a type τ of width k and let $\rho = \tau^S$ for some proper $S \subseteq \{1, \dots, k\}$. Assume moreover that we are given a proper coloring $\varphi : V(G(n, \rho)) \to [c]$ of $G(n, \rho)$. For every vertex $v = \{v_1, \dots, v_k\}_<$ of $G(n, \tau)$, let $v^S = \{v_i : i \in S\}$. The mapping $f : V(G(n, \tau)) \to V(G(n, \rho))$ defined by $f(v) = v^S$ is a homomorphism. Indeed we can easily check that if the type $\tau(v,w)$ is τ, then $\rho = \tau(v^S, w^S)$. Consequently, the mapping $\varphi \circ f : V(G(n,\tau)) \to [c]$ defined by $\varphi \circ f(v) = \varphi(f(v))$ is a proper coloring. ∎

LEMMA 3.8. *For $n \geq 6$, $\chi(G(n,\delta_3)) \leq 2\log_2 n$.*

Proof. Since $\chi(G(6,\delta_3)) = 2$, the statement is true for $n = 6$. We use induction for $n > 6$. Suppose the statement is true for the integer n and that χ is a proper coloring of $G(\delta_3, n)$ using $2\log_2 n$ colors. We give a proper coloring $\bar{\chi}$ of $G(2n,\delta_3)$ using two additional colors c_1 and c_2. Setting $A = [1,n]$ and $B = [n+1, 2n]$ we color the triples of the form (a,a,a) using the coloring χ and similarly the triples of the form (b,b,b). We color all triples of the form (a,a,b) using the new color c_1 and all the triples of the form (a,b,b) using the new color c_2. Observe that the type of any two (a,a,b) triples is never δ_3 and that the same holds for any (a,b,b) triples. Consequently, the new coloring is a proper coloring as well.

We have just showed that if $2^k 3 < n \leq 2^{k+1}3$, then $\chi(G(n,\delta_k)) = 2k+2$. Consequently, $\chi(G(n,\delta_k)) \leq 2\lceil \log_2 n/3 \rceil \leq 2\log_2 n$. ∎

Combined with Proposition 3.2, the next proposition shows that for a disjoint type τ, the chromatic number either grows linearly with n, or is bounded by a logarithm.

PROPOSITION 3.9. *A disjoint type τ is either simple, or $\chi(G(n,\tau)) \leq 2\log_2 n$.*

Proof. Let τ be a non-simple type. We can write $\tau = l\tau'$ where l is a simple type of maximal length (possibly empty, in which case $\tau = \tau'$). Assuming that τ' begins with 1, τ' must be of the following form:

$$1\ldots 12\ldots 21\ldots$$

where the initial string of 1's is of length a and the following string of 2's is of length b. Since l is simple of maximum length, we have $b < a$. Let $S = \{1, a, a+1\}$. Since $a < b$, the following represents the type τ' where the 1's and 2's of indices in S are respectively boxed and circled:

$$\boxed{1}\ldots\boxed{1}2\ldots 2\boxed{1}\ldots 2\ldots 2\ldots$$

From the above picture, it is clear that the subtype τ'^S is 112122, and by Lemma 3.7, $\chi(G(\tau',n)) \leq \chi(G(112122,n)) \leq 2\log_2 n$, the last equality following from Lemma 3.8. ∎

The next proposition is a strengthening of Theorem 1.4 for irreducible disjoint types.

PROPOSITION 3.10. *For each irreducible disjoint type τ of width k there exists a positive integer $m(\tau) = m \leq (k+1)/2$ such that*

$$\chi(G(n,\tau)) \geq \log_{(m-1)} \frac{n}{(2k)^m}.$$

Proof. Let us assume that the type τ begins with 1. We define for some integer $m = m(\tau)$ a partition $P_1 \cup \cdots \cup P_{m+1} = [1, 2k]$ into sets of consecutive integers as follows. Let $P_1 = [1, p_1]$, where p_1 is the size of the maximal consecutive block of 1's. In other words, p_1 is the largest integer satisfying $\tau_1 = \tau_2 = \cdots = \tau_{p_1} = 1$. Set P_2 to be the largest consecutive set of indices following P_1 and with as many 2's in as there are 1's in P_1. In other words, $P_2 = [p_1 + 1, p_2]$, where p_2 is the largest integer such that $|\{\tau_{p_1+1}, \tau_{p_1+2}, \ldots, \tau_{p_2}\} \cap T_2| = |P_1|$. We then define $P_3 = [p_2 + 1, p_3]$ similarly: p_3 is the largest integer such that $|\{\tau_{p_2+1}, \tau_{p_2+2}, \ldots, \tau_{p_3}\} \cap T_2| = |P_2|$. And by induction $P_{m+1} = [p_m + 1, p_{m+1}]$ with p_{m+1} the largest integer such that $|\{\tau_{p_m+1}, \tau_m + 2, \ldots, \tau_{p_{m+1}}\} \cap T_2| = |P_m|$. For example, for the type 1121112121212222 the partition looks as follows:

$$\begin{array}{cccc} P_1 & P_2 & P_3 & P_4 \\ \longleftrightarrow & \longleftrightarrow & \longleftrightarrow & \longleftrightarrow \\ 11 & 211121 & 212122 & 22 \end{array}$$

Note that by construction $|P_i \cap T_1| = |P_{i+1} \cap T_2|$, $1 \le i \le m$, and that due to the maximality of p_i, each P_i for $i \ge 2$ begins with a 2.

Claim 3.11. *For every* $1 \le i \le m - 1$, $|P_i \cap T_1| \ge 2$.

Proof. If not, let $1 \le i_0 \le m - 1$ be the smallest integer such that $|P_{i_0} \cap T_1| = 1$. Note that by construction,

$$\begin{aligned} |(P_1 \cup P_2 \cup \cdots \cup P_{i_0}) \cap T_1| &= |(P_2 \cup P_3 \cup \cdots \cup P_{i_0+1}) \cap T_2| \\ &= |(P_2 \cup P_3 \cup \cdots \cup P_{i_0}) \cap T_2| + 1. \end{aligned}$$

In other words, the number of 1's in the first $p_1 + \cdots + p_{i_0}$ entries is exactly one more than the number of 2's. As noted earlier, each P_i for $i \ge 2$ begins with a 2, i.e. $\tau_{1+p_1+\cdots+p_{i_0}} = 2$. Consequently, the number of 1's in the first $1 + p_1 + p_2 + \cdots + p_{i_0}$ entries is the same as the number of 2's, contradicting the assumption that τ is irreducible. ∎

By the claim above and the fact that $|P_i \cap T_1| = |P_{i+1} \cap T_2|$ for $1 \le i \le m$, we conclude that $|P_i \cap T_2| \ge 2$ for $2 \le i \le m$, and consequently P_1 contains at least two elements, P_2 through P_{m-1} at least four elements, P_m (which contains at least one 1) at least three and P_{m+1} at least one. In particular, we have $2 + 4(m - 2) + 3 + 1 \le 2k$, where k is the length of the type τ, i.e.

$$m \le \frac{k+1}{2}. \tag{1}$$

Remark 3.12. In (1) above, equality can be achieved in the case P_1 contains exactly two elements, P_2 through P_{m-1} exactly four elements, P_m exactly three and P_{m+1} exactly one.

Set $r = (2k)^m$ (which, as we will see shortly, is a bit wasteful). We are going to find an embedding $\varphi : G(n, \sigma_m) \to G(rn, \tau)$ which will establish that $\chi(G(nr,\tau)) \geq \chi(G(n,\sigma_m))$. To this end we divide $[rn]$ into n consecutive intervals $A_1,\ldots,A_n$, $|A_i| = r$ for all $i = 1,\ldots,n$, and for each m-tuple $A_{i_1},\ldots,A_{i_m}$ define $B_{i_j} \subset A_{i_j}$, $j = 1,\ldots,m$, with $|\bigcup_{j=1}^m B_{i_j}| = k$ so that $\varphi : \{i_1,\ldots,i_m\} \to \bigcup_{j=1}^m B_{i_j}$ gives the desired embedding.

Although the idea behind defining the B_{i_j}'s is simple, the formal description will require some more definitions. We will find it convenient to imagine a linear order on each of the intervals A_i, $i = 1,\ldots,n$, as a lexicographic order of m-tuples over the alphabet $\{0,1,\ldots,2k-1\}$. (Recall $r = (2k)^m$.) To this end we introduce $Q(2k,m)$ to be the set of lexicographically ordered m-tuples over $\{0,1,\ldots,2k-1\}$ and let $Q^j(2k,m) \subset Q(2k,m)$ be the set of those m-tuples that have zeroes on entries $j+1, j+2,\ldots,m$.

We define m subsets $Q_1,\ldots,Q_m$ of $Q(2k,m)$ that will be used in the definition of the B_{i_j}'s, $j = 1,\ldots,m$. Recalling the partition $P_1 \cup \cdots \cup P_{m+1} = [2k]$ considered above let

$$Q_1 = \{(1,0,\ldots,0),(2,0,\ldots,0),\ldots,(|P_1|,0,\ldots,0)\} \subset Q^1(2k,m).$$

Suppose that $Q_j \subset Q^j(2k,m)$ was already defined and that $|Q_j| = |P_j \cap T_1|$. In order to define Q_{j+1} we first identify the 2's in $P_{j+1} \cap T_2$ with the elements of Q_j (observe that this is possible in view of the fact that $|Q_j| = |P_j \cap T_1| = |P_{j+1} \cap T_2|$). Let $Q_{j+1} \subset Q^{j+1}(2k,m+1)$ be a set of vectors whose order type with respect to Q_j (by identifying Q_{j+1} with the 1's and Q_j with the 2's) will be identical to the order type of the 2's and 1's in P_{j+1}. Observe that this set exists due to the fact that between any two elements of Q_j there are enough (precisely $2k$) elements of $Q^{j+1}(2k,m+1)$.

Having defined the sets $Q_1,\ldots,Q_m$ we can now establish the promised embedding $\{i_1,\ldots,i_m\} \to \bigcup_{j=1}^m B_{i_j}$. Recall that each of the intervals $A_{i_1},\ldots,A_{i_m}$ has length $r = (2k)^m$, therefore its elements can be identified with the elements of $Q(2k,m)$ respecting the order of each of the intervals and lexicographic order of $Q(2k,m)$. For $j = 1,\ldots,m$, let $B_{i_j} \subset A_{i_j}$ be the sets of precisely those elements which correspond to Q_j. With this definition, one can verify that $\{i_1,\ldots,i_m\} \to \bigcup_{j=1}^m B_{i_j}$ is an embedding as required. ■

Proof of Theorem 1.4. In the proof above, for an irreducible type τ we construct an embedding of $G(n(2k)^{-m},\sigma_{m(\tau)})$ into $G(n,\tau)$, thus showing a lower bound on the chromatic number of $G(n,\tau)$. Next we consider the simplest reducible disjoint case, i.e. $\tau = \tau_1\tau_2$ is a type of width k with τ_1 and τ_2 irreducible. Let k_1 and k_2 ($k_1 + k_2 = k$) be the width of τ_1 and τ_2 respectively. Set $c_1 = (2k_1)^{-m_1}$, $c_2 = (2k_2)^{-m_2}$ and $n_1 = c_1 n$, $n_2 = c_2 n$. Consider the graphs $G(n,\tau_1)$, $G(n,\tau_2)$ and let $G(n,\tau_1) \times G(n,\tau_2)$ be a graph formed by k-tuples K of $[1,2n]$ with $K \cap [n] = k_1$ and such that K, K' form an edge in the product graph if and only if both $(K \cap [n], K' \cap [n]) \in G(n,\tau_1)$ and $(K - n \cap [n], K' - n \cap [n]) \in G(n,\tau_2)$ (here $K - n = \{x, x+n \in K\}$).

Clearly there is an embedding $G(n,\tau_1)\times G(n,\tau_2)\to G(2n,\tau)$. Since τ_1,τ_2 are irreducible it follows that there is an embedding

$$G(n_1,\sigma_{m_1})\times G(n_2,\sigma_{m_2})\to G(n,\tau_1)\times G(n,\tau_2). \tag{2}$$

Without loss of generality, we assume now that $m_1\ge m_2$. We will show that there is an embedding

$$G(n_1,\sigma_{m_1})\to G(n_1,\sigma_{m_1})\times G(n_2,\sigma_{m_2}). \tag{3}$$

In order to observe (3), to each m_1-tuple $i_1,\dots,i_{m_1}$ assign the m_1+m_2-tuple

$$i_1,i_2,\dots,i_{m_1},i_1+n,i_2+n,\dots,i_{m_2}+n.$$

Indeed, if $\{i_1,\dots,i_{m_1}\}$ and $\{j_1,\dots,j_{m_1}\}$ are joined in $G(n_1,\sigma_{m_1})$, then $i_2=j_1,\dots,i_{m_1}=j_{m_1-1}$ and hence $i_1,\dots,i_{m_2}$ and $j_1,\dots,j_{m_2}$ are joined in $G(n_2,\sigma_{m_2})$, establishing the required embedding.

By (2) and (3),

$$\chi(G(2n,\tau))\ge\chi(G(n_1,\sigma_{m_1})).$$

Since $m_1-1\le(k_1-1)/2\le(k-1)/2$, Proposition 3.10 implies that

$$\chi(G(2n,\tau))\ge\log_{(\lfloor(k-1)/2\rfloor)}n_1.$$

Thus, since $k\ge3$ and $n_1=c_1n$ for large n we have

$$\chi(G(2n,\tau))\ge(1+o(1))\log_{(\lfloor(k-1)/2\rfloor)}n=(1+o(1))\log_{(\lfloor(k-1)/2\rfloor)}2n.$$

Following this, a similar reasoning to the case $t=2$ shows that

$$\chi(G(n,\tau))\ge(1+o(1))\log_{(m-1)}\frac{n}{(2k)^m}. \tag{4}$$

Finally, note that by Remark 3.12, if τ is any irreducible type of length k then $m(\tau)\le m(\delta_k)$. Similarly, in order to achieve the lowest lower bound in (4), $m=\max\{m(\tau_i):1\le i\le t\}$ has to be largest. Since $m(\tau_i)$ is at most $m(\delta_{k_i})$, it is easily seen that m is largest when $t=1$ and $\tau=\delta_k$, proving

$$\chi(G(n,\tau))\ge(1+o(1))\log_{(\lfloor(k-1)/2\rfloor)}n. \blacksquare$$

4. Proof of Theorem 1.5. This proof is the most involved part of the paper. We begin by defining two modifications of a type τ which will play an important role in this section: overlapping and reduction.

DEFINITION 4.1. A 12 [21] *overlap* of a type τ is a type τ' which is obtained from τ by replacing a pair of consecutive 1 and 2 [2 and 1] by a single 3. We say that a type τ' is an *overlap* of τ if it is obtained from τ by a sequence (possibly empty) of 12 and 21 overlaps.

For example, the overlaps of the type 112122 are: 112122, 13122, 11322, 11232 and 1332. Observe that if τ is a type of width k, then any of its overlaps τ' has width k as well.

Definition 4.2. Let $1 \le i_1 < \cdots < i_m \le k-1$ be natural numbers such that $i_r \le i_{r+1} - 2$ for $r = 1, \ldots, m$ and let τ be a type obtained from δ_k by 'merging' the i_rth and (i_r+1)th ones and the i_rth and (i_r+1)th twos. More precisely, we replace any sequences 11 and 22 of merged ones and twos by 1 and 2 respectively. Furthermore, a sequence 121 where the ones are merged (but the two is not) is replaced by 3. Similarly a sequence 212 where the twos are merged (but the one is not) is replaced by 3. Finally, sequences of the form 1212 or 2121 for which both pairs of ones and twos are merged are each replaced by a 3. We call each type τ obtained in this way a *(k, m)-reduct of index $(i_1, \ldots, i_m)$*.

Thus, for instance, if $k = 10$ and $i_1 = 2$, $i_2 = 4$, and $i_3 = 9$, then $\tau = 1333121232$. Note that a (k, m)-reduct has width $k - m$, and a $(k, 0)$-reduct is δ_k. Our first result states that the chromatic number of a type obtained after performing an overlap, or a reduction, on a type τ is not significantly larger than the chromatic number of τ itself.

Lemma 4.3.

(a) *If τ is an irreducible type of width k, and $\widetilde{\tau}$ is obtained from τ by a single overlap, then*

$$G(n, \widetilde{\tau}) \subseteq G(2n, \tau).$$

(b) *If $\widetilde{\tau}$ is a (k, m)-reduct, then*

$$G(n, \widetilde{\tau}) \subseteq G(4n, \delta_k).$$

Before giving the proof of the lemma above, we extend Definition 3.5 to non-disjoint types in the following way.

Definition 4.4. Given a type $\tau = \tau_1 \ldots \tau_l$ of width k, let $t_1^{(1)} < \cdots < t_k^{(1)}$ be the indices of the 1's and 3's, i.e.

$$\tau_{t_i^{(1)}} = 1 \text{ or } 3, \quad 1 \le i \le k.$$

Similarly let $t_1^{(2)} < \cdots < t_k^{(2)}$ be the indices of the 2's and 3's. We set $T_1 = T_1(\tau) = \{t_1^{(1)}, \ldots, t_k^{(1)}\}$ and $T_2 = T_2(\tau) = \{t_1^{(2)}, \ldots, t_k^{(2)}\}$.

Proof of Lemma 4.3. We begin with the proof of (a). For a type τ consider two sequences τ^{13} and τ^{23} obtained by deleting the 2's and 1's from τ respectively. Clearly each of these sequences has length k. Let $\xi_1, \ldots, \xi_r$ and $\eta_1, \ldots, \eta_r$ be the positions of the 3's in τ^{13} and τ^{23} respectively. Due to the irreducibility of τ, we have $\eta_s < \xi_s$ for all $s = 1, \ldots, r$. Note that a 21 overlap of a type $\tau = \tau_1\tau_2 \ldots \tau_l$ is a 12 overlap of $\bar{\tau} = \tau_l \ldots \tau_2\tau_1$, and if τ is irreducible, so is $\bar{\tau}$. Since $\chi(n, \tau) = \chi(n, \bar{\tau})$, it is sufficient to prove the statement for a 12 overlap. Assume now that $\widetilde{\tau}$ arises from τ by merging 12 to 3 which in $\widetilde{\tau}$ appears on the position s_0 of $\widetilde{\tau}^{13}$. Consequently, $\widetilde{\tau}_{\widetilde{t}_{s_0}^{(1)}} = 3$. Consider the ground set of $G(2n, \tau)$ broken into n consecutive intervals $I_1 < \cdots < I_n$, each of size 2. Set $I_i = \{l_i, r_i\}$ for its "left" and "right" point. We describe an embedding $G(n, \widetilde{\tau}) \to G(2n, \tau)$ as follows: for each k-tuple $I_{a_1}, \ldots, I_{a_k}$, select $\alpha_j \in I_{a_j}$, $1 \le j \le k$, in such a way that

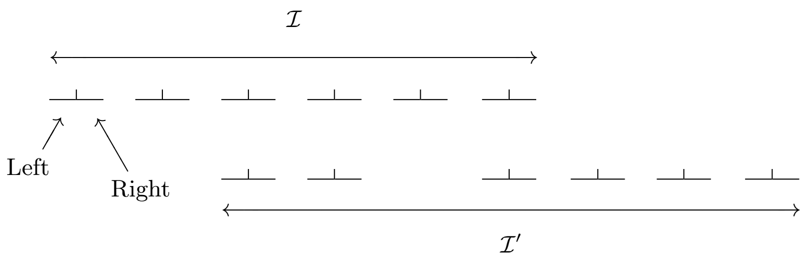
$$\alpha_j = r_{a_j} \quad \text{for } j < s_0 \quad \text{and} \quad \alpha_j = l_{a_j} \quad \text{for } j = s_0. \tag{5}$$

For $j > s_0$ and if $j = \xi_s$ we define the embedding inductively using the fact that $\eta_s < \xi_s$ for all s. We set

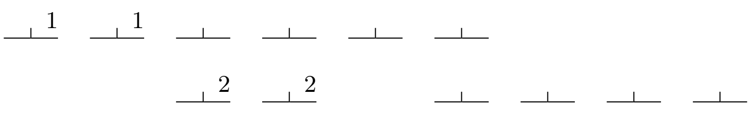
$$\alpha_j = l_j \quad \text{if } \alpha_{\eta_s} = l_{\eta_s} \quad \text{and} \quad \alpha_j = r_j \quad \text{if } \alpha_{\eta_s} = r_{\eta_s}. \tag{6}$$

On the other hand, if $j > s_0$ and $j \neq \xi_s$ we choose α_j arbitrarily to be l_j or r_j.

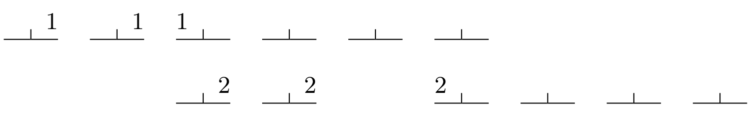
This way we have established a 1-1 mapping $\binom{n}{k} \to \binom{2n}{k}$, $(a_1, \ldots, a_k) \mapsto (\alpha_1, \ldots, \alpha_k)$. It remains to show that $(a_1, \ldots, a_k) \mapsto (\alpha_1, \ldots, \alpha_k)$ is an embedding $G(n, \widetilde{\tau}) \to G(2n, \tau)$. We omit the somewhat tedious verification of the general case and instead demonstrate the embedding by an example below. We will consider in our example the type $\tau = 1112313222$, which after a 12 overlap on the only occurrence of 12 yields $\widetilde{\tau} = 113313222$. Here $s_0 = 3$. In the successive pictures, we represent the different stages described in (5) and (6) on two elements $\mathcal{I}$ and $\mathcal{I}'$ of $\binom{2n}{k}$ of type $\widetilde{\tau}$.



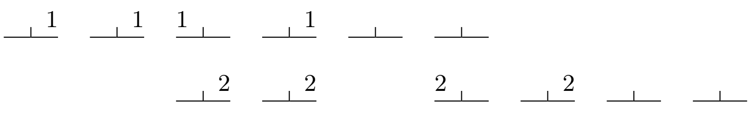
We first choose the right element of each set I_j for $j < s_0$:



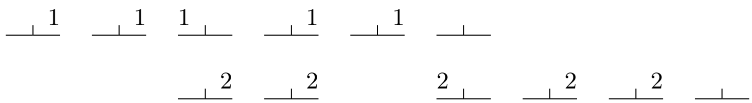
Then we choose the left element of I_j for $j = s_0$:



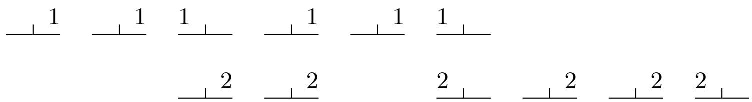
For $j > s_0$ and if $j = \xi_s$ we choose the element according to (6):



Then, for $j > s_0$ and $j \neq \xi_s$ we choose α_j arbitrarily to be l_j or r_j:

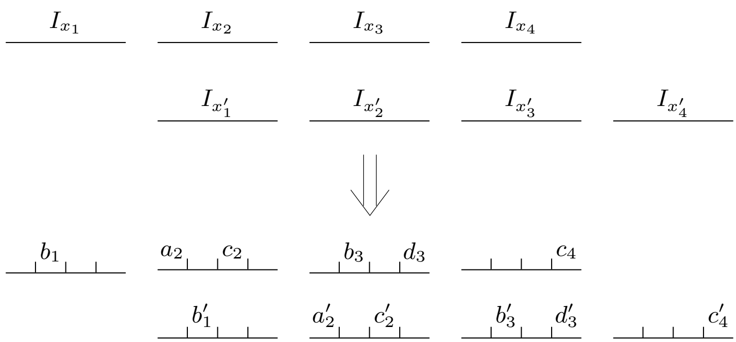


Finally, for $j > s_0$ and if $j = \xi_s$ we choose the element according to (6):



As we can see from the example above, the procedure describing the embedding leaves the 1's and 2's unchanged as well as the 3's present in the original type τ. The unique 3 arising from the 12 overlap is replaced by a 12, thus "undoing" the overlap and concluding the proof of part (a). Note that for this procedure to be guaranteed to work, it is important that the type τ (and thus $\widetilde{\tau}$) is irreducible.

The proof of part (b) is similar though simpler. In order to avoid complicated notation, we will illustrate the idea of the proof by an example. Consider $\delta_6 = 112121212122$ and its $(6,2)$-reduct of index $(2,4)$, namely $\rho = 13332$. We break the ground set of $G(4n,\tau)$ into n consecutive intervals $I_1 < \cdots < I_n$, each of size 4. Set $I_i = \{a_i, b_i, c_i, d_i\}$. We describe an embedding $G(n,\rho) \to G(4n,\delta_6)$ as follows: For each 4-tuple $I_{x_1}, I_{x_2}, I_{x_3}, I_{x_4}$, set $\alpha_{x_1} = \{b_1\}$, $\alpha_{x_2} = \{a_2, c_2\}$, $\alpha_{x_3} = \{b_3, d_3\}$ and $\alpha_{x_4} = \{c_4\}$. The process is illustrated in the following two drawings. The first picture represents two 4-tuples forming a type ρ, and the second represents the corresponding choices of a_i, b_i, c_i, d_i's:



As we can see in the picture above, the process described earlier is an embedding $G(n,\rho) \to G(4n,\delta_6)$. This method straightforward generalizes to δ_k for any k and to any reduct of δ_k. ■

The following result is a direct consequence of Lemma 4.3(a).

COROLLARY 4.5. *If τ is an irreducible type and $\widetilde{\tau}$ is obtained after m successive overlaps of τ, then*

$$\chi(G(n,\widetilde{\tau})) \le \chi(G(2^m n,\tau)).$$

Proof. Apply Lemma 4.3 m times to obtain $G(n,\widetilde{\tau}) \subseteq G(2^m n,\tau)$. ■

In the proof of Theorem 1.5, we have to properly color simultaneously all possible overlaps of each reduct of δ_k. Unfortunately, the argument we have leads to an additional power in the estimates of the chromatic number.

PROPOSITION 4.6. *Let τ be a fixed (k,m)-reduct. There exists a coloring of $[n]^k$ which is proper for simultaneously all overlaps of τ and using at most $\chi(G(2^{k-1}n,\tau))^{3^{k-1}}$ colors.*

Proof. By Corollary 4.5, each individual type ρ obtained after m successive overlaps of τ can be properly colored using no more than $\chi(G(2^m n,\tau))$ colors. Since one can perform at most $k-1$ overlaps on δ_k and thus on τ, each overlap can be colored using no more than $\chi(G(2^{k-1}n,\tau))$ colors. We will color all overlaps simultaneously using a product coloring. We first find an estimate of the number of possible overlaps.

Let M_k be the set of all types which can arise after successive overlaps of the type τ, and let $\mu_k = |M_k|$. All but the first 1 of τ can be overlapped with adjacent 2. So for each of these 1's, there are at most three possibilities: merging with the left, merging with the right or not merging. Thus, $\mu_k \le 3^{k-1}$.

We now define more in detail the product coloring. Consider the coloring φ of $[n]^k$ defined as follows. As mentioned earlier, each overlap ρ of τ arises after at most $k-1$ overlaps of 12's and 21's and Corollary 4.5 implies that $G(n,\rho) \subseteq G(2^{k-1}n,\tau)$. Consequently there exists a proper coloring φ_ρ of $G(n,\rho)$ using at most $\chi(G(2^{k-1}n,\tau))$ colors. Define φ to be the product of all the colorings φ_ρ when ρ ranges over all possible overlaps of τ, i.e. for each k-tuple $K \in [n]^k$,

$$\varphi(K) = \prod_{\rho} \varphi_\rho(K).$$

Clearly, φ is a proper vertex coloring of simultaneously all the overlaps of τ using at most

$$\chi(G(2^{k-1}n,\tau))^{m_k} \le \chi(G(2^{k-1}n,\tau))^{3^{k-1}}$$

colors. ■

In our argument we use the notion of planted rooted trees. Let $\mathbf{T}$ denote a complete binary tree in which all leaves have the same height and are "naturally" ordered from left to right. We shall measure the height of a tree from its leaves up, i.e. in $\mathbf{T}$ all leaves have height zero, and the root of $\mathbf{T}$ has the largest height. In this paper by a *planted rooted tree*, or briefly a *tree*, we mean a subtree T of $\mathbf{T}$ all of whose leaves have height zero. Note that the leaves of T are ordered by the order inherited from the order of leaves of $\mathbf{T}$. The root of T is its vertex of the largest height. A *left branch* of a rooted tree T, denoted by $P_L(T)$, is the path joining the root v of T with the smallest leaf. After removing the edges of $P_L(T)$ the tree T naturally decomposes into a forest which, counting from the right, consists of trees $T_{L,1},\dots,T_{L,s}$, where the last tree $T_{L,s}$ consists of one vertex only. We denote the roots of those trees by $v_{L,1},\dots,v_{L,s}$ respectively, and let $h_{L,i}$ denote the height of $v_{L,i}$ for $i=1,\dots,s$. In an analogous way we define the *right branch* $P_R(T)$ of T, trees $T_{R,i}$, vertices $v_{R,i}$ and heights $h_{R,i}$ for $i=1,\dots,t$. Thus, the root of T can be denoted as either $v_{L,1}$ or $v_{R,1}$ and clearly we have

$$h_{L,1} > h_{L,2} > \cdots > h_{L,s}, \tag{7}$$

$$h_{R,1} > h_{R,2} > \cdots > h_{R,t}. \tag{8}$$

An important role in this part of the paper is played by the notions of a comb and the shape of a tree.

Definition 4.7 (Left and right k-combs). We say that a planted tree T is a *left k-comb* if each of the trees $T_{L,1},\dots,T_{L,s}$ has at most k leaves. A tree T is a *right k-comb* if each of the trees $T_{R,1},\dots,T_{R,t}$ has at most k leaves. If we do not specify whether a k-comb is left or right, we refer to it as a *k-comb*.

Definition 4.8 (Shape of a tree). The *shape* $\mathrm{sh}(T)$ of a planted rooted tree T is a planted rooted tree whose vertex set consists of the root of T and of all vertices of degree at least three of T, and all leaves of T are ordered with the order inherited from T. Two vertices of $\mathrm{sh}(T)$ are adjacent if they are joined by a path in T. (See Figure 1.)

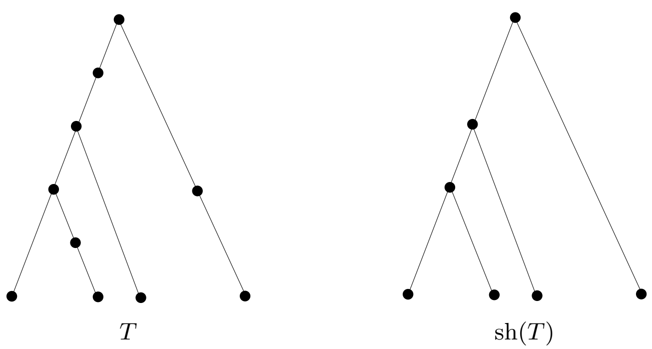


Fig. 1

Thus, roughly speaking, $\mathrm{sh}(T)$ is obtained from T by ignoring all vertices of degree two (except, possibly, the root).

Now we sketch the idea of our argument. In order to prove Theorem 1.5, we will view the elements of $[n]$ as sequences of zeros and ones of length $\lceil \log_2 n \rceil$. These sequences are naturally ordered and correspond to leaves of a binary tree $\mathbf{T}$ of height $\lceil \log_2 n \rceil$. Let us recall that our goal is to color all the elements of $[n]^k$ in such a way that if two of them are in the relative position described by the canonical type δ_k, then they are colored differently. To this end we will color each k-tuple $(x_1, \ldots, x_k)$ by considering the corresponding, uniquely determined tree $T(x_1, \ldots, x_k)$ with set of leaves $\{x_1, \ldots, x_k\}$. Each shape of tree will be colored using a set of colors disjoint from the other tree shapes. Note that the number of different tree shapes depends only on k, not on n, and thus does not contribute much to the estimate of the chromatic number of $G(n, \delta_k)$.

Hence, we will only have to ensure that every pair of trees $T(x_1, \ldots, x_k)$ and $T(y_1, \ldots, y_k)$ of the same shape and with their set of leaves forming the canonical type are colored differently. As we will show in Lemma 4.10, the only shapes for which this can happen are 2-combs. For a given shape of a 2-comb T, say, a left 2-comb, we color each tree of that shape by looking at the heights $h_{L,1}, \ldots, h_{L,s'}$ for some appropriately chosen s' (see Definition 4.11 below for the precise statement). We shall call such a sequence $\{h_1, \ldots, h_r\}$ a *certificate* of a tree. This will be done in such a way that if the leaves of two trees of the same shape form a canonical type δ_k, then their certificates form some $(k-2, m)$-reduct (more precisely, an overlap of some $(k-2, m)$-reduct). Since all heights h_i which form the certificate for a shape are smaller than $\lceil \log_2 n \rceil$, we can use our induction assumption along with Lemma 4.3 to deduce the assertion.

Before we state our main result concerning certificates of pairs of trees of the same shape whose leaves are in the canonical position (see Lemma 4.10), let us introduce some notation and make some simple observations. In what follows we consider two trees T^1 and T^2 of the same shape whose leaves are in the canonical position δ_k for some $k \geq 3$. Moreover we assume that the smallest among the leaves belongs to T^1, and consequently the largest leaf is in T^2. It is easy to see that then the following holds.

Claim 4.9. *Let T^1, T^2 be two subtrees of $\mathbf{T}$ of the same shape such that their sets of leaves form a canonical type of width $k \geq 3$. Then the roots of these trees are different and the root of one tree must lie on the left or right branch of the other one. More specifically, if the smallest leaf belongs to T^1, then either $v^1_{L,1}$ lies on $P_L(T^2)$, or $v^2_{R,1}$ belongs to $P_R(T^1)$.*

Proof. We first argue that the roots of T^1 and T^2 are not the same. Indeed, suppose that both trees have the same root v. Then, after removing v,

T^1 would be partitioned into two rooted trees T_1^1 and T_2^1, and T^2 would split into T_1^2 and T_2^2, where

(i) $\mathrm{sh}(T_1^1) = \mathrm{sh}(T_1^2)$ and $\mathrm{sh}(T_2^1) = \mathrm{sh}(T_2^2)$;
(ii) all leaves of T_1^1 are smaller than all leaves of T_2^1 and T_2^2 (since $\mathbf{T}$ contains no cycles);
(iii) all leaves of T_1^2 are smaller than all leaves of T_2^1 and T_2^2 (since the opposite would lead to a cycle in $\mathbf{T}$).

But this would mean that the canonical type is reducible, which is clearly not the case. Hence, the roots of T^1 and T^2 are different. Now, take a vertex v which dominates the roots of both T^1 and T^2, and connect it to these roots by paths P^1 and P^2. Consider paths $P_L(T^1) < P_L(T^2) < P_R(T^1) < P_R(T^2)$; we ordered them by the order of their leaves. It is easy to see that the only case where the union of the paths P^1, P^2, $P_L(T^1)$, $P_L(T^2)$, $P_R(T^1)$, $P_R(T^2)$ does not contain a cycle is when either the root of T^1 belongs to $P_L(T^2)$, or the root of T^2 belongs to $P_R(T^1)$. ■

The following lemma characterizes quite precisely those pairs of trees of the same size whose leaves form δ_k.

LEMMA 4.10. *Let T^1, T^2 be two subtrees of $\mathbf{T}$ of the same shape such that their sets of leaves form the canonical type of width $k \geq 3$ starting with a leaf of T^1. Moreover, assume that the root of T^2 lies on the right branch $P_R(T^1)$ of T^1 and, after removing the right branch, T^2 (and so also T^1) splits into t trees. Then:*

(i) *The roots of the trees $T_{R,1}^2, \ldots, T_{R,t-2}^2$ lie on the branch $P_R(T^1)$. If $T_{R,t-1}^2$ has at least two leaves, then its root belongs to $P_R(T^1)$ as well.*
(ii) *$h_{R,i+2}^1 \leq h_{R,i}^2 \leq h_{R,i+1}^1$ for each $i = 1, \ldots, t-3$. If $T_{R,t-1}^2$ has at least two leaves, then the above bound holds for $i = t-2$ as well.*
(iii) *If $T_{R,i}^2$ has at least two leaves for some $i = 2, \ldots, t-3$, then $h_{R,i}^2 = h_{R,i+1}^1$ and $h_{R,i}^1 = h_{R,i-1}^2$. If $T_{R,1}^2$ has at least two leaves, then also $h_{R,1}^2 = h_{R,2}^2$. If $T_{R,s-1}^2$ has at least two leaves, then $h_{R,s-1}^1 = h_{R,t-1}^2$.*
(iv) *Both T_1 and T_2 are left 2-combs.*

Furthermore, if the root of T^1 lies on the left branch $P_L(T^2)$ of T^2, then (i)–(iv) *holds with 'right' replaced by 'left'* (*and subsequently, 'R' by 'L'*), *and with T^2 and T^1 interchanged.*

Proof. Note that if the root of $T_{R,i}^2$ does not belong to the branch $P_R(T^1)$, then it must lie to the right of it and so, since we are dealing with subtrees of $\mathbf{T}$, all its leaves must be larger than all the leaves of T^1. Since the canonical type ends with just two 2's, the roots of at most two trees from the family $T^2_{R,1}, T^2_{R,2}, \dots, T^2_{R,t}$ do not belong to $P_R(T^1)$. If $T^2_{R,t-1}$ has two leaves, the only tree from among $T^2_{R,1}, T^2_{R,2}, \dots, T^2_{R,t}$ which does not start at $P_R(T^1)$ is $T^2_{R,t}$. This proves (i).

In order to show (ii) let us assume that $h^2_{R,i} > h^1_{R,i+1}$. Since from (i) the roots of both trees $T^2_{R,i}$ and $T^1_{R,i+1}$ lie on the right branch $P_R(T^1)$, it means that the canonical type formed by the leaves of T^2 and T^1 starts with the leaves of the trees $T^2_{R,1}, \dots, T^2_{R,i}$ immediately followed by the leaves of $T^1_{R,1}, \dots, T^1_{R,i}$, i.e. at some point the number of 1's matches the number of 2's in the pattern created by leaves of T^1 and T^2. However, this is clearly not the case for δ_k, which is irreducible. Similarly, suppose that $h^2_{R,i} < h^1_{R,i+2}$. Then to the left of $T^2_{R,i}$ lie all trees $T^1_{R,1}, \dots, T^1_{R,i+2}$, as well as all trees $T^2_{R,1}, \dots, T^2_{R,i-1}$. Thus, the type created by leaves of T^2 and T^1 starts with a sequence in which the number of 1's is at least three larger than the number of 2's. This is impossible for the canonical type, showing (ii).

We start the proof of (iii) with a simple observation that between the leaves of $T^2_{R,i}$ must lie a leaf of some tree $T^1_{R,j}$, so, because of (i) and (ii), we must have either $j = i + 1$ or $j = i + 2$. The latter case is impossible, since then the leaf pattern starts with the leaves of the trees $T^2_{R,1}, \dots, T^2_{R,i-1}$ and $T^1_{R,1}, \dots, T^1_{R,i+1}$. Since the tree $T^1_{R,i}$ has at least two leaves, the number of leaves in the second group is at least three larger than in the first group, a contradiction.

A similar argument proves (iv). Consider a tree $T^1_{R,i}$ with at least two leaves for some $i = 1, \dots, s-1$. Then, by (i)–(iii), the leaf pattern starts with the leaves of the trees $T^2_{R,1}, \dots, T^2_{R,i-1}$ and $T^1_{R,1}, \dots, T^1_{R,i}$. Since the number of 1's in the starting subsequence of the canonical type cannot differ by more than two from the number of 2's, $T^1_{R,i}$ has precisely two leaves.

The second part of the statement can be proved by a symmetric argument. ■

Now we can give a precise definition of the certificate $c(T)$ of a 2-comb T.

DEFINITION 4.11. Let T be a 2-comb. Then the *certificate* of T, denoted as $c(T) = \{h_1, \dots, h_r\}$, is defined as follows.

(i) If T has fewer than five leaves, then we put $c(T) = \{h_{L,1}\} = \{h_{R,1}\}$.
(ii) Let T be a left 2-comb such that removing the left branch results in the forest consisting of the trees $T_{L,1}, \dots, T_{L,s}$. Then $c(T) = \{h_{L,1}, \dots, h_{L,s-2}\}$ if $T_{L,s-1}$ has only one leaf, and $c(T) = \{h_{L,1}, \dots, h_{L,s-1}\}$ if $T_{L,s-1}$ has two leaves.
(iii) Let T be a right 2-comb such that removing the left branch results in the forest consisting of the trees $T_{R,1}, \dots, T_{R,t}$. Then

$c(T) = \{h_{R,1}, \ldots, h_{R,t-2}\}$ if $T_{R,t-1}$ has only one leaf, and $c(T) = \{h_{R,1}, \ldots, h_{R,t-1}\}$ if $T_{R,t-1}$ has two leaves.

The main ingredient of the proof of Theorem 1.5 is the following lemma.

LEMMA 4.12. *Let T^1 and T^2 be two different 2-combs of* $\mathbf{T}$ *of the same shape and of certificates $\{h_1^1, \ldots, h_r^1\}$ and $\{h_1^2, \ldots, h_r^2\}$ respectively, whose sets of leaves form a canonical type δ_k of width $k \geq 5$. Then there exists a $(k-2, k-2-r)$-reduct ρ of δ_k, which depends only on the shape of the trees T^1 and T^2, such that*

$$\tau(\{h_1^1, \ldots, h_r^1\}, \{h_1^2, \ldots, h_r^2\})$$

is an overlap of ρ.

Proof. The result follows from Lemma 4.10. Instead of giving its formal proof which would involve a lot of technical notation, we describe its idea, so the argument will become clear.

Let us assume first that both trees T^1 and T^2 are right 1-combs with k leaves. Since after removing the left branch the trees decompose into paths, their certificates $\{h_{R,1}^1, \ldots, h_{R,k-1}^1\}$, $\{h_{R,1}^2, \ldots, h_{R,k-1}^2\}$ have $k-1$ elements each. This means in particular that $r = k-2$ in the statement of the claim.

From Lemma 4.10 it follows that

$$\begin{aligned}(9)\qquad h_{R,1}^1 > h_{R,2}^1 \geq h_{R,1}^2 \geq h_{R,3}^1 \geq h_{R,2}^2 \geq \cdots \\ \geq h_{R,k-4}^2 \geq h_{R,k-2}^1 \geq h_{R,k-3}^2 > h_{R,k-2}^1,\end{aligned}$$

where (recall (8)) we also have

$$(10)\qquad h_{R,1}^1 > h_{R,2}^1 > \cdots > h_{R,k-2}^1 \quad \text{and} \quad h_{R,1}^2 > h_{R,2}^2 > \cdots > h_{R,k-2}^2.$$

Thus, if the trees T^1 and T^2 are in 'generic position', i.e. in (9) all the inequalities are strict, then the certificates $\{h_{R,1}^1, \ldots, h_{R,k-2}^1\}$ and $\{h_{R,1}^2, \ldots, h_{R,k-2}^2\}$ form a canonical type δ_{k-2}, i.e., using the terminology from the statement of the lemma, in this case ρ is a $(k-2, 0)$-reduct of δ_{k-2}, i.e. $\rho = \delta_{k-2}$. Note also that in this case $\tau(\{h_{R,1}^1, \ldots, h_{R,k-2}^1\}, \{h_{R,1}^2, \ldots, h_{R,k-2}^2\}) = \delta_{k-2}$, i.e. there are no overlaps. Figure 2 illustrates this case with $k = 5$.

However, some 'spontaneous' equalities in (9) can happen, e.g. some $T_{R,i}^1$ may share some vertices (and thus the root) with $T_{R,i-1}^2$. However, in this case the type $\tau(\{h_{R,1}^1, \ldots, h_{R,k-2}^1\}, \{h_{R,1}^2, \ldots, h_{R,k-2}^2\})$ is clearly an overlap of the canonical type δ_{k-2}, i.e. ρ is still δ_{k-2} but this time $\tau(\{h_1, \ldots, h_{k-2}\}, \{h'_1, \ldots, h'_{k-2}\})$ is an overlap of δ_{k-2}. Figure 3 corresponds to $k = 5$ and $h_3 = h'_2$.

Thus, the 1-combs result in the certificates being overlaps of δ_k. In this case $r = k-2$ and the reducts do not play any role. In order to show the assertion when T^1 and T^2 are 2-combs (say, right 2-combs) of the same shape

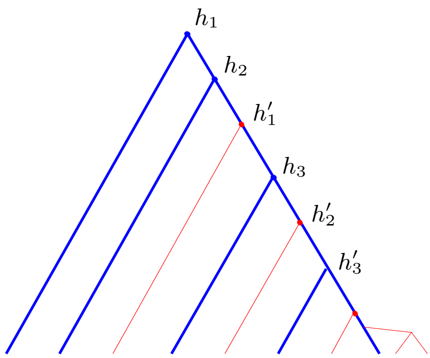


Fig. 2. 1-combs with $k = 5$ and strict inequalities in (9)

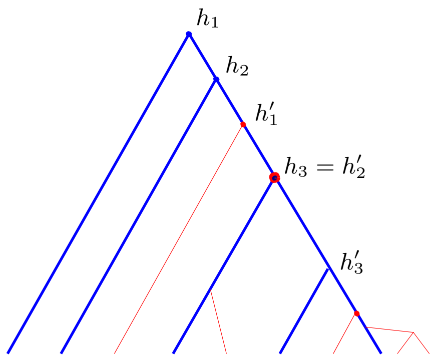


Fig. 3. 1-combs with $k = 5$ and $h_3 = h'_2$

we need basically to repeat the above argument in a slightly more general setting.

For a right comb T let us define the *extended certificate* as the sequence $(\hat{h}_{R,1}, \ldots, \hat{h}_{R,k-2})$, where $\hat{h}_{R,i}$ is the height of a tree $T_{R,i}$ which contains the ith leaf of T counting from the right. Clearly, the extended certificate can be obtained from the usual certificate of T by repeating some terms according to the structure of the tree (say, if $T_{R,2}$ has two leaves, then $\hat{h}_{R,2} = \hat{h}_{R,3} = h_{R,2}$). Furthermore, since we define the certificate for a tree of a given shape, both the certificate and the extended certificate basically carry the same information.

Note that since the leaves of T_1 and T_2 form the canonical type δ_k, for their extended certificates we have

$$\begin{aligned}(11)\qquad \hat{h}^1_{R,1} \geq \hat{h}^1_{R,2} \geq \hat{h}^2_{R,1} \geq \hat{h}^1_{R,3} \geq \hat{h}^2_{R,2} \geq \cdots \\ \geq \hat{h}^2_{R,k-4} \geq \hat{h}^1_{R,k-2} \geq \hat{h}^2_{R,k-3} \geq \hat{h}^1_{R,k-2},\end{aligned}$$

However, equality between $\hat{h}^t_{R,i}$ and $\hat{h}^t_{R,i+1}$ happens if the ith and the $(i+1)$th leaves belong to the same tree. Consequently, we need to apply the reduction operation on the above sequence to reduce it to a $(k-2, m)$-reduct, where m is the number of trees among $T^1_{R,1}, \ldots, T^1_{R,s-1}$ which have two leaves. In the generic position, i.e. when all equalities are forced by the condition described in Lemma 4.10(iii), and the other inequalities in Lemma 4.10(ii) are strict, the type $\tau(\{h^1_{R,1}, \ldots, h^1_{R,k-2}\}, \{h^2_{R,1}, \ldots, h^2_{R,k-2}\})$ is just a $(k-2, m)$-reduct (see Figure 4).

In the general case, however, some spontaneous equalities may occur and some trees $T^1_{R,i}$ may share a root with either $T^2_{R,i-1}$ or $T^2_{R,i-2}$ provided each of the glued trees has only one leaf. Hence, $\tau(\{h^1_{R,1}, \ldots, h^1_{R,k-2}\}, \{h^2_{R,1}, \ldots, h^2_{R,k-2}\})$ is just an overlap of some $(k-2, m)$-reduct ρ (see Figure 5). ■

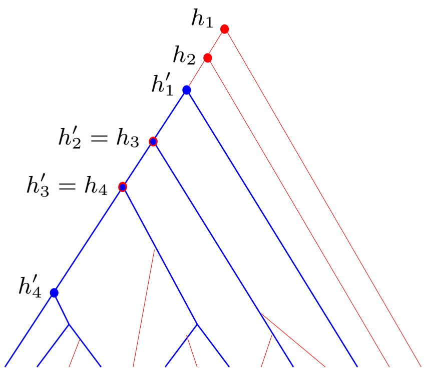


Fig. 4. $(5,1)$-reduct

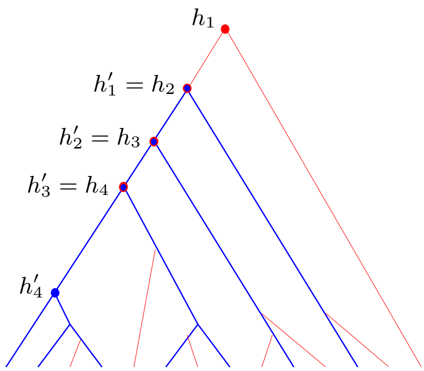


Fig. 5. Overlap of a $(5,1)$-reduct

Proof of Theorem 1.6. Recall that Lemma 3.7 asserts that if ρ is a subtype of τ, then $\chi(G(n,\tau)) \le \chi(G(n,\rho))$. Since for any $k \ge 3$ the canonical type δ_k is a subtype of δ_{k+1}, we obtain $\chi(n,\delta_{k+1}) \le \chi(n,\delta_k)$, which is part (ii) of Theorem 1.6.

Now let $k = 2l+1$ be odd. We shall show by induction on l that

$$\chi(G(n,\delta_{2l+1})) \le c_{2l+1}(\log_{(l)} n)^{a_l} \tag{12}$$

for some constant c_{2l+1} and $a_l = \binom{l+1}{2}$.

Lemma 3.8 states $\chi(G(n,\delta_3)) \le 2\log n$, which implies (12) for $l = 1$. We now assume that the assertion holds up to $l-1$ and prove it for $l \ge 2$. Consider the set of $(2l+1)$-tuples which correspond to a tree $T(S)$ of a certain fixed shape S. Let $G_S(n,\delta_{2l+1})$ be the subgraph of $G(n,\delta_{2l+1})$ induced by these $(2l+1)$-tuples. We will show that

$$\chi(G_S(n,\delta_{2l+1})) \le c'_{2l+1}(\log_{(l)} n)^{a_l}$$

for some constant c'_{2l+1}.

If $T(S)$ is not a 2-comb then, by Lemma 4.10, no two $(2l+1)$-tuples of the same shape S can be of type δ_{2l+1}. Thus, the graph $G_S(n,\delta_{2l+1})$ contains no edges and $\chi(G_S(n,\delta_{2l+1})) = 1$.

Now let us look at the more interesting case, when $T(S)$ is a 2-comb. We color each $(2l+1)$-tuple $\{x_1,\dots,x_{2l+1}\}$ of tree shape S by considering the certificate $\{h_1,\dots,h_r\}$ of the tree with leaves at $\{x_1,\dots,x_{2l+1}\}$. Since $k_0 = 2l+1 \ge 5$, it follows from Lemma 4.12 that if two $(2l+1)$-tuples $\{x^1_1,\dots,x^1_{2l+1}\}$ and $\{x^2_1,\dots,x^2_{2l+1}\}$ have the same shape S and form a δ_{2l+1} type, the type of their certificates $\{h^1_1,\dots,h^1_r\}$ and $\{h^2_1,\dots,h^2_r\}$ is an overlap ρ of some $(2l+1-2, 2l+1-2-r)$-reduct $\widetilde{\tau}$. Assigning to each $\{x_1,\dots,x_{2l+1}\}$ its certificate $\{h_1,\dots,h_r\}$ defines a homomorphism $\varphi : G_S(n,\delta_{2l+1}) \to \bigcup_\rho G(\log n,\rho)$ where the union is taken over all possible overlaps of $\widetilde{\tau}$. By Proposition 4.6, all overlaps ρ of $\widetilde{\tau}$ can be colored by at most

$$\big(\chi(G(2^{2l}\log n,\widetilde{\tau}))\big)^{3^{2l}}$$

colors. Thus,

$$\chi(G_S(n,\delta_{2l+1})) \le \big(\chi(G(2^{2l}\log n,\widetilde{\tau}))\big)^{3^{2l}}. \tag{13}$$

On the other hand, by Lemma 4.3,

$$\chi(G(2^{2l}\log n,\widetilde{\tau})) \le \chi(4\cdot 2^l\log n,\delta_{2l-1}) \tag{14}$$

and by the induction hypothesis,

$$\begin{aligned}\chi(4\cdot 2^l\log n,\delta_{2l-1}) &\le c_{2l-1}\big(\log_{(l-1)}(4\cdot 2^{2l}\log n)\big)^{a_{l-1}}\\ &\le c'_{2l-1}(\log_{(l)} n)^{a_{l-1}}\end{aligned} \tag{15}$$

for an appropriate constant c'_{2l-1}. Combining (13)–(15) with an appropriate constant c''_{2l-1} we obtain

$$\chi(G_S(n,\delta_{2l+1})) \le \big(c'_{2l-1}(\log_{(l)} n)^{a_{l-1}}\big)^{3^{2l}} \le \big(c''_{2l-1}(\log_{(l)} n)\big)^{3^{2l}a_{l-1}}. \tag{16}$$

Note that $a_{l-1}\cdot 3^{2l} = 9^{\binom{(l-1)+1}{2}}\cdot 9^l = 9^{\binom{l+1}{2}} = a_l$. Replacing c''_{2l-1} by an appropriate constant c'''_{2l+1}, we can rewrite the last expression as

$$c'''_{2l+1}\cdot(\log_{(l-1)} n)^{a_l}$$

Consequently, $\chi(G(n,\delta_{2l+1})) \subseteq \sum_S \chi(G_S(n,\delta_{2l+1})) \le c_{2l+1}(\log_{(l-1)} n)^{a_l}$ for some constant c_{2l+1}. ■

5. Extension to other types. In this section we discuss the chromatic number of $G(n,\tau)$ for some types τ other than δ_k. We begin by considering types of the form $1\ldots13\ldots32\ldots2$. It is a natural generalization of the type $\sigma_k = 133\ldots32$ mentioned in the introduction which has been used to define so-called shift graphs.

Definition 5.1. For $a,b\ge 1$, we define $\sigma_{a,b}$ as the type consisting of a 1's, followed by b 3's, followed by a 2's.

For instance, $\sigma_{4,3} = 11113332222$ and $\sigma_k = \sigma_{k-1,1}$. Note that $\sigma_{a,b}$ is a type of length $2a+b$.

Theorem 5.2. *For any $a\ge 1$, $b\ge 1$ and $n\in\mathbb{N}$ we have*

$$\log_{(\lceil b/a\rceil)}\frac{n}{a} \le \chi(G(\sigma_{a,b},n)) \le \log_{(\lceil b/a\rceil)} n.$$

Proof. Let $a,b\ge 1$ and consider $\sigma_{a,b} = \tau_1\ldots\tau_{2a+b}$. We begin by showing the upper bound. We construct a new type $\tilde{\sigma}_{a,b}$ by deleting τ_i from $\sigma_{a,b}$ whenever $i\equiv 3,4,\ldots,a \pmod a$. For example, $\tilde{\sigma}_{3,4} = 1\cancel{1}\cancel{1}33\cancel{3}3\cancel{3}3\cancel{2}2\cancel{2} = 1332$. Since we are deleting all but one of the initial 1's and all but one of the terminal 2's, it is clear that in this way we obtain a type σ_c for some c.

Furthermore, one can observe that exactly $\lceil b/a \rceil$ of the original 3's remain, consequently

$$\tilde{\sigma}_{a,b} = \sigma_{\lceil b/a \rceil + 1}.$$

Finally, note that the deletion is consistent in the sense that whenever $\tau_i^{(1)}$ is deleted, so is $\tau_i^{(2)}$ and vice versa. In other words, $\tilde{\sigma}_{a,b} = \sigma_{\lceil b/a \rceil + 1}$ is a subtype of $\sigma_{a,b}$ and Lemma 3.7 yields

$$\chi(G(n,\sigma_{a,b})) \le \chi(G(n,\sigma_{\lceil b/a \rceil + 1})) \le \log_{(\lceil b/a \rceil)} n.$$

We now turn to the lower bound. Set $r = \lceil b/a \rceil + 1$. In order to estimate $\chi(G(n,\sigma_{a,b}))$ from below we will find an embedding $\varphi : G(n,\sigma_r) \to G(an,\sigma_{a,b})$. To this end we divide $[an]$ into n consecutive intervals $A_1,\dots,A_n$, where $|A_i| = a$ for all $i = 1,\dots,n$. For each r-tuple $A_{i_1},\dots,A_{i_r}$ we define $B_{i_j} \subset A_{i_j}$, $j = 1,\dots,r$, with $|\bigcup_{j=1}^r B_{i_j}| = a+b$ so that $\varphi : \{i_1,\dots,i_r\} \to \bigcup_{j=1}^r B_{i_j}$ will be the desired embedding. Given an r-tuple $A_{i_1},\dots,A_{i_r}$ set $B_{i_j} = A_{i_j}$ for every $j \le r-1 = \lceil b/a \rceil$, and B_{i_r} to be the first $a+b-a\lceil b/a \rceil$ elements of A_{i_r}. We then have $|\bigcup_{j=1}^r B_{i_j}| = (r-1)a + a + b - a\lceil b/a \rceil = a+b$ and one can check that whenever $\tau(\{i_1,\dots,i_r\},\{i'_1,\dots,i'_r\}) = \sigma_r$ then $\tau(\bigcup_{j=1}^r B_{i_j}, \bigcup_{j=1}^r B_{i'_j}) = \sigma_{a,b}$. Consequently,

$$\chi(G(\sigma_{a,b},n)) \le \chi(G(\sigma_r,n)) = \log_{(r-1)} n = \log_{\lceil b/a \rceil} n. \ \blacksquare$$

One can mimic the above argument to extend Erdős and Hajnal's [3] result for $\chi(G(\kappa,\sigma_k))$ to the types $\sigma_{a,b}$.

THEOREM 5.3. *For any $a, b \ge 1$ and infinite κ we have*

$$\chi(G(\kappa,\sigma_{a,b})) = \min\{\alpha : \exp_{(\lceil b/a \rceil)}(\alpha) \ge \kappa\}. \ \blacksquare$$

Using a similar reasoning to the proof of Theorem 5.3, we extend Theorems 1.4 and 1.5 to the following larger class of irreducible disjoint types.

DEFINITION 5.4. For $a \ge 2$ and $b \ge 1$, the type $\delta_{a,b}$ consists of a 1's, followed by b copies of 21 and followed by a copies of 2.

For example, $\delta_{3,2} = 1112121222$ and $\delta_{2,k-2} = \delta_k$, the canonical type of width k.

THEOREM 5.5. *For any $a \ge 2$ and $b \ge 1$, we have*

$$\log_{(\lceil b/a \rceil)} \frac{n}{2^b} \le \chi(G(\delta_{a,b},n)) \le C(\log_{(\lceil b/a \rceil)} n)^{\binom{\lceil b/a \rceil + 1}{2}}$$

for some constant C depending on a and b.

Proof. Fix $a \ge 2$ and $b \ge 1$ and let $\delta_{a,b} = \tau_1 \dots \tau_{2a+2b} \in \{1,2\}^{2a+2b}$. For $k = a+b$, $j = 1,2$, and $1 \le \alpha \le a+b$, let $t_\alpha^{(j)}$ and $T_j(\tau)$ be as in Definition 3.5. We construct a new type $\tilde{\delta}_{a,b}$ by deleting those τ_i from $\delta_{a,b}$ which correspond to $t_\alpha^{(j)}$ with $\alpha = 3,4,\dots,a \pmod a$, i.e. we leave only those

1's (and 2's) which among all 1's (or 2's) appear on places $ar+1$ and $ar+2$ for some $r \geq 0$.

For example, for $\delta_{4,3} = 11112121212222$, $a = 4$ and consequently $\tilde{\delta}_{4,3} = 11\cancel{11}2121\cancel{212}22\cancel{2} = \delta_4$. The deletion is consistent in the sense that whenever $t_\alpha^{(1)}$ is deleted, so is $t_\alpha^{(2)}$ and vice versa, and consequently the type $\tilde{\delta}_{a,b}$ is a subtype of $\delta_{a,b}$.

Since the deletion process eliminates all but the first two 1's among the initial string of 1's, and then eliminates consecutive 21's and all but two 2's, it is clear that the resulting type is a canonical type. Furthermore the proportion of deleted 21's is $(a-2)/a$ so that we are left with $\lceil (2/a)b \rceil$ 21's. Consequently, $\tilde{\delta}_{a,b} = \delta_{\lceil 2b/a \rceil + 2}$ is a subtype of $\delta_{a,b}$, and by Lemma 3.7,

$$\chi(G(n, \delta_{a,b})) \leq \chi(G(n, \delta_{\lceil 2b/a \rceil + 2})).$$

Recall that by Theorem 1.5, for the canonical type δ_k of width k we have

$$\chi(G(n, \delta_k)) \leq C (\log_{(\lfloor (k-1)/2 \rfloor)} n)^{\binom{\lfloor (k+1)/2 \rfloor}{2}}$$

for some $C > 0$. Since $\lfloor (\lceil 2b/a \rceil + 2 - 1)/2 \rfloor = \lceil b/a \rceil$, we obtain

$$\chi(G(\delta_{a,b}, n)) \leq \chi(G(n, \delta_{\lceil 2b/a \rceil + 2})) \leq C (\log_{(\lceil b/a \rceil)} n)^{\binom{\lceil b/a \rceil + 1}{2}}$$

for some constant C.

We now turn to the lower bound. For each pair 12 in the type $\delta_{a,b}$, we perform a 12 overlap so that after b overlaps we obtain the type $\sigma_{a,b}$. For example, the type $\delta_{4,2} = 111121212222$ becomes $\sigma_{4,2} = 1111332222$. Using Lemma 4.3 we obtain $\chi(G(n, \sigma_b)) \subset \chi(G(2^b n, \sigma_{a,b}))$, which by Theorem 5.3 implies

$$\log_{(\lceil b/a \rceil)} n \leq \chi(G(2^b n, \sigma_{a,b})),$$

or equivalently

$$\log_{(\lceil b/a \rceil)} n/2^b \leq \chi(G(n, \sigma_{a,b})). \ \blacksquare$$

6. Final remarks and comments. The first natural question which emerges on $\chi(G(n, \delta_k))$ is about its order. We conjecture that the lower bound given by Theorem 1.4 is the correct one up to a constant factor, and the power of iterated logarithm in Theorem 1.5 is caused by the method we used in our argument (a consequence of Proposition 4.6). In fact, it is possible that the following stronger conjecture holds.

CONJECTURE 1. *For every irreducible type τ there exists an integer $f(\tau)$ such that*

$$\chi(G(n, \tau)) = \Theta_\tau(\log_{(f(\tau))} n),$$

where the hidden constants may depend on τ.

The following stronger (perhaps too strong) conjecture anticipates the correct value of $f(\tau)$.

Conjecture 2. *For every irreducible type τ we have*

$$\chi(G(n,\tau)) = \Theta_\tau\big(\min \chi(G(n,\delta_{k,r}))\big),$$

where the minimum is taken over all k and r such that τ can be obtained from some $\delta_{k,r}$ by the process of overlapping and reduction similar to that described in Definitions 4.1 *and* 4.2.

A conjecture similar to Conjecture 1 can also be made for an infinite case.

Conjecture 3. *For every type τ there exists an integer $g(\tau)$ such that for every infinite cardinal κ,*

$$\chi(G(\kappa,\tau)) = \min\{\alpha : \exp_{g(\tau)}(\alpha) \geq \kappa\}.$$

Note however that, due to Theorem 2.1, disjoint types such as $\delta_{k,r}$ do not play any role in finding $g(\tau)$ for a non-disjoint type τ.

Acknowledgements. The second author was partly supported by NCN grant 2012/06/A/ST1/00261 and NSF grant DMS 13-01698, while the third author acknowledges partial support of NSF grants DMS 13-01698 and DMS 11-02086.

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Received 12 November 2013;
in revised form 28 February 2014