

## On ultrapowers of Banach spaces of type $\mathcal{L}_\infty$

by

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**Abstract.** We prove that no ultraproduct of Banach spaces via a countably incomplete ultrafilter can contain  $c_0$  complemented. This shows that a “result” widely used in the theory of ultraproducts is wrong. We then amend a number of results whose proofs have been infected by that statement. In particular we provide proofs for the following statements: (i) All  $M$ -spaces, in particular all  $C(K)$ -spaces, have ultrapowers isomorphic to ultrapowers of  $c_0$ , as also do all their complemented subspaces isomorphic to their square. (ii) No ultrapower of the Gurarii space can be complemented in any  $M$ -space. (iii) There exist Banach spaces not complemented in any  $C(K)$ -space having ultrapowers isomorphic to a  $C(K)$ -space.

**1. Introduction.** The Banach space ultraproduct construction has been, and still continues to be, the main bridge between model theory and the theory of Banach spaces and its ramifications. Ultraproducts of Banach spaces, even at a very elementary level, proved very useful in the “local theory”, the study of Banach lattices, and also in some nonlinear problems, such as the uniform and Lipschitz classification of Banach spaces. We refer the reader to Heinrich’s survey paper [19] and Sims’ notes [37] for two complementary accounts. While the study of the isometric properties of ultraproducts goes back to their inception in Banach space theory and produced a rather coherent set of results very early (see for instance [24]), not much is known about their isomorphic theory.

The purpose of this paper is to study the interplay between the isomorphic theory of Banach spaces and ultraproducts, placing emphasis on spaces of type  $\mathcal{L}_\infty$ . To do this we need first to clarify the status of a number of “results” in the theory of ultraproducts of Banach spaces. Let us explain this point in detail as it might be the most interesting feature of the paper to

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some readers. We refer the reader to Section 2 for precise definitions and all unexplained notation.

The following statement appears, without proof, as Lemma 4.2(ii) in Stern's paper [38]:

- If  $\mathcal{U}$  is a countably incomplete ultrafilter and  $H$  is the corresponding ultrapower of  $c_0$ , then  $H$  contains a complemented subspace isometric to  $c_0(H)$ .

Here,  $c_0$  is the space of scalar sequences converging to zero and  $c_0(H)$  is the space of sequences converging to zero in  $H$ , with the sup norm. This statement, however, turns out to be false (see below). Unfortunately, Stern's lemma has infected the proofs of a number of results in the nonstandard theory and ultraproduct theory of Banach spaces. We can mention:

- (a) If  $E$  is isomorphic to a complemented subspace of a  $C$ -space, then  $E$  has an ultrapower isomorphic to a  $C$ -space (Stern [38, Theorem 4.5(ii)] and also Henson–Moore [26, Theorem 6.6(c)]).
- (b) If  $E$  is isomorphic to a complemented subspace of an  $M$ -space, then  $E$  has an ultrapower isomorphic to a  $C$ -space (Heinrich–Henson [21, Theorem 12(c)]).
- (c) If  $E$  is an  $M$ -space then  $E$  has an ultrapower isomorphic to an ultrapower of  $\ell_\infty$  (Henson–Moore [26, Theorem 6.7]).
- (d) Ultrapowers of the Gurariĭ space with respect to countably incomplete ultrafilters are not complemented in any  $C$ -space (Henson–Moore [26, Theorem 6.8]).

(Here, a  $C$ -space is a Banach space isometrically isomorphic to  $C(K)$ , the space of all continuous functions on the compact space  $K$  with the sup norm, while an  $M$ -space is a sublattice of a  $C$ -space; see Section 2.4.)

With this background in mind let us explain the organization of the paper and summarize its main results. Section 2 is preliminary and it mostly consists of definitions and conventions about notation. Section 3 contains a few general results on the structure of ultraproducts of Banach spaces—we invariably assume they are built over countably incomplete ultrafilters. We will show that ultraproducts of Banach spaces are Grothendieck spaces as long as they are  $\mathcal{L}_\infty$ -spaces (Proposition 3.2). A *Grothendieck* space is a Banach space where  $c_0$ -valued operators are weakly compact: in particular no Grothendieck space can contain a complemented copy of  $c_0$ . This already shows that Stern's lemma is wrong. And indeed more is true:  $c_0$  is never complemented in ultraproducts (Proposition 3.3). Interesting sideways can be taken to arrive at these results. In [3] we have shown that ultraproducts of  $\mathcal{L}_\infty$ -spaces are universally separably injective ( $E$  is *universally separably injective* if  $E$ -valued operators extend from separable subspaces) and that

universally separably injective spaces are always Grothendieck. To complete those results we have added here a proof that infinite-dimensional ultraproducts via countably incomplete ultrafilters are never injective spaces, a result basically due to Henson and Moore [25, Theorem 2.6].

In Section 4 we consider the problem of whether two given Banach spaces have isomorphic (not necessarily isometric) ultrapowers. Regarding the statements (a) to (d) we show that (c) and (d) are true and we provide amendments for (a) and (b) by proving that they hold under the additional hypothesis that  $E$  is isomorphic to its square. The closing Section 5 contains some additional results, together with some open problems that we found interesting.

## 2. Preliminaries

**2.1. Filters.** A family  $\mathcal{U}$  of subsets of a given set  $I$  is said to be a *filter* if it is closed under finite intersections, does not contain the empty set, and  $A \in \mathcal{U}$  provided  $B \subset A$  and  $B \in \mathcal{U}$ . An *ultrafilter* on  $I$  is a filter which is maximal with respect to inclusion. If  $X$  is a (Hausdorff) topological space,  $f : I \rightarrow X$  is a function, and  $x \in X$ , one says that  $f(i)$  *converges* to  $x$  *along*  $\mathcal{U}$  (written  $x = \lim_{\mathcal{U}} f(i)$  for short) if whenever  $V$  is a neighborhood of  $x$  in  $X$  then the set  $f^{-1}(V) = \{i \in I : f(i) \in V\}$  belongs to  $\mathcal{U}$ . The obvious compactness argument shows that if  $X$  is compact and Hausdorff, and  $\mathcal{U}$  is an ultrafilter on  $I$ , then for every function  $f : I \rightarrow X$  there is a unique  $x \in X$  such that  $x = \lim_{\mathcal{U}} f(i)$ .

DEFINITION 1. An ultrafilter  $\mathcal{U}$  on a set  $I$  is *countably incomplete* if there is a sequence  $(I_n)$  of subsets of  $I$  such that  $I_n \in \mathcal{U}$  for all  $n$ , and  $\bigcap_{n=1}^{\infty} I_n = \emptyset$ .

Throughout this paper all ultrafilters will be assumed to be countably incomplete. Notice that  $\mathcal{U}$  is countably incomplete if and only if there is a function  $n : I \rightarrow \mathbb{N}$  such that  $n(i) \rightarrow \infty$  along  $\mathcal{U}$  (equivalently, there is a family  $\varepsilon(i)$  of strictly positive numbers converging to zero along  $\mathcal{U}$ ). It is obvious that any countably incomplete ultrafilter is *free* (contains no singleton) and also that every free ultrafilter on  $\mathbb{N}$  is countably incomplete. Assuming all free ultrafilters are countably incomplete is consistent with ZFC, the usual setting of set theory, with the axiom of choice.

**2.2. Ultraproducts of Banach spaces.** Let us briefly recall the definition and some basic properties of ultraproducts of Banach spaces. Let  $(X_i)_{i \in I}$  be a family of Banach spaces indexed by the set  $I$  and let  $\mathcal{U}$  be an ultrafilter on  $I$ . The space of bounded families  $\ell_{\infty}(I, X_i)$  endowed with the supremum norm is a Banach space, and  $c_0^{\mathcal{U}}(X_i) = \{(x_i) \in \ell_{\infty}(I, X_i) : \lim_{\mathcal{U}} \|x_i\| = 0\}$  is a closed subspace of  $\ell_{\infty}(I, X_i)$ . The *ultraproduct* of the spaces  $(X_i)_{i \in I}$

following  $\mathcal{U}$  is defined as the quotient space

$$[X_i]_{\mathcal{U}} = \ell_{\infty}(I, X_i) / c_0^{\mathcal{U}}(X_i),$$

with the quotient norm. We denote by  $[(x_i)]$  the element of  $[X_i]_{\mathcal{U}}$  which has the family  $(x_i)$  as a representative. It is easy to see that  $\|[(x_i)]\| = \lim_{\mathcal{U}} \|x_i\|$ . In the case  $X_i = X$  for all  $i$ , we denote the ultraproduct by  $X_{\mathcal{U}}$ , and call it the *ultrapower* of  $X$  following  $\mathcal{U}$ . If  $T_i : X_i \rightarrow Y_i$  is a uniformly bounded family of operators, the *ultraproduct operator*  $[T_i]_{\mathcal{U}} : [X_i]_{\mathcal{U}} \rightarrow [Y_i]_{\mathcal{U}}$  is given by  $[T_i]_{\mathcal{U}}[(x_i)] = [T_i(x_i)]$ . Quite clearly,  $\|[T_i]_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|T_i\|$ .

**2.3. Banach spaces of type  $\mathcal{L}_{\infty}$  and Lindenstrauss spaces.** We shall write  $X \sim Y$  to indicate that the Banach spaces  $X$  and  $Y$  are linearly isomorphic. If they are isometric we write  $X \approx Y$ . The ground field is  $\mathbb{R}$ .

A Banach space  $X$  is said to be an  $\mathcal{L}_{\infty, \lambda}$ -space (with  $\lambda \geq 1$ ) if every finite-dimensional subspace  $F$  of  $X$  is contained in another finite-dimensional subspace of  $X$  whose Banach–Mazur distance to the corresponding  $\ell_{\infty}^n$  is at most  $\lambda$ . The *Banach–Mazur distance* between two isomorphic Banach spaces  $X$  and  $Y$  is defined as  $d(X, Y) = \inf_T \|T\| \|T^{-1}\|$ , where  $T$  runs over all isomorphisms between  $X$  and  $Y$ .

An  $\mathcal{L}_{\infty}$ -space is just an  $\mathcal{L}_{\infty, \lambda}$ -space for some  $\lambda \geq 1$ ; we will say that it is an  $\mathcal{L}_{\infty, \lambda+}$ -space when it is an  $\mathcal{L}_{\infty, \mu}$ -space for all  $\mu > \lambda$ . The  $\mathcal{L}_{\infty, 1+}$ -spaces are usually called *Lindenstrauss spaces* and coincide with the isometric pre-duals of  $L_1(\mu)$ -spaces [41, Theorem 4.1]. The classes of  $\mathcal{L}_{\infty, \lambda+}$ -spaces are stable under ultraproducts [11, Proposition 1.22]. In the opposite direction, a Banach space is an  $\mathcal{L}_{\infty, \lambda+}$ -space if and only if some (or every) ultrapower is. In particular, a Banach space is an  $\mathcal{L}_{\infty}$ -space or a Lindenstrauss space if and only if so are its ultrapowers (see, e.g., [20]). However it is possible to obtain Lindenstrauss spaces as ultraproducts of families of reflexive spaces: indeed, if  $p(i) \rightarrow \infty$  along  $\mathcal{U}$ , then the ultraproduct  $[L_{p(i)}]_{\mathcal{U}}$  is a Lindenstrauss space—in fact, an abstract  $M$ -space (see [13, Lemma 3.2]).

**2.4. Some classes of Lindenstrauss spaces.** We list some distinguished classes of Lindenstrauss spaces that we shall consider along the paper:

- *C-spaces*: Banach spaces of the form  $C(K)$  for some compact Hausdorff space  $K$ , with the sup norm.
- *C<sub>0</sub>-spaces*: maximal ideals of  $C$ -spaces.
- *G-spaces*: Banach spaces of the form  $X = \{f \in C(K) : f(x_i) = \lambda f(y_i) \text{ for all } i \in I\}$  for some compact space  $K$  and some family of triples  $(x_i, y_i, \lambda_i)$ , where  $x_i, y_i \in K$  and  $\lambda_i \in \mathbb{R}$ .
- *M-spaces*:  $G$ -spaces where  $\lambda_i \geq 0$  for every  $i \in I$ ; equivalently, closed sublattices of  $C$ -spaces.

It is perhaps worth noticing that all these classes admit quite elegant characterizations:  $C_0$ -spaces ( $C$ -spaces) are exactly those real Banach algebras  $X$  (with unit) satisfying the inequality  $\|x\|^2 \leq \|x^2 + y^2\|$  for all  $x, y \in X$ , a classical result by Arens (see [1, Theorem 4.2.5]). Also, a Banach lattice  $X$  is representable as a concrete  $M$ -space if and only if one has  $\|x + y\| = \max(\|x\|, \|y\|)$  whenever  $x$  and  $y$  are disjoint, that is,  $|x| \wedge |y| = 0$ . Finally,  $G$ -spaces are exactly those Banach spaces that are contractively complemented in  $M$ -spaces. The preceding classes are closed under ultraproducts (see [20, Proposition 1]). In particular, if  $(K_i)$  is a family of compact spaces indexed by  $I$  and  $\mathcal{U}$  is an ultrafilter on  $I$ , then there is a compact space  $K$  such that  $[C(K_i)]_{\mathcal{U}}$  is isometric to  $C(K)$ . This compact space  $K$  is often called the *ultraproduct* of the family  $(K_i)$  with respect to  $\mathcal{U}$  and is denoted by  $(K_i)_{\mathcal{U}}$ . We refer the interested reader to [19, Section 4] or [37, Section 8] for a description of  $(K_i)_{\mathcal{U}}$  based on Banach algebras techniques and to [6, Section 5] for a purely topological construction of the ultraproduct.

**3. Around Stern’s lemma.** Throughout this section,  $[X_i]_{\mathcal{U}}$  will denote the ultraproduct of a family of Banach spaces  $(X_i)_{i \in I}$  with respect to a countably incomplete ultrafilter  $\mathcal{U}$ . We begin with the following result about the structure of separable subspaces of ultraproducts of type  $\mathcal{L}_{\infty}$ .

LEMMA 3.1. *Suppose  $[X_i]_{\mathcal{U}}$  is an  $\mathcal{L}_{\infty, \lambda+}$ -space. Then each separable subspace of  $[X_i]_{\mathcal{U}}$  is contained in a subspace of the form  $[F_i]_{\mathcal{U}}$ , where  $F_i \subset X_i$  is finite-dimensional and  $\lim_{\mathcal{U}(i)} d(F_i, \ell_{\infty}^{k(i)}) \leq \lambda$ , with  $k(i) = \dim F_i$ .*

*Proof.* Let us assume  $S$  is an infinite-dimensional separable subspace of  $[X_i]_{\mathcal{U}}$ . Let  $(s^n)$  be a linearly independent sequence spanning a dense subspace in  $S$  and, for each  $n$ , let  $(s_i^n)$  be a fixed representative of  $s^n$  in  $\ell_{\infty}(I, X_i)$ . Let  $S^n = \text{span}\{s^1, \dots, s^n\}$ . Since  $[X_i]_{\mathcal{U}}$  is an  $\mathcal{L}_{\infty, \lambda+}$ -space, there is, for each  $n$ , a finite-dimensional  $F^n \subset [X_i]_{\mathcal{U}}$  containing  $S^n$  with  $d(F^n, \ell_{\infty}^{\dim F^n}) \leq \lambda + 1/n$ . For fixed  $n$ , let  $(f^m)$  be a basis for  $F^n$  containing  $s^1, \dots, s^n$ . Choose representatives  $(f_i^m)$  such that  $f_i^m = s_i^l$  if  $f^m = s^l$ . Moreover, let  $F_i^n$  be the subspace of  $X_i$  spanned by  $f_i^m$  for  $1 \leq m \leq \dim F^n$ . Let  $(I_n)$  be a decreasing sequence of subsets  $I_n \in \mathcal{U}$  such that  $\bigcap_{n=1}^{\infty} I_n = \emptyset$ . For each integer  $n$  put

$$J'_n = \{i \in I : d(F_i^n, \ell_{\infty}^{\dim F^n}) \leq \lambda + 2/n\} \cap I_n$$

and  $J_m = \bigcap_{n \leq m} J'_n$ . All these sets are in  $\mathcal{U}$ . We define a function  $k : I \rightarrow \mathbb{N}$  as

$$k(i) = \sup\{n : i \in J_n\}.$$

For each  $i \in I$ , take  $F_i = F_i^{k(i)}$ . This is a finite-dimensional subspace of  $X_i$  whose Banach–Mazur distance to the corresponding  $\ell_{\infty}^k$  is at most  $\lambda + 2/k(i)$ . It is clear that  $[F_i]_{\mathcal{U}}$  contains  $S$  and also that  $k(i) \rightarrow \infty$  along  $\mathcal{U}$ . ■

Recall that a Banach space  $X$  is said to be a *Grothendieck space* if every  $c_0$ -valued operator is weakly compact; equivalently, if weak\* and weak convergence for sequences in the dual space coincide. For every set  $\Gamma$  the space  $\ell_\infty(\Gamma)$  is Grothendieck. One has:

PROPOSITION 3.2. *If  $[X_i]_{\mathcal{U}}$  is an  $\mathcal{L}_\infty$ -space, then it is a Grothendieck space.*

*Proof.* It is fairly obvious that a Banach space  $X$  in which every separable subspace is contained in a Grothendieck subspace of  $X$  must be a Grothendieck space. Thus, in view of Lemma 3.1, one needs to show that all spaces  $[\ell_\infty^{n(i)}]_{\mathcal{U}}$  are Grothendieck spaces. But this follows from the definition of the ultraproduct space  $[\ell_\infty^{n(i)}]_{\mathcal{U}}$  as a quotient of  $\ell_\infty(\ell_\infty^{n(i)}) = \ell_\infty(\Gamma)$  and the simple fact that quotients of Grothendieck spaces are Grothendieck spaces. ■

Therefore, ultraproducts which are  $\mathcal{L}_\infty$ -spaces cannot contain infinite-dimensional separable complemented subspaces, in particular,  $c_0$ . This shows that Stern's claim that  $c_0((c_0)_{\mathcal{U}})$  is isometric to a complemented subspace of  $(c_0)_{\mathcal{U}}$  cannot be true since  $c_0((c_0)_{\mathcal{U}})$  obviously contains complemented copies of  $c_0$ . If we focus our attention on copies of  $c_0$ , we can present a much more general result, which improves Corollary 3.14 of Henson and Moore [26].

PROPOSITION 3.3. *No ultraproduct of Banach spaces over a countably incomplete ultrafilter contains a complemented subspace isomorphic to  $c_0$ .*

*Proof.* Assume  $[X_i]_{\mathcal{U}}$  has a subspace isomorphic to  $c_0$ , complemented or not, and let  $\iota : c_0 \rightarrow [X_i]_{\mathcal{U}}$  be the corresponding embedding.

Let  $f^n = \iota(e_n)$ , where  $(e_n)$  denotes the traditional basis of  $c_0$ , and let  $(f_i^n)$  be a representative of  $f^n$  in  $\ell_\infty(I, X_i)$ , with  $\|(f_i^n)\|_\infty = \|f^n\|$ . Then we have

$$\|\iota^{-1}\|^{-1}\|(t_n)\|_\infty \leq \left\| \sum_n t_n f^n \right\| \leq \|\iota\| \|(t_n)\|_\infty,$$

for all  $(t_n)$  in  $c_0$ . Fix  $0 < c < \|\iota^{-1}\|^{-1}$  and  $\|\iota\| < C$ , and for  $k \in \mathbb{N}$  define

$$J_k = \left\{ i \in I : c\|(t_n)\|_\infty \leq \left\| \sum_{n=1}^k t_n f_i^n \right\|_{X_i} \leq C\|(t_n)\|_\infty \text{ for all } (t_n) \in \ell_\infty^k \right\}.$$

It is easily seen that  $J_k$  belongs to  $\mathcal{U}$  for all  $k$ . Moreover,  $J_1 = I$  and  $J_{k+1} \subset J_k$  for all  $k \in \mathbb{N}$ . Now, for each  $i \in I$ , define  $k : I \rightarrow \mathbb{N} \cup \{\infty\}$  by setting  $k(i) = \sup\{n : i \in J_n\}$ .

Let us consider the ultraproduct  $[c_0^{k(i)}]_{\mathcal{U}}$ , where  $c_0^k = \ell_\infty^k$  when  $k$  is finite and  $c_0^k = c_0$  for  $k = \infty$ . We define operators  $j_i : c_0^{k(i)} \rightarrow X_i$  taking  $j_i(e_n) = f_i^n$  for  $1 \leq n \leq k(i)$  for finite  $k(i)$  and for all  $n$  if  $k(i) = \infty$ . These are uniformly bounded and so they define an operator  $j : [c_0^{k(i)}]_{\mathcal{U}} \rightarrow [X_i]_{\mathcal{U}}$ . Also, we define

$\kappa : c_0 \rightarrow [c_0^{k(i)}]_{\mathcal{U}}$  taking  $\kappa(x) = [(\kappa_i(x))]$ , where  $\kappa_i$  is the obvious projection of  $c_0$  onto  $c_0^{k(i)}$ . We claim that  $j\kappa = \iota$ . Indeed, for  $n \in \mathbb{N}$ , we have  $\kappa_i(e_n) = e_n$  (at least) for all  $i \in J_n$  and since  $J_n \in \mathcal{U}$  we have  $j \circ \kappa(e_n) = \iota(e_n)$  for all  $n \in \mathbb{N}$ . Now, if  $p : [X_i]_{\mathcal{U}} \rightarrow c_0$  is a projection for  $\iota$ , that is,  $pi$  is the identity on  $c_0$ , then  $pj$  is a projection for  $\kappa : c_0 \rightarrow [c_0^{k(i)}]_{\mathcal{U}}$ , which cannot be true since the latter is a Grothendieck space. ■

As we mentioned in the Introduction, if an ultraproduct  $E$  is an  $\mathcal{L}_\infty$ -space then it is universally separably injective in the following sense: for every Banach space  $X$  and each separable subspace  $Y \subset X$ , every operator  $t : Y \rightarrow E$  extends to an operator  $T : X \rightarrow E$  (see [3, Theorem 4.10]). In spite of this fact, infinite-dimensional ultraproducts via a countably incomplete ultrafilters are never injective (a Banach space  $E$  is said to be *injective* when  $E$ -valued operators can be extended to any superspace). We give the proof here because Henson–Moore’s proof in [25, Theorem 2.6] is written in the language of nonstandard analysis, and Sims’ version for ultraproducts along Section 8 of [37] is not easily accessible.

**THEOREM 3.4** (Henson and Moore). *Ultraproducts via countably incomplete ultrafilters are never injective, unless they are finite-dimensional.*

*Proof.* Recalling that injective Banach spaces are  $\mathcal{L}_\infty$ -spaces, assume that  $[X_i]_{\mathcal{U}}$  is an  $\mathcal{L}_\infty$ -space. According to Lemma 3.1, if  $[X_i]_{\mathcal{U}}$  is infinite-dimensional, it contains some infinite-dimensional complemented subspace isomorphic to  $[\ell_\infty^{k(i)}]_{\mathcal{U}}$ . Thus, it suffices to see that the latter is not an injective space.

Let  $(S_i)_{i \in I}$  be a family of sets and  $\mathcal{U}$  an ultrafilter on  $I$ . The *set-theoretic ultraproduct*  $\langle S_i \rangle_{\mathcal{U}}$  is the product set  $\prod_i S_i$  factored by the equivalence relation

$$(s_i) \equiv (t_i) \Leftrightarrow \{i \in I : s_i = t_i\} \in \mathcal{U}.$$

The class of  $(s_i)$  in  $\langle S_i \rangle_{\mathcal{U}}$  is denoted  $\langle (s_i) \rangle$ . Let thus  $\langle \{1, \dots, k(i)\} \rangle_{\mathcal{U}}$  denote the set-theoretic ultraproduct of the sets  $\{1, \dots, k(i)\}$ . We have

$$(1) \quad c_0(\langle \{1, \dots, k(i)\} \rangle_{\mathcal{U}}) \subset [\ell_\infty^{k(i)}]_{\mathcal{U}} \subset \ell_\infty(\langle \{1, \dots, k(i)\} \rangle_{\mathcal{U}}).$$

This should be understood as follows: each  $[(f_i)] \in [\ell_\infty^{k(i)}]_{\mathcal{U}}$  defines a function on  $\langle \{1, \dots, k(i)\} \rangle_{\mathcal{U}}$  by the formula  $f \langle (x_i) \rangle_{\mathcal{U}} = \lim_{\mathcal{U}(i)} f_i(x_i)$ . In this way,  $[\ell_\infty^{k(i)}]_{\mathcal{U}}$  embeds isometrically as a subspace of  $\ell_\infty(\langle \{1, \dots, k(i)\} \rangle_{\mathcal{U}})$  containing  $c_0(\langle \{1, \dots, k(i)\} \rangle_{\mathcal{U}})$ . Write  $\Gamma = \langle \{1, \dots, k(i)\} \rangle_{\mathcal{U}}$  and  $U = [\ell_\infty^{k(i)}]_{\mathcal{U}}$ , so that (1) becomes  $c_0(\Gamma) \subset U \subset \ell_\infty(\Gamma)$ . We will prove that the inclusion of  $c_0(\Gamma)$  into  $U$  cannot be extended to  $\ell_\infty^c(\Gamma)$ , the space of all countably supported bounded families on  $\Gamma$ .

Recall that an *internal* subset of  $\Gamma$  is one of the form  $\langle A_i \rangle_{\mathcal{U}}$ , where  $A_i \subset \{1, \dots, k(i)\}$  for each  $i \in I$ . Infinite internal sets must have cardinality at least  $\mathfrak{c}$ —just use an almost disjoint family. This is the basis of the ensuing argument: as  $U$  is spanned by the characteristic functions of internal sets, if  $f \in U$  is not in  $c_0(\Gamma)$ , then there is  $\delta > 0$  and an infinite internal  $A \subset \Gamma$  such that  $|f| \geq \delta$  on  $A$ .

Suppose  $I : \ell_\infty^c(\Gamma) \rightarrow U$  is an operator extending the inclusion of  $c_0(\Gamma)$  into  $U$ . Given a countable  $S \subset \Gamma$ , let us consider  $\ell_\infty(S)$  as the subspace of  $\ell_\infty^c(\Gamma)$  consisting of all functions vanishing outside  $S$  and let us write  $I_S$  for the endomorphism of  $\ell_\infty(S)$  given by  $I_S(f) = 1_S I(f)$ , where  $1_S$  is the characteristic function of  $S$ . Notice that  $I_S$  cannot map  $\ell_\infty(S)$  to  $c_0(S)$  since  $c_0$  is not complemented in  $\ell_\infty$ . Thus, given an infinite countable  $S \subset \Gamma$ , there is a norm one  $f \in \ell_\infty(S)$  (the characteristic function of a countable subset of  $S$ , if you prefer), a number  $\delta > 0$  and an infinite internal  $A \subset \Gamma$  such that  $|I(f)| \geq \delta$  on  $A$ , with  $|A \cap S| = \aleph_0$ . Let  $\beta(S)$  denote the supremum of the numbers  $\delta$  arising in this way. Also, if  $T$  is any subset of  $\Gamma$ , put  $\beta[T] = \sup\{\beta(S) : S \subset T, |S| = \aleph_0\}$ .

Let  $S_1$  be a countable set such that  $\beta(S_1) > \frac{1}{2}\beta[\Gamma]$  and let us take  $f_1 \in \ell_\infty(S_1)$  such that  $|I(f_1)| > \frac{1}{2}\beta(S_1)$  on an infinite internal set  $A^1$  with  $|A^1 \cap S_1| = \aleph_0$ .

Let  $S_2$  be a countable subset of  $A^1 \setminus S_1$  (notice  $|A^1 \setminus S_1| \geq \mathfrak{c}$ ) such that  $\beta(S_2) > \frac{1}{2}\beta[A^1 \setminus S_1]$  and take a norm one  $f_2 \in \ell_\infty(S_2)$  such that  $|I(f_2)| \geq \frac{1}{2}\beta(S_2)$  on an infinite internal set  $A^2 \subset A^1$  with  $|A^2 \cap S_2| = \aleph_0$ . Let  $S_3$  be an infinite countable subset of  $A^2 \setminus (S_1 \cup S_2)$  such that  $\beta(S_3) > \frac{1}{2}\beta[A^2 \setminus (S_1 \cup S_2)]$  and take a normalized  $f_3 \in \ell_\infty(S_3)$  such that  $|If_3| > \frac{1}{2}\beta(S_3)$  on certain internal  $A^3 \subset A^2$  such that  $|A^3 \cap S_3| = \aleph_0$ .

Continuing in this way we get sequences  $(S_n)$ ,  $(f_n)$  and  $(A^n)$ , where:

- Each  $A^n$  is an infinite internal subset of  $\Gamma$ .
- $A^0 = \Gamma$  and  $A^{n+1} \subset A^n$  for all  $n$ .
- $S_{n+1}$  is a countable subset of  $A^n \setminus \bigcup_{m=1}^n S_m$ , and

$$\beta(S_{n+1}) > \frac{1}{2}\beta\left[A^n \setminus \bigcup_{m=1}^n S_m\right].$$

- $f_n$  is a normalized function in  $\ell_\infty(S_n)$ .
- $|If_n| > \frac{1}{2}\beta(S_n)$  on  $A^n$ .
- For each  $n$  one has  $|A^n \cap S_n| = \aleph_0$ .

Our immediate aim is to see that  $\beta(S_n)$  converges to zero. Fix  $n$  and select any  $a \in A^{n+1}$  to define

$$h_n = \sum_{m=1}^n \text{sign}(If_m(a))f_m.$$



Clearly,  $\|h_n\| = 1$  since the  $f_m$ 's have disjoint supports. On the other hand,

$$\|I\| \geq \|Ih_n\| \geq Ih_n(a) = \sum_{m=1}^n |If_m(a)| \geq \frac{1}{2} \sum_{m=1}^n \beta(S_m),$$

so  $(\beta(S_n))$  is even summable.

For each  $n \in \mathbb{N}$ , choose a point  $a_n \in S_n$  and consider the set  $S = \{a_n : n \in \mathbb{N}\}$ . We achieve the final contradiction by showing that  $I_S$  maps  $\ell_\infty(S)$  to  $c_0(S)$ , thus completing the proof. Indeed, pick  $f \in \ell_\infty(S)$  and compute  $\text{dist}(1_S I(f), c_0(S))$ . For each  $n \in \mathbb{N}$ , set  $R_n = \{a_m : m \geq n\}$ . Then  $f = 1_{R_n} f + (1_S - 1_{R_n})f$  and since  $S \setminus R_n$  is finite, we have  $If = I1_{R_n} f + I((1_S - 1_{R_n})f) = I1_{R_n} f + (1_S - 1_{R_n})f$ . Moreover, the function  $1_{R_n} f$  has countable support contained in  $A^n \setminus \bigcup_{m=1}^n S_m$ . So,

$$\begin{aligned} \text{dist}(1_S If, c_0(S)) &= \text{dist}(1_S I1_{R_n} f, c_0(S)) \\ &\leq \text{dist}(1_{R_n} I1_{R_n} f, c_0(R_n)) + \text{dist}(1_{S \setminus R_n} I1_{R_n} f, c_0(S \setminus R_n)) \\ &= \text{dist}(1_{R_n} I1_{R_n} f, c_0(R_n)) \leq \|1_{R_n} f\| \beta(R_n) \\ &\leq \|f\| \beta \left[ A^n \setminus \bigcup_{m=1}^n S_m \right] \leq 2\|f\| \beta(S_{n+1}). \end{aligned}$$

And since  $\beta(S_{n+1}) \rightarrow 0$  we are done. ■

REMARKS 3.5. (a) Let us give a simpler proof of Theorem 3.4 for “countable” ultraproducts. The ensuing argument relies on Rosenthal’s result [35, Corollary 1.5] asserting that an injective Banach space containing  $c_0(\Gamma)$  contains  $\ell_\infty(\Gamma)$  as well. Suppose  $I$  countable. Then  $[\ell_\infty^{k(i)}]_{\mathcal{U}}$  is a quotient of  $\ell_\infty$ , and so its density character is (at most) the continuum. On the other hand, if  $[\ell_\infty^{k(i)}]_{\mathcal{U}}$  is infinite-dimensional, then  $\lim_{\mathcal{U}} k(i) = \infty$ , and using an almost disjoint family we see that the cardinality of  $\Gamma = \langle \{1, \dots, k(i)\} \rangle_{\mathcal{U}}$  equals the continuum. Thus, if  $[\ell_\infty^{k(i)}]_{\mathcal{U}}$  were injective, as it contains  $c_0(\Gamma)$ , it should contain a copy of  $\ell_\infty(\langle \{1, \dots, k(i)\} \rangle_{\mathcal{U}})$ , which is not possible, because the latter space has density character  $2^{\mathfrak{c}}$ .

(b) Leung and Rábiger proved in [29] that given a family  $(E_i)_{i \in I}$  of Banach spaces containing no complemented copy of  $c_0$ , the space  $\ell_\infty(I, E_i)$  does not contain a complemented copy of  $c_0$  if the cardinal of  $I$  is not real-valued measurable (that is, every countably additive measure defined on the power set of  $I$  and vanishing on every singleton is zero), in particular if  $I$  is countable. This implies that when  $I$  has non-real-valued measurable cardinal, the ultraproduct  $(E_i)_{\mathcal{U}}$  of a family  $(E_i)_{i \in I}$  of Lindenstrauss Grothendieck spaces is a Grothendieck space.

(c) It is a challenging problem in set theory to decide if measurable cardinals exist, that is, if some set can ever support a countably complete, free ultrafilter. In any case such a cardinal should be very, very large (see

[16, Section 4.2]). However ultraproducts based on countably complete ultrafilters should not be very interesting to us. In fact, if  $\mathcal{U}$  is countably complete and  $|X|$  is less than the least uncountable measurable cardinal, then  $X_{\mathcal{U}} = X$  in the sense that the diagonal embedding is onto. This is so because if  $\mathcal{U}$  is countably complete, one has  $\langle X_i \rangle_{\mathcal{U}} = [X_i]_{\mathcal{U}}$  for all families of Banach spaces in view of the remark following Definition 1, and the diagonal embedding of  $X$  into  $\langle X \rangle_{\mathcal{U}}$  is onto according to [16, Corollary 4.2.8].

**4. Isomorphic equivalence.** As we mentioned before, the study of isometric equivalence of ultrapowers goes back to the inception of the ultraproduct construction in Banach space theory and has produced many interesting results in the “model theory of Banach spaces”. In this section we will rather consider the isomorphic variation introduced by Henson and Moore [26, p. 106].

DEFINITION 2. We say that two Banach spaces  $X$  and  $Y$  are *ultra-isomorphic* (respectively, *ultra-isometric*), and we write  $X \overset{u}{\sim} Y$  (respectively,  $X \overset{u}{\approx} Y$ ) for short, if there is an ultrafilter  $\mathcal{U}$  such that  $X_{\mathcal{U}}$  and  $Y_{\mathcal{U}}$  are isomorphic (respectively, isometric).

Sometimes we will say that  $X$  and  $Y$  *have the same ultratype* when they are ultra-isomorphic. The following observation shows that “having the same ultratype” provides a true equivalence relation.

LEMMA 4.1.  *$X$  and  $Y$  are ultra-isomorphic if (and only if) there are ultrafilters  $\mathcal{U}$  and  $\mathcal{V}$  such that  $X_{\mathcal{U}}$  and  $Y_{\mathcal{V}}$  are isomorphic.*

*Proof.* The iteration of ultrapowers produces new ultrapowers. Indeed, suppose that  $\mathcal{U}$  and  $\mathcal{V}$  are ultrafilters on  $I$  and  $J$  respectively. Let  $\mathcal{W}$  denote the family of all subsets  $W$  of  $K = I \times J$  for which the set  $\{j \in J : \{i \in I : (i, j) \in W\} \in \mathcal{U}\}$  belongs to  $\mathcal{V}$ . Then  $\mathcal{W}$  is an ultrafilter, often denoted by  $\mathcal{U} \times \mathcal{V}$ , and moreover  $Z_{\mathcal{W}} = (Z_{\mathcal{U}})_{\mathcal{V}}$  for all Banach spaces  $Z$ . On the other hand, the Banach space version of the Keisler–Shelah isomorphism theorem due to Stern [38, Theorem 2.1] establishes that given a Banach space  $X$  and two ultrafilters  $\mathcal{U}, \mathcal{V}$  there is an ultrafilter  $\mathcal{W}$  on some index set  $K$  such that  $(X_{\mathcal{U}})_{\mathcal{W}} \approx (X_{\mathcal{V}})_{\mathcal{W}}$ .

Now, if  $X_{\mathcal{U}} \sim Y_{\mathcal{V}}$ , taking an ultrafilter  $\mathcal{W}$  such that  $(Y_{\mathcal{U}})_{\mathcal{W}} \approx (Y_{\mathcal{V}})_{\mathcal{W}}$  we have

$$X_{\mathcal{U} \times \mathcal{W}} = (X_{\mathcal{U}})_{\mathcal{W}} \sim (Y_{\mathcal{V}})_{\mathcal{W}} \approx (Y_{\mathcal{U}})_{\mathcal{W}} = Y_{\mathcal{U} \times \mathcal{W}}. \blacksquare$$

Recall that a Banach space is an  $\mathcal{L}_{\infty}$ -space if and only if some (or every) ultrapower is. The question of the classification of  $\mathcal{L}_{\infty}$ -spaces appears in [26, p. 106] and [21, p. 315] and was already considered in [23].

PROBLEM 1. How many ultratypes of  $\mathcal{L}_{\infty}$ -spaces are there?

We will support Henson–Moore’s assertion [26, p. 106] that there are at least two different ultratypes: one is that of  $C$ -spaces and the other is that of the Gurariĭ space. The following result was proved by Henson long time ago [23, Corollary 3.11] for nonstandard hulls of  $C$ -spaces (instead of ultrapowers of  $M$ -spaces). We give a proof based on ideas of [38] that can be easily modified so as to prove Theorem 4.3 below. To simplify the exposition let us write  $X \triangleleft Y$  to mean that  $X$  is isomorphic to a complemented subspace of  $Y$ .

**PROPOSITION 4.2.** *All infinite-dimensional  $M$ -spaces have the same ultratype.*

*Proof.* The key of the reasoning is the following nice result of Stern [38, Theorem 2.2]: *Let  $F$  be a separable subspace of the Banach space  $E$ . There exists a separable subspace  $L$  of  $E$  containing  $F$  and an ultrafilter  $\mathcal{U}$  such that  $L_{\mathcal{U}} \approx E_{\mathcal{U}}$ . If  $E$  is a Banach lattice then  $L$  can be chosen to be a sublattice of  $E$ .* This implies that every  $M$ -space  $X$  has an ultrapower isometric to an ultrapower of some separable  $M$ -space  $Y$ . It is therefore enough to prove the assertion for separable  $M$ -spaces and we will prove that if  $X$  is an infinite-dimensional separable  $M$ -space, then  $X \overset{u}{\sim} c_0$ .

We first observe that  $c_0 \triangleleft X$ : all Lindenstrauss spaces contain copies of  $c_0$  and all copies of  $c_0$  are complemented in separable spaces. On the other hand, by the very definition of a separable  $\mathcal{L}_{\infty}$ -space we see that  $X$  embeds into an ultraproduct  $(\ell_{\infty}^n)_{\mathcal{U}}$ , where  $\mathcal{U}$  is any free ultrafilter on the integers. Therefore  $X$  embeds as a subspace of  $(c_0)_{\mathcal{U}}$ . By Stern’s result quoted above there is a separable sublattice  $L$  of  $(c_0)_{\mathcal{U}}$  which contains a copy of  $X$  and an ultrafilter  $\mathcal{V}$  such that  $L_{\mathcal{V}} \approx (c_0)_{\mathcal{U} \times \mathcal{V}}$ . But  $X$  and  $L$  are  $M$ -spaces and separable  $M$ -spaces are isomorphic to  $C$ -spaces (Benyamini [7]). This implies that:

- $X$  is isomorphic to its square (Bessaga–Pełczyński [10, Theorem 3]);
- $L$  contains a complemented copy of  $X$  (Pełczyński [33, Theorem 1]).

We have arrived at the following situation:

$$X_{\mathcal{V}} \triangleleft L_{\mathcal{V}} \approx (c_0)_{\mathcal{U} \times \mathcal{V}} \triangleleft X_{\mathcal{U} \times \mathcal{V}}.$$

Now we can apply the ultrapower theorem to get an ultrafilter  $\mathcal{W}$  such that  $(X_{\mathcal{V}})_{\mathcal{W}} \approx (X_{\mathcal{U} \times \mathcal{V}})_{\mathcal{W}}$ . Letting  $\mathcal{T} = (\mathcal{U} \times \mathcal{V}) \times \mathcal{W}$  we have

$$X_{\mathcal{T}} \approx (X_{\mathcal{V}})_{\mathcal{W}} \triangleleft ((c_0)_{\mathcal{U} \times \mathcal{V}})_{\mathcal{W}} = (c_0)_{\mathcal{T}}.$$

Recalling that  $c_0 \triangleleft X$  one also has  $(c_0)_{\mathcal{T}} \triangleleft X_{\mathcal{T}}$ . Since both spaces  $X$  and  $c_0$  are isomorphic to their squares, the same is true for their ultrapowers, and Pełczyński’s decomposition method (see [32]) yields  $X_{\mathcal{T}} \approx (c_0)_{\mathcal{T}}$ . ■

**THEOREM 4.3.** *Let  $X$  be either an  $M$ -space or a complemented subspace of an  $M$ -space that is moreover isomorphic to its square. Then  $X \overset{u}{\sim} \ell_{\infty}$ .*

*Proof.* If  $X$  is an  $M$ -space, the statement is contained in the preceding proposition. Suppose  $X$  is isomorphic to its square and complemented in an  $M$ -space  $E$ . As  $E$  has the same ultratype as  $\ell_\infty$ , there is an ultrafilter  $\mathcal{U}$  such that  $E_{\mathcal{U}} \sim (\ell_\infty)_{\mathcal{U}}$  and so  $X_{\mathcal{U}} \triangleleft (\ell_\infty)_{\mathcal{U}}$ . But  $X$  is an infinite-dimensional  $\mathcal{L}_\infty$ -space and so  $\ell_\infty$  embeds as a subspace of  $X_{\mathcal{U}}$ . Hence  $\ell_\infty \triangleleft X_{\mathcal{U}} \triangleleft (\ell_\infty)_{\mathcal{U}}$ . Let  $\mathcal{V}$  be an ultrafilter such that  $(\ell_\infty)_{\mathcal{V}} \approx (\ell_\infty)_{\mathcal{U} \times \mathcal{V}}$ . One has

$$(\ell_\infty)_{\mathcal{U} \times \mathcal{V}} \approx (\ell_\infty)_{\mathcal{V}} \triangleleft X_{\mathcal{U} \times \mathcal{V}} \triangleleft (\ell_\infty)_{\mathcal{U} \times \mathcal{V}}$$

and since  $X$  and  $\ell_\infty$  and their ultrapowers are all isomorphic to their squares we can apply Pełczyński's decomposition method again and we are done. ■

Regarding the statements quoted in the Introduction, this provides a proof for (c) and amends (a) and (b): both are true (at least) under the additional hypothesis that  $E$  is isomorphic to its square. We show now that the Gurariĭ space has a different ultratype.

Let us recall a few basic facts about this space. A Banach space  $U$  is said to be of *almost universal disposition* if, given isometric embeddings  $u : A \rightarrow U$  and  $v : A \rightarrow B$ , where  $A$  and  $B$  are finite-dimensional, and  $\varepsilon > 0$ , there is an  $(1 + \varepsilon)$ -isometric embedding  $u' : B \rightarrow U$  such that  $u = u'v$ . Gurariĭ shows that there exists a separable Banach space of almost universal disposition [18, Theorem 2]. This space was shown by Lusky [30] to be unique, up to isometries; see [28] for an elementary proof. We will thus call it the *Gurariĭ space* and denote it by  $\mathcal{G}$ .

Henson and Moore [25, Theorem 6.5] show that a Banach space is of almost universal disposition if and only if some (or every) ultrapower of it is of almost universal disposition (see [4, Proposition 5.7] for an improvement of this result).

The Gurariĭ space is a Lindenstrauss space and, moreover, every separable Lindenstrauss space is isometric to a complemented subspace of  $\mathcal{G}$  [40] whose complement is isomorphic to  $\mathcal{G}$  itself [31] (see also [34]). This implies that  $\mathcal{G}$  is isomorphic (not isometric) to its square and also that  $\mathcal{G}$  is complemented in no  $C$ -space (Benyamini and Lindenstrauss [9, Corollary 2]). With all these prolegomena one has:

**PROPOSITION 4.4.** *No ultrapower of the Gurariĭ space is isomorphic to a complemented subspace of an  $M$ -space.*

*Proof.* Assume that some ultrapower of  $\mathcal{G}$  is isomorphic to a complemented subspace of an  $M$ -space. As  $\mathcal{G}$  is isomorphic to its square, Theorem 4.3 implies that there is a compact space  $K$ , an ultrafilter  $\mathcal{U}$  and a linear isomorphism  $u : \mathcal{G}_{\mathcal{U}} \rightarrow C(K)$ . Let  $G_1$  be a linear subspace of  $\mathcal{G}_{\mathcal{U}}$  isometric to  $\mathcal{G}$ , for instance that lying on the diagonal. Let  $A_1$  be the (separable) unital subalgebra that  $u(G_1)$  generates in  $C(K)$ . By Stern's result quoted in the proof of Proposition 4.2, there is a separable subspace  $G_2$  containing

$u^{-1}(A_1)$  and having an ultrapower isometric to an ultrapower of  $\mathcal{G}_{\mathcal{U}}$ . This implies that  $G_2$  is a space of almost universal disposition. Continuing in this way we get two sequences  $(G_n)$  and  $(A_n)$  such that:

- Every  $G_n$  is a separable space of almost universal disposition.
- Every  $A_n$  is a separable unital subalgebra of  $C(K)$ .
- For every  $n \in \mathbb{N}$  one has  $u(G_n) \subset A_n \subset u(G_{n+1})$ .

Now, letting  $G = \overline{\bigcup_n G_n}$  and  $A = \overline{\bigcup_n A_n}$  we see that  $G$  is of almost universal disposition, hence  $G \approx \mathcal{G}$ ,  $A$  is a separable and unital closed subalgebra of  $C(K)$ , hence a  $C$ -space, and  $u$  is a linear isomorphism from  $G$  onto  $A$ , which contradicts the above mentioned result of Benyamini and Lindenstrauss. ■

This amends the statement quoted as (d) in the Introduction. A more direct proof for this fact appears in [4, Theorem 6.1]. It would however be a mistake to think that the reason for such behavior is that  $\mathcal{G}$  is not complemented in any  $C$ -space, as the following examples show.

EXAMPLE 4.5.

- (a) There is a (nonseparable) Lindenstrauss space which is complemented in no  $C$ -space but has an ultrapower isomorphic to a  $C$ -space.
- (b) Under CH, there is a separable space that is not even a quotient of a Lindenstrauss space and has an ultrapower isomorphic to a  $C$ -space.

*Proof.* (a) Benyamini constructed in [8] a nonseparable  $M$ -space which is complemented in no  $C$ -space. That space has an ultrapower isomorphic to a  $C$ -space, by Theorem 4.3.

(b) It is not hard to check that if  $0 \rightarrow Y \rightarrow X \rightarrow Z \rightarrow 0$  is an exact sequence and  $\mathcal{U}$  an ultrafilter then  $0 \rightarrow Y_{\mathcal{U}} \rightarrow X_{\mathcal{U}} \rightarrow Z_{\mathcal{U}} \rightarrow 0$  is also exact (see [15, Lemma 2.2.g]).

On the other hand, it has been shown in [14, Corollary 2.4] that there is an exact sequence  $0 \rightarrow C(\Delta) \rightarrow \Omega \rightarrow C(\Delta) \rightarrow 0$  in which  $\Omega$  is not even isomorphic to a quotient of a Lindenstrauss space. Here,  $\Delta = 2^{\mathbb{N}}$  is the Cantor set. Let  $\mathcal{U}$  be a free ultrafilter on the integers and let us consider the ultrapower sequence

$$(2) \quad 0 \rightarrow C(\Delta)_{\mathcal{U}} \rightarrow \Omega_{\mathcal{U}} \rightarrow C(\Delta)_{\mathcal{U}} \rightarrow 0.$$

We will see that this sequence does split if we assume CH. Indeed, Bankston observed in [5, Proposition 2.4.1] that, under CH, the ultracoproduct  $\Delta^{\mathcal{U}}$  is homeomorphic to  $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$ , the growth of the space of integers in its Stone–Čech compactification. Thus, under CH, the sequence (2) has the form  $0 \rightarrow C(\mathbb{N}^*) \rightarrow \Omega_{\mathcal{U}} \rightarrow C(\mathbb{N}^*) \rightarrow 0$ . But we have proved in [3, Proposition 5.6] that every exact sequence of the form  $0 \rightarrow C(\mathbb{N}^*) \rightarrow X \rightarrow C(\mathbb{N}^*) \rightarrow 0$  splits and so (2) does. Therefore,  $\Omega_{\mathcal{U}} \sim C(\mathbb{N}^*) \times C(\mathbb{N}^*) \approx C(\mathbb{N}^*)$  is a  $C$ -space. ■

### 5. Further remarks and open problems

**5.1. More ultratypes, please.** We have obtained so far only two different ultratypes of  $\mathcal{L}_\infty$ -spaces: that of  $C$ -spaces and that of the Gurariĭ space. It would be interesting to add some new classes here. Reasonable candidates could be the recently constructed hereditarily indecomposable  $\mathcal{L}_\infty$ -spaces [2, 39], the preduals of  $\ell_1$  in [9, 17], or some Bourgain–Pisier spaces [12]. Since both  $\mathcal{G}$  and  $C$ -spaces are Lindenstrauss spaces, one may wonder whether every  $\mathcal{L}_\infty$ -space has an ultrapower isomorphic to a Lindenstrauss space.

The following problem was considered by Henson and Moore [25, Problem 21]. An affirmative answer would imply that the hypothesis of being isomorphic to its square is superfluous in Theorem 4.3.

**PROBLEM 2.** Does every (infinite-dimensional, separable) Banach space  $X$  have an ultrapower isomorphic to its square? What if  $X$  is an  $\mathcal{L}_\infty$ -space?

It is perhaps worth noticing that Semadeni proved in [36] that the space of continuous functions on the first uncountable ordinal is not isomorphic to its square. Needless to say, this space has an ultrapower which is isomorphic to its own square.

**5.2. Ultrasplitting.** As we already mentioned, if  $0 \rightarrow Y \rightarrow X \rightarrow Z \rightarrow 0$  is an exact sequence and  $\mathcal{U}$  an ultrafilter then  $0 \rightarrow Y_{\mathcal{U}} \rightarrow X_{\mathcal{U}} \rightarrow Z_{\mathcal{U}} \rightarrow 0$  is exact again. No criterion however is known to determine when the ultrapower sequence of a nontrivial exact sequence splits. Let us say that an exact sequence *ultrasplits* if some of its ultrapower sequences split. Applications of the previous results yield:

**PROPOSITION 5.1.** *Let  $0 \rightarrow Y \rightarrow X \rightarrow Z \rightarrow 0$  be an exact sequence.*

- *If  $X$  is a  $C$ -space and either  $Y$  or  $Z$  is the Gurariĭ space, then the sequence does not ultrasplit.*
- *Under CH, if  $Y$  an  $\mathcal{L}_\infty$ -space and  $Z$  is a separable Banach space complemented in a  $C$ -space, then the sequence ultrasplits.*

*Proof.* The first part obviously follows from Proposition 4.4. As for the second part, we may clearly assume that  $Z$  is complemented in  $C(\Delta)$ . If  $\mathcal{U}$  is a free ultrafilter on  $\mathbb{N}$ , then  $Z_{\mathcal{U}}$  is complemented in  $C(\Delta)_{\mathcal{U}}$  and, under CH, the latter space is isometric to  $C(\mathbb{N}^*)$  which is isometric to  $\ell_\infty/c_0$ —in ZFC. On the other hand  $Y_{\mathcal{U}}$  is universally separably injective, by [3, Theorem 4.10], and so every sequence

$$0 \rightarrow Y_{\mathcal{U}} \rightarrow E \rightarrow \ell_\infty/c_0 \rightarrow 0$$

splits. Therefore  $0 \rightarrow Y_{\mathcal{U}} \rightarrow X_{\mathcal{U}} \rightarrow Z_{\mathcal{U}} \rightarrow 0$  splits. ■

An interesting case occurs when one puts  $\mathcal{G}$  as the quotient space. Recall that Johnson and Zippin proved in [27] that every separable Lindenstrauss space is a quotient of  $C(\Delta)$ ; therefore, there exists an exact sequence

$$(3) \quad 0 \rightarrow \ker q \rightarrow C(\Delta) \xrightarrow{\pi} \mathcal{G} \rightarrow 0$$

which does not ultrasplit, by the preceding proposition. Pełczyński posed on the blackboard to us the question of whether it is possible to identify the kernel(s) of the preceding sequence(s) and in particular if some kernel can be a  $C$ -space. Observe that the structure of  $\ker \pi$  effectively depends on the quotient map  $\pi$ . It is not hard to check that  $\ker \pi$  is an  $\mathcal{L}_\infty$ -space when  $\pi$  is an “isometric” quotient—this means that  $\pi$  maps the open unit ball of  $C(\Delta)$  onto that of  $\mathcal{G}$ . On the other hand, Bourgain has shown that  $\ell_1$  does contain an uncomplemented subspace isomorphic to itself; this implies that there is an exact sequence  $0 \rightarrow E \rightarrow F \rightarrow G \rightarrow 0$  in which both  $F$  and  $G$  are isomorphic to  $c_0$  but  $E$  is not an  $\mathcal{L}_\infty$ , as can be seen in [11, Appendix 1]. Since both  $C(\Delta)$  and  $\mathcal{G}$  have (complemented) subspaces isomorphic to  $c_0$  we see that there are quotient mappings  $\pi : C(\Delta) \rightarrow \mathcal{G}$  whose kernels are not  $\mathcal{L}_\infty$ -spaces.

**5.3. Lindenstrauss spaces with isometric ultrapowers.** As we already mentioned, Heinrich undertook in [20] the classification of Lindenstrauss spaces up to ultra-isometry. Amongst the many interesting results he proved, one finds that the class of  $C$ -spaces is closed under “isometric ultraroots”: this just means that if a Banach space  $X$  has an ultrapower isometric to a  $C$ -space then  $X$  is itself isometric to a  $C$ -space. A similar result holds for  $G$ -spaces (see [20, Theorems 2.7 and 2.10]).

The result by Henson and Moore [25, Theorem 6.5] that a Banach space is of almost universal disposition if and only if some (or every) ultrapower is of almost universal disposition shows that the class of Lindenstrauss spaces of almost universal disposition is also closed under “isometric ultraroots”. One can deduce from this that a Banach space  $E$  has some ultrapower isometric to an ultrapower of the Gurariĭ space if and only if every separable subspace of  $E$  is contained in a Gurariĭ space contained in  $E$ .

At the end of [20] Heinrich asks whether the classes of  $C_0$ -spaces and  $M$ -spaces enjoy the same property. In a subsequent paper [22, Section 4] (and also in [26], around Problem 4) it is claimed that there is a Banach space  $X$  which fails to be isometric to a Banach lattice and such that  $X \overset{u}{\approx} c_0$ . Since  $c_0$  is both a  $C_0$ -space and an  $M$ -space this would imply a negative solution for both questions. Unfortunately, a close inspection of the example reveals that it is indeed a  $C_0$ -space since it is a subalgebra of  $\ell_\infty$ . Indeed, if  $\mathcal{F}$  is any almost disjoint family of subsets of  $\mathbb{N}$ , then the closed linear span of the characteristic functions of all sets of  $\mathcal{F}$  and  $c_0$  is always a subalgebra of  $\ell_\infty$ .

Thus, the following should be considered as an open problem.

**PROBLEM 3.** Are the classes of  $C_0$ -spaces and  $M$ -spaces closed under “isometric ultraroots”?

The following problem appears both in [21] (see Problem 2 on p. 316) and [26] (see Problems 5 and 7 on pp. 103 and 104).

**PROBLEM 4** (Heinrich, Henson, Moore). Does the Gurarii space have an ultrapower isometric (or isomorphic) to an ultraproduct of finite-dimensional spaces?

Of course the hypothesized finite-dimensional spaces could not be at uniform distance from the corresponding  $\ell_\infty^n$ -spaces.

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