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Almost Abelian regular dessins d'enfants

by

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Abstract. A regular dessin d'enfant, in this paper, will be a pair (S,β), where S is a closed Riemann surface and $\beta : S \to \widehat{\mathbb{C}}$ is a regular branched cover whose branch values are contained in the set $\{\infty, 0, 1\}$. Let $\mathrm{Aut}(S,\beta)$ be the group of automorphisms of (S,β), that is, the deck group of β. If $\mathrm{Aut}(S,\beta)$ is Abelian, then it is known that (S,β) can be defined over $\mathbb{Q}$. We prove that, if A is an Abelian group and $\mathrm{Aut}(S,\beta) \cong A \rtimes \mathbb{Z}_2$, then (S,β) is also definable over $\mathbb{Q}$. Moreover, if $A \cong \mathbb{Z}_n$, then we provide explicitly these dessins over $\mathbb{Q}$.

1. Introduction and statement of results. A *dessin d'enfant* (or just a *dessin*), as defined by Grothendieck [G], corresponds to a pair (X,D), where X is a closed orientable surface and $D \subset X$ is a bipartite graph (vertices are colored black or white and adjacent vertices have different colors) such that $X - D$ consists of a finite collection of topological discs (called the *faces* of the dessin). The *genus* of (X,D) is by definition the genus of X. Two dessins d'enfants, say (X_1,D_1) and (X_2,D_2), are said to be *equivalent* if there exists an orientation preserving homeomorphism $H : X_1 \to X_2$ inducing an isomorphism between D_1 and D_2 as bipartite graphs (i.e., an isomorphism of the graphs sending black (resp. white) vertices to black (resp. white) vertices).

A *Belyĭ pair* is a pair (S,β), where S is a closed Riemann surface, called a *Belyĭ curve*, and $\beta : S \to \widehat{\mathbb{C}}$ is a non-constant meromorphic map whose branch values are contained in the set $\{\infty, 0, 1\}$, called a *Belyĭ map*. The genus of (S,β) is the genus of S. Two Belyĭ pairs, say (S_1,β_1) and (S_2,β_2), are said to be equivalent if there is a biholomorphism $F : S_1 \to S_2$ such that $\beta_2 \circ F = \beta_1$. If $S_1 = S_2 = S$ and $\beta_1 = \beta_2 = \beta$, then the above provides the definition of an *automorphism* of (S,β). We denote by $\mathrm{Aut}(S)$ the full group of conformal automorphisms of S and by $\mathrm{Aut}(S,\beta)$ its subgroup of

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automorphisms of (S,β). We say that the Belyĭ pair (S,β) is *regular* if β is a regular branch cover; in that case its deck group is $\mathrm{Aut}(S,\beta)$. The group of Möbius transformations keeping invariant the set $\{\infty,0,1\}$ is the symmetric group $\mathfrak{S}_3$ on three letters generated by $A(z)=1/z$ and $B(z)=1/(1-z)$. If we have a (regular) Belyĭ pair (S,β) and $M\in\mathfrak{S}_3$, then $(S,M\circ\beta)$ is again a (regular) Belyĭ pair and, moreover, $\mathrm{Aut}(S,\beta)=\mathrm{Aut}(S,M\circ\beta)$. In this paper we are interested in regular dessins d'enfants.

As defined above, dessins d'enfants are combinatorial (2-dimensional) objects and Belyĭ pairs are analytic objects. By the uniformization theorem, a dessin d'enfant (X,D) determines a Belyĭ pair (S,β) (unique up to equivalence) so that $D=\beta^{-1}([0,1])$ (the black vertices being the preimages of 0 and the white vertices being the preimages of 1). Conversely, a Belyĭ pair (S,β) defines a dessin d'enfant as just described above. This provides a bijection between equivalence classes of dessins d'enfants and equivalence classes of Belyĭ pairs; so we may work indistinctly with dessins d'enfants and Belyĭ pairs. In this paper we consider Belyĭ pairs as the objects under study.

A *field of definition* of a Belyĭ pair (S,β) is a subfield $\mathbb{K}$ of $\mathbb{C}$ for which there is an equivalent Belyĭ pair (C,η), where C is a smooth complex algebraic curve and η is a rational map, both defined over $\mathbb{K}$; we also say that (S,β) is *definable* over $\mathbb{K}$. Belyĭ 's theorem [B] asserts that every Belyĭ pair is definable over the field $\overline{\mathbb{Q}}$ of algebraic numbers. The *field of moduli* of (S,β) is the intersection of all its fields of definition [K].

If a regular Belyĭ pair (S,β) has genus zero, then, up to conformal equivalence, $S=\widehat{\mathbb{C}}$ (which is defined over $\mathbb{Q}$) and $\mathrm{Aut}(S,\beta)$ is any of the finite groups of Möbius transformations (finite cyclic groups, dihedral groups, the alternating groups $\mathcal{A}_4$, $\mathcal{A}_5$ and the symmetric group $\mathfrak{S}_4$). Explicit regular branched cover maps, for each of these cases, are provided in [Ho]. It can be checked that all of these are definable over $\mathbb{Q}$.

Let us assume, from now on, that the regular Belyĭ pair (S,β) has genus $g\geq 1$. In this case, the branch values of β are 0, 1 and ∞, say with branch orders k_1, k_2 and k_3; we say that S/H has *signature* $(0;k_1,k_2,k_3)$ and that the associated dessin d'enfant has *type* (k_1,k_2,k_3). By the Riemann–Hurwitz formula, the condition $g\geq 1$ is equivalent to $k_1^{-1}+k_2^{-1}+k_3^{-1}\leq 1$ (strict inequality if and only if $g\geq 2$). As a consequence of the uniformization theorem, there is a surjective homomorphism $\theta:\Gamma\to\mathrm{Aut}(S,\beta)$, with a torsion free kernel $\ker(\theta)$, where $\Gamma=\langle x,y : x^{k_1}=y^{k_2}=(yx)^{k_3}=1\rangle$ is a triangular Kleinian group uniformizing the orbifold $S/\mathrm{Aut}(S,\beta)$ and $\ker(\theta)$ uniformizing S (the converse also holds). In particular, $\mathrm{Aut}(S,\beta)$ is generated by two elements, say a and b, with a of order k_1, b of order k_2 and $c=(ab)^{-1}$ of order k_3. This, and the fact that any two generators of a dihedral group are either both of order two or one of order two with product of order two, ensures that $\mathrm{Aut}(S,\beta)$ cannot be isomorphic to a dihedral group.

In [W] Wolfart proved that (S,β) and S can both be defined over their corresponding fields of moduli (this fact can also be obtained from Dèbes–Emsalem's results in [DE]). In [Hi] it was noticed that if $\mathrm{Aut}(S,\beta)$ is Abelian, then these two fields are equal to the field $\mathbb{Q}$ of rational numbers. We may wonder for other cases ensuring a regular Belyĭ pair (S,β) to be definable over $\mathbb{Q}$. A class of groups which are close to being Abelian groups (in some rough sense) are the semi-direct products $A \rtimes B$, where A and B are both Abelian groups. In [SW] Streit–Wolfart studied the family of those regular Belyĭ pairs with $\mathrm{Aut}(S,\beta) = \langle a,b : a^p = b^q = 1,\ bab^{-1} = a^m\rangle \cong \mathbb{Z}_p \rtimes \mathbb{Z}_q$ (i.e. $A \cong \mathbb{Z}_p$ and $B \cong \mathbb{Z}_q$), where $p > 3$ and $q > 3$ are primes and $m^q \equiv 1 \bmod p$, and they exhibit explicit curves and the corresponding fields of moduli (which result to be different from $\mathbb{Q}$).

In this paper we consider the case $\mathrm{Aut}(S,\beta) \cong A \rtimes \mathbb{Z}_2$, where A is an Abelian group. The case $\mathrm{Aut}(S,\beta) \cong A \rtimes \mathbb{Z}_3$ will be considered elsewhere (see also Remark 2.2). Our first result is the following.

THEOREM 1.1. *Let (S,β) be a regular Belyĭ pair of genus $g \geq 1$ with $\mathrm{Aut}(S,\beta) \cong A \rtimes \mathbb{Z}_2$. If A is an Abelian group, then (S,β) is definable over $\mathbb{Q}$.*

If in Theorem 1.1, $A \cong \mathbb{Z}_n$, that is, $\mathrm{Aut}(S,\beta) = \langle a,b : a^n = b^2 = baba^{-m} = 1\rangle \cong \mathbb{Z}_n \rtimes \mathbb{Z}_2$, where $n \in \{2,3,\dots\}$, $m \in \{1,2,\dots,n-1\}$, $m^2 \equiv 1 \bmod n$ and $\gcd(n,m) = 1$, then the next result describes explicit models over $\mathbb{Q}$. As already noticed above, the case $m = n-1$ (the dihedral case) is not possible.

THEOREM 1.2. *Let (S,β) be a regular Belyĭ pair of genus $g \geq 1$ with $\mathrm{Aut}(S,\beta) = \langle a,b : a^n = b^2 = baba^{-m} = 1\rangle$, where $n \in \{2,3,\dots\}$, $m \in \{1,2,\dots,n-2\}$, $m^2 \equiv 1 \bmod n$ and $\gcd(n,m) = 1$. Then the following hold:*

(1) *There exist integers $\alpha,\rho,\gamma \in \{1,\dots,n-1\}$ and non-negative integers $\vartheta_1,\vartheta_2,\vartheta_3$ satisfying*
   (1.1) $\gcd(n,\alpha,\rho,\gamma) = 1$;
   (1.2) $1+\gamma-\rho = m$;
   (1.3) $(\alpha+\rho+\gamma)(2+\gamma-\rho) \equiv 0 \bmod n$;
   (1.4) $\alpha(\gamma-\rho) = n\vartheta_1$;
   (1.5) $(\rho-1)(\gamma-\rho) = n\vartheta_2$;
   (1.6) $(\gamma+1)(\gamma-\rho) = n\vartheta_3$; *and*
   (1.7) $(-1)^{(\alpha+\rho+\gamma)(2+\gamma-\rho)/n} = (-1)^{\vartheta_1+\vartheta_2+\vartheta_3}$,

   so that (S,β) is equivalent to the regular Belyĭ pair (C,η), where

$$C:\ y^n = x^{\alpha}(x-1)^{\rho}(x+1)^{\gamma},$$
$$\eta : C \to \widehat{\mathbb{C}} : (x,y) \mapsto x^2,$$

and $\mathrm{Aut}(C,\eta) = \langle a,b\rangle$ *with*

$$a(x,y) = (x,\omega y), \qquad b(x,y) = \left(-x, \frac{\delta y^{1+\gamma-\rho}}{x^{\vartheta_1}(x-1)^{\vartheta_2}(x+1)^{\vartheta_3}}\right),$$
$$\delta = (-1)^{(\alpha+\rho+\gamma)/n}, \qquad \omega = e^{2\pi i/n}.$$

(2) *If* $\xi \in \mathbb{Z}$ *is such that* $(\xi-1)n < \alpha+\rho+\gamma \le \xi n$ *and we set* $\eta = \xi n - \alpha - \rho - \gamma$, *then* (*setting* $\gcd(n,0) := n$)
$$g = 1 + n - (1/2)(\gcd(n,\alpha) + \gcd(n,\rho) + \gcd(n,\gamma) + \gcd(n,\eta)).$$
(3) *If* $\alpha+\rho+\gamma \equiv 0 \bmod n$, *then* S/β *has signature* $(0;2,p,2q)$, *where* $p = n/\gcd(n,\rho) = n/\gcd(n,\gamma)$ *and* $q = n/\gcd(n,\alpha)$.
(4) *If* $\alpha+\rho+\gamma \not\equiv 0 \bmod n$, *then* S/β *has signature* $(0;p,2q,2u)$, *where* p *and* q *are as above and* $u = n/\gcd(n,\eta)$.

In the particular case $m = 1$ (the Abelian situation), Theorem 1.2 can be written as follows.

COROLLARY 1.3. *Let* (S,β) *be a regular Belyĭ pair of genus* $g \ge 1$, *with deck group* $\mathrm{Aut}(S,\beta) \cong \mathbb{Z}_n \times \mathbb{Z}_2$. *Then there exist integers* $\alpha,\rho \in \{1,\dots,n-1\}$, *with* $\gcd(n,\alpha,\rho) = 1$ *and* $\alpha + 2\rho \equiv 0 \bmod n$, *such that* (S,β) *is equivalent to* (C,η), *where*

$$C:\ y^n = x^{\alpha}(x^2-1)^{\rho}, \qquad \eta : C \to \widehat{\mathbb{C}} : (x,y) \mapsto x^2.$$

Moreover, $\mathrm{Aut}(C,\eta) = \langle a,b\rangle$ *with*

$$a(x,y) = (x,\omega y), \qquad b(x,y) = (-x,\delta y), \qquad \delta = (-1)^{(\alpha+2\rho)/n},\ \omega = e^{2\pi i/n}.$$

Proof. This follows from Theorem 1.2 taking $m = 1$; so $\gamma - \rho = 0$, $\vartheta_1 = \vartheta_2 = \vartheta_3 = 0$, $\gcd(n,\alpha,\rho) = 1$ and $\alpha + 2\rho \equiv 0 \bmod n$. ∎

2. Proof of Theorems 1.1 and 1.2.

Let us fix a regular Belyĭ pair (S,β) of genus $g \ge 1$ with $\mathrm{Aut}(S,\beta) \cong A \rtimes \mathbb{Z}_2$, where A is an Abelian group. Let $b \in \mathrm{Aut}(S,\beta)$ be the conformal automorphism that generates the $\mathbb{Z}_2$ component.

The quotient orbifold S/A has a signature $(h;n_1,\dots,n_r)$, that is, its underlying Riemann surface is a closed Riemann surface, say R, of genus h and it has exactly r cone points of orders $n_1,\dots, n_r$, respectively. Let $P : S \to R$ be a regular branched cover with deck group A.

As A is a normal subgroup of $\mathrm{Aut}(S,\beta)$, there is a conformal automorphism $\overline{b}$ of R, of order two, such that $\overline{b} \circ P = P \circ b$. The involution $\overline{b}$ permutes the cone points of S/A and it respects their orders. Let $Q : R \to \widehat{\mathbb{C}}$ be a regular branched cover with deck group $\langle \overline{b}\rangle$ and such that $Q \circ P = \beta$.

Since $S/\mathrm{Aut}(S,\beta)$ is the Riemann sphere $\widehat{\mathbb{C}}$ with cone points at ∞, 0 and 1, it follows that $R/\langle \overline{b}\rangle$ is $\widehat{\mathbb{C}}$ and that its cone points (that is, the branch values of Q) are contained in the set $\{\infty, 0, 1\}$. So, (R,Q) is a regular Belyĭ pair with $\mathrm{Aut}(R,Q)=\langle \overline{b}\rangle$.

By the Riemann–Hurwitz formula, the number of fixed points of the involution $\overline{b}$ is even, say $2s$.

Claim 2.1. *$s=1$, $h=0$ and $r\in\{3,4\}$.*

Proof. We first prove that $s=1$. In fact, if $s=0$, then necessarily $h\geq 1$ (since on the Riemann sphere every involution has two fixed points). Now, the Riemann–Hurwitz formula ensures that $R/\langle \overline{b}\rangle$ has positive genus, a contradiction. If $s\geq 2$, then Q will have $2s\geq 4$ branch values, again a contradiction.

Now, as $\overline{b}$ has exactly two fixed points and $R/\langle \overline{b}\rangle$ has genus zero, it follows from the Riemann–Hurwitz formula that $h=0$.

The above means that the signature of S/A is of the form $(0; n_1,\ldots,n_r)$. Since the genus of S is at least one, it again follows from the Riemann–Hurwitz formula that $r\geq 3$. Now, as the cone points of S/A are permuted by the involution $\overline{b}$ and $S/\mathrm{Aut}(S,\beta)$ has exactly three cone points, it follows that $r\in\{3,4\}$. ∎

The above claim ensures that $R=\widehat{\mathbb{C}}$ and that $\overline{b}$ is a Möbius transformation of order 2. So, up to composition of P on the left with a suitable Möbius transformation, we may assume that $\overline{b}(x)=-x$; so $Q(x)=x^2$.

If $r=3$, then one of the cone points is a fixed point of $\overline{b}$ and the other two are permuted by $\overline{b}$.

If $r=4$, then two of the cone points are fixed by $\overline{b}$ and the other two are permuted by it.

Up to composition of P on the left with a Möbius transformation of the form $T(x)=dx$, for a suitable $d\in\mathbb{C}-\{0\}$, we may also assume that the cone points of S/A are ± 1 (the ones which are permuted by $\overline{b}$), 0 (and ∞ for $r=4$).

2.1. Proof of Theorem 1.1

2.1.1. If $r=3$, then the branch values of $P: S\to\widehat{\mathbb{C}}$ are given by the points ± 1 and 0. If $M(x)=(1-x)/(1+x)$, then $P_M=M\circ P$ is a Belyĭ map with deck group A. By the results in [Hi], we may assume both S and P_M to be defined over $\mathbb{Q}$. The induced involution by b, under P_M, is $\widehat{b}(x)=M\circ\overline{b}\circ M^{-1}(x)=1/x$. The two-fold branch cover $\widehat{Q}(x)=(1-x)^2/(1+x)^2$ has deck group $\langle\widehat{b}\rangle$ and $Q=\widehat{Q}\circ M$. It follows that $\beta=\widehat{Q}\circ P_M$ is defined over $\mathbb{Q}$.

2.1.2. If $r = 4$, then we may proceed as follows (see [Hi]). Let $\mu \geq 2$ be the least common multiple of the orders of the four cone points (∞, 0, 1 and -1) of S/A. Let us consider the generalized Fermat curve [GHL] (a closed Riemann surface of genus $g_C = (\mu - 1)(\mu^2 + \mu - 1) \geq 5$)

$$C = \left\{ \begin{matrix} x_1^\mu + x_2^\mu + x_3^\mu = 0 \\ -x_1^\mu + x_2^\mu + x_4^\mu = 0 \end{matrix} \right\} \subset \mathbb{P}^3_{\mathbb{C}}.$$

The group $K = \langle a_1, a_2, a_3 \rangle \cong \mathbb{Z}_\mu^3$, where a_j is multiplication by $e^{2\pi i/\mu}$ on the x_j-coordinate, is a group of conformal automorphisms of C. If $L : C \to \widehat{\mathbb{C}}$ is defined by $L([x_1 : x_2 : x_3 : x_4]) = -(x_2/x_1)^\mu$, then L is a regular branched cover with deck group K and branch values ± 1, 0 and ∞, each of order μ.

If $\Gamma = \langle y_1, y_2, y_3, y_4 : y_1^\mu = y_2^\mu = y_3^\mu = y_4^\mu = y_1 y_2 y_3 y_4 = 1 \rangle$ is a Fuchsian group acting on the hyperbolic plane $\mathbb{H}^2$ so that $\mathbb{H}^2/\Gamma = C/K$, then $\mathbb{H}^2/\Gamma' = C$, where Γ' is the derived subgroup of Γ.

It follows that there is a normal subgroup Γ_S of Γ (containing Γ') whose uniformized orbifold $\mathbb{H}^2/\Gamma_S$ has underlying Riemann surface structure isomorphic to S (and $A = \Gamma/\Gamma_S$). In particular, there is a subgroup $K_0 = \Gamma_S/\Gamma'$ of K so that the underlying Riemann surface structure of C/K_0 is isomorphic to S.

In [Hi, Section 6] we described (using geometric invariant theory) how to compute a curve model E for C/K_0 (we do not need the explicit form). By [Hi, Lemma 5.1] such a curve E, and the regular branched cover $U : C \to E$ (with deck group K_0), are both defined over $\mathbb{Q}$. As $L = P \circ U$ we may see that P is also defined over $\mathbb{Q}$ and, in particular, that $\beta = Q \circ P$ is defined over $\mathbb{Q}$.

Remark 2.2. If $\mathrm{Aut}(S, \beta) \cong A \rtimes \mathbb{Z}_3$, then one may try to follow the same ideas as in the above proof. We will have the conformal automorphism $\overline{b}$, of order 3, of the quotient orbifold S/A induced by the $\mathbb{Z}_3$ component of $\mathrm{Aut}(S, \beta)$. As $\overline{b}$ permutes the cone points of S/A and $(S/A)/\langle \overline{b} \rangle = S/\mathrm{Aut}(S, \beta)$, we necessarily have that S/A is either of genus zero or of genus one. If S/A has genus zero, then we may assume that $\overline{b}(z) = e^{2\pi i/3} z$ and that the cone points are inside the set $\{\infty, 0, 1, e^{2\pi i/3}, e^{4\pi i/3}\}$. In this case, it is not clear how to ensure that S can be defined over $\mathbb{Q}$. If the quotient S/A has genus one, then $\overline{b}$ must have exactly three fixed points; in other words, the Riemann surface structure of S/A is given by the curve $C : y^3 = x(x-1)$ and cone points being $(0,0)$, $(1,0)$ and ∞. Let us consider $Q : C \to \widehat{\mathbb{C}}$ defined by $Q(x, y) = x$ (a regular branched cover with deck group $\langle \overline{b} \rangle$) and a regular branched cover map $P : S \to C$, with deck group A; then $\beta = Q \circ P : S \to \widehat{\mathbb{C}}$, up to post-composition with a Möbius transformation in $\mathfrak{S}_3$. In order to see if the result in Theorem 1.1 holds or not for this case, we need to check if it is possible to find an algebraic curve for S and a regular branched cover $P : S \to C$, both defined over $\mathbb{Q}$.

2.2. Proof of Theorem 1.2. Let $a \in \mathrm{Aut}(S,\beta)$ be a conformal automorphism that generates the cyclic group $A = \mathbb{Z}_n$, that is, $\mathrm{Aut}(S,\beta) = \langle a, b\rangle$.

If $r = 3$, then (as the involution $\overline{b}$ permutes two of the cone points and preserves the orders) the signature of $S/\langle a\rangle$ must of the form $(0; p, p, q)$ (where p and q are divisors of n) and the signature of $S/\mathrm{Aut}(S,\beta)$ is $(0; 2, p, 2q)$. The two cone points ± 1 of S/A have order p and the other cone point 0 has order q.

If $r = 4$, then the signature of $S/\langle a\rangle$ must be of the form $(0; p, p, q, u)$ (where p, q and u are divisors of n) and the signature of $S/\mathrm{Aut}(S,\beta)$ is $(0; p, 2q, 2u)$. The cone points ± 1 have order p, the cone point 0 has order q and the cone point ∞ has order u.

It follows from [BW] that S can be described by a cyclic n-gonal curve of the form

$$C : y^n = x^{\alpha}(x-1)^{\rho}(x+1)^{\gamma},$$

where $\alpha, \rho, \gamma \in \{1, \ldots, n-1\}$ and $\gcd(n, \alpha, \rho, \gamma) = 1$.

We should note that $r = 3$ if and only if $\alpha + \rho + \gamma \equiv 0 \bmod n$ and that $r = 4$ otherwise. Moreover, also from [BW], $p = n/\gcd(n,\rho) = n/\gcd(n,\gamma)$ and $q = n/\gcd(n,\alpha)$ and, if $r = 4$, then $u = n/\gcd(n,\eta)$, where $\eta = \xi n - \alpha - \rho - \gamma$ and $\xi \in \mathbb{Z}$ is so that $(\xi - 1)n < \alpha + \rho + \gamma \leq \xi n$ (setting $\gcd(n, 0) := n$).

In this model, we have $P(x,y) = x$, $a(x,y) = (x, \omega y)$, where $\omega = e^{2\pi i/n}$, and (as $Q(x) = x^2$) $\beta(x,y) = x^2$.

As $\overline{b}(x) = -x$ and $P \circ b = \overline{b} \circ P$, it follows that the involution b must be of the form

$$b(x,y) = \left(-x, \delta y \left(\frac{x-1}{x+1}\right)^{(\gamma-\rho)/n}\right), \qquad \delta^n = (-1)^{\alpha+\rho+\gamma}.$$

We will distinguish the cases (i) $\rho = \gamma$ and (ii) $\rho \neq \gamma$.

2.3. If $\gamma = \rho$, then $\vartheta_1 = \vartheta_2 = \vartheta_3 = 0$,

$$C : y^n = x^{\alpha}(x-1)^{\rho}(x+1)^{\rho},$$

and $b(x,y) = (-x, \delta y)$. As b^2 is the identity, it follows that $\delta^2 = 1$, that is, $(-1)^{2(\alpha+2\rho)/n} = 1$, from which we see that n necessarily divides $\alpha + 2\rho$. In this case $m = 1$ and $\mathrm{Aut}(S,\beta) \cong \mathbb{Z}_n \times \mathbb{Z}_2$.

2.4. If $\gamma \neq \rho$, then we may assume without loss of generality that $\rho \leq \gamma$. As

$$x-1=\frac{y^n}{x^{\alpha}(x-1)^{\rho-1}(x+1)^{\gamma}}$$

we have

$$b(x,y)=\left(-x,\frac{\delta y^{1+\gamma-\rho}}{x^{\alpha(\gamma-\rho)/n}(x-1)^{(\rho-1)(\gamma-\rho)/n}(x+1)^{(\gamma+1)(\gamma-\rho)/n}}\right).$$

It follows from the above that there exist non-negative integers $\vartheta_1,\vartheta_2,\vartheta_3$ such that

$$\alpha(\gamma-\rho)=n\vartheta_1,\qquad (\rho-1)(\gamma-\rho)=n\vartheta_2,\qquad (\gamma+1)(\gamma-\rho)=n\vartheta_3.$$

In this way

$$b(x,y)=\left(-x,\frac{\delta y^{1+\gamma-\rho}}{x^{\vartheta_1}(x-1)^{\vartheta_2}(x+1)^{\vartheta_3}}\right).$$

As b^2 is the identity, the equality

$$(x,y)=b^2(x,y)=\left(x,\frac{\delta^{2+\gamma-\rho}(-1)^{\vartheta_1+\vartheta_2+\vartheta_3}yy^{(1+\gamma-\rho)^2-1}}{x^{\vartheta_1(2+\gamma-\rho)}(x-1)^{\vartheta_3+\vartheta_2(2+\gamma-\rho)}(x+1)^{\vartheta_3+\vartheta_2(2+\gamma-\rho)}}\right)$$

ensures that

$$y^{(1+\gamma-\rho)^2-1}=x^{\vartheta_1(2+\gamma-\rho)}(x-1)^{\vartheta_3+\vartheta_2(2+\gamma-\rho)}(x+1)^{\vartheta_3+\vartheta_2(2+\gamma-\rho)}$$

and

$$\delta^{2+\gamma-\rho}=(-1)^{\vartheta_1+\vartheta_2+\vartheta_3}.$$

In particular, n necessarily divides $(\alpha+\rho+\gamma)(2+\gamma-\rho)$ and $bab=a^{1+\gamma-\rho}$, that is, $m=1+\gamma-\rho$.

The formula for g is just a consequence of the Riemann–Hurwitz formula (see also [BW]).

Remark 2.3. As already noted in the Introduction, there is no regular Belyĭ pair (S,β) of genus at least one with $\mathrm{Aut}(S,\beta)$ isomorphic to a dihedral group. This also follows directly from the first part of the proof of Theorem 1.2. In fact, assume there is a regular Belyĭ pair (S,β) with $\mathrm{Aut}(S,\beta)=\langle a,b : a^n=b^2=1, bab=a^{-1}\rangle$. The involution $\overline{b}$ has as one of its fixed points a cone point of $S/\langle a\rangle$. This means that there is some a^k (where $k\in\{1,\dots,n-1\}$) and some a^lb (where $l\in\{0,1,\dots,n-1\}$) which have a common fixed point. This implies that $\langle a^k,a^lb\rangle$ should be a cyclic group (the stabilizer of any point of S in $\mathrm{Aut}(S)$ is known to be a cyclic group), a contradiction.

3. An example in genus two. Let us describe those regular Belyĭ pairs (S,β) with

$$\mathrm{Aut}(S,\beta)=\langle a,b : a^8=b^2=1,\ bab=a^3\rangle\cong\mathbb{Z}_8\rtimes\mathbb{Z}_2.$$

By Theorem 1.2, taking $n=8$ and $m=3$, we know that there are $\alpha,\rho,\gamma\in\{1,\dots,7\}$ and non-negative integers $\vartheta_1,\vartheta_2,\vartheta_3$ such that

$\gcd(8,\alpha,\rho,\gamma)=1$, $\rho\leq\gamma$, $\gamma-\rho=m-1=2$, $\alpha+\rho+\gamma$ is even, $\alpha=4\vartheta_1$, $\rho-1=4\vartheta_2$ and $\gamma+1=4\vartheta_3$, and (S,β) is equivalent to (C,η), where $\eta(x,y)=x^2$ and

$$C: y^8=x^{\alpha}(x-1)^{\rho}(x+1)^{\gamma}.$$

By checking all possibilities, we only obtain the following two cases:

$$(\alpha,\rho,\gamma)\in\{(4,1,3),(4,5,7)\},$$

that is, C must be one of the following two curves of genus 2:

$$\begin{aligned} C_1 &: y^8=x^4(x-1)(x+1)^3, & (\alpha,\rho,\gamma)&=(4,1,3),\\ C_2 &: y^8=x^4(x-1)^5(x+1)^7, & (\alpha,\rho,\gamma)&=(4,5,7). \end{aligned}$$

The group $\mathrm{Aut}(C_j,\eta)$ is generated by

$$a(x,y)=(x,\omega y)\quad(\omega=e^{\pi i/4})$$

and

$$b(x,y)=\begin{cases}\left(-x,\dfrac{-y^3}{x(x+1)}\right) & \text{for } C_1,\\[2ex] \left(-x,\dfrac{y^3}{x(x-1)(x+1)^2}\right) & \text{for } C_2.\end{cases}$$

In both cases, $r=3$, $p=8$, $q=2$, $C_j/\langle a\rangle$ has signature $(0;2,8,8)$ and the regular Belyĭ pair (S,β) has type $(0;2,4,8)$. There is only one, up to isomorphism, Riemann surface of genus 2 whose reduced group of automorphisms contains a group of order 8 (the quotient of $\mathrm{Aut}(S,\beta)$ by the cyclic group generated by the hyperelliptic involution, [Ig]; in particular, C_1 and C_2 are isomorphic). That surface has as full reduced group the symmetric group $\mathfrak{S}_4$ and it is described by the hyperelliptic curve

$$E: w^2=u(u^4-1).$$

The Belyĭ pair (S,β) is equivalent to (E,θ), where

$$\theta(u,w)=(u^8-2u^4+1)/(-4u^4),$$

and $\mathrm{Aut}(E,\theta)$ is generated by the element

$$A(u,w)=(iu,\sqrt{i}\,w)$$

of order 8 and the involution

$$B(u,w)=(i/u,i\sqrt{i}\,w/u^3).$$

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