

Shadowing and internal chain transitivity

by

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Abstract. The main result of this paper is that a map $f : X \rightarrow X$ which has shadowing and for which the space of ω -limit sets is closed in the Hausdorff topology has the property that a set $A \subseteq X$ is an ω -limit set if and only if it is closed and internally chain transitive. Moreover, a map which has the property that every closed internally chain transitive set is an ω -limit set must also have the property that the space of ω -limit sets is closed. As consequences of this result, we show that interval maps with shadowing have the property that every internally chain transitive set is an ω -limit set of a point, and we also show that topologically hyperbolic maps and certain quadratic Julia sets have a closed space of ω -limit sets.

1. Introduction. Let $f : X \rightarrow X$ be a continuous map on a compact metric space. For $x \in X$ define $\omega(x)$ to be the set of limit points of the orbit of x . Let $\omega(f) = \{A \subseteq X : \text{there exists } x \in X \text{ such that } \omega(x) = A\}$ be the space of ω -limit sets of f with the Hausdorff topology. The structure of $\omega(f)$ has been extensively studied, and a key question is “What types of dynamical systems (X, f) have the property that $\omega(f)$ is closed?”. Blokh et al. [9] proved that maps of the interval have the property that $\omega(f)$ is closed. It has also been shown that maps of finite graphs have this property [15]. But there are also many interesting examples with the property that $\omega(f)$ is not closed. Often these are maps on topologically complicated spaces, such as *dendrites* (locally connected tree-like spaces with branch points of infinite order) [14]. The main theorem of the present paper implies that many seemingly exotic dynamical systems (such as a family of locally connected quadratic Julia sets) have the property that $\omega(f)$ is closed.

A closed set A is *internally chain transitive* provided for every $\epsilon > 0$ and for every $x, y \in A$ there is an ϵ -pseudo-orbit $\{x_i\}_{i=0}^n \subseteq A$ with $x_0 = x$ and $x_n = y$ and $d(f(x_i), x_{i+1}) < \epsilon$ for $0 \leq i < n$. It is known that every ω -limit

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set is internally chain transitive [12], and in several settings the converse has been established. Specifically, for several types of dynamical systems it is known that

(†) A closed set A is internally chain transitive if, and only if, there is some x with $\omega(x) = A$.

It is known that (†) holds for shifts of finite type, topologically hyperbolic maps, a family of quadratic Julia sets, and certain interval maps [5, 8, 7, 3]. In fact, in [4], Barwell et al. show that there are certain unimodal maps of the unit interval for which (†) holds and certain other ones for which it fails. They conjecture that the key to (†) on the interval is the property of *shadowing* (defined in the next section). We prove this conjecture by showing that if $f : X \rightarrow X$ is a map of a compact metric space with the shadowing property and $\omega(f)$ is closed then f satisfies (†). Since interval maps have the property that $\omega(f)$ is closed, the conjecture is true since in that setting shadowing implies (†).

The main result of this paper has two broad implications. On one hand we use property (†) to establish $\omega(f)$ is closed in several new settings, and on the other we show that in the presence of shadowing, $\omega(f)$ being closed implies (†).

In the next section we give some preliminary definitions and results. In Section 3 we prove our main theorem, and we end in Section 4 with a list of corollaries to the main result.

2. Preliminaries. For the purposes of this paper, a *dynamical system* consists of a compact metric space X with metric d and a continuous map $f : X \rightarrow X$. For each $x \in X$, the ω -limit set of x under f is the set

$$\omega(x) = \bigcap_{n \in \mathbb{N}} \overline{\{f^i(x) : i \geq n\}},$$

i.e. the set of limit points of the sequence $\langle f^i(x) \rangle_{i \in \mathbb{N}}$. The ω -limit space of f is the collection of subsets of X which are the ω -limit sets of some points $x \in X$. We will use the symbol $\omega(f)$ to denote this set.

The properties of ω -limit sets have been extensively studied. It is well-known that ω -limit sets for dynamical systems on compact metric spaces are compact. The hyperspace of compact subsets of X is equipped with the *Hausdorff metric*, which defines the distance between compact subsets A and B of X as follows:

$$d_H(A, B) = \max \left\{ \sup_{a \in A} \inf_{b \in B} d(a, b), \sup_{b \in B} \inf_{a \in A} d(a, b) \right\}.$$

In [9], Blokh et al. demonstrated that for an interval map $f : I \rightarrow I$, the set $\omega(f)$ is closed with respect to this metric. It has also been shown

that dynamical systems on circles [16] and on graphs [15] have the property that $\omega(f)$ is closed. It is not, however, the case that $\omega(f)$ is always closed. Examples of systems for which $\omega(f)$ is not closed include certain maps on dendrites [14] and on the unit square [13].

For a dynamical system $f : X \rightarrow X$, a sequence (finite or infinite) $\langle x_i \rangle$ in X is a δ -pseudo-orbit provided that for each i , $d(f(x_i), x_{i+1}) < \delta$. A point $z \in X$ ϵ -shadows a δ -pseudo-orbit $\langle x_i \rangle$ provided that $d(f^i(z), x_i) < \epsilon$ for all i .

We say that a dynamical system $f : X \rightarrow X$ has shadowing provided that for all $\epsilon > 0$ there exists a $\delta > 0$ such that each δ -pseudo-orbit is ϵ -shadowed by some $z \in X$.

This property has also been studied extensively. It is known that there are interval maps both with and without shadowing, and partial classifications exist in this context [11]. It has also been demonstrated that shifts of finite type [5] and Julia sets for certain quadratic maps [7, 8] all exhibit shadowing.

An interesting subject of inquiry is finding characteristics that classify the ω -limit sets of a dynamical system. Characterizations of ω -limit sets exist for a variety of classes of maps, including interval maps, circle maps, shifts of finite type, and many others.

For a dynamical system $f : X \rightarrow X$, a closed set $A \subseteq X$ is said to be *internally chain transitive* (ICT) if it has the property that for all $a, b \in A$ and all $\epsilon > 0$, there exists a finite ϵ -pseudo-orbit $\langle x_i \rangle_{i=0}^n$ in A with $x_0 = a$ and $x_n = b$. We will use $\text{ICT}(f)$ to denote the collection of all ICT sets for f .

Hirsch et al. [12] showed that for any dynamical system $f : X \rightarrow X$, the ω -limit sets of f are ICT. It has also been demonstrated that in certain types of dynamical systems, each ICT set is an ω -limit set. In particular, it has been demonstrated that $\omega(f) = \text{ICT}(f)$ in shifts of finite type [5], Julia sets for certain quadratic maps [7, 8], and certain classes of interval maps [3].

While it is known that there are systems which do not exhibit this $\omega(f) = \text{ICT}(f)$ equality [5], in all of these systems, shadowing is either absent or unknown. This has led to the conjecture that in systems which exhibit shadowing, $\text{ICT}(f) = \omega(f)$ [4].

In the next section, we will prove that shadowing, with the additional assumption that $\omega(f)$ is closed, is sufficient to ensure that $\omega(f) = \text{ICT}(f)$. We will also demonstrate that there are systems exhibiting $\omega(f) = \text{ICT}(f)$ for which $\omega(f)$ is closed but which do not exhibit shadowing.

3. Main results

THEOREM 1. *Let $f : X \rightarrow X$ be a dynamical system which exhibits shadowing and with $\omega(f)$ closed in the Hausdorff metric. Then $f : X \rightarrow X$ has the property that $\omega(f) = \text{ICT}(f)$.*

Proof. Let $f : X \rightarrow X$ be a dynamical system with shadowing such that $\omega(f)$ is closed. As stated earlier, it is always the case that $\omega(f) \subseteq \text{ICT}(f)$. Hence, we need only demonstrate that each ICT set is the ω -limit set of some $x \in X$.

Let $A \subseteq X$ be ICT. Since A is internally chain transitive we can construct a sequence $\langle a_i \rangle_{i=0}^\infty$ in A with the following properties:

- (1) For all $\epsilon > 0$ there exists $M \in \mathbb{N}$ such that $\langle a_i \rangle_{i=M}^\infty$ is an ϵ -pseudo-orbit.
- (2) For all $a \in A$, $\epsilon > 0$ and $M \in \mathbb{N}$ there exists $n > M$ such that $d(a_n, a) < \epsilon$.

Such a sequence can be constructed as follows. Fix $a \in A$. Since A is internally chain transitive, for all $\epsilon > 0$ we can find a finite ϵ -pseudo-orbit $\langle a_i^\epsilon \rangle_{i=0}^{n_\epsilon}$ in A with $a_0 = a = a_{n_\epsilon}$ and with the property that for all $b \in A$ there is an element in the pseudo-orbit within ϵ of b . Since each such pseudo-orbit begins and ends at a , we can concatenate a sequence of $1/2^i$ -pseudo-orbits to construct the desired sequence. For a more detailed and explicit construction, see [6].

Notice that by construction,

$$A = \bigcap_{n \in \mathbb{N}} \overline{\{a_i : i \geq n\}}.$$

Now, since f has shadowing, for each $\epsilon > 0$ let $\delta_\epsilon > 0$ be such that each δ_ϵ -pseudo-orbit is ϵ -shadowed. Without loss of generality, we may assume that $\delta_\epsilon < \epsilon$. As mentioned, there exists M_ϵ such that $\langle a_i \rangle_{i=M_\epsilon}^\infty$ is a δ_ϵ -pseudo-orbit. Let $z_\epsilon \in X$ be a point that ϵ -shadows this pseudo-orbit.

We claim that $d_H(\omega(z_\epsilon), A) < 2\epsilon$. Consider $x \in \omega(z_\epsilon)$. Then there exists a sequence $\langle n_k \rangle$ of natural numbers with $x = \lim f^{n_k}(z_\epsilon)$. But for each $f^{n_k}(z_\epsilon)$ there exists an element $a_{m_k} \in A$ in the sequence $\langle a_i \rangle$ with $d(f^{n_k}(z), a_k) < \epsilon$. Then

$$\sup_{x \in \omega(z_\epsilon)} \inf_{a \in A} d(x, a) < 2\epsilon.$$

Now, let $a \in A$. Then there exists a sequence $\langle n_k \rangle$ of natural numbers with $n_k > M_\epsilon$ for all ϵ and $d(a_{n_k}, a) < \epsilon/2$. Additionally, since $n_k > M_\epsilon$, $d(f^{n_k - M_\epsilon}(z_\epsilon), a_{n_k}) < \epsilon$. Then $d(a, f^{n_k}(z_\epsilon)) < 3\epsilon/2$ for all k . Finally, we can find $N \in \mathbb{N}$ such that for every $i > N$ there exists $z'_i \in \omega(z_\epsilon)$ such that $d(f^i(z_\epsilon), z'_i) < \epsilon/2$. Putting all this together, we see that there exists a natural number $j > M_\epsilon + N$ for which

$$d(a, z'_j) \leq d(a, f^j(z_\epsilon)) + d(f^j(z_\epsilon), z'_j) < 2\epsilon.$$

In particular, then

$$\sup_{a \in A} \inf_{x \in \omega(z_\epsilon)} d(x, a) < 2\epsilon,$$

and hence $d_H(\omega(z_\epsilon), A) < 2\epsilon$.

This establishes that A is a limit point of $\omega(f)$. Since $\omega(f)$ is closed by assumption, it follows that A is the ω -limit set of some $z \in X$. ■

It is important to note that the map $\omega : X \rightarrow \omega(f)$ defined by $x \mapsto \omega(x)$ is often not continuous, and so the z for which $A = \omega(z)$ in the proof above is not explicitly related to the z_ϵ 's.

The converse of this theorem is not true, as seen in the following example.

Let $f : [-1, 1] \rightarrow [-1, 1]$ be given as follows:

$$f(x) = \begin{cases} x^3, & -1 \leq x \leq 0, \\ 2x, & 0 \leq x \leq 1/2, \\ 2(1-x), & 1/2 \leq x \leq 1. \end{cases}$$

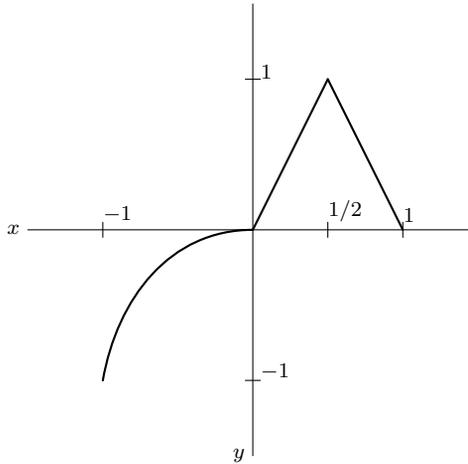


Fig. 1. The graph of a function $f : [-1, 1] \rightarrow [-1, 1]$ which satisfies $\text{ICT}(f) = \omega(f)$ but does not exhibit shadowing

Notice that f is the one-point union of two distinct simple dynamical systems, $g = f|_{[-1,0]}$ and $h = f|_{[0,1]}$, with well-understood dynamics.

In fact, since $f([0, 1]) = [0, 1]$ and $f([-1, 0]) = [-1, 0]$, it follows that W is an ω -limit set for f if and only if it is an ω -limit set for g or h . The map g is the slope 2 tent map, which is known to have shadowing and a closed ω -limit space. Thus any ICT subset of $[0, 1]$ is an ω -limit set. Furthermore, it is clear by inspection that the only ICT subsets of $[-1, 0]$ are $\{-1\}$ and $\{0\}$, both of which are ω -limit sets.

Finally, it is straightforward to check that an ICT subset of $[-1, 1]$ must be contained in either $[0, 1]$ or $[-1, 0]$, and so it follows that $\text{ICT}(f) = \omega(f)$. Furthermore, as f is an interval map, by [9], $\omega(f)$ is closed.

However, f does not exhibit shadowing. Let $\epsilon = 1/2$. For all $\delta > 0$, we can construct a δ -pseudo-orbit that is not ϵ -shadowed as follows. Fix $\delta > 0$. Let $x_0 = -3/4$, and let $N \in \mathbb{N}$ be such that $f^N(-3/4) \in (-\delta/2, 0)$. Also, choose $k \in \mathbb{N}$ such that $1/2^k < \delta/2$.

For $i \leq N$, let $x_i = f^i(x)$. Then choose $x_{N+1} = 1/2^k \in (0, \delta/2)$, and for $i > N + 1$ define $x_i = f^{i-(N+1)}(x_{N+1})$. By construction, this is a δ -pseudo-orbit. Furthermore, $x_{N+1+k} = 1$. However, for any point $z \in [-1, 1]$ with $d(z, 3/4) < 1/2$, the entire orbit of z lies in $[-1, 0]$, and thus $d(f^{N+1+k}(z), x_{N+1+k}) > 1/2$.

Although the converse of Theorem 1 is false, we do have a partial converse.

LEMMA 2. *Let $f : X \rightarrow X$ be a dynamical system. Then $\text{ICT}(f)$ is closed.*

Proof. Let A be a limit point of $\text{ICT}(f)$. Then for all $\delta > 0$ there exists an ICT set A_δ for which $d_H(A, A_\delta) < \delta$.

Let $a, b \in A$ and fix $\epsilon > 0$. By uniform continuity, let $\delta > 0$ be such that if $d(p, q) < \delta$, then $d(f(p), f(q)) < \epsilon/3$. Without loss of generality, assume $\delta < \epsilon/3$.

Let $a' \in A_\delta \cap B_\delta(a)$ and $b' \in A_\delta \cap B_\delta(a)$. Since A_δ is ICT, let $\langle y_i \rangle_{i=0}^n$ be an $\epsilon/3$ -pseudo-orbit in A_δ with $y_0 = a'$ and $y_n = b'$. Now, let $x_0 = a$, $x_n = b$ and for all $1 < i < n$ choose $x_i \in A \cap B_\delta(y_i)$.

We claim that $\langle x_i \rangle_{i=0}^n$ is an ϵ -pseudo-orbit in A with $a = x_0$ and $b = x_n$. By construction, we need only verify that for all $i < n$, $d(f(x_i), x_{i+1}) < \epsilon$. Indeed,

$$\begin{aligned} d(f(x_i), x_{i+1}) &\leq d(f(x_i), f(y_i)) + d(f(y_i), y_{i+1}) + d(y_{i+1}, x_{i+1}) \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3. \end{aligned}$$

Since such a pseudo-orbit exists for all $a, b \in A$ and each $\epsilon > 0$, we see that A is ICT. ■

Since in all dynamical systems $\text{ICT}(f)$ is closed and contains $\omega(f)$, it is reasonable to ask if $\text{ICT}(f)$ is the closure of $\omega(f)$. This turns out not to be the case, as exhibited by the map on $[-1, 1]$ obtained by the union of the slope 2 tent map and its reflection across the origin as pictured in Figure 2. As with the previous example, any ω -limit set of this map is contained in either $[-1, 0]$ or $[0, 1]$. However, the set $\{0\} \cup \{\pm 1/2^i : i \in \mathbb{N}\}$ belongs to $\text{ICT}(f)$, and is clearly not a limit point of $\omega(f)$.

As an immediate consequence of Lemma 2, we have the following.

LEMMA 3. *Let $f : X \rightarrow X$ be a dynamical system for which $\text{ICT}(f) = \omega(f)$. Then $\omega(f)$ is closed.*

Combining this result with Theorem 1 yields the following corollary.

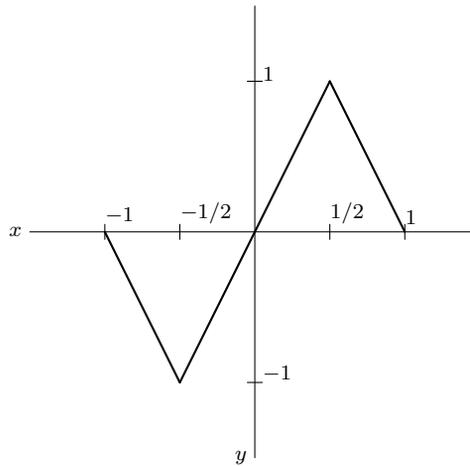


Fig. 2. The graph of a function $g : [-1, 1] \rightarrow [-1, 1]$ for which $\text{ICT}(g)$ is not the closure of $\omega(g)$

COROLLARY 4. *Let $f : X \rightarrow X$ be a dynamical system with shadowing. Then $\omega(f)$ is closed if and only if $\omega(f) = \text{ICT}(f)$.*

4. Implications. Since the properties of shadowing and $\omega(f)$ being closed have been studied in a variety of contexts, the results of Corollary 4 can be applied to achieve a number of new results. These results fall into one of two categories: systems for which it is known that $\omega(f)$ is closed, and systems which are known to satisfy $\text{ICT}(f) = \omega(f)$.

As mentioned in Section 2, it has been shown that for continuous maps on graphs, $\omega(f)$ is closed.

COROLLARY 5. *Let $f : G \rightarrow G$ be a dynamical system on the graph G which exhibits shadowing. Then $\omega(f) = \text{ICT}(f)$.*

In particular, this applies to interval maps, so that Conjecture 1.2 of [4] can be answered in the affirmative.

COROLLARY 6. *Let $f : I \rightarrow I$ be a continuous map of the interval which exhibits shadowing. Then $\omega(f) = \text{ICT}(f)$.*

In that same paper, the authors conjecture that if $f : X \rightarrow X$ is a dynamical system on a compact metric space with shadowing, then $\omega(f) = \text{ICT}(f)$. This conjecture remains open, but as an application of Corollary 4, it has the following equivalent formulation: if $f : X \rightarrow X$ is a dynamical system on a compact metric space with shadowing, then $\omega(f)$ is closed. Additionally, there are known sufficient conditions for shadowing in interval maps (see [11]), so that this corollary can be applied directly to maps satisfying these conditions.

For maps on spaces more topologically complex than interval maps, it is not true in general that $\omega(f)$ is closed. However, several authors have shown that certain categories of systems satisfy $\text{ICT}(f) = \omega(f)$. In these systems, we can apply Corollary 4 as well.

For example, it has been shown in [5] that shifts of finite type satisfy $\text{ICT}(\sigma) = \omega(\sigma)$.

COROLLARY 7. *In shifts of finite type, $\omega(\sigma)$ is closed.*

Based on the work of Baldwin [1, 2], Barwell et al. [7, 8] demonstrated that for certain parameters $c \in \mathbb{C}$, the the map $f_c : \mathbb{C} \rightarrow \mathbb{C}$ defined by $f(z) = z^2 + c$ restricted to its Julia set J_c has $\text{ICT}(f_c) = \omega(f_c)$.

COROLLARY 8. *Let $c \in \mathbb{C}$ be such that $f_c : \mathbb{C} \rightarrow \mathbb{C}$ is a quadratic map for which J_c is a dendrite. Then $f_c|_{J_c} : J_c \rightarrow J_c$ has $\omega(f_c)$ closed.*

COROLLARY 9. *Let $c \in \mathbb{C}$ be such that f_c has an attracting or parabolic periodic point, and kneading sequence τ which is not an n -tupling. Then $f_c|_{J_c} : J_c \rightarrow J_c$ has $\omega(f_c)$ closed.*

In both of these cases, J_c is hereditarily locally connected. While it is not true that every dynamical system on a hereditarily locally connected space satisfies $\text{ICT}(f) = \omega(f)$, it seems likely that these results will extend to Julia sets of higher degree maps with similar structure.

Another application of Corollary 4 is to topologically hyperbolic maps. A map $f : X \rightarrow X$ is *topologically hyperbolic* if it has shadowing and there exists a constant c such that if $\langle x_i \rangle_{i \in \mathbb{Z}}$ and $\langle y_i \rangle_{i \in \mathbb{Z}}$ are full orbits through x and y satisfying $d(x_i, y_i) < c$ for all $i \in \mathbb{Z}$, then $x = y$. In [6], the authors prove that topologically hyperbolic systems satisfy $\text{ICT}(f) = \omega(f)$. Applying Corollary 4 gives us the following:

COROLLARY 10. *Let $f : X \rightarrow X$ by topologically hyperbolic. Then $\omega(f)$ is closed.*

The class of topologically hyperbolic maps includes Axiom A diffeomorphisms restricted to their non-wandering sets, shifts of finite type and topologically Anosov maps [10], [17].

References

- [1] S. Baldwin, *Continuous itinerary functions and dendrite maps*, Topology Appl. 154 (2007), 2889–2938.
- [2] S. Baldwin, *Julia sets and periodic kneading sequences*, J. Fixed Point Theory Appl. 7 (2010), 201–222.
- [3] A. D. Barwell, *A characterization of ω -limit sets of piecewise monotone maps of the interval*, Fund. Math. 207 (2010), 161–174.

- [4] A. D. Barwell, G. Davies, and C. Good, *On the ω -limit sets of tent maps*, Fund. Math. 217 (2012), 35–54.
- [5] A. D. Barwell, C. Good, R. Knight, and B. E. Raines, *A characterization of ω -limit sets in shift spaces*, Ergodic Theory Dynam. Systems 30 (2010), 21–31.
- [6] A. D. Barwell, C. Good, P. Oprocha, and B. E. Raines, *Characterizations of ω -limit sets of topologically hyperbolic spaces*, Discrete Contin. Dynam. Systems 33 (2013), 1819–1833.
- [7] A. D. Barwell, J. Meddaugh and B. E. Raines, *Shadowing and ω -limit sets of circular Julia sets*, Ergodic Theory Dynam. Systems, to appear.
- [8] A. D. Barwell and B. E. Raines, *The ω -limit sets of quadratic Julia sets*, Ergodic Theory Dynam. Systems, to appear.
- [9] A. Blokh, A. M. Bruckner, P. D. Humke, and J. Smítal, *The space of ω -limit sets of a continuous map of the interval*, Trans. Amer. Math. Soc. 348 (1996), 1357–1372.
- [10] R. Bowen, *ω -Limit sets for Axiom A diffeomorphisms*, J. Differential Equations 18 (1975), 333–339.
- [11] E. M. Coven, I. Kan, and J. A. Yorke, *Pseudo-orbit shadowing in the family of tent maps*, Trans. Amer. Math. Soc. 308 (1988), 227–241.
- [12] M. W. Hirsch, H. L. Smith and X.-Q. Zhao, *Chain transitivity, attractivity, and strong repellers for semidynamical systems*, J. Dynam. Differential Equations 13 (2001), 107–131.
- [13] V. Jiménez López and J. Smítal, *ω -limit sets for triangular mappings*, Fund. Math. 167 (2001), 1–15.
- [14] Z. Kočan, V. Kornecká-Kurková and M. Málek, *On the centre and the set of ω -limit points of continuous maps on dendrites*, Topology Appl. 156 (2009), 2923–2931.
- [15] J.-H. Mai and S. Shao, *Spaces of ω -limit sets of graph maps*, Fund. Math. 196 (2007), 91–100.
- [16] D. Pokluda, *Characterization of ω -limit sets of continuous maps of the circle*, Comment. Math. Univ. Carolin. 43 (2002), 575–581.
- [17] R. Yang, *Topological Anosov maps of non-compact metric spaces*, Northeast Math. J. 17 (2001), 120–126.

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