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The Banach–Tarski paradox for the hyperbolic plane (II)

by

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Abstract. The second author found a gap in the proof of the main theorem in [J. Mycielski, Fund. Math. 132 (1989), 143–149]. Here we fill that gap and add some remarks about the geometry of the hyperbolic plane $\mathbb{H}^2$.

In order to prove the Banach–Tarski paradox for the hyperbolic plane $\mathbb{H}^2$, in Section 3 of [1], a free group F of piecewise isometries of a certain bounded set $B \subset \mathbb{H}^2$ is defined, and it is assumed that

(A) *Each transformation of F other than the identity has at most countably many fixed points.*

But the argument of Section 3 in [1] fails to prove (A), and without (A) Lemma 2.4 cannot be applied, whence the proof of the main result is incomplete. Here follows a proof much simpler than the vagaries in Section 3 of [1] of the following statement slightly stronger than (A):

(B) *Each element of $F \setminus \{e\}$ has finitely many fixed points.*

Proof. As in [1], all isometries considered here are assumed to be sense preserving. Let the set B and the piecewise isometries φ and ψ be defined as in Section 3 of [1]. Hence φ preserves the real axis and ψ preserves the imaginary axis. Furthermore, φ consists of three disjoint pieces of three isometries of $\mathbb{H}^2$. The first of these isometries is of the form $\varphi_1\varphi_2$, the second is φ_1 and the third is φ_2, where each φ_i preserves the real axis. It follows that the φ_i commute and for every integer $n > 0$ and every $z \in B$ there exist two integers $n_1(z) \geq 0$ and $n_2(z) \geq 0$, with $n \leq n_1(z) + n_2(z) \leq 2n$, such that $\varphi^n(z) = \varphi^{n_1(z)}\varphi^{n_2(z)}(z)$ (see [1, Lemma 3.1]). Mutatis mutandis, the same is true for ψ and certain isometries ψ_1 and ψ_2 of $\mathbb{H}^2$ preserving the imaginary axis. Moreover the parameters defining the φ_i and ψ_i can be chosen such

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that the group G which they generate is a free product (non-abelian) of the abelian groups $\langle\varphi_1,\varphi_2\rangle$ and $\langle\psi_1,\psi_2\rangle$ and the latter are free abelian of rank 2 (see [1, Lemma 2.2]). Therefore every $\xi\in F\setminus\{e\}$ is a transformation of B which splits into a disjoint union of finitely many pieces of isometries in $G\setminus\{e\}$. Since every isometry of $\mathbb{H}^2$ other than e has at most one fixed point, ξ has only finitely many fixed points. This concludes the proof of (B) and fills the gap in [1]. ∎

Remarks. 1. The above proof is very similar to the proof in [3] of a conjecture (C) stated in [2] on page 155.

2. Our construction of $F\setminus\{e\}$ does not guarantee that its transformations are fixed-point-free. Let α and β be two distinct, small translations of $\mathbb{H}^2$ along perpendicular axes k and l, respectively. Then the commutator $\alpha\beta\alpha^{-1}\beta^{-1}$ has a fixed point in $\mathbb{H}^2$, i.e., is a rotation of $\mathbb{H}^2$.

Indeed, we can choose two straight lines l_1 and l_2 on opposite sides of l both perpendicular to k and both at the same distance to l such that $\beta(l_1)=l_2$, and a similar pair of lines k_1 and k_2 perpendicular to l, at the same distance to k, such that $\alpha(k_1)=k_2$. These lines k_i and l_i form a quadrilateral whose corners can be labeled a, b, c and d, such that $\beta^{-1}(a)=b$, $\alpha^{-1}(b)=c$, $\beta(c)=d$ and $\alpha(d)=a$. Hence $\alpha\beta\alpha^{-1}\beta^{-1}(a)=a$.

3. We do not know if B admits a free non-abelian group of piecewise isometries whose elements other than e have no fixed points. Without such a group we have no tool to construct, say, a decomposition of B into three disjoint sets X,Y,Z such that $X\equiv Y\equiv Z\equiv X\cup Y\equiv X\cup Z\equiv Y\cup Z$, where $\equiv$ denotes equivalence of sets by finite decomposition. Such decompositions of the sphere $\mathbb{S}^2$ exist if sense reversing isometries are allowed (see [4, Th. 4.16]).

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