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A Z-set unknotting theorem for Nöbeling spaces

by

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Abstract. We prove a Z-set unknotting theorem for Nöbeling spaces. The theorem is proved for a certain model of Nöbeling spaces.

1. Introduction. All spaces are assumed to be separable metrizable. A *manifold* means a manifold with (possibly empty) boundary, and a *triangulated space* means a locally finite simplicial complex which we identify with the underlying space. For a triangulated space we consider only triangulations compatible with the PL-structure of the space. All triangulated manifolds are assumed to be combinatorial.

A complete n-dimensional metric space X is said to be an *n-dimensional Nöbeling space* if the following conditions are satisfied:

(i) X is an absolute extensor in dimension n, that is, every map $f : A \to X$ from a closed subset A of a space Y of dimension $\leq n$ extends over Y;

(ii) every map $f : Y \to X$ from a complete metric space Y of dimension $\leq n$ can be arbitrarily closely approximated by a closed embedding, that is, for every open cover $\mathcal{U}$ of X there is a closed embedding $g : Y \to X$ which is $\mathcal{U}$-close to f (*$\mathcal{U}$-close* means that for every $y \in Y$ there is an element of $\mathcal{U}$ that contains both $f(y)$ and $g(y)$).

Examples of Nöbeling spaces can be constructed as follows.

Let us say that a point of a triangulated space is *rational* if it has rational barycentric coordinates with respect to the triangulation of the space. By a *rational map* $f : M_1 \to M_2$ between triangulated spaces M_1 and M_2 with the triangulations $\mathcal{T}_1$ and $\mathcal{T}_2$ respectively we mean a PL-map that sends the rational points of M_1 to the rational points of M_2. Note that if f is a rational embedding and $f(x)$ is a rational point in M_2 then x is rational in M_1.

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Indeed, let Δ_1 be a simplex of $\mathcal{T}_1$ containing x. Since f is a PL-embedding there are a simplex Δ'_1 linearly embedded into Δ_1 and a simplex $\Delta_2 \in \mathcal{T}_2$ such that $x \in \Delta'_1$, $\dim \Delta'_1 = \dim \Delta_1$ and f linearly sends Δ'_1 into Δ_2. Then the rational points of M_1 contain a subset which is dense in Δ'_1 and therefore we can choose points $x_0, \dots, x_n \in \Delta'_1$, $n = \dim \Delta'_1$, which are rational in M_1 and in general position in Δ'_1. Since f is rational and linear on Δ'_1, we see that $f(x_0), \dots, f(x_n)$ are rational in M_2 and in general position in $\Delta'_2 = f(\Delta'_1)$. Hence, since $f(x)$ is rational in M_2, $f(x)$ can be represented as $f(x) = \lambda_0 f(x_0) + \dots + \lambda_n f(x_n)$, $\lambda_0 + \dots + \lambda_n = 1$, with $\lambda_0, \dots, \lambda_n$ (not necessarily non-negative) rational numbers. Then $x = \lambda_0 x_0 + \dots + \lambda_n x_n$ and hence x is rational in M_1.

Two triangulations of a space are said to be *rationally equivalent* if the identity map is a rational map with respect to these triangulations. Let M be a triangulated space. Every triangulation of M which is rationally equivalent to the given triangulation of M is said to be a *rational triangulation*, and the class of all rational triangulations is said to be the *rational structure* of M. Denote by $M(k)$ the subspace of M which is the complement of the union of all the triangulated spaces of dimension $\leq k$ which are rationally embedded in M.

The space $M(k)$ admits the following interpretation. Let H be a Hilbert space. A point in H is said to be *rational* if it has rational coordinates and only finitely many of them are non-zero. A k-dimensional plane in H is said to be *rational* if it is spanned by $k+1$ rational points. Fix a rational triangulation of M and embed M in H by an embedding which sends the vertices to rational points and which is linear on every simplex of the triangulation. Denote by K the union of all rational k-dimensional planes in H. Then $M(k) = M \setminus K$. Indeed, for every simplex Δ of M and every rational k-dimensional plane L, $\Delta \cap L$ admits a triangulation for which it is rationally embedded in M and therefore $M(k) \subset M \setminus K$. Now let $e : \Delta' \to M$ be a rational embedding of a simplex Δ' of dimension $\leq k$. Then there is a (not necessarily rational) triangulation $\mathcal{T}$ of Δ' such that e is linear on every simplex of $\mathcal{T}$. Since every $\Delta'' \in \mathcal{T}$ with $\dim \Delta'' = \dim \Delta'$ has a dense subset of points with rational barycentric coordinates with respect to Δ', we conclude that $e(\Delta'')$ is contained in a k-dimensional rational plane in H. Thus $e(\Delta') \subset K$ and hence $M(k) = M \setminus K$.

Let us state the following important fact, leaving its proof to the reader.

THEOREM 1.1. *Let M be a triangulated m-dimensional manifold, let $k \geq 0$ be an integer and let $n = m - k - 1$. If M is $(n-1)$-connected and $m \geq 2n+1$ then $M(k)$ is an n-dimensional Nöbeling space.*

A space $M(k)$ satisfying the assumptions of Theorem 1.1 will be called a Nöbeling space *modeled* on a triangulated manifold.

A subset A of a space X is called a *Z-set* if A is closed in X and the identity map of X can be arbitrarily closely approximated by a map $f : X \to X$ with $f(X) \cap A = \emptyset$. Note that if X is an n-dimensional Nöbeling space modeled on a manifold M and $A \subset X$ is a Z-set in X then $X \setminus A$ is also an n-dimensional Nöbeling space modeled on the manifold $N = M \setminus$ the closure of A in M (the rational structure of N is defined such that the inclusion is a rational map; see the beginning of Section 3 for details).

The main result of this paper is the following Z-set unknotting theorem.

THEOREM 1.2. *Let X_1 and X_2 be n-dimensional Nöbeling spaces and let A_1 and A_2 be Z-sets in X_1 and X_2 respectively such that $X_1 \setminus A_1$ and $X_2 \setminus A_2$ are homeomorphic to n-dimensional Nöbeling spaces modeled on triangulated manifolds. If A_1 and A_2 are homeomorphic then any homeomorphism between A_1 and A_2 can be extended to a homeomorphism between X_1 and X_2.*

In fact, we will prove a slightly stronger version of Theorem 1.2 which is presented in Theorem 3.1. Z-set unknotting theorems similar to Theorem 1.2 were also proved using different approaches by A. Nagórko [9] and for a restricted class of Z-sets by S. Ageev [1], [2], [3]. The proof of Theorem 1.2 is self-contained and relies only on well-known facts of PL-topology [6], [11], Nöbeling spaces [5] and elementary properties of partitions presented at the very beginning of [4]. Some ideas of the proof of Theorem 1.2 came from [7].

The results of this paper along with some additional arguments apply to validate the characterization theorem for Nöbeling saying that any two Nöbeling spaces of the same dimension are homeomorphic (see [8]). The characterization theorem implies that the assumptions on the complements $X_1 \setminus A_1$ and $X_2 \setminus A_2$ in Theorem 1.2 are automatically satisfied and therefore can be dropped.

The characterization theorem for Nöbeling spaces was also proved by Nagórko [9] and Ageev [1], [2], [3]. Note that both a Z-set unknotting theorem and the characterization theorem were proved by Nagórko in a more general setting of Nöbeling manifolds.

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2. Preliminaries

2.1. *General.* Let M be a triangulated manifold. The *geometric interior* $\operatorname{Int} M$ of M is the set of points having a neighborhood PL-homeomorphic to an Euclidean space of dimension $\dim M$. The set $\partial M = M \setminus \operatorname{Int} M$ is the *geometric boundary* of M.

A triangulated space (manifold) PL-embedded in M is said to be a *PL-subspace* (*PL-submanifold*) of M. An open subset of a triangulated space is always considered with the induced PL-structure for which the inclusion is a PL-map. Note that an open subset of a PL-submanifold of M is also a PL-submanifold of M.

A subset P of M is said to be a *PL-subcomplex* of M if there is a triangulation of M for which P is a subcomplex. A subset R of M is said to be a *PL-presented* subset of M if there are closed subsets $R_1 \subset \cdots \subset R_n$ of M such that $R_n = R$, R_1 is a PL-subcomplex of M and $R_{i+1} \setminus R_i$ is a PL-subcomplex of $M \setminus R_i$, $i = 1, \ldots, n-1$.

A collection $\mathcal{P}$ of subsets of M is said to be a *decomposition* of M if $\mathcal{P}$ is a locally finite cover of M and the elements of $\mathcal{P}$ are PL-subcomplexes of M. By a *finite intersection* of a decomposition $\mathcal{P}$ we mean an intersection of finitely many elements of $\mathcal{P}$ (the elements of $\mathcal{P}$ are also considered as finite intersections). Note that since $\mathcal{P}$ is locally finite, any non-empty intersection of elements of $\mathcal{P}$ must be a finite intersection of $\mathcal{P}$. It is clear that for a PL-submanifold N of M, the restriction $\mathcal{P}|N = \{P \cap N : P \in \mathcal{P}\}$ of $\mathcal{P}$ to N is a decomposition of N.

A decomposition $\mathcal{P}$ of M is said to be a *partition* of M if each finite intersection of $\mathcal{P}$ is a PL-manifold, the geometric interiors of any two non-equal finite intersections are disjoint and for every non-empty finite intersection $P = P_0 \cap \cdots \cap P_t$ of distinct elements $P_0, \ldots, P_t \in \mathcal{P}$, $\dim P = \dim M - t$.

A decomposition $\mathcal{P}$ of M is said to be a *partition on a PL-submanifold* N of M if $\mathcal{P}|N$ is a partition of N (in this case we also say that $\mathcal{P}$ *forms a partition on* N or $\mathcal{P}$ *restricted to* N *is a partition*).

The PL-notions defined above can be translated to the corresponding rational notions by referring to the rational structure of M instead of the PL-structure (see Section 1). Thus $P \subset M$ is a *rational subspace* if there is a triangulation of P for which the inclusion is a rational map, P is a *rational subcomplex* if there is a rational triangulation of M for which P is a subcomplex, $\mathcal{P}$ is a *rational decomposition* of M if the elements of $\mathcal{P}$ are rational subcomplexes etc. Note that an open subset of M is a rational subset of M and it is always considered with the rational structure for which the inclusion is a rational map. Thus the rationally presented sets are well-defined as well.

A subset P of M is *q-connected*, $q \geq 0$, if $\pi_i(P)$ is trivial for every $0 \leq i \leq q$. We will say that P is *l-co-connected*, $l \geq 0$, if P is $(\dim P - l)$-connected (we assume that P is q-connected for every $q < 0$ and P is l-co-connected for every $l > \dim P$). A partition $\mathcal{P}$ is said to be *l-co-connected* if every finite intersection of $\mathcal{P}$ is l-co-connected.

A map $f : X \to M$ is said to be *in general position* with a triangulation of M if, for every simplex Δ of the triangulation, $\dim f^{-1}(\Delta) \leq$

$\dim X + \dim \Delta - \dim M$. Every map from X to M can be arbitrarily closely approximated by a general position map. Moreover, let F be closed in X and a map $f : X \to M$ restricted to F be in general position. Then f can be arbitrarily closely approximated by a general position map $g : X \to M$ such that g coincides with f on F and $g(X \setminus F) \subset \operatorname{Int} M$. A map $f : X \to M$ is said to be in general position with a decomposition of M if it is in general position with a triangulation underlying this decomposition (a triangulation of M for which the elements of the decomposition are subcomplexes).

An *n-dimensional cube* B^n is a set of the form $B^n = \{(x_1, \ldots, x_n) : -r_i \le x_i \le r_i,\ r_i > 0,\ i = 1, \ldots, n\}$ in the Euclidean space $\mathbb{R}^n$. Thus we always assume that the origin O is at the center of B^n. Considering the product $B^n \times B^m$ of two cubes we identify B^n and B^m with $B^n \times O$ and $O \times B^m$ respectively.

Let $\mathcal{A}$ and $\mathcal{B}$ be collections of subsets of a set X and let $C \subset X$. We define $\operatorname{st}(C, \mathcal{A}) = \bigcup\{A : A \in \mathcal{A},\ A \cap C \ne \emptyset\}$, $\operatorname{st}(\mathcal{A}, \mathcal{B}) = \{\operatorname{st}(A, \mathcal{B}) : A \in \mathcal{A}\}$, $\operatorname{st} \mathcal{A} = \operatorname{st}^1 \mathcal{A} = \operatorname{st}(\mathcal{A}, \mathcal{A})$ and by induction $\operatorname{st}^{i+1} \mathcal{A} = \operatorname{st}(\operatorname{st}^i \mathcal{A})$.

2.2. *Elementary properties of partitions.* If $\mathcal{P}$ is a partition of a triangulated manifold M then $\mathcal{P}|V$ is a partition of V for every open $V \subset M$. Let $\mathcal{P}$ be a decomposition of a triangulated manifold M and let $\mathcal{V}$ be an open cover M such that $\mathcal{P}|V$ is a partition of V for every $V \in \mathcal{V}$. Then $\mathcal{P}$ is a partition of M.

Assume that M is a triangulated space. One can show that $M \times (0, 1)$ or $M \times [0, 1)$ is a PL-manifold if and only if M is a PL-manifold. This implies that for a triangulated manifold N, the product $M \times N$ is a PL-manifold if and only if M is a PL-manifold. Thus if M and N are triangulated manifolds and $\mathcal{P}$ is a decomposition of M, then the decomposition $\mathcal{P} \times N = \{P \times N : P \in \mathcal{P}\}$ is a partition of $M \times N$ if and only if $\mathcal{P}$ is a partition of M.

The following properties are proved in [4] for compact manifolds, but their proof also applies for the non-compact case.

Let $\mathcal{P}$ be a partition of a triangulated manifold M, let $\mathcal{P} = \bigcup\{\mathcal{P}_i : i = 1, 2, \ldots\}$ be a splitting of $\mathcal{P}$ into disjoint subfamilies $\mathcal{P}_i$ and let $Q_i = \bigcup\{P : P \in \mathcal{P}_i\}$. Then $\mathcal{Q} = \{Q_1, Q_2, \ldots\}$ is a partition of M (see 1.1.5 of [4]). In particular, any union of elements of a partition is a PL-manifold.

Let $\mathcal{P}$ be a partition of a triangulated manifold M. Then for every finite intersection P of $\mathcal{P}$, $P \cap \partial M \subset \partial P$ (see 1.1.9 of [4]).

Let M be an m-dimensional triangulated manifold, $\mathcal{P}$ a partition of M and $\mathcal{Q}$ a decomposition of M such that for every finite intersection P of $\mathcal{P}$, $\mathcal{Q}|P$ is a partition of P. Then $\mathcal{Q}$ is a partition of M (see 1.1.11 of [4]). In particular, if $M = M_1 \cup M_2$ is a decomposition of M into two m-dimensional PL-submanifolds M_1 and M_2 such that $N = M_1 \cap M_2$ is an $(m-1)$-dimensional PL-submanifold of both ∂M_1 and ∂M_2, and $\mathcal{Q}$ is a decomposition of M such that $\mathcal{Q}|M_1$, $\mathcal{Q}|M_2$ and $\mathcal{Q}|N$ are partitions, then $\mathcal{Q}$ is a partition of M.

2.3. *A matching of partitions.* Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be partitions of triangulated manifolds M_1 and M_2 respectively. A one-to-one correspondence $\mu : \mathcal{P}_1 \to \mathcal{P}_2$ is said to be a *matching* of $\mathcal{P}_1$ and $\mathcal{P}_2$ if it induces a one-to-one correspondence between non-empty intersections. This means that for every finite intersection $P = P_0 \cap \cdots \cap P_t$ of distinct elements $P_0, \ldots, P_t \in \mathcal{P}_1$, $\mu(P) = \mu(P_0) \cap \cdots \cap \mu(P_t) = \emptyset$ if and only if $P = \emptyset$.

Assume that $\mu : \mathcal{P}_1 \to \mathcal{P}_2$ is a matching of partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ such that $\mathcal{P}_2$ is l-co-connected. Let F be a closed subset of X such that $\dim X \leq m_i - l + 1$, $m_i = \dim M_i$, $i = 1, 2$, and let $f_i : F \to M_i$ be maps such that $f_1^{-1}(P) = f_2^{-1}(\mu(P))$ for every finite intersection P of $\mathcal{P}_1$. Assume that a map $f_1' : X \to M_1$ extends f_1 so that f_1' is in general position with $\mathcal{P}_1$. Then there is a map $f_2' : X \to M_2$ such that f_2' extends f_2 and $(f_1')^{-1}(P) = (f_2')^{-1}(\mu(P))$ for every finite intersection P of $\mathcal{P}_1$.

The required extension of f_2 can be constructed by induction on the codimension $t = m_2, m_2 - 1, m_2 - 2, \ldots, 0$ of the intersections of $\mathcal{P}_2$. Assume that we have already extended f_2 to f_2' on $X_t =$ the union of the preimages of all the intersections of $\mathcal{P}_2$ of dimension $\leq m_2 - t$ such that f_2' on X_t has the required properties and let an intersection P of $\mathcal{P}_2$ be of dimension $m_2 - t + 1$. Then for $X_P = (f_1')^{-1}(\mu^{-1}(P))$, $\dim X_P \leq \dim X + \dim \mu^{-1}(P) - m_1 = \dim X + \dim P - m_2$. Since P is $(\dim P - l)$-connected and $\dim P - l \geq \dim P - m_2 + \dim X - 1 \geq \dim X_P - 1$, f_2' restricted to $X_P \cap X_t$ can be extended over X_P so that $f_2'(X_P \setminus X_t) \subset \operatorname{Int} P$.

Thus we can extend f_2 to a map f_2' with the required properties. This extension will be called a *transfer* of the extension f_1' via the matching μ.

Assume that $\mu : \mathcal{P}_1 \to \mathcal{P}_2$ is a matching of partitions on manifolds M_1 and M_2 respectively such that $\mathcal{P}_i$ is l_i-co-connected and $m_1 - l_1 = m_2 - l_2$ where $m_i = \dim M_i$. Let $n \leq m_1 - l_1$ and let $P_1 \subset P_1'$ be subsets of M_1 such that P_1, P_1' are unions of elements of $\mathcal{P}_1$ and the inclusion $P_1 \subset P_1'$ induces the zero-homomorphism of the homotopy groups in dimensions $\leq n$. Then the inclusion $P_2 = \mu(P_1) \subset P_2' = \mu(P_1')$ also induces the zero-homomorphism of the homotopy groups in dimensions $\leq n$. Indeed, take a map $f_2 : S^p \to P_2$, $p \leq n$, from a p-dimensional sphere S^p into P_2. Since P_2 is a manifold we can homotope f_2 inside P_2 to a general position map. Thus we assume that f_2 is a general position map and transfer this map via the matching μ to $f_1 : S^p \to P_1$. Extend f_1 to a general position map $f_1' : B^{p+1} \to P_1'$ of a $(p+1)$-dimensional ball B^{p+1} such that $S^p = \partial B^{p+1}$ and once again transfer this extension to an extension $f_2' : B^{p+1} \to P_2'$ of f_2. Thus f_2 is null-homotopic in P_2'.

2.4. *Improving connectivity of intersections—a summary.* Assume that $m \geq 2q+1$ and $l = m-q+2$, M is a triangulated m-dimensional $(q-1)$-connected ($= (l-1)$-co-connected) manifold and F is a PL-subcomplex of M lying in $\operatorname{Int} M$. Let $\mathcal{P}$ be a decomposition of M which forms an l-co-connected partition on $U = M \setminus F$ and let $0 \leq t \leq m-l+1$. Suppose that the finite intersections of $\mathcal{P}|U$ of dimension $> m-t$ are $(l-1)$-co-connected. We will describe in Section 4 a procedure how to improve the connectivity of the intersections of $\mathcal{P}|U$ of dimension $m-t$.

Namely, we will show how to modify M to an open subset $M' \subset M$, F to a PL-subcomplex F' of M' lying in $\operatorname{Int} M'$ and each $P \in \mathcal{P}$ to a PL-subcomplex P' of M' so that $R = M \setminus M'$ is a PL-presented subset of M, $\dim R \leq q$, $\dim F' \leq \max\{m-q-1, \dim F\}$, $\mathcal{P}' = \{P' : P \in \mathcal{P}\}$ is a decomposition of M' which forms an l-co-connected partition on $U' = M' \setminus F'$, the finite intersections of dimension $\geq m-t$ of $\mathcal{P}'|U'$ are $(l-1)$-co-connected and the correspondence between $\mathcal{P}$ and $\mathcal{P}'$ defined by sending $P \in \mathcal{P}$ to its modification $P' \in \mathcal{P}'$ induces a matching of partitions when $\mathcal{P}$ and $\mathcal{P}'$ are restricted to U and U' respectively.

Moreover, if $\mathcal{W}$ is an open cover of M such that for every $P \in \mathcal{P}$ there is $W \in \mathcal{W}$ such that $\operatorname{st}(P, \mathcal{P}) \subset W$ and the inclusion of $\operatorname{st}(P, \mathcal{P})$ into W induces the zero-homomorphism of the homotopy in dimensions $\leq q-1$, then $\mathcal{P}'$ can be constructed so that for each $P \in \mathcal{P}$ and its modification $P' \in \mathcal{P}'$, $P' \subset \operatorname{st}(P, \operatorname{st}^2 \mathcal{W})$.

A detailed description of the procedure of improving connectivity of intersections is given in Subsections 4.1–4.4.

2.5. *Improving the total connectivity of a partition.* Let M be a triangulated $(q-1)$-connected m-dimensional manifold with $m \geq 2q+1$ and let $l = m-q+2$. Assume that F is a PL-subcomplex of M lying in $\operatorname{Int} M$ and $\mathcal{P}$ is a decomposition of M such that $\dim F \leq m-q$ and $\mathcal{P}$ is an l-co-connected partition on $U = M \setminus F$.

Apply 2.4 to improve the connectivity of the elements of $\mathcal{P}|U$ to the $(l-1)$-co-connectivity, again apply 2.4 to the modified decomposition to improve the connectivity of the intersections of dimension $m-1$ (of the modified decomposition restricted to the complement of the modified F) to the $(l-1)$-co-connectivity and thus proceed on the dimension of the intersections $\geq l-1$ until we modify M to an open subset $M' \subset M$, $\mathcal{P}$ to a decomposition $\mathcal{P}'$ of M' and F to a PL-subcomplex F' of M' lying in $\operatorname{Int} M'$ such that $M \setminus M'$ is a closed PL-presented subset of M, $\dim(M \setminus M') \leq q$, $\dim F' \leq \max\{m-q-1, \dim F\} \leq m-q$, $\mathcal{P}'$ is an $(l-1)$-co-connected partition on $U' = M' \setminus F'$ and $\mathcal{P}$ admits a natural one-to-one correspondence to $\mathcal{P}'$ which sends each element of $\mathcal{P}$ to its modification in $\mathcal{P}'$ and which becomes a matching of partitions when $\mathcal{P}$ and $\mathcal{P}'$ are restricted to U and U' respectively.

2.6. *Absorbing simplexes—a summary.* Let M be a triangulated $(q-1)$-connected m-dimensional manifold with $m \geq 2q+1$ and let $l = m-q+1$. Assume that F is a PL-subcomplex of M lying in $\operatorname{Int} M$ such that $U = M \setminus F$ is l-co-connected ($= (q-1)$-connected) and $\dim F \leq m-q$, and assume that $\mathcal{P}$ is a decomposition of M such that $\mathcal{P}$ is an l-co-connected partition on U. In Section 4 we will describe a procedure of reducing the dimension of F by 1 by absorbing the $(m-q)$-dimensional simplexes of F into U.

Namely, we will show how to modify M to an open subset M' of M, F to a PL-subcomplex F' of M' lying in $\operatorname{Int} M'$ and each $P \in \mathcal{P}$ to a PL-subcomplex P' of M' so that $F' =$ the $(m-q-1)$-skeleton of F with respect to some triangulation of F, $R = M \setminus M'$ is a PL-subcomplex of M, $\dim R \leq q$, $\mathcal{P}' = \{P' : P \in \mathcal{P}\}$ is a decomposition of M', $\mathcal{P}'$ restricted to $U' = M' \setminus F'$ is an l-co-connected partition and the natural one-to-one correspondence from $\mathcal{P}$ to $\mathcal{P}'$ defined by sending $P \in \mathcal{P}$ to its modification $P' \in \mathcal{P}'$ becomes a matching of partitions when $\mathcal{P}$ and $\mathcal{P}'$ are restricted to U and U' respectively.

Moreover, let $\mathcal{W}$ be an open cover of M having the following property: for every $P \in \mathcal{P}$ there are $W \in \mathcal{W}$ and a set H such that $\operatorname{st}(P, \mathcal{P}) \subset H \subset W$, H is a union of elements of $\mathcal{P}$ and the inclusion $\operatorname{st}(P, \mathcal{P}) \cap U \subset H \cap U$ induces the zero-homomorphism of the homotopy groups in dimensions $\leq q-1$. Then $\mathcal{P}'$ can be constructed so that for every $P \in \mathcal{P}$ and its modification $P' \in \mathcal{P}'$, $P' \subset \operatorname{st}(P, \operatorname{st}^2 \mathcal{W})$.

A detailed description of the procedure of absorbing simplexes is given in Subsection 4.5 (which is based on Subsections 4.1 and 4.3).

2.7. *Improving connectivity via a matching.* Let M_i be l_i-co-connected triangulated manifolds such that $m_i = \dim M_i \geq 2(m_i - l_i) + 3$, $i = 1, 2$, and $m_1 - l_1 = m_2 - l_2$. Suppose that $\mathcal{P}_1$ and $\mathcal{P}_2$ are partitions of M_1 and M_2 respectively such that there is a matching between $\mathcal{P}_1$ and $\mathcal{P}_2$, and $\mathcal{P}_1$ is l_1-co-connected. We will show how to modify M_2 to M_2' and $\mathcal{P}_2$ to $\mathcal{P}_2'$ so that M_2' is an open l_2-co-connected submanifold of M_2, $M_2 \setminus M_2'$ is a PL-presented closed subset of M_2, $\dim(M_2 \setminus M_2') \leq l_2 - 2$ and $\mathcal{P}_2'$ is an l_2-co-connected partition of M_2' which admits a natural matching to $\mathcal{P}_2$ defined by sending each element of $\mathcal{P}_2$ to its modification in $\mathcal{P}_2'$.

By 2.5 modify M_2 to $M_2^0 \subset M_2$, $\mathcal{P}_2$ to $\mathcal{P}_2^0$ and construct a subset F^0 of M_2^0 so that $M_2 \setminus M_2^0$ is a PL-presented closed subset of M_2, $\dim M_2 \setminus M_2^0 \leq 1$, F^0 is a PL-subcomplex of M_2^0, $\dim F^0 \leq m_2 - 2$, and $\mathcal{P}_2^0$ is a decomposition of M_2^0 such that $\mathcal{P}_2^0$ restricted to $M_2^0 \setminus F^0$ is an m_2-co-connected partition that admits a matching to $\mathcal{P}_2$ and therefore to $\mathcal{P}_1$.

Now assume that for $0 \leq t < m_2 - l_2$ we have constructed M_2^{2t}, $\mathcal{P}_2^{2t}$ and F^{2t} such that that $M_2 \setminus M_2^{2t}$ is a PL-presented closed subset of M_2, $\dim(M_2 \setminus M_2^{2t}) \leq t+1$, F^{2t} is a PL-subcomplex of M_2^{2t}, $\dim F^{2t} \leq m_2 - t - 2$, and $\mathcal{P}_2^{2t}$ is a decomposition of M_2^{2t} such that $\mathcal{P}_2^{2t}$ restricted to $M_2^{2t}\setminus F^{2t}$ is an (m_2-t)-co-connected partition that admits a matching to $\mathcal{P}_2$ and therefore to $\mathcal{P}_1$. Proceed to $t+1$ as follows.

By 2.5 modify M_2^{2t} to $M_2^{2t+1}\subset M_2^{2t}$, $\mathcal{P}_2^{2t}$ to $\mathcal{P}_2^{2t+1}$, F^{2t} to F^{2t+1} so that $M_2^{2t}\setminus M_2^{2t+1}$ is a PL-presented closed subset of M_2^{2t}, $\dim(M_2^{2t}\setminus M_2^{2t+1})\le t+2$, F^{2t+1} is a PL-subcomplex of M_2^{2t+1}, $\dim F^{2t+1}\le m_2-t-2$, $\mathcal{P}_2^{2t+1}$ is a decomposition of M_2^{2t+1}, and $\mathcal{P}_2^{2t+1}$ restricted to $M_2^{2t+1}\setminus F^{2t+1}$ is an (m_2-t-1)-co-connected partition that admits a matching to $\mathcal{P}_2^{2t}$ restricted to $M_2^{2t}\setminus F^{2t}$ and therefore to $\mathcal{P}_2$ and $\mathcal{P}_1$.

Then since $\mathcal{P}_1$ is l_1-co-connected on the l_1-co-connected manifold M_1 we conclude by 2.3 that $M_2^{2t+1}\setminus F^{2t+1}$ is (m_2-t-1)-co-connected. Therefore, by 2.6, M_2^{2t+1}, F^{2t+1} and $\mathcal{P}_2^{2t+1}$ can be modified to M_2^{2t+2}, F^{2t+2} and $\mathcal{P}_2^{2t+2}$ respectively such that $M_2^{2t+2}\subset M_2^{2t+1}$, $M_2^{2t+1}\setminus M_2^{2t+2}$ is a PL-subcomplex of M_2^{2t+1}, $\dim(M_2^{2t+1}\setminus M_2^{2t+2})\le t+2$, F^{2t+2} is a PL-subcomplex of M_2^{2t+2}, $\dim F^{2t}\le m_2-t-3$, $\mathcal{P}_2^{2t+2}$ is a decomposition of M_2^{2t+2}, and $\mathcal{P}_2^{2t+2}$ restricted to $M_2^{2t+2}\setminus F^{2t+2}$ is an (m_2-t-1)-co-connected partition that admits a matching to $\mathcal{P}_2^{2t+1}$ restricted to $M_2^{2t+1}\setminus F^{2t+1}$ and therefore to $\mathcal{P}_2$ and $\mathcal{P}_1$.

Then for $t=m_2-l_2$ we have $M_2'=M_2^{2t}\setminus F^{2t}$, and $\mathcal{P}_2'=\mathcal{P}_2^{2t}$ restricted to M_2' will have the required properties (note that $m_2-l_2+1\le l_2-2$ and therefore $\dim(M_2\setminus M_2')\le l_2-2$).

2.8. *Moving to a rational position.* Let M be a triangulated manifold with the rational structure determined by a triangulation $\mathcal{T}$ (see Section 1). Assume that $\mathcal{T}'$ is a (not necessarily rational) triangulation of M. We will show that the identity map of M can be arbitrarily closely approximated by a PL-homeomorphism $f: M\to M$ such that for very $\Delta'\in\mathcal{T}'$, $f(\Delta')$ is a rational subcomplex of M.

Embed M into a Hilbert space by a map which is linear on every simplex of $\mathcal{T}$ and refer to this Hilbert space when properties of linearity are used. Let $\mathcal{T}''$ be a triangulation of M such that $\mathcal{T}''$ is a subdivision of both $\mathcal{T}$ and $\mathcal{T}'$, and the simplexes of $\mathcal{T}''$ are linear. Approximate every vertex v of $\mathcal{T}''$ by a rational point p_v (= a point with rational barycentric coordinates with respect to $\mathcal{T}$) such that for every simplex Δ of $\mathcal{T}$, $p_v\in\Delta$ if and only if $v\in\Delta$. The approximation of the vertices of $\mathcal{T}''$ can be extended to the PL-map $f: M\to M$ sending each vertex v of $\mathcal{T}''$ to p_v so that f is linear on each simplex of $\mathcal{T}''$. If p_v is sufficiently close to v for every vertex $v\in\mathcal{T}''$ then the map f is a PL-homeomorphism that can be chosen to be arbitrarily close to the identity map. Clearly, f sends every simplex of $\mathcal{T}''$ to a rational simplex and therefore f sends every simplex of $\mathcal{T}'$ to a rational subcomplex of M.

Let M be an open subset of a space Y and d a metric on Y. We can assume that the homeomorphism $f: M\to M$ constructed above satisfies

$d(y, f(y)) \leq d(y, Y \setminus M)$ for every $y \in M$. Then f extends to the homeomorphism $g : Y \to Y$ such that $g(y) = y$ if $y \in Y \setminus M$ and $g(y) = f(y)$ if $y \in M$. Now assume that R is a PL-presented subset of M and let $R_1 \subset R_2 \subset \cdots \subset R_n$ be closed subsets of M such that $R = R_n$, R_1 is a PL-subcomplex of M and $R_{i+1} \setminus R_i$ is a PL-subcomplex of $M \setminus R_i$, $i = 1, \ldots, n-1$. Let us show that the identity map of Y can be arbitrarily closely approximated by a homeomorphism $g : Y \to Y$ such that $g(y) = y$ for every $y \in Y \setminus M$, $g(R)$ is a closed rationally presented subset of M, and g restricted to $M \setminus R$ is a PL-homeomorphism to $M \setminus g(R)$ sending $\mathcal{P}$ to a rational decomposition of $M \setminus g(R)$ (note that the rational structure of M induces the corresponding rational structure on open subsets of M such that the inclusions are rational maps).

Approximate the identity map of Y by a homeomorphism $g_1 : Y \to Y$ such that g_1 does not move the points of $Y \setminus M$, g_1 restricted to M is a PL-homeomorphism and $g_1(R_1)$ is a rational subcomplex of M. Approximate g_1 by a map $g_2 : Y \to Y$ such that g_2 does not move the points of $(Y \setminus M) \cup g_1(R_1)$, g_2 restricted to $M \setminus g_1(R_1)$ is a PL-homeomorphism and $g_2(g_1(R_2 \setminus R_1))$ is a rational subcomplex of $M \setminus g_1(R_1)$. Proceed by induction and construct for every $i = 1, \ldots, n-1$ an approximation of the identity map of Y by a homeomorphism $g_{i+1} : Y \to Y$ such that g_{i+1} does not move the points of $(Y \setminus M) \cup (g_i \circ \cdots \circ g_1)(R_i)$, g_{i+1} restricted to $M \setminus (g_i \circ \cdots \circ g_1)(R_i)$ is a PL-homeomorphism and $(g_{i+1} \circ \cdots \circ g_1)(R_{i+1} \setminus R_i)$ is a rational subcomplex of $M \setminus (g_i \circ \cdots \circ g_1)(R_i)$. Finally, construct g_n with the additional property that $g = g_n \circ \cdots \circ g_1$ sends $\mathcal{P}$ to a rational decomposition of $M \setminus g(R)$. Then g will have the required properties.

3. Proof of the Z-set unknotting theorem. A map $\psi : X \to \Gamma$ between two spaces X and Γ is said to be UV^{n-1} if $\psi(X)$ is dense in Γ and, for every point $p \in \Gamma$ and every neighborhood U of p, there is a smaller neighborhood V of p such that the inclusion of $\psi^{-1}(V)$ into $\psi^{-1}(U)$ induces the zero-homomorphism of the homotopy groups in dimensions $\leq n-1$. In this section we will prove the following Z-set unknotting theorem which is slightly stronger than Theorem 1.2.

THEOREM 3.1. *Let X_1 and X_2 be n-dimensional Nöbeling spaces and let A_1 and A_2 be homeomorphic Z-sets in X_1 and X_2 respectively such that $X_1 \setminus A_1$ and $X_2 \setminus A_2$ are homeomorphic to Nöbeling spaces modeled on triangulated manifolds. Then every homeomorphism $f_A : A_1 \to A_2$ extends to a homeomorphism $f_X : X_1 \to X_2$.*

Moreover, for any UV^{n-1}-maps $\psi_1 : X_1 \to \Gamma$ and $\psi_2 : X_2 \to \Gamma$ to a space Γ and every open cover $\mathcal{O}^+$ of Γ there is an open cover $\mathcal{O}^-$ of Γ such that $\mathcal{O}^-$ does not depend on A_1, A_2 and f_A and $\mathcal{O}^-$ has the following *property: if ψ_1 restricted to A_1 and $\psi_2 \circ f_A$ are $\mathcal{O}^-$-close then f_X can be constructed so that ψ_1 and $\psi_2 \circ f_X$ are $\mathcal{O}^+$-close.*

The proof of Theorem 3.1 is based on a few propositions; before each proposition or its proof we present notions, notations and properties used in that proposition.

Let M be a triangulated space and let P be either a rational subcomplex or an open subset of M. We always consider P with the induced rational structure for which the inclusion of P into M is a rational map. Then there are rational triangulations of P and M respectively such that the simplexes of P are linearly and rationally embedded in the simplexes of M. Now we can derive from the interpretation of $M(k)$ given in Section 1 that $P \cap M(k) = P(k)$.

Let M be a triangulated manifold. It can also be derived from the interpretation of $M(k)$ given in Section 1 that $M \setminus M(k)$ is a countable union of simplexes of dimension $\leq k$ which are PL-embedded in M. Then for an integer q such that $\dim M \geq q + k$ every map f from a $(q-1)$-dimensional sphere S^{q-1} to M can be arbitrarily closely approximated by a map into $M(k)$ and therefore f can be homotoped into $M(k)$. In particular, this implies that if $M(k)$ is $(q-1)$-connected and M' is an open subset of M such that $M(k) \subset M'$ then, since $M'(k) = M(k)$, we deduce that M' is $(q-1)$-connected as well.

Let A be a closed subset of a space X. A collection $\mathcal{C}$ of subsets $X \setminus A$ is said to *properly approach* A if for every sequence $\{C_j\}$ of elements of $\mathcal{C}$ such that there is a sequence of points $x_j \in C_j$ converging to a point $a \in A$ we have $\lim_{j\to\infty} C_j = a$ (that is, $\lim_{j\to\infty} y_j = a$ for every sequence $\{y_j\}$ with $y_j \in C_j$). Note that if $\mathcal{C}$ properly approaches A then the closure in X of no $C \in \mathcal{C}$ intersects A. Also if collections $\mathcal{C}$ and $\mathcal{C}'$ of subsets of $X \setminus A$ properly approach A then $\mathrm{st}(\mathcal{C}, \mathcal{C}')$ properly approaches A as well.

Let X be a subspace of a space Y. Saying that $\mathcal{U}$ is an open cover of X we mean that $\mathcal{U}$ is a cover of X by subsets open in X.

Let X be a dense subset of a space Y and U an open subset of X. By the *extension* U_Y of U to Y we understand the largest open subset U_Y of Y such that $U = U_Y \cap X$. By the *extension* to Y *of a collection* $\mathcal{U}$ of open subsets of X we mean the collection of the extensions to Y of the elements of $\mathcal{U}$. Note that if A is a closed subset of X such that A is also closed in Y, and $\mathcal{U}$ is a collection of open subsets of $X \setminus A$ such that $\mathcal{U}$ properly approaches A, then for the extension $\mathcal{U}_Y$ of $\mathcal{U}$ to Y we deduce that $\mathcal{U}_Y$ properly approaches A as well.

Let A_1 and A_2 be homeomorphic subsets of spaces Y_1 and Y_2 respectively, let $f_A : A_1 \to A_2$ be a homeomorphism and let $\mu : \mathcal{C}_1 \to \mathcal{C}_2$ be a one-to-one correspondence between collections of subsets of $Y_1 \setminus A_1$ and $Y_2 \setminus A_2$

respectively. We say that μ *agrees* with $f_A : A_1 \to A_2$ if for every sequence $C_j, j = 1, 2, \ldots$, of elements of $\mathcal{C}_1$, C_j converge to $a \in A_1$ in Y_1 (as $j \to \infty$) if and only $\mu(C_j)$ converge to $f_A(a)$ in Y_2. It is easy to check that if $\mathcal{C}_1$ and $\mathcal{C}_2$ properly approach A_1 and A_2 respectively, μ agrees with f_A and a collection $\mathcal{C}$ of subsets of $Y_1 \setminus A_1$ properly approaches A_1, then $\mu(\mathrm{st}(\mathcal{C}, \mathcal{C}_1))$ properly approaches A_2, where $\mu(\mathrm{st}(\mathcal{C}, \mathcal{C}_1)) = \{\mu(\mathrm{st}(C, \mathcal{C}_1)) : C \in \mathcal{C}\}$ and $\mu(\mathrm{st}(C, \mathcal{C}_1)) = \bigcup\{\mu(C') : C' \in \mathcal{C}_1,\ C' \cap C \neq \emptyset\}$.

Let X_i, $i = 1, 2$, be subsets $X_i \subset Y_i$ such that X_i is dense in Y_i and let $\psi_i : X_i \to \varGamma$ be maps to a space $\varGamma$ and $\mathcal{O}$ an open cover of $\varGamma$. We say that μ, ψ_1 and ψ_2 *agree with respect to* $\mathcal{O}$ if for every $C \in \mathcal{C}_1$ there is an element $\varOmega \in \mathcal{O}$ such that C is contained in the extension of $\psi_1^{-1}(\varOmega)$ to Y_1 and $\mu(C)$ is contained in the extension of $\psi_2^{-1}(\varOmega)$ to Y_2

Let M be a triangulated manifold. Then $(\partial M)(k)$ is a Z-set in $M(k)$. Indeed, take an open cover $\mathcal{U}$ of $M(k)$ and let $\mathcal{U}_M$ be the extension of $\mathcal{U}$ to M and M' the union of the elements of $\mathcal{U}_M$. The identity map of M' can be arbitrarily closely approximated by a PL-embedding f of M' into $\operatorname{Int} M'$ such that $f(M')$ is a PL-subcomplex of M'. Then, by 2.8, we may assume that f is rational and $\mathcal{U}_M$-close to the identity map of M', and hence f induces a $\mathcal{U}$-close map from $M(k)$ into $(\operatorname{Int} M)(k)$.

Suppose that A is a closed subset of X and $X \setminus A$ is embedded in a space M. Let us show that there are an open subset V of M containing $X \setminus A$ and a space Y such that X and V embed into Y so that $Y = X \cup V$, $A = Y \setminus V$ and A is closed in Y.

Let H be a Hilbert space. Consider the space $H \times [0, 1]$. Let $\mathcal{U}$ be an open cover of $H \times (0, 1]$ such that $\mathcal{U}$ properly approaches $H \times \{1\}$. Take any embedding $e : X \to H \times [0, 1]$ such that $e(A) \subset H \times \{1\}$ and $e(X \setminus A) \subset H \times (0, 1]$. By Walsh's lemma $e|X \setminus A$ can be $\mathcal{U}$-closely approximated by a map $g_{X \setminus A} : X \setminus A \to H \times (0, 1]$ such that $g_{X \setminus A}$ extends over an open subset $X \setminus A \subset V$ of M to a map $g_V : V \to H \times (0, 1]$. Approximate g_V by a $\mathcal{U}$-close embedding $h_V : V \to H \times (0, 1]$ and define $h : Y = A \cup V \to H \times [0, 1]$ by $h(y) = e(y)$ if $y \in A$ and $h(y) = h_V(y)$ if $y \in V$. Now we can transfer the topology of $h(Y)$ to Y and it is easy to see that then Y has the required properties.

Assume that X_i and A_i, $i = 1, 2$, satisfy the assumptions of Theorem 3.1, let $f_A : A_1 \to A_2$ be a homeomorphism and let $X_i \setminus A_i$ be homeomorphic to $M_i(k_i)$, where M_i is an l_i-co-connected triangulated manifold such that $n = m_i - l_i + 1$, $k_i = l_i - 2$ and $m_i \geq 2(m_i - l_i) + 3$ for $n = \dim X_1 = \dim X_2$ and $m_i = \dim M_i$ (note that by Theorem 1.1, $M_i(k_i)$ is an n-dimensional Nöbeling space). Identify $X_i \setminus A_i$ with $M_i(k_i)$ and, by the property proved above, replace M_i by an open subset of M_i containing $M_i(k_i)$ and assume that X_i and M_i are subspaces of a space Y_i such that $Y_i = X_i \cup M_i$, $A_i = Y_i \setminus M_i$ and A_i is closed in Y_i.

PROPOSITION 3.2. *Let $X_i, A_i, f_A, Y_i, M_i, k_i, l_i, m_i, n$ be as above. Then for every l_1-co-connected rational partition $\mathcal{P}_1$ of an open subset M_1' of M_1 such that $M_1(k_1) \subset M_1'$, $\mathcal{P}_1$ properly approaches A_1 and $\mathcal{P}_1$ contains no finite intersections of dimension $\leq l_1-2$, there are an open subset M_2' of M_2 and an l_2-co-connected rational partition $\mathcal{P}_2$ of M_2' such that $M_2(k_2) \subset M_2'$, $\mathcal{P}_2$ properly approaches A_2, $\mathcal{P}_2$ contains no finite intersections of dimension $\leq l_2-2$ and $\mathcal{P}_1$ and $\mathcal{P}_2$ admit a matching $\mu : \mathcal{P}_1 \to \mathcal{P}_2$ which agrees with f_A.*

Moreover, for any UV^{n-1}-maps $\psi_1 : X_1 \to \Gamma$ and $\psi_2 : X_2 \to \Gamma$ to a space Γ and every open cover $\mathcal{O}^+$ of Γ there is an open cover $\mathcal{O}^-$ of Γ which refines $\mathcal{O}^+$ such that $\mathcal{O}^-$ does not depend on A_1, A_2 and f_A and $\mathcal{O}^-$ has the following property: if $\psi_1|A_1$ and $\psi_2\circ f_A$ are $\mathcal{O}^-$-close and $\mathcal{P}_1$ refines the extension of $\psi_1^{-1}(\mathcal{O}^-)$ to Y_1 then M_2', $\mathcal{P}_2$ and μ can be constructed so that μ, ψ_1 and ψ_2 agree with respect to $\mathcal{O}^+$.

Let us show how Theorem 3.1 can be derived from Proposition 3.2.

Proof of Theorem 3.1. Assume that the conclusion holds in dimensions $\leq n-1$ and let us prove it in dimensions n. Fix complete metrics on X_1 and X_2 and let $\varepsilon > 0$. Take an open cover $\mathcal{U}$ of $X_1 \setminus A_1$ of $\operatorname{mesh}\mathcal{U} < \varepsilon$ such that $\mathcal{U}$ properly approaches A_1. Denote by $\mathcal{U}_M$ the extension to M_1 of $\mathcal{U}$ and let M_1' be the union of the elements of $\mathcal{U}_M$. Take a rational triangulation of M_1' such that for the partition $\mathcal{P}_1$ of M_1' formed by the stars of the vertices with respect to the first barycentric subdivision of the triangulation, $\mathcal{P}_1$ refines $\mathcal{U}_M$. Then $\mathcal{P}_1$ is a rational partition, $\mathcal{P}_1$ properly approaches A_1 and for every $P \in \mathcal{P}_1$, $\operatorname{diam} P \cap X_1 < \varepsilon$. Since the intersections of $\mathcal{P}_1$ of dimension $\leq l_1-2$ do not intersect $M_1(k_1)$ we can remove them from M_1' and the elements of $\mathcal{P}_1$, and assume that $\mathcal{P}_1$ has no non-empty intersections of dimension $\leq l_1-2$. Note that all the intersections of $\mathcal{P}_1$ remain contractible and hence $\mathcal{P}_1$ is l_1-co-connected.

By Proposition 3.2 there are an open submanifold M_2' of M_2 and an l_2-co-connected rational partition of M_2' such that $M_2(k_2) \subset M_2'$, $\mathcal{P}_2$ properly approaches A_2 and $\mathcal{P}_2$ admits a matching $\mu : \mathcal{P}_1 \to \mathcal{P}_2$ which agrees with $f_A : A_1 \to A_2$.

We are going to construct homeomorphisms $f_P : P(k_1) \to \mu(P)(k_2)$ for all finite intersections P of $\mathcal{P}_1$ of dimension $\leq m_1 - 1$ which will agree on the common intersections. Let $t \leq m_1 - 1$ and assume that for every finite intersection P of dimension $\leq t-1$ we already constructed f_P. Take an intersection P of $\mathcal{P}_1$ such that $\dim P = t$. By Theorem 1.1, $P(k_1)$ is a Nöbeling space of dimension $\leq n-1$. Note that for the union P' of the intersections of $\mathcal{P}_1$ of dimension $< t$ that are contained in P, P' lies in ∂P and hence $P'(k_1)$ is a Z-set in $P(k_1)$. Define the homeomorphism $f_{P'} : P'(k_1) \to \mu(P')(k_2)$ by the homeomorphisms of the intersections forming P'. According to our assumption Theorem 3.1 holds in dimensions $\leq n-1$. Therefore $f_{P'}$ can be extended to a homeomorphism $f_P : P(k_1) \to \mu(P)(k_2)$.

Recall that $\mathcal{P}_1$ and $\mathcal{P}_2$ properly approach A_1 and A_2 respectively and the matching μ agrees with f_A. Then a homeomorphism extending f_X between X_1 and X_2 can be obtain by pasting homeomorphisms from $P(k_1)$ to $\mu(P)(k_2)$ for $P \in \mathcal{P}_1$ which extend the already defined homeomorphisms on intersections of $\mathcal{P}_1$ of dimension $\leq m_1 - 1$.

Fix $P \in \mathcal{P}_1$ and let P' be the union of the intersections of $\mathcal{P}_1$ of dimension $\leq m_1 - 1$ that are contained in P. The homeomorphism $f_{P'} : P'(k_1) \to \mu(P')(k_2)$ is a homeomorphism of Z-subsets of $P(k_1)$ and $\mu(P)(k_2)$ respectively. Therefore we can repeat for P and $\mu(P)$ the same procedure that we did for M_1 and M_2 but this time in the opposite direction from $\mu(P)$ to P, first "splitting" $\mu(P)$ into small pieces and then defining the corresponding splitting of P.

Thus going back and forth we are able after each iteration to extend f_A to a partial homeomorphism of larger and larger parts of X_1 and X_2 and simultaneously to restrict for each point of X_1 and X_2 the set to which this point can be sent under a possible extension of f. Finally, passing to the limit and using the completeness of X_1 and X_2 we get the desired homeomorphism $f_X : X_1 \to X_2$. It is clear from the construction that $f_X(P \cap X_1) = \mu(P) \cap X_2$ for every $P \in \mathcal{P}_1$.

Now let $\psi_i : X_i \to \Gamma$, $i = 1, 2$, be UV^{n-1}-maps to a space Γ and $\mathcal{O}^+$ an open cover of Γ. Assume that an open cover $\mathcal{O}^-$ of Γ satisfies the conclusions of Proposition 3.2 and assume that $\psi_1|A$ and $\psi_2 \circ f_A$ are $\mathcal{O}^-$-close. Then we can take the cover $\mathcal{U}$ of $X_1 \setminus A_1$ such that $\mathcal{U}$ refines $\psi_1^{-1}(\mathcal{O}^-)$ and will find that $\mathcal{P}_1$ refines the extension of $\psi_1^{-1}(\mathcal{O}^-)$ to Y_1. Now, by Proposition 3.2, we can construct M_2', $\mathcal{P}_2$ and μ so that μ, ψ_1 and ψ_2 agree with respect to $\mathcal{O}^+$. Then for every $P \in \mathcal{P}_1$ there is an element of $\mathcal{O}^+$ containing both $\psi_1(P \cap X_1)$ and $\psi_2(\mu(P) \cap X_2)$ and, since $f_X(P \cap X_1) = \mu(P) \cap X_2$ for every $P \in \mathcal{P}_1$, we see that the maps ψ_1 and $\psi_2 \circ f_X$ are $\mathcal{O}^+$-close. ■

In the proof of Proposition 3.2 we will use the following propositions and facts.

A map $\phi : Y \to X$ is said to be a Z-*embedding* if ϕ is a closed embedding of Y into X and $\phi(Y)$ is a Z-set in X. Let X be a Nöbeling space. It is well-known that if Y is a complete space of dimension $\leq n$, A is a closed subset of Y and $f : A \to X$ is a Z-embedding, then every extension of f over Y can be arbitrarily closely approximated by an extension which is a Z-embedding. It is also well-known (and can be derived from the previous fact) that for every Z-set $A \subset X$ the identity map on X can be arbitrarily closely approximated by a Z-embedding whose image misses A. Another simple property of Z-sets says that if a closed subset A of X is a countable union of Z-sets then A is a Z-set as well.

We say that for two collections $\mathcal{V}$ and $\mathcal{U}$ of subsets of a space X, $\mathcal{V}$ is an $(n-1)$-*refinement* of $\mathcal{U}$, written $\mathcal{V} \prec_{n-1} \mathcal{U}$, if for every $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ such that $V \subset U$ and the inclusion of V into U induces the zero-homomorphism of the homotopy groups in dimensions $\le n-1$.

PROPOSITION 3.3. *Let X be an n-dimensional Nöbeling space and let $\psi : X \to \Gamma$ be a UV^{n-1}-map. Then for every open cover $\mathcal{U}^+$ of Γ there is an open cover $\mathcal{U}^-$ of Γ having the following property: for every complete space Y of dimension $\le n$, a closed subset A of Y, a map $f : Y \to \Gamma$ and a Z-embedding $\phi_A : A \to X$ such that f restricted to A and $\psi \circ \phi_A$ are $\mathcal{U}^-$-close, we can extend ϕ_A to a Z-embedding $\phi : Y \to X$ such that f and $\psi \circ \phi$ are $\mathcal{U}^+$-close.*

Proof. Let open covers $\mathcal{U}^0, \mathcal{U}^1, \ldots, \mathcal{U}^{n+2}$ of Γ be such that $\operatorname{st}\mathcal{U}^i$ refines $\mathcal{U}^{i+1}$, $\operatorname{st}\psi^{-1}(\mathcal{U}^i) \prec_{n-1} \psi^{-1}(\mathcal{U}^{i+1})$, $0 \le i < n+2$, and $\mathcal{U}^{n+2} = \mathcal{U}^+$. Note that since $\psi(X)$ is dense in Γ we have $\operatorname{st}\psi^{-1}(\mathcal{U}^i) = \psi^{-1}(\operatorname{st}\mathcal{U}^i)$. Set $\mathcal{U}^- = \mathcal{U}^0$ and assume that $f|A$ and $\psi \circ \phi_A$ are $\mathcal{U}^0$-close. Since $\dim Y \le n$ there is a surjective map $p : Y \to Y'$ such that $p^{-1}(p(A)) = A$, $p|A$ is a homeomorphism between A and $p(A)$, $K = Y' \setminus p(A)$ is a simplicial complex of dimension $\le n$, the collection $\mathcal{T}$ of the simplexes of K properly approaches A and $p^{-1}(\mathcal{T})$ refines $f^{-1}(\mathcal{U}^0)$. Identify A with $p(A)$ and consider A as a subset of Y'.

We will construct a continuous extension ϕ' of ϕ_A over Y'. Fix a metric d on Y'. For every 0-dimensional simplex (= vertex) Δ of K choose a point $a_\Delta \in A$ such that $d(a_\Delta, \Delta) \le 2d(A, \Delta)$. If there is $U \in \mathcal{U}^0$ such that $p^{-1}(\Delta) \subset f^{-1}(U)$ and $\phi_A(a_\Delta) \in \psi^{-1}(U)$ then define $\phi'(\Delta) = \phi_A(a_\Delta)$, otherwise choose $U \in \mathcal{U}^0$ such that $p^{-1}(\Delta) \subset f^{-1}(U)$ and define $\phi'(\Delta)$ as any point in $\psi^{-1}(U)$. It is easy to see that, since $f|A$ and $\psi \circ \phi_A$ are $\mathcal{U}^0$-close, ϕ' is continuous on $A \cup K^0$, where K^i is the *i-skeleton* of K (= the union of the simplexes of dimension $\le i$). Assume that ϕ' is already defined on $A \cup K^i$, $i \le n-1$, and such that the images of the simplexes of K of dimension $\le i$ refine $\psi^{-1}(\mathcal{U}^i)$.

Let Δ be an $(i+1)$-dimensional simplex of K. Fix a metric on X and define $d_\Delta = \inf\{\operatorname{diam} C : \phi'(\partial\Delta) \subset C,\ \phi'|\partial\Delta \text{ is null-homotopic in } C \text{ and there is } U \in \mathcal{U}^{i+1} \text{ such that } C \subset \psi^{-1}(U)\}$. Since $\operatorname{st}\psi^{-1}(\mathcal{U}^i) \prec_{n-1} \psi^{-1}(\mathcal{U}^{i+1})$ we conclude that d_Δ is well-defined. Extend ϕ' from $\partial\Delta$ to Δ so that $\operatorname{diam}\phi'(\Delta) \le d_\Delta + d(A, \Delta)$ and $\phi'(\Delta)$ is contained in an element of $\psi^{-1}(\mathcal{U}^{i+1})$. Thus we extend ϕ' over $A \cup K^{i+1}$ and this extension is continuous. Indeed, if a sequence Δ_j of $(i+1)$-dimensional simplexes of K converges to a point $a \in A$ then $\phi'(\partial\Delta_j)$ converges to $\phi'(a)$ and hence d_{Δ_j} converges to 0 because X is locally $(n-1)$-connected. Then $\phi'(\Delta_j)$ converges to $\phi'(a)$ and hence ϕ' is continuous on $A \cup K^{i+1}$. Clearly, the images of the $(i+1)$-dimensional simplexes of K refine $\psi^{-1}(\mathcal{U}^{i+1})$. Thus, by induction, we can extend ϕ_A to $\phi' : Y' = A \cup K \to X$ such that $\phi'(\mathcal{T})$ refines $\psi^{-1}(\mathcal{U}^n)$.

Since $f(p^{-1}(\Delta))$ and $\psi(\phi'(\Delta))$ are $\mathcal{U}^0$-close for every 0-dimensional simplex Δ of K, and since $p^{-1}(\mathcal{T})$ refines $f^{-1}(\mathcal{U}^0)$ and $\phi'(\mathcal{T})$ refines $\psi^{-1}(\mathcal{U}^n)$, we deduce that f and $\psi \circ \phi' \circ p$ are $\operatorname{st}(\mathcal{U}^0,\mathcal{U}^n)$-close, and hence f and $\psi \circ \phi' \circ p$ are $\mathcal{U}^{n+1}$-close. Now we can approximate $\phi' \circ p$ by a Z-embedding $\phi : Y \to X$ which coincides with ϕ_A on A such that f and $\psi \circ \phi$ are $\mathcal{U}^{n+2}$-close ($= \mathcal{U}^+$-close) and we are done. ∎

Let X be an n-dimensional Nöbeling space and $\mathcal{U}$ an open cover of X. It is well-known (and can be easily shown) that if two maps from a space Y of dimension $\leq n-1$ into X are sufficiently close then they are $\mathcal{U}$-homotopic ($=$ there is a homotopy $H : Y \times [0,1] \to X$ which connects the maps such that for every $y \in Y$, $H(y \times [0,1])$ is contained in an element of $\mathcal{U}$).

Proposition 3.4. *Let X be an n-dimensional Nöbeling space, let A be a Z-set in X and let $\mathcal{C}$ be a cover of $X \setminus A$ that properly approaches A. Then there is an open cover $\mathcal{V}$ of $X \setminus A$ such that $\mathcal{V}$ properly approaches A and $\mathcal{C} \prec_{n-1} \mathcal{V}$. Moreover, if $\mathcal{W}$ is an open cover of X such that $\mathcal{C} \prec_{n-1} \mathcal{W}$ then $\mathcal{V}$ can be constructed so that $\mathcal{V}$ refines* $\operatorname{st}^3 \mathcal{W}$.

Proof. Fix a metric d on X such that $d(x,y) \leq 1$ for every $x, y \in X$. If $C \in \mathcal{C}$ is not a singleton, write d_C for the infimum of $\operatorname{diam}(G)$ for open subsets G of X such that $C \subset G$, G is contained in an element of $\mathcal{W}$ and the inclusion $C \subset G$ induces the zero-homomorphism of the homotopy groups in dimensions $\leq n-1$. If $C \in \mathcal{C}$ is a singleton, define $d_C = d(C, A)$. For every $C \in \mathcal{C}$ fix an open set G_C such that $\operatorname{diam}(G_C) \leq 2d_C$, G is contained in an element of $\mathcal{W}$ and the inclusion $C \subset G_C$ induces the zero-homomorphism of the homotopy groups in dimensions $\leq n-1$.

Define $\mathcal{C}_i = \{C \in \mathcal{C} : 1/i+1 < d_C \leq 1/i\}$. Note that the closure of the union of the elements of $\mathcal{C}_i$ does not intersect A since otherwise there is a point of A to which a sequence of elements of $\mathcal{C}_i$ converges and therefore $\mathcal{C}_i$ would contain elements C with arbitrarily small d_C because X is locally $(n-1)$-connected.

Let $\mathcal{U}$ be an open cover of $X \setminus A$ such that $\mathcal{U}$ properly approaches A and $\mathcal{U}$ refines $\mathcal{W}$. Then for every i we can approximate the identity map of X by a closed embedding $e_i : X \to X$ such that $e_i(X) \subset X \setminus A$, e_i is $\mathcal{W}$-close to the identity map of X and, for every $C \in \mathcal{C}_i$, $\operatorname{diam}(e_i(G_C)) \leq \operatorname{diam}(G_C) + 1/i$, $e_i(C) \subset \operatorname{st}(C,\mathcal{U})$ and every map $f : S^p \to C$ from a p-dimensional sphere S^p, $p \leq n-1$, can be homotoped into $e_i(C)$ inside $\operatorname{st}(C,\mathcal{U})$. Write $Y_C = \operatorname{st}(C,\mathcal{U}) \cup e_i(G_C)$ for $C \in \mathcal{C}_i$. Clearly, the inclusion $C \subset Y_C$ induces the zero-homomorphism of the homotopy groups in dimensions $\leq n-1$.

Define $\mathcal{G}_i = \{e_i(G_C) : C \in \mathcal{C}_i\}$ and $\mathcal{G} = \bigcup\{\mathcal{G}_i : i = 1, 2, \ldots\}$. Recall that $\operatorname{diam}(e_i(G_C)) \leq \operatorname{diam}(G_C) + 1/i \leq 3/i$ for $C \in \mathcal{C}_i$. Then, since the elements of $\mathcal{G}_i$ are contained in the closed subset $e_i(X)$ which does not meet A, we can conclude that the collection $\mathcal{G}$ properly approaches A. Since each e_i is

$\mathcal{W}$-close to the identity map of X and each G_C is contained in an element of $\mathcal{W}$, we deduce that $\mathcal{G}$ refines $\operatorname{st}\mathcal{W}$.

Clearly, the cover $\mathcal{Y} = \{Y_C : C \in \mathcal{C}\}$ of $X \setminus A$ refines $\operatorname{st}(\operatorname{st}(\mathcal{C},\mathcal{U}),\mathcal{G})$. Then $\mathcal{Y}$ refines $\operatorname{st}^2\mathcal{W}$ and, since $\mathcal{C}$, $\mathcal{U}$ and $\mathcal{G}$ properly approach A, we see that $\mathcal{Y}$ properly approaches A as well. Set $V_C = \operatorname{st}(Y_C,\mathcal{U})$. Then $\mathcal{V} = \{V_C : C \in \mathcal{C}\}$ properly approaches A, $\mathcal{V}$ refines $\operatorname{st}^3\mathcal{W}$ and therefore $\mathcal{V}$ has all the required properties. ∎

In the proof of Proposition 3.2 we will use the following notion of maps witnessing intersections. Let $\mathcal{P}$ be a partition of an m-dimensional manifold M. Let $P_0,\dots,P_t$ be distinct elements of $\mathcal{P}$ and let $P = P_0\cap\dots\cap P_t$ be the intersection of $\mathcal{P}$. Denote by Δ_P a t-dimensional simplex with vertices $v_0,\dots,v_t$. A map $e_P : \Delta_P \to M_1'$ is called a map *witnessing the intersection* P if $e_P(\Delta_P) \subset \operatorname{Int}(P_0\cup\dots\cup P_t)$, $e_P(v_i) \in \operatorname{Int} P_i$ and P_i does not intersect $e_P(\Delta_P^i)$, where Δ_P^i is the face of Δ_P spanned by the vertices $\{v_0,\dots,v_t\}\setminus\{v_i\}$, $i = 0,\dots,t$. It is clear that any sufficiently close approximation of a map witnessing the intersection P also witnesses the intersection P.

Note that if there is a map e_P witnessing the intersection P then $P \neq \emptyset$. Indeed, aiming at a contradiction assume that $P = \emptyset$. Enlarge each $e_P^{-1}(P_i)$ to an open subset $e_P^{-1}(P_i) \subset G_i$ of Δ_P such that G_i does not meet Δ_P^i and $G_0\cap\dots\cap G_t = \emptyset$. Let $f_0, f_1,\dots,f_t$ be a partition of unity subordinated to the cover $\mathcal{G} = \{G_0,\dots,G_t\}$ of Δ_P. Define $f : \Delta_P \to \Delta_P$ by $f(x) = f_0(x)v_0 + \dots + f_t(x)v_t$, $x \in \Delta_P$. Then $f(\Delta_P) \subset \partial\Delta_P$ and, for every i and $x \in \Delta_P^i$, $f(x) \in \Delta_P^i$. Hence f restricted to $\partial\Delta_P$ is homotopic to the identity map of $\partial\Delta_P$ and this contradicts the fact that $\partial\Delta_P$ is not a retract of Δ_P.

Assume that $P \neq \emptyset$ and let us show how to construct a map $e_P : \Delta_P \to M$ which witnesses the intersection P. Fix a point $x_P \in \operatorname{Int} P$ and a neighborhood W of x_P in M such that $W \subset \operatorname{Int}(P_0\cup\dots\cup P_t)$. Replacing W by a smaller neighborhood of x_P we may assume that for every intersection P' of $\mathcal{P}$ containing x_P, $P'\cap W$ is contractible.

Let F_i = the star of v_i with respect to the first barycentric subdivision of Δ_P. Clearly, $\mathcal{F} = \{F_0,\dots,F_t\}$ is a partition of Δ_P. Let us say that a map $e_P : \Delta_P \to W$ *preserves the intersection* $F' = F_{i_0}\cap\dots\cap F_{i_j}$ if $F' = e_P^{-1}(P_{i_0}\cap\dots\cap P_{i_j})$. It is easy to see that a map preserving all the intersections of $\mathcal{F}$ witnesses the intersection P.

A map e_P preserving all the intersections of $\mathcal{F}$ can be constructed as follows. First send the barycenter of Δ_P to x_P. Assume that we already constructed e_P on the union C_j of the finite intersections of $\mathcal{F}$ of dimension $\le j$ such that e_P preserves the intersections of $\mathcal{F}$ of dimension $\le j$. Take a $(j+1)$-dimensional intersection $F' = F_{i_0}\cap\dots\cap F_{i_{j+1}}$ of $\mathcal{F}$ and let $P' = P_{i_0}\cap\dots\cap P_{i_{j+1}}$ be the corresponding intersection of $\mathcal{P}$. Then $e_P(C_j\cap F') \subset$

$\partial(P' \cap W)$ and, since $P' \cap W$ is contractible, we can extend e_P over F' so that $e_P(F' \setminus C_j) \subset \operatorname{Int} P' \cap W$. Thus we construct the desired map $e_P : \Delta_P \to W$.

Proof of Proposition 3.2. We only need to prove the second part of the proposition because the first part follows from the second one if we assume that Γ is a singleton. Let $\psi_i : X_i \to \Gamma$ be UV^{n-1}-maps, $\mathcal{O}^+$ an open cover of Γ and ω a positive integer which depends only on n and will be determined later. Then there is a sequence $\mathcal{O}^0, \mathcal{O}^1, \ldots, \mathcal{O}^\omega = \mathcal{O}^+$ of open covers of Γ such that for the open covers $\mathcal{O}_i^j = \psi_i^{-1}(\mathcal{O}^j)$ of X_i we have $\operatorname{st} \mathcal{O}_i^j \prec_{n-1} \mathcal{O}_i^{j+1}$ for $i = 1, 2$ and $j = 0, 1, \ldots, \omega - 1$.

Apply Proposition 3.3 for X, ψ and $\mathcal{U}^+$ replaced by X_2, ψ_2 and $\mathcal{O}^0$ respectively in order to get an open cover $\mathcal{O}^{-1}$ of Γ which corresponds to $\mathcal{U}^-$. Once again apply Proposition 3.3 for X, ψ and $\mathcal{U}^+$ replaced by X_1, ψ_1 and $\mathcal{O}^{-1}$ respectively in order to get an open cover $\mathcal{O}^{-2}$ of Γ which corresponds to $\mathcal{U}^-$. Clearly, we can assume that $\mathcal{O}^{-2}$ refines $\mathcal{O}^{-1}$ and that $\mathcal{O}^{-1}$ refines $\mathcal{O}^0$.

Set $\mathcal{O}^- = \mathcal{O}^{-2}$, $\mathcal{O}_i^j = \psi_i^{-1}(\mathcal{O}^j)$ for $i = 1, 2$ and $j = -1, -2$ and assume that $\psi_1|A$ and $\psi_2 \circ f_A$ are $\mathcal{O}^-$-close. The proof of the proposition splits into two independent parts.

Constructing an initial partition $\mathcal{P}_2$. Let an open subset M_1' of M_1 and a rational partition $\mathcal{P}_1$ of M_1' satisfy the assumptions of the proposition. In each non-empty finite intersection P of $\mathcal{P}$ fix a point $x_P \in \operatorname{Int} P$ such that $x_P \in P(k_1)$ and take $e_P : \Delta_P \to M_1'$ witnessing the intersection P with $e_P(\Delta_P)$ so close to x_P that the collection of the images of $e_P(\Delta_P)$ for all non-empty finite intersections of $\mathcal{P}_1$ will form a discrete family in M_1'. Replace each e_P by a sufficiently close Z-embedding into $M_1(k_1) = X_1 \setminus A_1$ preserving the other properties of e_P. Denote by Z_1 the union of A_1 with the union of the images of all the maps e_P. Then Z_1 is a Z-set in X_1.

Assume that $\psi_1|A_1$ and $\psi_2 \circ f_A$ are $\mathcal{O}^-$-close and $\mathcal{P}_1$ refines the extension of $\psi_1^{-1}(\mathcal{O}^-)$ to Y_1. By Proposition 3.3 extend $f_A : A_1 \to A_2$ to a Z-embedding $g_1 : X_1 \to X_2$ such that ψ_1 and $\psi_2 \circ g_1$ are $\mathcal{O}^{-1}$-close. Once again by Proposition 3.3 extend the map $g_1^{-1}|_{\ldots} : Z_2 = g_1(Z_1) \to Z_1$ to a Z-embedding $g_2 : X_2 \to X_1$ such that ψ_2 and $\psi_1 \circ g_2$ are $\mathcal{O}^0$-close.

Take an open cover $\mathcal{U}_1$ of $X_1 \setminus A_1$ having the following properties:

(1) $\mathcal{U}_1$ properly approaches A_1 and refines $\mathcal{O}_1^0$;
(2) for every non-empty intersection $P = P_0 \cap \cdots \cap P_t$ of distinct elements $P_0, \ldots, P_t$ of $\mathcal{P}_1$ and the map $e_P : \Delta_P = [v_0, \ldots, v_t] \to M_1(k_1) \subset M_2'$ we have $\operatorname{st}(e_P(\Delta_P), \mathcal{U}_1) \subset \operatorname{Int}(P_0 \cup \cdots \cup P_t)$, and for every P_i we have $\operatorname{st}(P_i, \mathcal{U}_1) \cap \operatorname{st}(e_P(\Delta_P^i), \mathcal{U}_1) = \emptyset$ and $\operatorname{st}(e_P(v_i), \mathcal{U}_1) \subset \operatorname{Int} P_i$;
(3) for any finite intersections P and P' of $\mathcal{P}_1$ such that $P \cap P' = \emptyset$ we have $\operatorname{st}(P, \mathcal{U}_1) \cap \operatorname{st}(P', \mathcal{U}_1) = \emptyset$.

Define $\mathcal{U}_2 = g_2^{-1}(\mathcal{U}_1)$. Since g_2 is a closed embedding extending f_A^{-1} and $\mathcal{U}_1$ properly approaches A_1, we find that $\mathcal{U}_2$ properly approaches A_2. Since ψ_2 and $\psi_1 \circ g_2$ are $\mathcal{O}^0$-close and $\mathcal{U}_1$ refines $\mathcal{O}^0_1$, we see that $\mathcal{U}_2$ refines $\mathcal{O}^1_2$. Let $\mathcal{U}_2^M$ be the extension of $\mathcal{U}_2$ to M_2 and let M'_2 be the union of the elements of $\mathcal{U}_2^M$. Note that M'_2 is $(n-1)$-connected ($= l_i$-co-connected) since $M_2(k_2) \subset M'_2$. Take a rational triangulation of M'_2 such that, for the partition $\mathcal{B}$ of M'_2 formed by the stars of the vertices with respect to the first barycentric subdivision of the triangulation, $\mathcal{B}$ refines $\mathcal{U}_2^M$.

Consider g_2 as a map into Y_1. Arrange the elements of $\mathcal{P}_1 = \{P^1, P^2, \ldots\}$ into a sequence and define $\mu(P^1) =$ the union of the elements of $\mathcal{B}$ that intersect $g_2^{-1}(P^1)$, $\mu(P^2) =$ the union of the elements of $\mathcal{B}$ that intersect $g_2^{-1}(P^2)$ but do not intersect $g_2^{-1}(P^1)$, $\mu(P^3) =$ the union of the elements of $\mathcal{B}$ that intersect $g_2^{-1}(P^3)$ but do not intersect $g_2^{-1}(P^1) \cup g_2^{-1}(P^2)$ and so on. Let $\mathcal{P}_2 = \{\mu(P^1), \mu(P^2), \ldots\}$. Since each element of $\mathcal{B}$ intersects $M_2(k_2)$ we see that $\mathcal{P}_2$ covers M'_2. The property (2) guarantees that for every $P \in \mathcal{P}_1$, $g_1(e_P(\Delta_P)) \subset \mu(P)$ and therefore $\mu(P) \neq \emptyset$. Then, by 2.2, $\mathcal{P}_2$ is a partition of M'_2. The property (2) also guarantees that for every non-empty intersection $P = P_0 \cap \cdots \cap P_t$ of distinct elements $P_0, \ldots, P_t \in \mathcal{P}_1$, the map $g_1 \circ e_P$ witnesses the intersection of $\mu(P_0), \ldots, \mu(P_t)$ and therefore $\mu(P_0) \cap \cdots \cap \mu(P_t) \neq \emptyset$. The property (3) implies that we do not create additional intersections in $\mathcal{P}_2$ and hence $\mu : \mathcal{P}_1 \to \mathcal{P}_2$ is a matching of partitions. It follows from the construction that

(4) $\mu(P) \subset \mathrm{st}(g_2^{-1}(P), \mathcal{B}) \subset \mathrm{st}(g_2^{-1}(P), \mathcal{U}_2^M)$ for every $P \in \mathcal{P}_1$.

Since $\mathcal{P}_1$ restricted to $X_1 \setminus A_1$ refines $\mathcal{O}^0_1$ and the maps ψ_2 and $\psi_1 \circ g_2$ are $\mathcal{O}^0$-close, we find that $g_2^{-1}(\mathcal{P}_1)$ refines $\mathcal{O}^1_2$. Then, since $\mathcal{B}$ refines $\mathcal{U}_2^M$ and $\mathcal{U}_2$ refines $\mathcal{O}^1_2$, we deduce by (4) that $\mathcal{P}_2$ refines the extension of $\mathcal{O}^2_2$ to Y_2 and μ, ψ_1 and ψ_2 agree with respect to $\mathcal{O}^2$.

Since g_2 is a closed embedding which coincides with f_A^{-1} on A_2 and $g_2(X_2)$ intersects all the elements of $\mathcal{P}_1$, we conclude that $g_2^{-1}(\mathcal{P}_1)$ properly approaches A_2 and the correspondence between $\mathcal{P}_1$ and $g_2^{-1}(\mathcal{P}_1)$ defined by $P \mapsto g_2^{-1}(P)$, $P \in \mathcal{P}_1$, agrees with f_A. Since $\mathcal{U}_2$ properly approaches A_2, we see that $\mathcal{U}_2^M$ properly approaches A_2 as well. Then it is easy to derive from (4) that $\mathcal{P}_2$ properly approaches A_2 and μ agrees with f_A. Recall that $\mathcal{P}_2$ refines the extension of $\mathcal{O}^2_2$ to Y_2 and μ, ψ_1 and ψ_2 agree with respect to $\mathcal{O}^2$.

Improving connectivity of $\mathcal{P}_2$. Without loss of generality we may replace M_1 and M_2 by M'_1 and M'_2 respectively and assume that $M_1 = M'_1$ and $M_2 = M'_2$ for the partition $\mathcal{P}_1$ and the initial partition $\mathcal{P}_2$. Let us show that using 2.7 we can modify $\mathcal{P}_2$ into the required l_2-co-connected rational partition. The construction of 2.7 is a combination of 2.5 and 2.6, and 2.5 consists of 2.4 applied finitely many times. Abusing the notation we always denote by M_2, F and $\mathcal{P}_2$ the manifold, the PL-subcomplex of M_2 and the decomposition of M_2 which are the input of 2.4 and 2.6 (= the output of the previous applications of 2.4 and 2.6), and by M_2', F' and $\mathcal{P}_2'$ the output of 2.4 and 2.6 (= the modifications of M_2, F and $\mathcal{P}_2$ that are obtained after applying 2.4 and 2.6), and we denote again by μ the correspondence $\mu : \mathcal{P}_1 \to \mathcal{P}_2'$ which is the composition of $\mu : \mathcal{P}_1 \to \mathcal{P}_2$ with the natural one-to-one correspondence between $\mathcal{P}_2$ and its modification $\mathcal{P}_2'$ (defined by sending each element of $\mathcal{P}_2$ to its modification). Thus we start with the initial partition $\mathcal{P}_2$ that during 2.7 turns into a decomposition forming a partition on $U = M_2 \setminus F$ and, gradually reducing the dimension of F, $\mathcal{P}_2$ returns to be a partition at the end of 2.7. Recall that after applying 2.4 and 2.6 we always obtain an open subset M_2' of M_2 such that $R = M_2 \setminus M_2'$ is a PL-presented subset of M_2 with $\dim R \leq n = m_2 - l_2 + 1 \leq k_2$. By 2.8 we can choose a homeomorphism of $h : A_2 \cup M_2 \to A_2 \cup M_2$ such that h is arbitrarily close to the identity map, $h(a) = a$ for $a \in A_2$, $h(R)$ is a rationally presented closed subset of M_2, $h(F')$ is a rational subcomplex of $h(M_2')$, $h(\mathcal{P}_2')$ is a rational decomposition of $h(M_2')$ and h restricted to $M_2 \setminus R$ is a PL-homeomorphism to $M_2 \setminus h(R)$. Then replacing M_2', F' and $\mathcal{P}_2'$ by $h(M_2')$, $h(F')$ and $h(\mathcal{P}_2')$ respectively we can always assume that $M_2(k_2) \subset M_2'$, F' is a rational subcomplex of M_2' and $\mathcal{P}_2'$ is a rational decomposition. Thus at the end of 2.7 we deduce that $\dim F' \leq n = m_2 - l_2 + 1 \leq k_2$ and hence $M_2(k_2) \subset M_2' \setminus F'$. Now we can replace the final modification M_2' of M_2 by $M_2' \setminus F$ and get the final partition on M_2'.

We will apply 2.4 and 2.6 so that there will exist an open cover $\mathcal{W}$ of M_2 such that $\mathcal{W}$ properly approaches A_2 and for every $P \in \mathcal{P}_2$ and its modification $P' \in \mathcal{P}_2$ we have $P' \subset \mathrm{st}(P, \mathrm{st}^2 \mathcal{W})$. This property implies that $\mathcal{P}_2'$ will properly approach A_2 and $\mu : \mathcal{P}_1 \to \mathcal{P}_2'$ will agree with f_A. In addition, we will apply 2.4 and 2.6 so that there exists j such that the modification $\mathcal{P}_2'$ of $\mathcal{P}_2$ will refine the extension of $\mathcal{O}_2^j$ to Y_2 and $\mu : \mathcal{P}_1 \to \mathcal{P}_2'$, ψ_1 and ψ_2 will agree with respect to $\mathcal{O}^j$. Clearly, applying 2.8 as described above, we preserve the properties that $\mathcal{P}_2'$ properly approaches A_2 and μ agrees with f_A, and choosing the homeomorphism h sufficiently close to the identity map of M_2' we can increase j by 1 and assume that $\mathcal{P}_2'$ refines the extension of $\mathcal{O}_2^j$ to Y_2 and μ, ψ_1 and ψ_2 agree with respect to $\mathcal{O}^j$.

Let us first analyze 2.4. Assume that $\mathcal{P}_2$ properly approaches A_2, $\mathcal{P}_2$ refines the extension of $\mathcal{O}_2^j$ to Y_2 and μ, ψ_1 and ψ_2 agree with respect to $\mathcal{O}^j$. Take any open cover $\mathcal{U}$ of M_2 such that $\mathcal{U}$ refines the extension of $\mathcal{O}_2^j$ to Y_2 and $\mathcal{U}$ properly approaches A_2.

Set $\mathcal{C}_M = \mathrm{st}(\mathrm{st}\,\mathcal{P}_2, \mathcal{U})$. Then $\mathcal{C}_M$ is an open cover of M_2 such that $\mathcal{C}_M$ properly approaches A_2, $\mathrm{st}\,\mathcal{P}_2$ refines $\mathcal{C}_M$ and $\mathcal{C}_M$ refines the extension of $\mathcal{O}_2^{j+2}$ to Y_2. Let $\mathcal{C} = \mathcal{C}_M | M_2(k_2)$. Recall that $\mathcal{O}_2^{j+2} \prec_{n-1} \mathcal{O}_2^{j+3}$. Then by Proposition 3.4 there is an open cover $\mathcal{V}$ of $M_2(k_2)$ such that $\mathcal{V}$ properly approaches A_2, $\mathcal{C} \prec_{n-1} \mathcal{V}$, $\mathcal{V}$ refines $\mathrm{st}^3 \mathcal{O}_2^{j+3}$ and hence $\mathcal{V}$ refines $\mathcal{O}_2^{j+6}$. Let

$\mathcal{W} = \mathrm{st}(\mathcal{V}, \mathcal{U})$. Note that $\mathcal{W}$ is an open cover of M_2, $\mathcal{W}$ properly approaches A_2 and refines the extension of $\mathcal{O}_2^{j+7}$ to Y_2. Since every map from a sphere of dimension $\le n-1$ to an open subset G of M_2 can be homotoped inside G to a map into $G \cap M_2(k_2)$, we conclude that $\mathcal{C} \prec_{n-1} \mathcal{V}$ implies $\mathcal{C}_M \prec_{n-1} \mathcal{W}$ and therefore $\mathrm{st}\,\mathcal{P}_2 \prec_{n-1} \mathcal{W}$. Then 2.4 can be carried out so that for every $P \in \mathcal{P}_2$ and its modification $P' \in \mathcal{P}_2'$ we have $P' \subset \mathrm{st}(P, \mathrm{st}^2 \mathcal{W})$. The last property also implies that $\mathcal{P}_2'$ refines the extension of $\mathcal{O}_2^{j+10}$ to Y_2 and $\mu : \mathcal{P}_1 \to \mathcal{P}_2'$, ψ_1 and ψ_2 agree with respect to $\mathcal{O}^{j+10}$.

Now we will analyze 2.6. Once again we assume that $\mathcal{P}_2$ properly approaches A_2, $\mathcal{P}_2$ refines the extension of $\mathcal{O}_2^j$ to Y_2 and μ, ψ_1 and ψ_2 agree with respect to $\mathcal{O}^j$. Then the collection $\mathrm{st}\,\mathcal{P}_2$ properly approaches A_2. Since for $P \in \mathcal{P}_2$, $\mathrm{st}(P, \mathcal{P}_2)$ is a union of elements of $\mathcal{P}_2$, we can naturally define the set $\mu^{-1}(\mathrm{st}(P, \mathcal{P}_2))$ as the union of the corresponding elements of $\mathcal{P}_1$. Then the collection $\mu^{-1}(\mathrm{st}\,\mathcal{P}_2)$ properly approaches A_1 since μ agrees with f_A, and $\mu^{-1}(\mathrm{st}\,\mathcal{P}_2)$ refines the extension of $\mathcal{O}_1^{j+2}$ to Y_1 since μ, ψ_1 and ψ_2 agree with respect to $\mathcal{O}^j$. Let $\mathcal{U}$ be an open cover of M_1 such that $\mathcal{U}$ refines the extension of $\mathcal{O}_1^{j+2}$ to Y_1 and $\mathcal{U}$ properly approaches A_1.

Set $\mathcal{C}_M = \mathrm{st}(\mu^{-1}(\mathrm{st}\,\mathcal{P}_2), \mathcal{U})$. Then $\mathcal{C}_M$ is an open cover of M_1 such that $\mathcal{C}_M$ properly approaches A_1, $\mu^{-1}(\mathrm{st}\,\mathcal{P}_2)$ refines $\mathcal{C}_M$ and $\mathcal{C}_M$ refines the extension of $\mathcal{O}_1^{j+3}$ to Y_1. Set $\mathcal{C} = \mathcal{C}_M | M_1(k_1)$. Recall that $\mathcal{O}_1^{j+3} \prec_{n-1} \mathcal{O}_1^{j+4}$. Then, by Proposition 3.4, there is an open cover $\mathcal{V}$ of $M_1(k_1)$ such that $\mathcal{V}$ properly approaches A_1, $\mu^{-1}(\mathrm{st}\,\mathcal{P}_2)$ restricted to $M_1(k_1)$ is an $(n-1)$-refinement of $\mathcal{V}$, $\mathcal{V}$ refines $\mathrm{st}^3 \mathcal{O}_1^{j+4}$ and hence $\mathcal{V}$ refines $\mathcal{O}_1^{j+7}$.

Define $\mathcal{H} = \mathrm{st}(\mathcal{V}, \mu^{-1}(\mathrm{st}\,\mathcal{P}_2))$. It is clear that $\mu^{-1}(\mathrm{st}\,\mathcal{P}_2)$ refines $\mathcal{H}$, $\mu^{-1}(\mathrm{st}\,\mathcal{P}_2)$ restricted to $M_1(k_1)$ is an $(n-1)$-refinement of $\mathcal{H}$, $\mathcal{H}$ properly approaches A_1 and $\mathcal{H}$ refines $\mathcal{O}_1^{j+8}$. Since $\mu^{-1}(\mathrm{st}(P, \mathcal{P}_2))$ is a rational submanifold of M_1 of dimension m_1, every map from a space of dimension $\le n-1$ into $\mu^{-1}(\mathrm{st}(P, \mathcal{P}_2))$ can be homotoped inside $\mu^{-1}(\mathrm{st}(P, \mathcal{P}_2))$ into $\mu^{-1}(\mathrm{st}(P, \mathcal{P}_2)) \cap M_1(k_1)$ and therefore $\mu^{-1}(\mathrm{st}\,\mathcal{P}_2)$ is an $(n-1)$-refinement of $\mathcal{H}$. Note that $\mu(\mathcal{H})$ is well-defined because each element of $\mathcal{H}$ is a union of elements of $\mathcal{P}_1$. Also, since μ, ψ_1 and ψ_2 agree with respect to $\mathcal{O}^j$, we deduce that $\mu(\mathcal{H})$ refines the extension of $\mathcal{O}_2^{j+9}$ to Y_2.

Assume that F is a PL-subcomplex of M_2 such that for $U = M_2 \setminus F$ the decomposition $\mathcal{P}_2$ forms on U an l-co-connected partition, $l \ge l_2$, for which $\mu : \mathcal{P}_1 \to \mathcal{P}_2$ becomes a matching when $\mathcal{P}_2$ is restricted to U. Recall that $\mu^{-1}(\mathrm{st}\,\mathcal{P}_2)$ is an $(n-1)$-refinement of $\mathcal{H}$. Then, by 2.3, $\mathrm{st}(\mathcal{P}_2)|U$ is an $(m_2 - l)$-refinement of $\mu(\mathcal{H})|U$.

Take any open cover $\mathcal{G}$ of M_2 such that $\mathcal{G}$ properly approaches A_2 and $\mathcal{G}$ refines the extension of $\mathcal{O}_2^j$ to Y_2. Define $\mathcal{W} = \mathrm{st}(\mu(\mathcal{H}), \mathcal{G})$ and note that $\mathcal{W}$ refines the extension of $\mathcal{O}_2^{j+10}$ to Y_2. Then for every $P \in \mathcal{P}_2$ there are $H \in \mu(\mathcal{H})$ and $W \in \mathcal{W}$ with $\mathrm{st}(P, \mathcal{P}_2) \subset H \subset W$ and such that the inclusion

$\mathrm{st}(P, \mathcal{P}_2) \cap U \subset H \cap U$ induces the zero-homomorphism of the homotopy groups in dimensions $\leq m_2 - l$. Hence 2.6 can be carried out so that for every $P \in \mathcal{P}_2$ and its modification $P' \in \mathcal{P}'_2$ we get $P' \subset \mathrm{st}(P, \mathrm{st}^2 \mathcal{W})$. The last property also implies that $\mathcal{P}'_2$ refines the extension of $\mathcal{O}_2^{j+13}$ to Y_2 and $\mu : \mathcal{P}_1 \to \mathcal{P}'_2$, ψ_1 and ψ_2 agree with respect to $\mathcal{O}^{j+13}$.

Now we need to determine the maximal possible value of j which we assign to ω at the beginning of the proof. From the proof it is clear that this value depends only on the number of times we apply the constructions 2.4, 2.6, 2.8 and it is easy to see from 2.7 and 2.5 that this number depends only on n. ∎

4. Constructions of improving connectivity and absorbing simplexes

4.1. *A black hole modification.* Let B^q, B^{m-q}, B_*^{m-q} and $B^m = B^q \times B^{m-q}$ be cubes with the dimensions indicated by the superscripts such that B_*^{m-q} is contained in $\mathrm{Int}\, B^{m-q}$ and the centers of B_*^{m-q} and B^{m-q} coincide (recall that we assume that the centers of the cubes are located at O and we identify B^q and B^{m-q} with the subsets $B^q \times O$ and $O \times B^{m-q}$ of B^m respectively; see 2.1). Define $T = B^q \times (B^{m-q} \setminus \mathrm{Int}\, B_*^{m-q})$ and $S = \partial B^{m-q}$. Note that S is an $(m-q-1)$-dimensional sphere in ∂T and T is a subset of B^m which is PL-homeomorphic to the product $B^{q+1} \times S$, where S is identified with $a \times S$ for a point $a \in \partial B^{q+1}$.

Assume that B^m is PL-embedded in $\mathrm{Int}\, M$ of a triangulated m-dimensional manifold M. Set $L = M \setminus \mathrm{Int}\, T$ and assume that F_L is a PL-subcomplex of L and $\mathcal{P}_L$ is a decomposition of L such that $S \subset F_L$ and $\mathcal{P}_L$ restricted to $L \setminus F_L$ and to $\partial T \setminus F_L$ are partitions.

Fix a triangulation $\mathcal{T}$ of M that underlies S, F_L, L and the elements of $\mathcal{P}_L$, and denote by $\mathcal{A}$ the collection of subcomplexes of L with respect to this triangulation. We are going to extend each $A \in \mathcal{A}$ to a PL-subcomplex $A \subset A^\pi$ of M such that the following conditions are satisfied:

- $P_L^\pi = P_L$ for $P_L \in \mathcal{P}_L$ such that $P_L \cap \partial T = \emptyset$;
- $\mathcal{P}_L^\pi = \{P_L^\pi : P_L \in \mathcal{P}_L\}$ is a decomposition of M which forms a partition on $M \setminus F_L^\pi$;
- $F_L^\pi \cap L = F_L$ and $\mathcal{P}_L^\pi$ restricted to $L \setminus F_L^\pi$ coincides with $\mathcal{P}_L$ restricted to $L \setminus F_L$;
- the correspondence between $\mathcal{P}_L$ and $\mathcal{P}_L^\pi$ defined by $P_L \mapsto P_L^\pi$ induces a matching of partitions when $\mathcal{P}_L$ and $\mathcal{P}_L^\pi$ are restricted to $L \setminus F_L$ and $M \setminus F_L^\pi$ respectively;
- for every intersection $P_L = P_L^0 \cap \cdots \cap P_L^j$ of elements $P_L^0, \ldots, P_L^j$ of $\mathcal{P}_L$, $P_L^\pi \setminus F_L^\pi = ((P_L^0)^\pi \cap \cdots \cap (P_L^j)^\pi) \setminus F_L^\pi$ and $P_L \setminus F_L$ is a (strong) deformation retract of $P_L^\pi \setminus F_L^\pi$;

- $\dim(F_L^\pi \cap T) \setminus S \le \dim(F_L \cap \partial T) \setminus S + 1$;
- $F_L^\pi = F_L$ if $F_L \cap \partial T = S$.

Recall that T is homeomorphic to the product $B^{q+1} \times S$, where S is identified with $a \times S$ for a point $a \in \partial B^{q+1}$. Note that $B^{q+1} \setminus a$ is homeomorphic to $(\partial B^{q+1} \setminus a) \times [0,1)$ and hence $T \setminus S$ can be represented as the product $(\partial T \setminus S) \times [0,1)$, where $\partial T \setminus S$ corresponds to $(\partial T \setminus S) \times 0$. One of the ways to construct A^π is to find a retraction $\pi : T \setminus S \to \partial T \setminus S$ which is topologically equivalent to the projection of $(\partial T \setminus S) \times [0,1)$ to $(\partial T \setminus S) \times 0$ such that, for A^π defined by $A^\pi = A$ if $A \cap \partial T = \emptyset$ and $A^\pi = A \cup S \cup \pi^{-1}(A \cap \partial T)$ otherwise, A^π is a PL-subcomplex of M for every $A \in \mathcal{A}$. Such a retraction π can be constructed, but the author could not find an elementary and direct argument for this. Therefore, for the sake of elementariness (and at the expense of a geometric transparency), we adopt a slightly different approach for constructing A^π which is presented in 5.1. Note that all the properties of $\mathcal{P}_L^\pi$ and F_L^π that will be used subsequently are described above. Thus for understanding the rest of the paper the reader may assume that the construction is carried out on the base of an appropriate retraction π. Also note that S is added to A^π for A intersecting ∂T in order to make A^π a closed subset of M and $\mathcal{P}_L^\pi$ a cover of M.

We will call the decomposition $\mathcal{P}_L^\pi$ a *black hole modification* of $\mathcal{P}_L$ with the sphere S as the black hole of the modification.

4.2. *Improving connectivity of an intersection.* Assume that $m \ge 2q+1$ and $l = m - q + 2$, M is a triangulated m-dimensional $(q-1)$-connected ($= (l-1)$-co-connected) manifold and F is a PL-subcomplex of M lying in $\operatorname{Int} M$. Let $\mathcal{P}$ be a decomposition of M which forms an l-co-connected partition on $U = M \setminus F$ and let $P = P_0 \cap P_1 \cap \dots \cap P_t$, $0 \le t \le m - l + 1$, be an intersection of distinct elements $P_0, \dots, P_t$ of $\mathcal{P}$ such that $P \cap U \ne \emptyset$. Then $P \cap U$ is an $(m-t)$-dimensional and $(q-t-2)$-connected ($= l$-co-connected) manifold.

Assume that the intersections of $\mathcal{P}|U$ of dimension $> m-t$ are $(l-1)$-co-connected. Let us show how to improve the connectivity of $P \cap U$ preserving the level of connectivity of the rest of the intersections.

Let $f : S_P \to \operatorname{Int}(P \cap U)$ be a PL-embedding of a triangulated $(q-t-1)$-dimensional sphere S_P such that f is not null-homotopic in $P \cap U$.

Consider a t-dimensional simplex Δ with vertices $V(\Delta) = \{v_0, \dots, v_t\}$. Let Δ' be a (sub)simplex of Δ lying in $\partial\Delta$ and let $V(\Delta') \subset V(\Delta)$ be the set of vertices of Δ'. Define $P(\Delta') = \bigcap\{P_i : v_i \in V(\Delta) \setminus V(\Delta')\}$. Note that $P(\emptyset) = P$ and $P(\Delta') \subset P(\Delta'')$ if $\Delta' \subset \Delta''$.

For every simplex Δ' of $\partial\Delta$ we will define by induction on $\dim \Delta'$ a PL-embedding $f_{\Delta'} : S_P * \Delta' \to P(\Delta') \cap U$ such that $f_{\Delta''}$ extends $f_{\Delta'}$ if $\Delta' \subset \Delta''$. Setting $f_\emptyset = f$ we are done for $t = 0$. Let $t > 0$. Take a simplex

Δ' of dimension $i+1 \le t-1$ and assume that $f_{\Delta'}$ is defined for all Δ' of dimension $\le i$. Then the maps $f_{\Delta''}$ for $\Delta'' \subset \partial\Delta'$ define the corresponding PL-embedding $f_{\partial\Delta'} : S_P * \partial\Delta' \to \partial(P(\Delta') \cap U)$. Note that $S_P * \partial\Delta'$ is a $(q-t+i)$-dimensional sphere and $P(\Delta') \cap U$ is $(m-t+i+2)$-dimensional and $(q-t+i+1)$-connected $(= (l-1)$-co-connected) and $\dim(P(\Delta')\cap U) = m-t+i+2 \ge 2q+1-t+i+2 \ge 2(q-t+i+1)+1+(t-i-2) \ge 2(q-t+i+1)+1$ since $m \ge 2q+1$ and $t-i-2 \ge 0$. Then $f_{\partial\Delta'}$ can be extended to a PL-embedding $f_{\Delta'} : S_P * \Delta' \to P(\Delta') \cap U$ such that $f_{\Delta'}$ sends $\operatorname{Int}(S_P * \Delta')$ to $\operatorname{Int}(P(\Delta') \cap U)$.

The maps $f_{\Delta'}$ define the PL-embedding $f_{\partial\Delta}$ of $S_P * \partial\Delta$ to $\operatorname{Int}((P_0 \cup \cdots \cup P_t) \cap U)$. Identify the $(q-1)$-dimensional sphere $S^{q-1} = S_P * \partial\Delta$ with its image under $f_{\partial\Delta}$ and, by 5.2 below, extend the embedding of S^{q-1} to a PL-embedding of a q-dimensional cube B^q such that $S^{q-1} = \partial B^q$ and the embedding of B^q extends to a PL-embedding of an m-dimensional cube $B^m = B^q \times B^{m-q} \subset \operatorname{Int} M$ having the following properties: $P' \cap B^m = (B^q \cap P') \times B^{m-q}$ for every $P' \in \mathcal{P}$; $F \cap B^m = (B^q \cap F) \times B^{m-q}$; $\mathcal{P}$ restricted to $N \cap U$ and to $\partial B^m \cap U$ are partitions, where $N = M \setminus \operatorname{Int} B^m$ and for every finite intersection P' of $\mathcal{P}$ such that $P' \cap U \ne \emptyset$ we have $P' \cap N \cap U \ne \emptyset$ and $P' \cap N \cap U$ is a deformation retract of $(P' \cap U) \setminus \operatorname{Int} B^q$. Note that for every finite intersection P' of $\mathcal{P}$ that intersects U, $(\dim(P' \cap U) - (l-1) + 1) + q + 1 = \dim(P' \cap U) - m + 2q + 1 \le \dim(P' \cap U)$ and therefore the inclusion $(P' \cap U) \setminus \operatorname{Int} B^q \subset P' \cap U$ induces an isomorphism of the homotopy groups in dimensions $\le \dim(P' \cap U) - (l-1)$ $(=$ in co-dimensions $\ge l-1)$. Also note that since $F \cap \partial B^q = \emptyset$, we have $F \cap \partial B^m = (F \cap \operatorname{Int} B^q) \times \partial B^{m-q}$ and therefore $\dim F \cap \partial B^m \le \dim F - 1$.

Let B^{m-q}_* be a cube lying in the geometric interior of B^{m-q} such that the centers of B^{m-q}_* and B^{m-q} coincide and $\partial B^q \times B^{m-q}_* \subset \operatorname{Int}((P_0 \cup \cdots \cup P_t) \cap U)$.

Use the notations of 4.1. Define $F_L = (N \cap F) \cup S$, where $S = \partial B^{m-q}$. Consider B^q as the join $O * \partial B^q$, where O is the center of B^q, and define a decomposition $\mathcal{P}_L$ of L as the collection of the sets $\mathcal{P}_L = \{P'_L : P' \in \mathcal{P}\}$, where $P'_L = P' \cap N$ if $P' \in \mathcal{P}$ and $P' \ne P_0, \dots, P_t$ and $P'_L = (P' \cap N) \cup ((O * (P' \cap \partial B^q)) \times B^{m-q}_*)$ for $P' = P_i$, $0 \le i \le t$. Note that $\mathcal{P}_L$ restricted to N coincides with $\mathcal{P}$ restricted to N. It is easy to see that $\mathcal{P}_L$ restricted to $B^q \times B^{m-q}_*$, to $\partial B^q \times (B^{m-q} \setminus \operatorname{Int} B^{m-q}_*)$, to $\partial B^q \times B^{m-q}_*$, to $B^q \times \partial B^{m-q}_*$ and to $\partial B^q \times \partial B^{m-q}_*$ are partitions. Then, by 2.2, $\mathcal{P}_L$ is a partition on $L \setminus (N \cap F)$ and $\partial T \setminus (N \cap F)$ and hence $\mathcal{P}_L$ is a partition on $L \setminus F_L$ and $\partial T \setminus F_L$. For a finite intersection $P' = P'_0 \cap \cdots \cap P'_j$ of $P'_0, \dots, P'_j \in \mathcal{P}$ such that $P' \cap U \ne \emptyset$ define $P'_L = (P'_0)_L \cap \cdots \cap (P'_j)_L$.

Now apply the black hole modification to $\mathcal{P}_L$ and F_L with S as the black hole of the modification. Note that since S lies in ∂N, by 2.2, for every finite intersection P' of $\mathcal{P}$, $P' \cap U \cap N \cap S \subset \partial(P' \cap U \cap N)$ and hence the inclusion $(P' \cap U \cap N) \setminus S \subset P' \cap U \cap N$ induces an isomorphism of the homotopy groups in all dimensions. Recall that $f(S_P) \subset P \cap U \cap N$. By 4.1 and the construction of $\mathcal{P}_L$ one can easily verify that for every finite intersection P' of $\mathcal{P}$ such that $P' \cap U \neq \emptyset$ and $P \neq P'$, $(P' \cap U \cap N) \setminus S$ is a deformation retract of $(P'_L)^\pi \setminus F^\pi_L$ and, for the intersection P, the inclusion of $(P \cap U \cap N) \setminus S$ into $P^\pi_L \setminus F^\pi_L$ induces an isomorphism of the homotopy groups in dimensions $< q-t-1$ and an epimorphism in dimension $q-t-1$ such that the element of the homotopy group represented by the map f is in the kernel of this epimorphism.

Thus for every intersection P' of $\mathcal{P}$ such that $P' \cap U \neq \emptyset$ the connectivity of $(P'_L)^\pi \setminus F^\pi_L$ fits the connectivity of $P' \setminus F$ for $P' \neq P$ and we contributed toward improving the connectivity of $P \setminus F$ in its modification $P^\pi_L \setminus F^\pi_L$.

Define $F^\pi = F^\pi_L$, $(P')^\pi = (P'_L)^\pi$ for a finite intersection P' of $\mathcal{P}$ and let $\mathcal{P}^\pi = \{(P')^\pi : P' \in \mathcal{P}\} = \mathcal{P}^\pi_L$. It is easy to see that the one-to-one correspondence between $\mathcal{P}$ and $\mathcal{P}_L$ defined by $P' \mapsto (P')_L$, $P' \in \mathcal{P}$, becomes a matching of partitions when $\mathcal{P}$ and $\mathcal{P}_L$ are restricted to $M \setminus F$ and $L \setminus F_L$ respectively. Then by 4.1 the one-to-one correspondence between $\mathcal{P}$ and $\mathcal{P}^\pi$ defined by $P' \to (P')^\pi$, $P' \in \mathcal{P}$, becomes a matching of partitions when $\mathcal{P}$ and $\mathcal{P}^\pi$ are restricted to $M \setminus F$ and $M \setminus F^\pi$ respectively. Note that $F^\pi \subset \operatorname{Int} M$ and because $\dim F \cap \partial B^m \leq \dim F - 1$ we have $\dim F^\pi \leq \max\{\dim S, \dim F\} = \max\{m-q-1, \dim F\}$.

Finally, note that even if $F = \emptyset$ then $F^\pi = S$ and we improve the connectivity of P at the expense of an "irregular" behavior of $\mathcal{P}^\pi$ on the black hole S, in particular, we create additional intersections of $\mathcal{P}^\pi$ on S and the elements of $\mathcal{P}^\pi$ are no longer manifolds in M.

4.3. *A discretization of images of maps.* Assume that M is a triangulated manifold such that $\dim M \geq 2r+1$ and $\mathcal{U}$ is an open cover of M.

Let $\mathcal{F}$ be countable collection of maps $f : S^r \to M$ from a triangulated r-dimensional sphere S^r. Then there is a PL-subcomplex R of M of dimension $\leq r$ lying in $\operatorname{Int} M$ such that every map f in $\mathcal{F}$ admits a $\mathcal{U}$-close approximation $f' : S^r \to \operatorname{Int} M \setminus R$ such that f' is a PL-embedding and the images of the maps in $\mathcal{F}' = \{f' : f \in \mathcal{F}\}$ form a discrete family in $M \setminus R$. Moreover, given a decomposition $\mathcal{P}$ of M, the set R can be chosen so that R is nowhere dense on the finite intersections of $\mathcal{P}$, that is, the intersection of R with every finite intersection of $\mathcal{P}$ is nowhere dense in that intersection.

Indeed, arrange $\mathcal{F}$ into a sequence $\mathcal{F} = \{f_1, f_2, \ldots\}$ and take a sufficiently small triangulation $\mathcal{T}$ of M such that $\mathcal{T}$ underlies $\mathcal{P}$. Approximate each $f_i : S^r \to M$ by a map $f''_i : S^r \to M$ such that $f''_i(S^r)$ is contained in $R' =$ the r-skeleton of $\mathcal{T}$. Fix a metric d on M and approximate each f''_i by a PL-embedding $f'''_i : S^r \to M$ such that $f'''_i(S^r) \cap R' = \emptyset$, $d(f'''_i(x), R') \leq 1/i$ for every $x \in S^r$ and $f'''_i(S^r) \cap f'''_j(S^r) = \emptyset$ for $i \neq j$. Then the sets $f'''_i(S^r)$, $i = 1, 2, \ldots$, form a discrete family in $M \setminus R'$. Now take a PL-embedding

$g : M \to M$ such that $g(M)$ is a PL-subcomplex of M, $g(M) \subset \operatorname{Int} M$, the restriction of g to R' is in general position with $\mathcal{T}$, and g is close to the identity map of M. It is clear that the construction can be carried out so that $f_i' = g \circ f_i'''$ is $\mathcal{U}$-close to f_i. Then the maps f_i' and the set $R = g(R')$ will have the required properties.

In a similar way one can show that if $\mathcal{F}$ is a countable collection of maps $f : B^r \to M$ from a triangulated r-dimensional ball B^r such that, for every f in $\mathcal{F}$, f restricted to ∂B^r is a PL-embedding and the images of the maps in $\{f|_{\partial B^r} : f \in \mathcal{F}\}$ form a discrete family in M, then there is a PL-subcomplex R of M of dimension $\leq r$ lying in $\operatorname{Int} M$ such that every map f in $\mathcal{F}$ admits a $\mathcal{U}$-close approximation $f' : B^r \to M \setminus R$ such that f' is a PL-embedding, $f'|_{\partial B^r} = f|_{\partial B^r}$, $f'(\operatorname{Int} B^r) \subset \operatorname{Int} M$ and the images of the maps in $\mathcal{F}' = \{f' : f \in \mathcal{F}\}$ form a discrete family in $M \setminus R$. Moreover, for a given decomposition $\mathcal{P}$ of M, the set R can be chosen so that R is nowhere dense on the finite intersections of $\mathcal{P}$.

Note that if M is $(r-1)$-connected and R is a closed PL-presented subset of M of dimension $\leq r$, then $M' = M \setminus R$ is $(r-1)$-connected. In addition, if $\mathcal{P}$ is a decomposition of M which is an l-co-connected partition on an open subset U of M with $l \geq r+2$ and R is nowhere dense on the finite intersections of $\mathcal{P}$, then $\mathcal{P}' = \{P \setminus R : P \in \mathcal{P}\}$ is a decomposition of M' which is an l-co-connected partition on $U' = U \setminus R$ and the one-to-one correspondence between $\mathcal{P}$ and $\mathcal{P}'$ defined by $P \mapsto P \setminus R$, $P \in \mathcal{P}$, induces a matching of partitions when $\mathcal{P}$ and $\mathcal{P}'$ are restricted to U and U' respectively (the property that R is nowhere dense on the finite intersections of $\mathcal{P}$ is needed to guarantee that for every finite intersection P of $\mathcal{P}$, $(P \cap U) \setminus R \neq \emptyset$ provided $P \cap U \neq \emptyset$).

4.4. *Improving connectivity of intersections simultaneously.* Adopt the notations and the assumptions of 4.2. We are going to show how to improve simultaneously the connectivity of all the intersections of $\mathcal{P}|U$ of dimension $m-t$ to $(l-1)$-co-connectivity.

For an intersection P of $\mathcal{P}$ with $\dim P \cap U = m - t$ choose a countable collection $\mathcal{F}_P$ of PL-embeddings $f : S_P \to \operatorname{Int}(P \cap U)$ from a $(q-t-1)$-dimensional sphere S_P that generate the $(q-t-1)$-dimensional homotopy group of $P \cap U$. Since F is a PL-subcomplex of M we may assume that the images of the maps in $\mathcal{F}_P$ lie outside a neighborhood of F. Then by 4.3 one can find a PL-subcomplex R_P of M such that $R_P \subset \operatorname{Int}(P \cap U)$ and $\dim R_P \leq q - t - 1$ and assume that the images of the embeddings in $\mathcal{F}_P$ are contained in $\operatorname{Int}((P \cap U) \setminus R_P)$ and form a discrete family in $M \setminus R_P$.

Recall that in 4.2 the elements of $\mathcal{P}$ that form the intersection P are enumerated according to the vertices of a sample simplex $\Delta = \{v_0, \dots, v_t\}$ (we fix such an enumeration arbitrarily and independently for every intersection

P of $\mathcal{P}$ with $\dim P \cap U = m-t$). Following 4.2 and using 4.3 we can define for every $\Delta' \subset \Delta$ a set $R_P^{\Delta'}$ and collections of maps $\mathcal{F}_P^{\Delta'} = \{f_{\Delta'} : f \in \mathcal{F}_P\}$ and $\mathcal{F}_P^{\partial\Delta'} = \{f_{\partial\Delta'} : f \in \mathcal{F}_P\}$ such that $R_P^{\Delta''} \subset R_P^{\Delta'}$ if $\Delta'' \subset \Delta'$, $R_P^{\Delta'}$ is a closed subset of M lying in $P(\Delta') \cap U$, the images of the maps of $\mathcal{F}_P^{\Delta'}$ are contained in $G_P^{\Delta'} = (P(\Delta') \cap U) \setminus R_P^{\Delta'}$ and form a discrete family in $M \setminus R_P^{\Delta'}$, and $R_P^{\Delta'} \setminus R_P^{\partial\Delta'}$ is a PL-subcomplex of $M \setminus R_P^{\partial\Delta'}$ lying in $(P(\Delta') \cap U) \setminus R_P^{\partial\Delta'}$ such that $\dim R_P^{\Delta'} \setminus R_P^{\partial\Delta'} \leq q - t + \dim \Delta'$ where $R_P^{\partial\Delta'} = \bigcup\{R_P^{\Delta''} : \Delta'' \subset \partial\Delta'\}$. Here we assume that $\Delta' = \emptyset$ is a simplex of Δ and $\mathcal{F}_P^{\emptyset} = \mathcal{F}_P$ and $R_P^{\emptyset} = R_P$, and we also assume that $P(\Delta) = M$.

Denote by $\mathcal{F}^{\partial\Delta}$ and $R^{\partial\Delta}$ the union of $\mathcal{F}_P^{\partial\Delta}$ and $R_P^{\partial\Delta}$ respectively over all the intersections P of $\mathcal{P}$ with $\dim P \cap U = m - t$. In order to carry out the construction described above so that $R^{\partial\Delta}$ will be a closed PL-presented subset of M and the images of the maps in $\mathcal{F}^{\partial\Delta}$ will form a discrete family in $M \setminus R^{\partial\Delta}$, we need to take into account that the same intersection P' of $\mathcal{P}$ with $\dim P' \cap U > m - t$ can be involved in many (even countably many) intersections of dimension $m - t$. This can be done as follows. For a finite intersection P' of $\mathcal{P}$ with $\dim P' \cap U > m - t$ denote by $R_{\partial P'}$, $R_{P'}$, $\mathcal{F}_{\partial P'}$ and $\mathcal{F}_{P'}$ the union of $R_P^{\partial\Delta'}$, $R_P^{\Delta'}$, $\mathcal{F}_P^{\partial\Delta'}$ and $\mathcal{F}_P^{\Delta'}$ respectively over all finite intersections P of $\mathcal{P}$ such that $\dim P \cap U = m - t$ and all $\Delta' \subset \partial\Delta$ such that $P(\Delta') = P'$, and let $G_{P'} = (P' \cap U) \setminus R_{P'}$. It is obvious that if $\dim P' \cap U = m - t + 1$ then $R_{\partial P'}$ is closed in M, the images of the maps in $\mathcal{F}_{\partial P'}$ are contained in $\partial(P' \cap U) \setminus R_{\partial P'}$ and form a discrete family in $M \setminus R_{\partial P'}$. Now assume that the last properties hold for a finite intersection P' of $\mathcal{P}$ with $\dim P' \cap U > m - t$. Then applying 4.3 to all the maps of $\mathcal{F}_{\partial P'}$ we can enlarge $R_{\partial P'}$ to a closed subset $R_{P'}$ of M contained in $P' \cap U$ and find extensions $\mathcal{F}_{P'}$ of the maps in $\mathcal{F}_{\partial P'}$ such that $\dim R_{P'} \leq \dim(P' \cap U) - l + 1 = \dim(P' \cap U) + q - m - 1$, $R_{P'} \setminus R_{\partial P'}$ is a PL-subcomplex of $M \setminus R_{\partial P'}$, the images of the extensions $\mathcal{F}_{P'}$ of the maps in $\mathcal{F}_{\partial P'}$ are contained in $G_{P'}$ and form a discrete family in $M \setminus R_{P'}$. Thus we can define $R_P^{\Delta'} = R_{P'}$ for every finite intersection P of $\mathcal{P}$ with $\dim P \cap U = m - t$ and $\Delta' \subset \partial\Delta$ such that $P(\Delta') = P'$. Then proceeding by induction on $\dim P' \cap U$ we construct for every intersection P of $\mathcal{P}$ with $\dim P \cap U = m - t$ and $\Delta' \subset \partial\Delta$ the collection of maps $\mathcal{F}_P^{\Delta'}$ and the set $R_P^{\Delta'}$ such that $R^{\partial\Delta}$ is a closed PL-presented subset of M contained in U such that $\dim R^{\partial\Delta} \leq q - 1$ and the images of the maps in $\mathcal{F}^{\partial\Delta}$ are contained in $(M \cap U) \setminus R^{\partial\Delta}$ and form a discrete family in $M \setminus R^{\partial\Delta}$. From the construction it is clear that $R^{\partial\Delta}$ is nowhere dense on the finite intersections of $\mathcal{P}$.

By 4.3 the set $R^{\partial\Delta}$ can be enlarged to a closed PL-presented subset R of M and each map $f_{\partial\Delta} : S^{q-1} \to M \setminus R^{\partial\Delta}$ in $\mathcal{F}^{\partial\Delta}$ can be extended to a map $f'_\Delta : B^q \to M \setminus R$ such that $\dim R \leq q$, R is nowhere dense on the finite intersections of $\mathcal{P}$ and the images of the maps f'_Δ form a discrete family in $M \setminus R$. Hence for every $f_{\partial\Delta} \in \mathcal{F}^{\partial\Delta}$ there is an open subset $Q(f_{\partial\Delta})$ of $M \setminus R$ containing the image of $f_{\partial\Delta}$ such that $\mathcal{Q} = \{Q(f_{\partial\Delta}) : f_{\partial\Delta} \in \mathcal{F}^{\partial\Delta}\}$ is a discrete family in $M \setminus R$ and $f_{\partial\Delta}$ is null-homotopic in $Q(f_{\partial\Delta})$. Then, by 5.2 below, the black hole modification used in 4.2 and involving $f_{\partial\Delta}$ can be carried out inside $Q(f_{\partial\Delta})$.

Note that since R is a PL-presented set of dimension $\leq q$ and $m \geq 2q+1$, for every finite intersection P of $\mathcal{P}$ that intersects U we find that the inclusion $(P \cap U) \setminus R \subset P \cap U$ induces an isomorphism of the homotopy groups in dimensions $\leq \dim P \cap U - (l-1)$ (= co-dimensions $\geq l-1$).

Thus, after removing R from M, F and the elements of $\mathcal{P}$, the black hole modification 4.1 used in 4.2 can be applied independently for every map in $\mathcal{F}^{\partial\Delta}$ in order to modify $\mathcal{P}$ to a decomposition $\mathcal{P}'$ of $M' = M \setminus R$ and F to a set F' such that $\mathcal{P}'=$ the result of all the modifications $\mathcal{P}^\pi$ and $F'=$ the result of all the modifications F^π. Then it is easy to derive from 4.2 that F' is a PL-subcomplex of M' of dimension $\leq \max\{m-q-1, \dim F\}$ lying in $\operatorname{Int} M'$, $\mathcal{P}'$ is a partition on $M' \setminus F'$ and for every intersection P of $\mathcal{P}$ with $P \cap U \neq \emptyset$, P is modified to P' (= the result of all the modifications P^π) such that $P' \cap (M' \setminus F')$ is l-co-connected and $P' \cap (M' \setminus F')$ is $(l-1)$-co-connected if $\dim P \cap U \geq m-t$.

Note that the natural one-to-one correspondence between $\mathcal{P}$ and $\mathcal{P}'$ defined by sending $P \in \mathcal{P}$ to its modification $P' \in \mathcal{P}'$ turns into a matching of partitions when $\mathcal{P}$ and $\mathcal{P}'$ are restricted to $M \setminus F$ and $M' \setminus F'$.

It is clear that the images of the maps in $\mathcal{F}^{\partial\Delta}$ are contained in the elements of $\operatorname{st}\mathcal{P}$. Assume that $\mathcal{W}$ is an open cover of M such that for every $P \in \mathcal{P}$ there is $W \in \mathcal{W}$ such that $\operatorname{st}(P, \mathcal{P}) \subset W$ and the inclusion of $\operatorname{st}(P, \mathcal{P})$ into W induces the zero-homomorphism of the homotopy groups in dimensions $\leq q-1$. By 4.3 the collection $\mathcal{Q}$ can be chosen so that for every $f_{\partial\Delta} \in \mathcal{F}^{\partial\Delta}$, $Q(f_{\partial\Delta}) \subset \operatorname{st}(f_{\partial\Delta}(S^{q-1}), \operatorname{st}\mathcal{W})$.

Then it is easy to see for every $P' \in \mathcal{P}'$ that $P' \subset \operatorname{st}(P, \operatorname{st}^2\mathcal{W})$, where P' is the modification of $P \in \mathcal{P}$.

4.5. *Absorbing simplexes.* Let M be a triangulated $(q-1)$-connected m-dimensional manifold with $m \geq 2q+1$ and let $l = m-q+1$. Assume that F is a PL-subcomplex of M lying in $\operatorname{Int} M$ such that $U = M \setminus F$ is l-co-connected and $\dim F \leq m-q$, and assume that $\mathcal{P}$ is a decomposition of M such that $\mathcal{P}$ is an l-co-connected partition on U. Fix a triangulation $\mathcal{T}$ of M for which F is a subcomplex and let F' be the $(m-q-1)$-skeleton of F. We will show how to modify M to an open subset $M' \subset M$ and $\mathcal{P}$ to a decomposition $\mathcal{P}'$ of M' such that $M \setminus M'$ is a PL-subcomplex of M of dimension $\leq q$, $\mathcal{P}'$ restricted to $U' = M' \setminus F'$ is an l-co-connected partition and $\mathcal{P}$ admits a natural one-to-one correspondence to $\mathcal{P}'$ which becomes a matching of partitions when $\mathcal{P}$ and $\mathcal{P}'$ are restricted to U and U' respectively.

Assume that the triangulation $\mathcal{T}$ that we fixed in M is the second barycentric subdivision of a triangulation for which F and the elements of $\mathcal{P}$ are subcomplexes. Let $\mathcal{T}_F$ be the collection of all $(m-q)$-dimensional simplexes lying in F and for $\Delta \in \mathcal{T}_F$ let S_Δ and $\Delta * S_\Delta$ be the *link* of Δ and the *join* of Δ with S_Δ respectively which are defined as follows:

$$S_\Delta = \bigcup\{\Delta' \in \mathcal{T} : \Delta' \cap \Delta = \emptyset \text{ and there is } \Delta'' \in \mathcal{T} \text{ with } \Delta, \Delta' \subset \Delta''\},$$

$$\Delta * S_\Delta = \bigcup\{\Delta' \in \mathcal{T} : \Delta \subset \Delta'\}.$$

Let Q be the union of $\operatorname{Int}(\Delta * S_\Delta)$ for all Δ in $\mathcal{T}_F$, and $N = M \setminus (Q \cup F)$. Note that for every $\Delta \in \mathcal{T}_F$ the link S_Δ of Δ is a $(q-1)$-dimensional sphere lying in $\operatorname{Int} M$, and for different simplexes Δ_1 and Δ_2 in $\mathcal{T}_F$ the joins $\Delta_1 * S_{\Delta_1}$ and $\Delta_2 * S_{\Delta_2}$ do not intersect on U. Also note that for every subset H which is a union of a finite intersection of $\mathcal{P}$ such that $H \cap U \neq \emptyset$ and for every $\Delta \in \mathcal{T}_F$ we find that $(H \cap U) \setminus \operatorname{Int}(\Delta * S_\Delta)$ is a deformation retract of $H \cap U$ and hence $H \cap N$ is a deformation retract of $H \cap U$. Then N is an l-co-connected manifold (since N is a deformation retract of U) and $\mathcal{P}|N$ is an l-co-connected partition (since $\mathcal{P}|U$ is l-co-connected).

Fix $\Delta \in \mathcal{T}_F$ and let an open subset V_Δ of N be such that $S^{q-1} = S_\Delta \subset V_\Delta$ and the inclusion of S^{q-1} into V_Δ is null-homotopic in V_Δ. Then by 5.3 there are an element $P_\Delta \in \mathcal{P}$ and a PL-embedding of a cube $B^m = B^q \times B^{m-q}$ into $\operatorname{Int} M$ such that $B^m \subset (\Delta * S_\Delta) \cup V_\Delta$, $\Delta = O \times B^{m-q} = F \cap B^m$, $P \cap U \cap B^m = (P \cap U \cap B^q) \times B^{m-q}$ for every $P \in \mathcal{P}$, $\partial B^q \times B^{m-q} \subset \operatorname{Int}(P_\Delta \cap U)$, $\mathcal{P}$ restricted to $(M \setminus \operatorname{Int} B^m) \cap U$ and to $\partial B^m \cap U$ are partitions and $(P \setminus \operatorname{Int} B^m) \cap U$ is a deformation retract of $(P \setminus \operatorname{Int} B^q) \cap U$ for every finite intersection P of $\mathcal{P}$.

Let us briefly describe the general idea of the construction in 5.3. The inclusion of S^{q-1} can be extended to a PL-embedding of a q-dimensional cube $B^q_\#$ into V_Δ such that $S^{q-1} = \partial B^q_\#$ and the embedding of $B^q_\#$ extends to a PL-embedding of an m-dimensional cube $B^m_\# = B^q_\# \times B^{m-q}_\#$ into V_Δ such that $\partial B^q_\# \times B^{m-q}_\# \subset B^m_\# \cap \partial(\Delta * S_\Delta)$ and $P \cap B^m_\# = (P \cap B^q_\#) \times B^{m-q}_\#$ for every $P \in \mathcal{P}$. Let $P_\Delta \in \mathcal{P}$ be such that $\operatorname{Int}(P_\Delta \cap B^q_\#) \neq \emptyset$ and let a q-dimensional cube $\sigma_\#$ be PL-embedded in $\operatorname{Int}(P_\Delta \cap B^q_\#)$. Then the cube $B^m = (\Delta * S_\Delta) \cup ((B^q_\# \setminus \operatorname{Int} \sigma_\#) \times B^{m-q}_\#)$ can be represented as the product $B^m = B^q \times B^{m-q}$ with the required properties. See 5.3 for details.

Let B^{m-q}_* be a cube lying in the geometric interior of B^{m-q}. Use the notations of 4.1. Define $F_L = F \setminus \operatorname{Int} B^m$, $P_L = P \setminus \operatorname{Int} B^m$ if $P \neq P_\Delta$ and $P \in \mathcal{P}$, $(P_\Delta)_L = (P_\Delta \setminus \operatorname{Int} B^m) \cup (B^q \times B^{m-q}_*)$ and define $\mathcal{P}_L = \{P_L : P \in \mathcal{P}\}$. Then $S = \partial B^{m-q} = \partial \Delta \subset F_L$ and it is easy to see that $\mathcal{P}_L$ is an l-co-connected partition on $L \setminus F_L$ and $\mathcal{P}_L$ is a partition on $\partial T \setminus F_L$. Now apply the black hole modification with the black hole $S = \partial \Delta$ and write $F^\pi = F^\pi_L$,

$P^\pi = P^\pi_L$ for $P \in \mathcal{P}$ and $\mathcal{P}^\pi = \{P^\pi : P \in \mathcal{P}\} = \mathcal{P}^\pi_L$. Note that $F^\pi_L = F_L$ since $F \cap \partial B^m = \partial \Delta$. Then $\mathcal{P}^\pi$ is a decomposition of M which is an l-co-connected partition on the manifold $M \setminus F^\pi = (M \setminus F) \cup \operatorname{Int} \Delta$ in which we have absorbed the geometric interior of Δ in the modification of P_Δ. Note that the one-to-one correspondence between $\mathcal{P}$ and $\mathcal{P}_L$ defined by $P \mapsto P_L$, $P \in \mathcal{P}$, becomes a matching of partitions when $\mathcal{P}$ and $\mathcal{P}_L$ are restricted to $M \setminus F$ and $L \setminus F_L$ respectively. Then by 4.1 the one-to-one correspondence between $\mathcal{P}$ and $\mathcal{P}^\pi$ defined by $P \to P^\pi$, $P \in \mathcal{P}$, becomes a matching of partitions when $\mathcal{P}$ and $\mathcal{P}^\pi$ are restricted to $M \setminus F$ and $M \setminus F^\pi$ respectively.

Note that each S_Δ can be contracted to a point in N outside some neighborhood of F in M. Then by 4.3 there is a PL-subcomplex R of M contained in $\operatorname{Int} N$ and nowhere dense on the finite intersections of $\mathcal{P}$ with $\dim R \le q$ such that the collection $\mathcal{V}$ of the sets V_Δ, $\Delta \in \mathcal{T}_F$, can be chosen to be discrete in $N \setminus R$. Then replacing M by $M' = M \setminus R$ and removing R from the elements of $\mathcal{P}$ we can perform all the black hole modifications independently for all $\Delta \in \mathcal{T}_F$ and get from all the modifications $\mathcal{P}^\pi$ the corresponding decomposition $\mathcal{P}'$ of M' with the required properties. It is clear that the natural one-to-one correspondence between $\mathcal{P}$ and $\mathcal{P}'$ defined by sending each element of $\mathcal{P}$ to its modification becomes a matching of partitions when the decompositions are restricted to $M \setminus F$ and $M' \setminus F'$.

It is clear that the spheres S_Δ, $\Delta \in \mathcal{T}_F$, are contained in the elements of $\operatorname{st} \mathcal{P}$. Let $\mathcal{W}$ be an open cover of M having the following property: for every $P \in \mathcal{P}$ there are $W \in \mathcal{W}$ and a set H such that $\operatorname{st}(P, \mathcal{P}) \subset H \subset W$, H is a union of elements of $\mathcal{P}$ and the inclusion $\operatorname{st}(P, \mathcal{P}) \cap U \subset H \cap U$ induces the zero-homomorphism of the homotopy groups in dimensions $\le q-1$. Then the inclusion $\operatorname{st}(P, \mathcal{P}) \cap N \subset H \cap N$ also induces the zero-homomorphism of the homotopy groups in dimensions $\le q-1$ because $H \cap N$ is a deformation retract of $H \cap U$. Thus S_Δ can be contracted to a point in $\operatorname{st}(\Delta, \mathcal{W}) \cap N$. Then the collection $\mathcal{V}$ can be chosen so that for every $V_\Delta \in \mathcal{V}$, $V_\Delta \subset \operatorname{st}(\Delta, \operatorname{st} \mathcal{W})$, and this implies that for every $P' \in \mathcal{P}'$, $P' \subset \operatorname{st}(P, \operatorname{st}^2 \mathcal{W})$, where P' is the modification of $P \in \mathcal{P}$.

5. Appendix

5.1. *Extending partitions in the black hole modification.* This is a direct continuation of 4.1 and we adopt the notations and the assumptions of 4.1. This subsection is devoted to constructing the extensions A^π, $A \in \mathcal{A}$, the step that was omitted in 4.1.

Recall that T can be represented as the product $T = B^{q+1} \times S$, where S is identified with $a \times S$ for a point a in ∂B^{q+1}. We consider B^{q+1} and S as PL-embedded in Euclidean spaces. This induces the corresponding PL-embedding of T in the product of Euclidean spaces and we refer to these

Euclidean spaces when properties of linearity are used; thus we say that a simplex is *linear* if it is a simplex (linearly spanned by its vertices) in the corresponding Euclidean space, and a map of a linear simplex is *linear* if it is the linear extension of its values on the vertices.

Denote by $p_S : T \to S$ and $p_B : T \to B^{q+1}$ the projections. Let $\mathcal{T}_T$ be a triangulation of T such that the simplexes of $\mathcal{T}_T$ are linear, p_S and p_B restricted to every simplex of $\mathcal{T}_T$ are linear and $\mathcal{T}_T$ underlies S and the simplexes of $\mathcal{T}$ contained in T. Let $\mathcal{T}'_T$ be a subdivision of $\mathcal{T}_T$ such that the simplexes of $\mathcal{T}'_T$ are linear and, for every simplex Δ of $\mathcal{T}_T$, $p_S(\Delta)$ is a subcomplex of S with respect to $\mathcal{T}'_T$.

Denote by $\mathcal{T}''_T$ the (first) barycentric subdivision of $\mathcal{T}'_T$ and let $\mathcal{B}_T$ be the partition of T formed by the stars of the vertices of $\mathcal{T}'_T$ with respect to $\mathcal{T}''_T$. Then $\mathcal{B}_S = \mathcal{B}_T|S$ is a partition of S. For a non-empty finite intersection B of $\mathcal{B}_S$ which is the intersection of distinct elements $B_0, \ldots, B_t$ of $\mathcal{B}_S$ let the vertex v_B of $\mathcal{T}''_T$ be the barycenter of the simplex of $\mathcal{T}'_T$ spanned by the vertices contained in $B_0 \cup \cdots \cup B_t$.

For a finite intersection B of $\mathcal{B}_S$ define $K_B = p_S^{-1}(B)$, $N_B = p_S^{-1}(\partial B)$ and $M_B = K_B \cap \partial T$. Note that $K_B = B^{q+1} \times B$, $N_B = B^{q+1} \times \partial B$, $M_B = \partial B^{q+1} \times B$, $\partial K_B = N_B \cup M_B$ and $N_B \cap M_B = \partial B^{q+1} \cap \partial B$.

Let us show that $\mathcal{P}_L$ restricted to $M_B \setminus F$ and $\mathcal{P}_L$ restricted to $\partial M_B \setminus F$ are partitions. Assume that $B \in \mathcal{B}_S$ (that is, $\dim B = \dim S$). Note that $\mathcal{P}_L$ restricted to $\operatorname{Int} M_B \setminus F_L$ is a partition. Take a point $x \in \partial M_B$ and let Δ_x be the smallest simplex of $\mathcal{T}_T$ containing x. Then $v_B \in p_S(\Delta_x)$ and let $y \in \Delta_x$ be such that $p_S(y) = v_B$. Let G be a neighborhood of x in ∂M_B such that G is contained in the star of y with respect to $\mathcal{T}_T$. Then $(z,t) \mapsto z(1-t) + ty$, $z \in G$, $0 \le t < 1$, defines a PL-embedding of $G \times [0,1)$ into M_B such that $G \times [0,1)$ is a neighborhood of x in M_B and for every simplex $\Delta \in \mathcal{T}_T$ we have $\Delta \cap (G \times [0,1)) = (G \cap \Delta) \times [0,1)$. Recall that $\mathcal{T}_T$ underlies $\mathcal{P}_L|\partial T$ and $F_L \cap \partial T$, and note that $G \times (0,1)$ is an open subset of ∂T. Therefore $\mathcal{P}_L$ restricted to $(G \times (0,1)) \setminus F_L$ is a partition. All these facts together imply by 2.2 that $\mathcal{P}_L$ restricted $(G \times [0,1)) \setminus F_L$ and $\mathcal{P}_L$ restricted to $G \setminus F_L$ are partitions and hence $\mathcal{P}_L$ restricted to $M_B \setminus F_L$ and $\mathcal{P}_L$ restricted to $\partial M_B \setminus F_L$ are partitions.

Now assume that for a finite intersection $B' \in \mathcal{B}_S$ we already showed that $\mathcal{P}_L$ restricted to $M_{B'} \setminus F_L$ and $\mathcal{P}_L$ restricted to $\partial M_{B'} \setminus F_L$ are partitions, and let B be a finite intersection of $\mathcal{B}_S$ such that $B \subset \partial B'$ and $\dim B + 1 = \dim B'$. Then replacing S by $\partial B'$ and T by $B^{q+1} \times \partial B'$ we can repeat the above reasoning to show that $\mathcal{P}_L$ restricted to $M_B \setminus F_L$ and $\mathcal{P}_L$ restricted to $\partial M_B \setminus F_L$ are partitions. Thus we have shown that for every finite intersection of $\mathcal{B}_S$, $\mathcal{P}_L$ restricted to $M_B \setminus F_L$ and to $\partial M_B \setminus F_L$ are partitions.

Let B be a non-empty finite intersection of $\mathcal{B}_S$. Denote by V the set of vertices v of $\mathcal{T}''_T$ such that $v \neq v_B$, $v \in \partial T$ and $[v, v_B]$ is a simplex of $\mathcal{T}''_T$. For each $v \in V$ choose a point v' lying on the interval $[v, v_B]$ connecting v and v_B as follows. If $v \in S$ then set $v' = v$ and note that in this case $v' \in \partial B$. If $v \notin S$ then choose v' such that $v' \neq v_B$, $v' \neq v$ and $p_S([v', v]) \subset \operatorname{Int} B$. For a simplex Δ of $\mathcal{T}''_T$ spanned by vertices $v_B, v_1, \ldots, v_k$, with $v_1, \ldots, v_k \in V$ denote by Δ' the simplex spanned by the vertices $v_B, v'_1, \ldots, v'_k$ and denote by W_B the union of all such simplexes Δ'. Then W_B is a PL-ball which is a closed neighborhood of v_B in M_B such that $W_B \cap S = M_B \cap S = B$ and for every $x \in \partial W_B$ the interval $[v_B, x]$ is contained in W_B and $[v_B, x] \cap \partial W_B = x$. Let $\pi^W_B : W_B \setminus v_B \to \partial W_B$ be the radial projection. Using a PL-homeomorphism represent K_B as $W_B \times [0,1]$ so that W_B is identified with $W_B \times 0$ and extend π^W_B to the radial projection

$$\pi_B : K_B \setminus v_B = (W_B \times [0,1]) \setminus v_B \to \partial K_B \setminus \operatorname{Int} W_B = (\partial W_B \times [0,1]) \cup (W_B \times 1)$$

(π_B is defined with respect to the linear structure of $W_B \times [0,1]$ induced by the linear structures of W_B and $[0,1]$).

Let us show that $\mathcal{P}_L$ restricted $M_B \setminus (F_L \cup \operatorname{Int} W_B)$ is a partition. Take $x \in \partial W_B \setminus S$. Then there are an open neighborhood G of x in $\partial W_B \setminus S$ and $\varepsilon > 0$ such that $G \times [0, \varepsilon)$ is embedded into $M_B \setminus (S \cup \operatorname{Int} W_B)$ by the map $(z, t) \mapsto z(1 + t) - t v_B$, $z \in G$, so that for every simplex Δ of $\mathcal{T}''_T$, $\Delta \cap (G \times [0, \varepsilon)) = (\Delta \cap G) \times [0, \varepsilon)$. Recall that $\mathcal{T}''_T$ underlies $\mathcal{P}_L$, and F_L restricted to ∂T and $\mathcal{P}_L$ restricted to $M_B \setminus F_L$ and to $\partial M_B \setminus F_L$ are partitions. Since $G \times (0, \varepsilon)$ is an open subset of M_B, $\mathcal{P}_L$ restricted to $(G \times (0, \varepsilon)) \setminus F_L$ is a partition and therefore, by 2.2, $\mathcal{P}_L$ restricted to $G \setminus F_L$ is a partition. Then, again by 2.2, $\mathcal{P}_L$ restricted to $(G \times [0, \varepsilon)) \setminus F_L$ is a partition and hence $\mathcal{P}_L$ restricted $M_B \setminus (F_L \cup \operatorname{Int} W_B)$ is a partition (recall that $S \subset F_L$).

For every $A \in \mathcal{A}$ and every finite intersection B of $\mathcal{B}_S$ define sets A^B and $A^{\partial B}$ as follows. For $B = \emptyset$ set $A^\emptyset = A$ if $A \cap T = \emptyset$ and $A^\emptyset = A \cup S$ if $A \cap T \neq \emptyset$, and for $\dim B = 0$ set $A^{\partial B} = A^\emptyset$. Now by induction on $\dim B$ define A^B and $A^{\partial B}$ so that $A^B = A^{\partial B} \cup \pi_B^{-1}(A^{\partial B} \cap (\partial K_B \setminus \operatorname{Int} W_B))$ for $\dim B \geq 0$ and $A^{\partial B} = \bigcup \{A^{B'} : B' \subset \partial B\}$ for $\dim B \geq 1$. Clearly, A^B and $A^{\partial B}$ are PL-subcomplexes of M. Note that if $A \cap T = \emptyset$ then $A^B = A^{\partial B} = A$ for every B.

From the construction of W_B and π_B it follows that for every simplex $\Delta \in \mathcal{T}'_T$ intersecting W_B we have $\pi_B^{-1}(\Delta \cap \partial W_B) \setminus S = (\Delta \cap W_B) \setminus S$. Then it is easy to see that $S^{\partial B} = S^B = S$, $(A^{\partial B} \cap M_B) \setminus S = (A^B \cap M_B) \setminus S = (A \cap M_B) \setminus S$ and hence $(A^{B_1} \cap A^{B_2}) \setminus S = A^{B_1 \cap B_2} \setminus S$, $(A_1^B \cap A_2^B) \setminus S = (A_1 \cap A_2)^B \setminus S$ for any $A, A_1, A_2 \in \mathcal{A}$ and finite intersections B, B_1, B_2 of $\mathcal{B}_S$.

Define $\mathcal{P}^B = \{P^B : P \in \mathcal{P}_L\}$ and $\mathcal{P}^{\partial B} = \{P^{\partial B} : P \in \mathcal{P}_L\}$. Recall that $S \subset F_L$. Let us show by induction on $\dim B$ that $\mathcal{P}^B$ restricted to $K_B \setminus F_L^B$ and $\mathcal{P}^{\partial B}$ restricted to $N_B \setminus F_L^{\partial B}$ are partitions.

For $\dim B = 0$ we have $N_B = \emptyset$ and therefore $\mathcal{P}^{\partial B}$ restricted to $N_B \setminus F_L^{\partial B}$ is vacuously a partition. Let B be a finite intersection of $\mathcal{B}_S$ such that $\mathcal{P}^{\partial B}$ restricted to $N_B \setminus F_L^{\partial B}$ is a partition. Define $R_B = \partial K_B \setminus \operatorname{Int} W_B$. From the construction of π_B it follows that $K_B \setminus v_B$ is PL-homeomorphic to $R_B \times [0,1)$ so that R_B corresponds to $R_B \times 0$, $\mathcal{P}^B$ restricted to $K_B \setminus v_B$ corresponds to the decomposition $(\mathcal{P}^{\partial B}|R_B) \times [0,1)$ of $R_B \times [0,1)$ and $(F_L^B \cap K_B) \setminus v_B$ corresponds to $(F_L^{\partial B} \cap R_B) \times [0,1)$. Note that $F_L^{\partial B} \cap M_B = F_L \cap M_B$ and $\mathcal{P}^{\partial B}$ restricted to $M_B \setminus S$ coincides with $\mathcal{P}_L$ restricted $M_B \setminus S$, and recall that $\mathcal{P}_L$ restricted to $\partial M_B \setminus F_L$ and to $M_B \setminus (F_L \cup \operatorname{Int} W_B)$ are partitions. Then, by 2.2, $\mathcal{P}^{\partial B}$ restricted to $R_B \setminus F_L^{\partial B}$ is a partition and therefore $\mathcal{P}^B$ restricted to $K_B \setminus F_L^B$ is also a partition.

Now assume B is a finite intersection of $\mathcal{B}_S$ such that $\dim B > 0$ and for every finite intersection B' of $\mathcal{B}_S$ such that $B' \subset \partial B$ we knew that $\mathcal{P}^{B'}$ restricted to $K_{B'} \setminus F_L^{B'}$ is a partition. Then, by 2.2, $\mathcal{P}^{\partial B}$ is a partition of $N_B \setminus F_L^{\partial B}$ since $\{K_{B'} : B'$ is a finite intersection of $\mathcal{B}_S$ such that $B' \subset \partial B\}$ is a partition of N_B. The induction is complete.

Thus we have shown that $\mathcal{P}^B$ restricted to $K_B \setminus F_L^B$ is a partition for every finite intersection B of $\mathcal{B}_S$. For $A \in \mathcal{A}$ write $A^\pi = \bigcup\{A^B : B \in \mathcal{B}_S\}$ and define $\mathcal{P}_L^\pi = \{P^\pi : P \in \mathcal{P}_L\}$. Note that $F_L^B = K_B \cap F_L^\pi$ and $\mathcal{P}_L^\pi$ restricted to K_B coincides with $\mathcal{P}^B$. Then, by 2.2, $\mathcal{P}_L^\pi$ is a partition on $T \setminus F_L^\pi$ since $\{K_B : B$ is a finite intersection of $\mathcal{B}_S\}$ is a partition of T. It is obvious that $\mathcal{P}_L^\pi$ restricted to $L \setminus S$ coincides with $\mathcal{P}_L$ restricted to $L \setminus S$ and therefore $\mathcal{P}_L^\pi$ is a decomposition of M which is a partition on $M \setminus F_L^\pi$.

Now let us show that for $A_1, A_2 \in \mathcal{A}$, $(A_1 \setminus A_2) \setminus S$ is a strong deformation retract of $(A_1^\pi \setminus A_2^\pi) \setminus S$. To this end it is enough to show that for every finite intersection B of $\mathcal{B}_S$, $(C_1' \setminus C_2') \setminus S$ is a strong deformation retract of $(C_1 \setminus C_2) \setminus S$ where $C_i' = A_i^{\partial B} \cap \partial K_B$ and $C_i = A_i^B \cap K_B$. From the construction it follows that $(C_1 \setminus C_2) \setminus S$ can be topologically represented as $C' \times [0,1)$ with $C' = (C_1' \setminus C_2') \setminus (S \cup \operatorname{Int} W_B)$ so that $(C_1' \setminus C_2') \setminus S$ is identified with $(C' \times 0) \cup ((C' \cap \partial W_B) \times [0,1))$. Note that C' is a space admitting a triangulation for which $C' \cap \partial W_B$ is a subcomplex of C' and therefore $(C' \times 0) \cup ((C' \cap \partial W_B) \times [0,1))$ is a strong deformation retract of $C' \times [0,1)$.

Let $\Delta \in \mathcal{T}_T$ be contained in ∂T. Since p_S is linear on Δ and $p_S(\Delta)$ is a subcomplex of $\mathcal{T}_T'$ we can conclude that $\dim \Delta \cap M_B \le \dim \Delta + \dim B - \dim S$ for every finite intersection B of $\mathcal{B}_S$. Then, as $\dim \Delta^B \setminus S \le \dim(\Delta \cap M_B) \setminus S + 1$ for $\dim B = 0$ and $\dim \Delta^B \setminus S \le \dim \Delta^{\partial B} \setminus S + 1$ for $\dim B > 0$, we get $\dim \Delta^\pi \setminus S \le \dim \Delta \setminus S + 1$. Thus $\dim (A^\pi \cap T) \setminus S \le \dim (A \cap \partial T) \setminus S + 1$ for every $A \in \mathcal{A}$.

The remaining properties required in 4.1 are easy to verify.

5.2. *Digging holes for improving connectivity of an intersection*

Summary. Here we present a construction used in 4.2. Let M be a triangulated m-dimensional manifold, F a PL-subcomplex of M and $\mathcal{P}$ a decomposition of M which is a partition on $M \setminus F$. Assume that M is $(q-1)$-connected, $m \geq 2q+1$ and let $S^{q-1} \subset \operatorname{Int} M \setminus F$ be a PL-embedded q-dimensional sphere so that $\mathcal{P}$ restricted to S^{q-1} is a partition and for every finite intersection P' of $\mathcal{P}$ with $P' \cap S^{q-1} \neq \emptyset$, $S^{q-1} \cap P'$ is properly embedded in $P' \setminus F$ (that is, $S^{q-1} \cap \partial(P' \setminus F) = \partial(S^{q-1} \cap P')$).

We will show that there is an m-dimensional cube $B^m = B^q \times B^{m-q} \subset \operatorname{Int} M$ such that B^m is PL-embedded in M, and $B^q = B^q \times O$ and B^m have the following properties: $S^{q-1} = \partial B^q$, $\mathcal{P}$ restricted to $\partial B^m \setminus F$ and to $M \setminus (\operatorname{Int} B^m \cup F)$ are partitions, $F \cap B^m = (F \cap B^q) \times B^{m-q}$, $P' \cap B^m = (P' \cap B^q) \times B^{m-q}$ for every $P' \in \mathcal{P}$ and $P' \setminus (F \cup \operatorname{Int} B^m)$ is a deformation retract of $P' \setminus (F \cup \operatorname{Int} B^q)$ for every finite intersection P' of $\mathcal{P}$.

Moreover, if $\Phi : B^q_\Phi \to M$ is a map from a q-dimensional cube B^q_Φ into M such that Φ restricted to ∂B^q_Φ is a PL-homeomorphism between ∂B^q_Φ and S^{q-1}, then B^q can be chosen to be arbitrarily close to $\Phi(B^q_\Phi)$ in the sense that B^q can be chosen to be the image of an arbitrarily close approximation of Φ by a PL-embedding which coincides with Φ on ∂B^q_Φ.

Clearly, replacing B^{m-q} by a smaller cube we ensure that B^m will be arbitrarily close to B^q. Hence, if S^{q-1} can be contracted to a point in an open subset $S^{q-1} \subset W$ of M, then we can assume that $B^m \subset W$.

Construction. The case $q = 1$ and $m = 3$ can be visualized directly and is left to the reader. Assume that $m \geq 4$ (in fact, we can always assume that $m \geq 4$ restricting ourselves to Nöbeling spaces modeled on manifolds of dimension ≥ 4). Let B^q be a q-dimensional ball PL-embedded in $\operatorname{Int} M$ such that $S^{q-1} = \partial B^q$. Using an unknotting theorem for manifolds (see Theorem 10.3 of [6]) find a PL-embedded Euclidean space $\mathbb{R}^m = \mathbb{R}^q \times \mathbb{R}^{m-q} \subset \operatorname{Int} M$ such that $B^q \subset \mathbb{R}^q$, where $\mathbb{R}^q$ and $\mathbb{R}^{m-q}$ are identified with the subsets $\mathbb{R}^q \times O \subset \mathbb{R}^m$ and $O \times \mathbb{R}^{m-q} \subset \mathbb{R}^m$ respectively.

Let Φ be as above. It is easy to see that Φ can be arbitrarily closely approximated by a PL-embedding into $\operatorname{Int} M$. Then replacing Φ by such an approximation and setting $B^q = \Phi(B^q_\Phi)$ we can assume without loss of generality that $\Phi(B^q_\Phi) \subset \mathbb{R}^m$.

For a block bundle ξ over a cell complex X, written ξ/X, denote by σ_ξ the cells of X, by β_ξ the blocks of the total space $E(\xi)$ and by (σ_ξ, β_ξ) the pairs such that β_ξ is the block over σ_ξ (see [10]). We say that ξ *underlies* $Y \subset E(\xi)$ if Y is a union of blocks of ξ, and ξ *underlies* a collection $\mathcal{Y}$ of subsets of $E(\xi)$ if ξ underlies each $Y \in \mathcal{Y}$.

Let us say that a block bundle ξ over S^{q-1} is *subordinated* to $\mathcal{P}$ if $E(\xi)$ is a regular neighborhood of S^{q-1} in $\mathbb{R}^m$, $E(\xi)$ does not intersect F and $E(\xi)$ underlies $\mathcal{P}$ restricted to $E(\xi)$.

A block bundle ξ subordinated to $\mathcal{P}$ can be constructed as follows. Fix a triangulation $\mathcal{T}$ of M which is the second barycentric subdivision of a triangulation that underlies S^{q-1}, F and $\mathcal{P}$. By a *simplicial neighborhood* of $A \subset M$ in a subcomplex X of M we mean the union of the simplexes of $\mathcal{T}$ intersecting A and contained in X. We may assume that $\mathcal{T}$ is chosen so that the simplicial neighborhood G of S^{q-1} in M is contained in $\mathbb{R}^m$ and $G \cap F = \emptyset$. For every finite intersection P' of $\mathcal{P}$ that meets S^{q-1} denote by $G_{P'}$ the simplicial neighborhood of $P' \cap S^{q-1}$ in P'. Note that $G_{P'} = G \cap P'$, $G_{P''} = G_{P'} \cap P''$ if $P'' \subset P'$, $G_{P'}$ is a regular neighborhood of $P' \cap S^{q-1}$ in $P' \setminus F$ and $P' \cap S^{q-1}$ is locally flat in $P' \setminus F$ because $\dim P' - \dim P' \cap S^{q-1} = m - (q-1) \geq q+2$.

For every finite intersection P' of $\mathcal{P}$ that meets S^{q-1} we are going to construct by induction on $\dim P' \cap S^{q-1}$ a block bundle $\xi_{P'}$ over $P' \cap S^{q-1}$ such that $E(\xi_{P'}) = G_{P'}$ and $\xi_{P'} | P'' \cap S^{q-1} = \xi_{P''}$ if $P'' \subset P'$. Obviously, such a block bundle exists if $\dim P' \cap S^{q-1} = 0$. Assume that for a finite intersection P' of $\mathcal{P}$ a block bundle $\xi_{P''}$ is already constructed for every finite intersection P'' of $\mathcal{P}$ such that $P'' \subset P'$ and $P'' \neq P'$. Then $\xi_{P''}$ defines the corresponding block bundle $\xi_{\partial P'}$ over $\partial(P' \cap S^{q-1})$ and by Theorem 4.3 of [10] this block bundle extends to a block bundle $\xi_{P'}$ over P' such that $E(\xi_{P'}) = G_{P'}$.

Thus we have constructed block bundles $\xi_{P'}$ for every finite intersection P' of $\mathcal{P}$ that meets S^{q-1}, and these block bundles define the corresponding block bundle ξ over S^{q-1}. Clearly, ξ is subordinated to $\mathcal{P}$ and $E(\xi) = G$.

Let B_1^{m-q} be a cube in $\mathbb{R}^{m-q}$. Fix $\varepsilon = 1/2$ and for each cell σ_ξ of S^{q-1} define the pair $(\sigma_\eta, \beta_\eta)$ with $\sigma_\eta = \sigma_\xi$ and $\beta_\eta = \gamma_\xi \times B_1^{m-q}$ where $\gamma_\xi = \{x : x = ts,\ 1-\varepsilon \leq t \leq 1+\varepsilon,\ s \in \sigma_\xi\} \subset \mathbb{R}^q$. Then the pairs $(\sigma_\eta, \beta_\eta)$ form a block bundle η over S^{q-1} such that $E(\eta)$ underlies a regular neighborhood of S^{q-1} in $\mathbb{R}^m$. Hence by Theorem 4.4 of [10] there is a PL-homeomorphism of $\mathbb{R}^m$ realizing an isomorphism from η to ξ.

Let $e : \mathbb{R}^m \to M$ be the composition of such a PL-homeomorphism with the original embedding of $\mathbb{R}^m$ in M. Then replacing the original embedding of $\mathbb{R}^m$ into M by the embedding e we may assume that ξ coincides with η (the meaning of such a replacement is that we leave $\mathbb{R}^m$ as the same subset of M but change the representation of $\mathbb{R}^m$ as the product of $\mathbb{R}^q$ and $\mathbb{R}^{m-q}$). Thus we deduce that η is subordinated to $\mathcal{P}$ and this is the only property of η with respect to $\mathcal{P}$ that will be used in what follows.

Replace the map Φ by a close PL-embedding which coincides with Φ on ∂B_Φ^q and sends a small neighborhood of ∂B_Φ^q in B_Φ^q onto a neighborhood of ∂B^q in B^q. Then there is a neighborhood Ω of S^{q-1} in $\mathbb{R}^m$ such that $\Phi(B_\Phi^q) \cap \Omega = B^q \cap \Omega$. It is easy to see from the definition of η that there is a PL-homeomorphism $\phi_1 : \mathbb{R}^m \to \mathbb{R}^m$ such that ϕ_1 does not move the points of S^{q-1}, $\phi_1(B^q) = B^q$, $\phi_1(E(\eta)) \subset E(\eta) \cap \Omega$ and for every block β_η we have $\phi_1(\beta_\eta) \subset \beta_\eta$. Then the block bundle $\phi_1(\eta)$ is also subordinated to $\mathcal{P}$ and we can replace the embedding e by $e \circ \phi_1$ and assume that $E(\eta) \subset \Omega$ and $\Phi(B^q_\Phi) \cap E(\eta) = B^q \cap E(\eta)$.

By an unknotting theorem for manifolds (see Theorem 10.1 of [6]) there is a PL-homeomorphism $\phi_2 : \mathbb{R}^m \to \mathbb{R}^m$ such that ϕ_2 does not move the points of $E(\eta)$ and $\Phi(B^q_\Phi) = \phi_2(B^q)$. Then replacing the embedding e by $e \circ \phi_2$ we may assume that $\Phi(B^q_\Phi) = B^q$.

Fix a triangulation $\mathcal{T}_\mathbb{R}$ of $\mathbb{R}^m$ such that the simplexes of $\mathcal{T}_\mathbb{R}$ are linear in $\mathbb{R}^m$ and $\mathcal{T}_\mathbb{R}$ underlies $\mathcal{P}|\mathbb{R}^m$, $F \cap \mathbb{R}^m$ and $B^q_1 \times B^{m-q}_1$ where $B^q_1 = (1+\varepsilon)B^q$. Let $p : \mathbb{R}^m = \mathbb{R}^q \times \mathbb{R}^{m-q} \to \mathbb{R}^{m-q}$ be the projection and let $a \in \operatorname{Int} B^{m-q}_1$ be such that $a \notin p(\Delta)$ for every $\Delta \in \mathcal{T}_\mathbb{R}$ with $\dim p(\Delta) < m - q$ and $a \notin \partial p(\Delta)$ for every $\Delta \in \mathcal{T}_\mathbb{R}$ with $\dim p(\Delta) = m - q$. Take a cube B^{m-q}_2 such that $a + B^{m-q}_2 \subset \operatorname{Int} B^{m-q}_1$ and $a + B^{m-q}_2 \subset \operatorname{Int} p(\Delta)$ for every $\Delta \in \mathcal{T}_\mathbb{R}$ such that $\Delta \cap (B^q_1 \times a) \neq \emptyset$. Then the pairs $(\sigma_\tau, \beta_\tau)$ with $\sigma_\tau = (B^q_1 \times a) \cap \Delta$ and $\beta_\tau = (B^q_1 \times (a + B^{m-q}_2)) \cap \Delta$, $\Delta \in \mathcal{T}_\mathbb{R}$, form a block bundle τ over $B^q_1 \times a$, and we assume that B^{m-q}_2 is so small that $E(\tau) = B^q_1 \times (a + B^{m-q}_2)$ and $E(\tau|\partial B^q_1) = \partial B^q_1 \times (a + B^{m-q}_2)$.

By Theorem 1.1 of [10] every block bundle over a cube is trivial. Hence τ is isomorphic to a trivial block bundle δ over B^q_1, that is, there is a PL-homeomorphism $h : E(\delta) \to E(\tau)$ where $E(\delta)$ is the product $E(\delta) = B^q_1 \times B^{m-q}_3 \subset \mathbb{R}^m$ of B^q_1 with a cube B^{m-q}_3 in $\mathbb{R}^{m-q}$ such that $h(x) = (x, a)$ for every $x \in B^q_1$, $(\sigma_\delta, \beta_\delta)$ is a pair of δ if and only if $(h(\sigma_\delta), h(\beta_\delta))$ is a pair of τ and, for every pair $(\sigma_\delta, \beta_\delta)$ of δ, $\beta_\delta = \sigma_\delta \times B^{m-q}_3$.

Identify $B^q_1 \times B^{m-q}_3$ with $h(B^q_1 \times B^{m-q}_3)$ (that is, re-embed $B^q_1 \times B^{m-q}_3$ according to h) and let B^{m-q} be a cube lying in $\operatorname{Int} B^{m-q}_3$. Then from the construction it follows that

(∗) $P' \cap (B^q_1 \times \operatorname{Int} B^{m-q}_3) = (P' \cap B^q_1) \times \operatorname{Int} B^{m-q}_3$ for every $P' \in \mathcal{P}$ and $F \cap (B^q_1 \times \operatorname{Int} B^{m-q}_3) = (F \cap B^q_1) \times \operatorname{Int} B^{m-q}_3$.

From the construction it also follows that $\mathcal{P}$ restricted to $\operatorname{Int} B^q_1 \setminus \operatorname{Int} B^q$, to ∂B^q and to $B^q \setminus F$ are partitions. Hence by (∗) and 2.2, $\mathcal{P}$ restricted to $(\operatorname{Int} B^q_1 \setminus \operatorname{Int} B^q) \times \operatorname{Int} B^{m-q}_3$, to $(B^q \setminus F) \times (\operatorname{Int} B^{m-q}_3 \setminus \operatorname{Int} B^{m-q})$, to $\partial B^q \times (\operatorname{Int} B^{m-q}_3 \setminus \operatorname{Int} B^{m-q})$, to $\partial B^q \times \partial B^{m-q}$, $\partial B^q \times B^{m-q}$ and to $(B^q \setminus F) \times \partial B^{m-q}$ are partitions. Then, once again by 2.2, $\mathcal{P}$ restricted to $\operatorname{Int}(B^q_1 \times B^{m-q}_3) \setminus (F \cup (\operatorname{Int}(B^q \times B^{m-q})))$ and to $\partial(B^q \times B^{m-q}) \setminus F$ are partitions.

Define $B^m = B^q \times B^{m-q}$. Thus we see that

(∗∗) $\mathcal{P}$ restricted to $M \setminus (F \cup \operatorname{Int} B^m)$ and to $\partial B^m \setminus F$ are partitions.

Let P' be a finite intersection of $\mathcal{P}$. Let us show that

(∗∗∗) $P' \setminus (F \cup \operatorname{Int} B^m)$ is a deformation retract of $P' \setminus (F \cup \operatorname{Int} B^q)$.

Note the following general fact: for a triangulated space A_1, a subcomplex A_2 of A_1 and a cube B, the space $(A_2 \times B) \cup (A_1 \times \partial B)$ is a strong deformation retract of $(A_2 \times B) \cup (A_1 \times (B \setminus O))$. Then for $A_1 = (P' \cap B^q) \setminus F$ and $A_2 = P' \cap \partial B^q$ we deduce by (∗) that $B^m \cap (P' \setminus (F \cup \operatorname{Int} B^m)) = (A_2 \times B^{m-q}) \cup (A_1 \times \partial B^{m-q})$ and $B^m \cap (P' \setminus (F \cup \operatorname{Int} B^q)) = (A_2 \times B^{m-q}) \cup (A_1 \times (B^{m-q} \setminus O))$. Thus $B^m \cap (P' \setminus (F \cup \operatorname{Int} B^m))$ is a strong deformation retract of $B^m \cap (P' \setminus (F \cup \operatorname{Int} B^q))$ and hence (∗∗∗) holds.

Let us compare the embeddings of B^q under h and e. For this purpose we assume that $\mathbb{R}^m \subset M$ is identified with $\mathbb{R}^q \times \mathbb{R}^{m-q}$ according to the embedding e and we disregard the identification induced by h. Then $h(x) = (x, a)$ for $x \in B^q$. One can easily observe that there is a PL-homeomorphism $\psi : M \to M$ such that $\psi \circ h$ does not move the points of S^{q-1}, ψ does not move the points outside $\operatorname{Int}(E(\eta))$ and $\psi(P') = P'$ for every $P' \in \mathcal{P}$. Moreover, choosing the point a arbitrarily close to O we may choose ψ to be arbitrarily close to the identity map of M. Note that since ψ does not move the points outside $\operatorname{Int}(E(\eta))$, we have $\psi(F) = F$. Now we can replace Φ by $\psi \circ h \circ \Phi$ and h by $\psi \circ h$, and after identifying $B_1^q \times B_3^{m-q}$ with its image under the new embedding h, we find that $\Phi(B_\Phi^q) = B^q$ and the properties (∗)–(∗∗∗) are preserved.

Note that (∗) trivially implies that $F \cap B^m = (F \cap B^q) \times B^{m-q}$ and $P' \cap B^m = (P' \cap B^q) \times B^{m-q}$ for every $P' \in \mathcal{P}$.

Thus we have replaced the original map Φ by an arbitrarily close PL-embedding which coincides with Φ on ∂B_Φ^q and we have found a PL-embedding of $B^m = B^q \times B^{m-q} \subset \operatorname{Int} M$ such that $B^q = \Phi(B_\Phi^q)$ and B^m has the required properties.

5.3. *Digging holes for absorbing simplexes*

Summary. Here we present a construction used in 2.6. Let M be a triangulated $(q-1)$-connected m-dimensional manifold with $m \geq 2q+1$ and let $l = m - q + 1$. Assume that F is a PL-subcomplex of M lying in $\operatorname{Int} M$ such that $U = M \setminus F$ is l-co-connected ($= (q-1)$-connected) and $\dim F \leq m - q$, and assume that $\mathcal{P}$ is a decomposition of M such that $\mathcal{P}$ is an l-co-connected partition on U.

Fix a triangulation $\mathcal{T}_0$ of M which underlies F and the elements of $\mathcal{P}$ and let the triangulation $\mathcal{T}$ be the second barycentric subdivision of $\mathcal{T}_0$. Let $\mathcal{T}_F$ be the collection of all $(m-q)$-dimensional simplexes of $\mathcal{T}$ lying in F and let S_Δ and $\Delta * S_\Delta$ be the link of Δ and the join of Δ with S_Δ respectively. Recall that

$$S_\Delta = \bigcup\{\Delta' \in \mathcal{T} : \Delta' \cap \Delta = \emptyset \text{ and there is } \Delta'' \in \mathcal{T} \text{ with } \Delta, \Delta' \subset \Delta''\},$$

$$\Delta * S_\Delta = \bigcup\{\Delta' \in \mathcal{T} : \Delta \subset \Delta'\}.$$

Define Q as the union of $\mathrm{Int}(\Delta * S_\Delta)$ for all Δ in $\mathcal{T}_F$ and $N = M \setminus (Q \cup F)$. Let $\Delta \in \mathcal{T}_F$ and let an open subset V_Δ of N be such that $S^{q-1} = S_\Delta \subset V_\Delta$ and S^{q-1} is contractible in V_Δ.

We will show that there are an element $P_\Delta \in \mathcal{P}$ and a PL-embedding of a cube $B^m = B^q \times B^{m-q}$ into $\mathrm{Int}\, M$ such that $B^m \subset (\Delta * S_\Delta) \cup V_\Delta$, $\Delta = O \times B^{m-q} = F \cap B^m$, $P \cap U \cap B^m = (P \cap U \cap B^q) \times B^{m-q}$ for every $P \in \mathcal{P}$, $\partial B^q \times B^{m-q} \subset \mathrm{Int}(P_\Delta \cap U)$, $\mathcal{P}$ restricted to $(M \setminus \mathrm{Int}\, B^m) \cap U$ and to $\partial B^m \cap U$ are partitions, and $(P \setminus \mathrm{Int}\, B^m) \cap U$ is a deformation retract of $(P \setminus \mathrm{Int}\, B^q) \cap U$ for every finite intersection P of $\mathcal{P}$.

Construction. Let us repeat a few simple observations noted in 4.5 which immediately follow from the fact that $\mathcal{T}$ is the second barycentric subdivision of a triangulation which underlies F and the elements of $\mathcal{P}$. Namely, we note that for every $\Delta \in \mathcal{T}_F$ the link S_Δ of Δ is a $(q-1)$-dimensional sphere lying in $\mathrm{Int}\, M$, and for different simplexes Δ_1 and Δ_2 in $\mathcal{T}_F$ the joins $\Delta_1 * S_{\Delta_1}$ and $\Delta_2 * S_{\Delta_2}$ do not intersect on U. We also note that for every subset H which is the union of a finite intersection of $\mathcal{P}$ such that $H \cap U \neq \emptyset$, and for every $\Delta \in \mathcal{T}_F$, we know that $(H \cap U) \setminus \mathrm{Int}(\Delta * S_\Delta)$ is a deformation retract of $H \cap U$ and hence $H \cap N$ is a deformation retract of $H \cap U$. Then N is an l-co-connected manifold (since N is a deformation retract of U) and $\mathcal{P}|N$ is an l-co-connected partition (since $\mathcal{P}|U$ is l-co-connected).

Consider M as embedded in a Hilbert space by an embedding which is linear on every simplex of $\mathcal{T}_0$.

Fix $\Delta \in \mathcal{T}_F$ and let $S^{q-1} = S_\Delta$ be the link of Δ with respect to $\mathcal{T}$, $S^{m-1} = \partial(\Delta * S_\Delta)$ and b be the barycenter of Δ.

Let us show that $\mathcal{P}$ restricted to N, to $S^{m-1} \setminus \partial\Delta$ and to $(\Delta * S_\Delta) \setminus \Delta$ are partitions. For a point $x \in S^{m-1} \setminus \partial\Delta$ choose an $\varepsilon > 0$ and a neighborhood G of x in $S^{m-1} \setminus \partial\Delta$ such that the map $(z,t) \mapsto z + tb$, $z \in G$, $t \in (-\varepsilon, \varepsilon)$, embeds $G \times (-\varepsilon, \varepsilon)$ into $M \setminus F$ and for every simplex Δ' of $\mathcal{T}_0$ that intersects $G \times (-\varepsilon, \varepsilon)$ we have $\Delta' \cap (G \times (-\varepsilon, \varepsilon)) = (\Delta' \cap G) \times (-\varepsilon, \varepsilon)$. Then for every $P \in \mathcal{P}$ that intersects $G \times (-\varepsilon, \varepsilon)$ we have $P \cap (G \times (-\varepsilon, \varepsilon)) = (P \cap G) \times (-\varepsilon, \varepsilon)$. Therefore, by 2.2, $\mathcal{P}$ restricted to G, to $G \times (-\varepsilon, 0]$ and to $G \times [0, \varepsilon)$ are partitions and hence $\mathcal{P}$ restricted to $S^{m-1} \setminus \partial\Delta$, to $(\Delta * S_\Delta) \setminus \Delta$ and to N are partitions as well.

Denote by $\mathcal{T}_\Delta$ the collection of the simplexes of $\mathcal{T}$ intersecting S^{q-1} and containing Δ. Let $\mathcal{T}'$ be the second barycentric subdivision of $\mathcal{T}$. For every $\Delta' \in \mathcal{T}_\Delta$ denote by $G_{\Delta'}$ the simplicial neighborhood of $\Delta' \cap S^{q-1}$ in $\Delta' \cap S^{m-1}$ with respect to $\mathcal{T}'$. In a way similar to constructing $\xi_{P'}$ in 5.2 we construct for every $\Delta' \in \mathcal{T}_\Delta$ a block bundle $\xi_{\Delta'}$ over $\Delta' \cap S^{q-1}$ such that $E(\xi_{\Delta'}) = G_{\Delta'}$ and $\xi_{\Delta'}|(\Delta'' \cap S^{m-1}) = \xi_{\Delta''}$ if $\Delta'' \subset \Delta'$. Then the block bundles $\xi_{\Delta'}$ define the corresponding block bundle ξ over S^{q-1} with $E(\xi)$

being the simplicial neighborhood of S^{q-1} in S^{m-1} with respect to $\mathcal{T}'$. Note that $E(\xi)$ is a regular neighborhood of S^{q-1} in ∂N.

Extend the embedding of S^{q-1} in N to a PL-embedding of a q-dimensional cube B_1^q into N such that $S^{q-1} = \partial B_1^q$ and $B_1^q \cap \partial N = S^{q-1}$, and note that the embedding of B_1^q is locally flat because $m \geq 2q+1$. Extend the regular neighborhood $E(\xi)$ of S^{q-1} in ∂N to a regular neighborhood of B_1^q in N, and by Theorem 4.3 of [10] represent this neighborhood as the total space $E(\nu)$ of a block bundle ν over B_1^q such that $\nu|S^{q-1} = \xi$. Since ν is a trivial block bundle, $E(\nu)$ is PL-homeomorphic to the product $E(\nu) = B_1^q \times B_1^{m-q}$ of B_1^q with an $(m-q)$-dimensional cube B_1^{m-q} such that for each pair (σ_ν, β_ν) of ν, $\beta_\nu = \sigma_\nu \times B_1^{m-q}$. Fix a triangulation $\mathcal{T}_\nu$ of $E(\nu)$ which underlies $\mathcal{T}$ restricted to $E(\nu)$. In a way similar to constructing the block bundle τ in 5.2 choose a point a in B_1^{m-q} and a cube B_2^{m-q} such that $a + B_2^{m-q} \subset \operatorname{Int} B_1^{m-q}$ and the pairs $(\sigma_\tau, \beta_\tau)$ defined by $\sigma_\tau = \Delta' \cap (B_1^q \times a)$, $\beta_\tau = \Delta' \cap (B_1^q \times (a + B_2^{m-q}))$, $\Delta' \in \mathcal{T}_\nu$, form a block bundle τ over $B_1^q \times a$ with $E(\tau) = B_1^q \times (a + B_2^{m-q})$ and $E(\tau|\partial(B_1^q \times a)) = E(\tau) \cap (\partial B_1^q \times (a + B_2^{m-q}))$ $(= E(\tau) \cap S^{m-1})$. Then the block bundle τ underlies $\mathcal{P}$ and $\mathcal{T}_\Delta$ restricted to $E(\tau)$.

Note that since the block bundle ν is trivial, the block bundle ξ is trivial as well and $E(\xi) = \partial B_1^q \times B_1^{m-q}$. Then it is obvious that the projection $\partial B_1^q \times a \to \partial B_1^q$ extends to a PL-homeomorphism $h : E(\xi) \to E(\xi)$ such that $h(x) = x$ for every $x \in \partial E(\xi)$, $h(\beta_\xi) = \beta_\xi$ for every block β_ξ of ξ and there is a PL-isotopy $H : E(\xi) \times [0,1] \to E(\xi) \times [0,1]$ relative to $\partial E(\xi)$ so that $H_0 = h$. Recall that the barycenter of Δ is denoted by b. Then, since $\partial\Delta \cap E(\xi) = \emptyset$, there is $\varepsilon > 0$ such that for every $x \in E(\xi)$, $P \in \mathcal{P}$ and $t \in [0,1]$, the point $b + (1 + t\varepsilon)(x - b)$ belongs to P if and only if x belongs to P. Embed $E(\xi) \times [0,1]$ into N by sending (x,t) to $b + (1 + t\varepsilon)(x - b)$ and define the PL-homeomorphism $g : N \to N$ such that g coincides with H on $E(\xi) \times [0,1]$ and g does not move the points outside $E(\xi) \times [0,1]$. From the properties described above it is clear that $g(\partial(B_1^q \times a)) = \partial B_1^q$ and the block bundle $g(\tau)$ underlies $\mathcal{P}$ and $\mathcal{T}_\Delta$ restricted to $E(g(\tau))$. Define $B_2^q = g(B_1^q \times a)$ and $\theta = g(\tau)$.

Thus we get a block bundle θ over a cube B_2^q such that $S^{q-1} = \partial B_2^q = \partial N \cap B_2^q$, θ underlies $\mathcal{P}$ and $\mathcal{T}_\Delta$ restricted to $E(\theta)$, and $E(\theta|\partial B_2^q) = S^{m-1} \cap E(\theta)$. Then, since θ is trivial, $E(\theta)$ is PL-homeomorphic to $B_2^q \times B_3^{m-q}$ such that $\partial N \cap (B_2^q \times B_3^{m-q}) = \partial B_2^q \times B_3^{m-q} \subset S^{m-1}$ and for each pair $(\sigma_\theta, \beta_\theta)$ of θ, $\beta_\theta = \sigma_\theta \times B_3^{m-q}$. Take a cube B_4^{m-q} such that $B_4^{m-q} \subset \operatorname{Int} B_3^{m-q}$ and define the block bundle ϱ over B_2^q by restricting the blocks of θ to $E(\varrho) = B_2^q \times B_4^{m-q}$. Using a reasoning similar to the one applied for proving the property $(**)$ in 5.2 we can show that $\mathcal{P}$ restricted $N \setminus (B_2^q \times \operatorname{Int} B_4^{m-q})$

and to $\partial((\Delta * S_\Delta) \cup (B_2^q \times B_4^{m-q})) \setminus \partial\Delta$ are partitions. It is clear that $E(\varrho) \subset \operatorname{Int} M$.

Take a pair $(\sigma_\varrho^\Delta, \beta_\varrho^\Delta)$ of ϱ such that $\dim \beta_\varrho^\Delta = m$ and let $P_\Delta \in \mathcal{P}$ be such that $\beta_\varrho^\Delta \subset P_\Delta$. Since ϱ is a trivial block bundle we can replace ϱ by another block bundle subdividing the cells of B_2^q into smaller cells and defining the corresponding blocks using the product structure. This way we may assume that the cells of B_2^q are so small that the pair $(\sigma_\varrho^\Delta, \beta_\varrho^\Delta)$ can be chosen so that $\beta_\varrho^\Delta \subset \operatorname{Int}(P_\Delta \cap N)$.

Let the cube $B_3^q \subset \Delta * S_\Delta$ be the join $B_3^q = b * S_\Delta$. Define a block bundle ψ over B_3^q as follows: the pairs $(\sigma_\varrho, \beta_\varrho)$ of ϱ with $\sigma_\varrho \subset S^{q-1}$ are the pairs of ψ, (b, Δ) is a pair of ψ and the rest of the pairs $(\sigma_\psi, \beta_\psi)$ of ψ are of the form $\sigma_\psi = \Delta' \cap B_3^q$, $\beta_\psi = \Delta'$, $\Delta' \in \mathcal{T}_\Delta$. One can easily check that ψ is indeed a block bundle and $E(\psi) = \Delta * S_\Delta$. Since ψ and ϱ coincide over S^{q-1}, they form the corresponding block bundle ϕ over the sphere $B_3^q \cup B_2^q$. Note that $B^q = (B_2^q \cup B_3^q) \setminus \operatorname{Int} \sigma_\varrho^\Delta$ is a q-dimensional cube. Define the block bundle η as ϕ restricted to B^q. Then (b, Δ) is a pair of η, $b \in \operatorname{Int} B^q$, $\partial B^q = \partial \sigma_\varrho^\Delta$, the intersections of the blocks of η with $U = M \setminus F$ underlie $\mathcal{P}$ restricted to $E(\eta) \cap U$, $E(\eta) \cap F = \Delta$ and hence η underlies F restricted to $E(\eta)$.

Thus assuming that the center O of B^q is located at b we can represent $E(\eta)$ as the product $B^m = B^q \times B^{m-q}$ such that $\Delta = 0 \times B^{m-q} = F \cap B^m$, $\partial B^q \times B^{m-q} \subset \operatorname{Int}(P_\Delta \cap U)$ and $P \cap U \cap B^m = (P \cap U \cap B^q) \times B^{m-q}$ for every $P \in \mathcal{P}$ that meets $B^m \cap U$. By a reasoning similar to the one applied in the proof of $({*}{*}{*})$ in 5.2 these properties imply that for every finite intersection P of $\mathcal{P}$, $P \setminus (F \cup \operatorname{Int} B^m)$ is a deformation retract of $P \setminus (F \cup \operatorname{Int} B^q)$. Since $B^m = (\Delta * S_\Delta) \cup ((B_2^q \setminus \operatorname{Int} \sigma_\varrho^\Delta) \times B_4^{m-q})$ and $\beta_\varrho^\Delta = \sigma_\varrho^\Delta \times B_4^{m-q}$ is contained in $\operatorname{Int}(P_\Delta \cap U)$, we derive from the properties of ϱ that $\mathcal{P}$ restricted to $M \setminus (F \cup \operatorname{Int} B^m)$ and to $\partial B^m \setminus F$ are partitions and $B^m \subset \operatorname{Int} M$.

Note that if S^{q-1} can be contracted to a point in an open subset V_Δ of N then the construction can be carried out so that B^m is contained in $(\Delta * S_\Delta) \cup V_\Delta$.

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