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An indecomposable Banach space of continuous functions which has small density

by

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Abstract. Using the method of forcing we construct a model for ZFC where CH does not hold and where there exists a connected compact topological space K of weight $\omega_1 < 2^\omega$ such that every operator on the Banach space of continuous functions on K is multiplication by a continuous function plus a weakly compact operator. In particular, the Banach space of continuous functions on K is indecomposable.

1. Introduction. In Banach space theory, several questions about complemented subspaces have been asked. Recall that a closed subspace Y of a Banach space X is *complemented* in X if there exists a closed subspace Z of X such that $X = Y \oplus Z$, where $\oplus$ means direct sum. For many years it remained an open problem if every infinite-dimensional Banach space X has infinite-dimensional closed subspaces Y and Z such that $X = Y \oplus Z$. When it occurs we say that X is *decomposable*. Since decompositions of Banach spaces are given by projections, indecomposable Banach spaces are related to the property of having few operators, in some sense.

In 1993 Gowers and Maurey [GM] constructed the first example of an indecomposable Banach space. Moreover, that space is *hereditarily indecomposable*, i.e., all its closed subspaces are indecomposable.

All operators on the space constructed by Gowers and Maurey have the form $cI + S$, where I is the identity operator, $c \in \mathbb{R}$ and S is *strictly singular*, i.e., the restriction of S to no infinite-dimensional closed subspace is an isomorphism onto its range.

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In [Ko2], Koszmider constructed the first example of an indecomposable Banach space of the form $C(K)$, i.e., the Banach space of continuous functions on a compact space K, with the supremum norm. All the operators on $C(K)$ are weak multipliers (see Definition 3.1). Assuming the continuum hypothesis, Koszmider constructed an indecomposable $C(K)$ space on which all operators have the form $gI + S$, where $g \in C(K)$, I is the identity operator and S is weakly compact. Plebanek [Pl] constructed in ZFC an indecomposable Banach space on which all operators have the above form. Operators of the form $gI + S$ with $g \in C(K)$ and S weakly compact are weak multipliers.

Unlike Gowers and Maurey's space, which is separable, the spaces built by Koszmider and Plebanek have density continuum. An indecomposable $C(K)$ cannot be separable, since a separable $C(K)$ contains a complemented copy of c_0. Neither can a $C(K)$ where all operators are weak multipliers be separable, since in [Ko2] it is shown that if all operators on $C(K)$ are weak multipliers then none of its proper subspaces or proper quotients is isomorphic to itself.

In this paper, using iterated forcing (see [Ku]) we prove that there exists consistently a Banach space $C(K)$ of density $\omega_1 < 2^\omega$ such that all operators on $C(K)$ have the form $gI+S$ for some $g \in C(K)$ and S weakly compact. The compact space K can be constructed either 0-dimensional or connected. In the latter case, $C(K)$ is indecomposable. We will only present the connected case, which is technically more complicated.

It is proved in [Fr] that under MA $+\neg$ CH every infinite compact space K of weight smaller than continuum contains a non-trivial converging sequence, which implies that $C(K)$ can be written as $c_0 \oplus Y$.

We say that a Banach space X has the *Grothendieck property* if a sequence in X^* converges in the weak topology iff it converges in the weak* topology. It is shown in [Sc] that a Banach space $C(K)$ has the Grothendieck property iff it does not contain a complemented copy of c_0. In particular, if $C(K)$ is indecomposable then it has the Grothendieck property. Furthermore, it is proved in [Ko2] that if $C(K)$ has few operators then it has the Grothendieck property.

The first consistent construction of a Banach space $C(K)$ of density smaller than continuum which has the Grothendieck property is due to Brech [Br]. The compact K constructed in [Br] is the Stone space of $\mathcal{P}(\omega) \cap M$, in a generic extension over a ground model M of ZFC. It is easy to verify that $C(K)$ has many operators.

2. Measures on a compact topological space. The purpose of this section is to fix some notations and to state some properties that we will use throughout this paper. Proposition 2.3 and Corollary 2.5 will be used in the proof of Lemma 5.7. Corollary 2.5 is an easy consequence of Proposition 2.4, rephrased for measures instead of functionals.

Throughout, by a *topological space* we mean a Hausdorff topological space.

Let K be a compact topological space. By a *measure on* K we mean a regular Borel measure of bounded variation. We define $M(K)$ to be the Banach space of measures on K with the norm $\|\mu\| = |\mu|(K)$.

If $\mathcal{B}$ is a topological basis for K, closed under finite unions and intersections, then a measure on K is uniquely determined by its restriction to $\mathcal{B}$. In other words, if $\mu|_{\mathcal{B}} = \nu|_{\mathcal{B}}$, then $\mu = \nu$.

For each $\alpha \le \omega_1$ we let $\mathcal{B}_\alpha$ be the set of all finite unions of open sets of the kind $\prod_{\beta<\alpha}(a_\beta, b_\beta) \cap [0,1]$, where $a_\beta, b_\beta \in \mathbb{Q}$ and $\{\beta < \alpha : (a_\beta, b_\beta) \cap [0,1] \neq [0,1]\}$ is finite. It is clear that $\mathcal{B}_\alpha$ is a basis for $[0,1]^\alpha$ and it is closed under finite unions and intersections. Moreover, if $K \subseteq [0,1]^\alpha$ is compact, then $\{V \cap K : V \in \mathcal{B}_\alpha\}$ is also a basis for K closed under finite unions and inteserctions. Since we may identify measures on K with measures on $[0,1]^\alpha$ whose variations are null on $[0,1]^\alpha \setminus K$, we may interpret a measure on a subspace of $[0,1]^\alpha$ as a function from $\mathcal{B}_\alpha$ into $\mathbb{R}$.

Definition 2.1. Let K be a compact topological space and let $\varepsilon > 0$. We say that a bounded set $S \subseteq M(K)$ is *ε-weakly relatively compact* if for every sequence $(V_n)_{n\in\omega}$ of pairwise disjoint open subsets of K,

$$\sup\{|\mu(V_n)| : \mu \in S\} \le \varepsilon$$

for all but finitely many $n \in \omega$.

We state the Dieudonné–Grothendieck theorem (see [Di, VII, 14]) rephrased by use of Definition 2.1.

Theorem 2.2 (Dieudonné–Grothendieck Theorem). *For any compact topological space K and bounded $S \subseteq M(K)$ we have:*

(a) *S is weakly relatively compact iff S is ε-weakly relatively compact for all $\varepsilon > 0$.*
(b) *Given $\varepsilon > 0$, S is not ε-weakly relatively compact iff there exist a sequence $(\mu_n)_{n\in\omega}$ in S and a pairwise disjoint sequence $(V_n)_{n\in\omega}$ of open subsets of K such that $|\mu_n(V_n)| > \varepsilon$ for all $n \in \omega$.*

We say that a sequence $(\mu_n)_{n\in\omega}$ of measures on K is *pairwise disjoint* iff there exists a pairwise disjoint sequence $(A_n)_{n\in\omega}$ of Borel sets such that $|\mu_n|(K \setminus A_n) = 0$ for all n. A sequence of L^1 functions is *pairwise disjoint* if they have pairwise disjoint supports.

Proposition 2.3 (Pełczyński). *Suppose that K is a compact topological space, $\varepsilon > 0$ and $(\mu_n)_{n\in\omega}$ is a pairwise disjoint sequence of probability measures on K. Then there exist a subsequence $(\mu'_n)_{n\in\omega}$ of $(\mu_n)_{n\in\omega}$ and a pair-*

wise disjoint sequence $(U_n)_{n\in\omega}$ *of open subsets of* K *such that* $\mu'_n(U_n) \geq 1-\varepsilon$ *for all* $n \in \omega$.

PROPOSITION 2.4 (Kadec, Pełczyński). *Suppose that* (X, Σ, μ) *is a measure space with a probability measure* μ. *Let* $(v_n)_{n\in\omega}$ *be a bounded sequence of elements of* $L^1(\mu)$. *Then there exist a subsequence* $(v'_n)_{n\in\omega}$ *of* $(v_n)_{n\in\omega}$, *a weakly converging sequence* $(g_n)_{n\in\omega} \subseteq L^1(\mu)$ *and a pairwise disjoint sequence* $(h_n)_{n\in\omega} \subseteq L^1(\mu)$ *such that* $v'_n = g_n + h_n$ *for all* n.

Propositions 2.3 and 2.4 follow from the proof of Lemma 1 of [Pe1] and Theorem 6 of [KP], respectively. We use the versions presented in [Ta] (Lemmas 1 and 2). The following result is a corollary of Proposition 2.4.

COROLLARY 2.5. *Let* $(\mu_n)_{n\in\omega}$ *be a bounded sequence in* $M(K)$ *for a compact space* K. *Then there exist a subsequence* $(\mu'_n)_{n\in\omega}$ *of* $(\mu_n)_{n\in\omega}$, *a weakly converging sequence* $(\lambda_n)_{n\in\omega}$ *and a pairwise disjoint sequence* $(\nu_n)_{n\in\omega}$ *such that* $\mu'_n = \nu_n + \lambda_n$ *for all* n.

Proof. Take $\mu = \sum_{n=0}^{\infty} |\mu_n|/\|\mu_n\|2^{n+1}$. It is clear that μ is a probability measure.

Define $i : L_1(\mu) \to M(K)$ as $i(h)(E) = \int_E h\,d\mu$ for all $h \in L_1(\mu)$ and all Borel $E \subseteq K$. One can easily verify that i is an isometry onto its range. Using the Radon–Nikodým theorem ([Ru, 6.10]), for each $n \in \omega$ we find a unique $h_n \in L^1(\mu)$ such that $i(h_n) = \mu_n$. It is also easy to see that i maps pairwise disjoint sequences in $L_1(\mu)$ into pairwise disjoint sequences in $M(K)$, and weakly converging sequences in $L_1(\mu)$ into weakly converging sequences in $M(K)$.

Hence, applying Proposition 2.4 for $(h_n)_{n\in\omega}$ concludes the proof. ∎

3. Weak multipliers. The definition of weak multipliers first appears in [Ko2]. In this section we cite the main results about weak multipliers.

DEFINITION 3.1 ([Ko2, 2.1]). An operator $T : C(K) \to C(K)$ is called a *weak multiplier* if for every bounded sequence $(e_n)_{n\in\omega}$ of pairwise disjoint elements of $C(K)$ (i.e., $e_n \cdot e_m = 0$ for $n \neq m$) and any sequence $(x_n)_{n\in\omega} \subseteq K$ such that $e_n(x_n) = 0$ we have

$$\lim_{n\to\infty} T(e_n)(x_n) = 0.$$

Let us recall that $Y \subseteq X$ is C^*-*embedded* in X iff every bounded continuous function on Y can be extended to a bounded continuous function on X.

LEMMA 3.2 ([Ko2, 2.8]). *Suppose that* K *is a compact space with no disjoint open subsets* U_1 *and* U_2 *such that* $\overline{U}_1 \cap \overline{U}_2$ *is singleton. Then for every* $x \in K$ *the space* $K \setminus \{x\}$ *is* C^*-*embedded in* K.

THEOREM 3.3 ([Ko2, 2.7]). *The following are equivalent for a compact space K:*

(a) *All operators $T : C(K) \to C(K)$ are of the form $gI + S$ where $g \in C(K)$ and S is weakly compact.*
(b) *All operators on $C(K)$ are weak multipliers and for every $x \in K$ the space $K \setminus \{x\}$ is C^*-embedded in K.*

The following lemma is an adaptation of Lemma 2.5 of [Ko2].

LEMMA 3.4. *Let K be a compact and connected space such that all operators on $C(K)$ have the form $gI + S$, where $g \in C(K)$ and S is a weakly compact operator. Then $C(K)$ is an indecomposable Banach space.*

Proof. Let K be as in the hypothesis and suppose that X and Y are closed subspaces of $C(K)$ such that $C(K) = X \oplus Y$. We will prove that X or Y is finite-dimensional.

Let $P : C(K) \to C(K)$ be a projection such that $\mathrm{Im}(P) = X$ and $\mathrm{Ker}(P) = Y$. Fix $g \in C(K)$ and a weakly compact operator S such that $P = gI + S$. Since $P^2 = P$ we have $P^2 I + S^2 + gS + Sg = gI + S$. Thus $S' = (g^2 - g)I$ is weakly compact, and therefore strictly singular (see [Pe2]). If $(g^2 - g)(x) \neq 0$ for some $x \in K$ we find an open neighbourhood V of x such that $|(g^2 - g)(y)| > \varepsilon$ for some $\varepsilon > 0$ and every $y \in \overline{V}$. Let Z be the subspace of $C(K)$ consisting of all continuous functions on K with supports in V. Since K is connected, it does not have isolated points and so Z is infinite-dimensional. But $S'|_Z$ is an isomorphism onto its range, since $(g^2 - g)^{-1}$ is well defined and continuous in $\overline{V}$ and determines the inverse operator of S', contradicting the fact that S' is strictly singular.

Thus $(g^2 - g)(x) = 0$ for all $x \in K$, which implies that $g(x) \in \{0, 1\}$ for all $x \in K$. By the connectedness of K we have $g \equiv 0$ or $g \equiv 1$. Therefore $P = S$ or $P = I + S$, which means that P or $I - P$ is weakly compact. In the first case $P|_{\mathrm{Im}(P)}$ is an isomorphism onto its range and hence $\mathrm{Im}(P)$ is finite-dimensional. In the second case $\mathrm{Ker}(P)$ is finite-dimensional. ∎

LEMMA 3.5. *Suppose that K is compact, D is a dense subset of K and $T : C(K) \to C(K)$ is not a weak multiplier. Then there exists a sequence $(x_n)_{n\in\omega}$ in D such that for every bounded Borel function $f : K \to \mathbb{R}$, the set $\{T^*(\delta_{x_n}) - f\delta_{x_n} : n \in \omega\}$ is not weakly relatively compact in $M(K)$, where $f\delta_x = f(x)\delta_x \in M(K)$.*

Proof. As T is not a weak multiplier, there exists a bounded pairwise disjoint sequence $(e_n)_{n\in\omega}$ of continuous functions from K into $\mathbb{R}$, a sequence $(x_n)_{n\in\omega}$ of distinct points of K and $\varepsilon > 0$ such that $e_n(x_n) = 0$ and $|T(e_n)(x_n)| > \varepsilon$ for all n. Since D is dense in K, we may assume that

$x_n \in D$. Set $\mu_n = T^*(\delta_{x_n}) - f\delta_{x_n}$. We have

$$T(e_n)(x_n) = \int e_n \, dT^*(\delta_{x_n}) = \int e_n \, d\mu_n + \int e_n \, d(f\delta_{x_n}).$$

Since $e_n(x_n) = 0$, we have $\int e_n \, d(f\delta_{x_n}) = 0$, and hence

$$|T(e_n)(x_n)| = \left| \int e_n \, d\mu_n \right| > \varepsilon \quad \text{for all } n.$$

Let M be a real number such that $\|e_n\| < M$ for all n. We have $|\mu_n|(\mathrm{supp}(e_n)) > \varepsilon/M$. By the definition of variation we find Borel sets $U_n \subseteq \mathrm{supp}(e_n)$ such that $|\mu_n(U_n)| > \varepsilon/2M$. By regularity of μ_n we may assume that U_n are pairwise disjoint open sets. Since T and f are bounded, $(\mu_n)_{n\in\omega}$ is a bounded sequence in $M(K)$. By the Dieudonné–Grothendieck theorem we conclude that $(\mu_n)_{n\in\omega}$ is not weakly relatively compact. ∎

4. Construction of a forcing. In this section we introduce a forcing $R(K)$ for every connected first countable compact space K. The idea of this construction is based on the forcing $R(A)$ defined in Section 6 of [Ko1], where A is a countable boolean subalgebra of $\mathcal{P}(\mathbb{N})$. The generic extension adds a new set $g \subseteq \mathbb{N}$ to obtain the boolean algebra generated by $A \cup \{g\}$ used in the next step of the iteration. Following the idea of [Ko2], we replace the countable boolean algebra A by a first countable compact space K, in order to obtain connectedness. Instead of adding a new element to the algebra A, we add a new continuous real function on K, taking the closure of the graph of a continuous function defined on a dense open subset of K. This new function separates a weak* converging sequence of measures in K (Lemma 4.6 makes it clear), eliminating one undesirable operator on $C(K)$.

Fix $\alpha < \omega_1$ and a compact set $K \subseteq [0,1]^\alpha$ with no isolated points. Let $\mathcal{B}_\alpha$ be a basis for $[0,1]^\alpha$ as defined previously, at the beginning of Section 2.

For $\alpha < \beta$ we interpret $\mathcal{B}_\alpha$ as a subset of $\mathcal{B}_\beta$, identifying $V \in \mathcal{B}_\alpha$ with $V \times [0,1]^{\beta\setminus\alpha} \in \mathcal{B}_\beta$. Thus, if μ is a measure on $[0,1]^\beta$ and ν is a measure on $[0,1]^\alpha$, we say that $\mu|_{\mathcal{B}_\alpha} = \nu$ when $\mu(\pi_\alpha^{-1}[E]) = \nu(E)$ for all Borel $E \subseteq [0,1]^\alpha$.

For a function f we denote by $\mathrm{Gr}(f)$ the graph of f.

We define a forcing $R(K)$ consisting of the conditions $p = (f_p, \Omega_p, M_p, \varepsilon_p, \Delta_p)$ such that:

A.1. $f_p : K \to [0,1]$ is continuous;
A.2. $\Omega_p \in \mathcal{B}_\alpha \setminus \{[0,1]^\alpha\}$;
A.3. $\mathrm{supp}(f_p) \subseteq K \cap \Omega_p$;
A.4. M_p is a finite set of positive measures on K;
A.5. $\varepsilon_p \in \mathbb{Q} \cap (0,\infty)$;
A.6. $\mu(\overline{\Omega}_p) < \varepsilon_p$ for all $\mu \in M_p$;
A.7. $\Delta_p \in \mathcal{B}_{\alpha+1}$;

A.8. $\pi_\alpha[\overline{\Delta}_p] = \overline{\Omega}_p$;
A.9. $\mathrm{Gr}(f_p|_{\overline{\Omega}_p\cap K}) \subseteq \overline{\Delta}_p$.

The order $\le$ on $R(K)$ is given by $q \le p$ if and only if

B.1. $\Omega_q \supseteq \Omega_p$;
B.2. $M_q \supseteq M_p$;
B.3. $\varepsilon_q \le \varepsilon_p$;
B.4. $\overline{\Delta}_q \cap (\overline{\Omega}_p \times [0,1]) \subseteq \overline{\Delta}_p$.

Given $p \in R(K)$ we define

$$\mathrm{diam}(\Delta_p) = \sup\{|y_1 - y_2| : \exists x \in \Omega_p\ (\{(x,y_1),(x,y_2)\} \subseteq \Delta_p)\}$$

when $\Delta_p \neq \emptyset$. Otherwise we define $\mathrm{diam}(\Delta_p) = 0$. It is easy to see that for every $p \in R(K)$ and every $q, r \le p$ we have

$$(*) \qquad \forall x \in \overline{\Omega}_p \cap K\ (|f_q(x) - f_r(x)| \le \mathrm{diam}(\Delta_p)).$$

LEMMA 4.1. *For every $p \in R(K)$ and $\varepsilon > 0$ there exists $q \le p$ such that $f_q = f_p$, $\Omega_q = \Omega_p$, $M_q = M_p$, $\varepsilon_q = \varepsilon_p$ and $\mathrm{diam}(\Delta_q) \le \varepsilon$.*

Proof. Using the Tietze theorem, we find $f : [0,1]^\alpha \to [0,1]$ continuous which extends f_p. Let L be the graph of f. By the continuity of f, for each $x \in L$ there exists $V_x \in \mathcal{B}_\alpha$ such that $x \in V_x$ and $|f(y) - f(x)| < \varepsilon/8$ for all $y \in V_x$. Fix an open interval I_x in $[0,1]$ with rational endpoints such that

$$(f(x) - \varepsilon/8, f(x) + \varepsilon/8) \cap [0,1] \subseteq I_x \subseteq (f(x) - \varepsilon/4, f(x) + \varepsilon/4) \cap [0,1].$$

Set $W_x = V_x \times I_x \in \mathcal{B}_{\alpha+1}$. By the construction, $(y, f(y)) \in W_x$ for all $y \in V_x$. So $\{W_x : x \in [0,1]^\alpha\}$ is an open cover of L. Since L is closed in $[0,1]^{\alpha+1}$, as the graph of a continuous function, and so is compact, we can take a finite set $F \subseteq [0,1]^\alpha$ such that $L \subseteq \bigcup_{x\in F} W_x$. Define $\Delta_q = \Delta_p \cap \bigcup_{x\in F} W_x$. Since $L \subseteq \bigcup_{x\in F} W_x$, condition A.9 is satisfied. From the construction of Δ_q we have $\Delta_q \in \mathcal{B}_{\alpha+1}$, proving A.7. From A.9 and the fact that $\Delta_q \subseteq \Delta_p$ we have A.8 and B.4. It remains to prove that $\mathrm{diam}(\Delta_q) \le \varepsilon$.

Let $(x,y_1),(x,y_2) \in \Delta_q$. Choose $x_1, x_2 \in F$ such that $(x,y_1) \in W_{x_1}$ and $(x,y_2) \in W_{x_2}$. Since $y_1 \in I_{x_1}$ we have $|y_1 - f(x_1)| < \varepsilon/4$, because $I_{x_1} \subseteq (f(x_1) - \varepsilon/4, f(x_1) + \varepsilon/4)$. Since $x \in V_{x_1}$, we have $|f(x) - f(x_1)| < \varepsilon/8$. So

$$|y_1 - f(x)| \le |y_1 - f(x_1)| + |f(x_1) - f(x)| \le \varepsilon/4 + \varepsilon/8 < \varepsilon/2.$$

Analogously we conclude that $|y_2 - f(x)| < \varepsilon/2$ and therefore $|y_1 - y_2| < \varepsilon$. ∎

LEMMA 4.2. *Suppose that we have a compact set $K \subseteq [0,1]^\alpha$, a forcing condition $p \in R(K)$, an open set $\Omega \in \mathcal{B}_\alpha$, a finite set M of positive measures on $[0,1]^\alpha$, a continuous function $f : K \to [0,1]$ with support in $\Omega \cap K$ and a positive rational number ε. Suppose that $\Omega \supseteq \Omega_p$, $\mathrm{Gr}(f|_{\Omega_p\cap K}) \subseteq \Delta_p$,*

$M \supseteq M_p$ and $\varepsilon \geq \varepsilon_p$, and $\mu(\Omega) < \varepsilon$ for all $\mu \in M$. Then there exists $q \leq p$ such that $f_q = f$, $\Omega_q = \Omega$, $M_q = M$ and $\varepsilon_q = \varepsilon$.

Proof. Define $\Delta_q = (\Omega_q \smallsetminus \overline{\Omega}_p) \times [0,1] \cup \Delta_p$. It is easy to verify that $\Omega_q \smallsetminus \overline{\Omega}_p \in \mathcal{B}_\alpha$, and conditions A.7, A.8, A.9 and B.4 are clearly satisfied. The other conditions follow immediately from the hypothesis. ∎

LEMMA 4.3. *Suppose that $K \subseteq [0,1]^\alpha$ is compact for $\alpha < \omega_1$. Then*

$$\forall \varepsilon > 0\, \forall p \in R(K)\, \exists q \leq p\, \forall p_1, p_2 \leq q\, \forall x \in \Omega_q \cap K\ (|f_{p_1}(x) - f_{p_2}(x)| < \varepsilon).$$

Proof. Fix $\varepsilon > 0$ and $p \in R(K)$. By Lemma 4.1, $D_\varepsilon = \{q \in P : \operatorname{diam}(\Delta_p) < \varepsilon\}$ is dense in $R(K)$. Hence there exists $q \leq p$ such that $\operatorname{diam}(\Delta_q) < \varepsilon$. From $(*)$ it follows that for all $p_1, p_2 \leq p$ and $x \in \Omega_p \cap K$ we have $|f_{p_1}(x) - f_{p_2}(x)| < \varepsilon$. ∎

Let M be a transitive standard model for ZFC and take $R(K) \in M$. Let G be an $R(K)$-generic over M. We note that, since $\mathbb{R}^M \neq \mathbb{R}^{M[G]}$, the space K may not be compact in $M[G]$, and the elements of $\mathcal{B}_\alpha$ may change too. But using the definition at the beginning of Section 2 each element of $\mathcal{B}_\alpha$ may be characterized by a finite collection of finite sets of triples (β, a, b), where $\beta < \alpha$ and a, b are rational numbers such that $0 \leq a < b \leq 1$. With this characterization, elements of $\mathcal{B}_\alpha$ are absolute sets (see [Ku] for absoluteness). Thus, for $V \in \mathcal{B}_\alpha$, we denote both V^M and $V^{M[G]}$ by V, unless it is not clear from context. We note that $V^{M[G]} \cap \overline{K} = V^M \cap K$.

Let $p \in R(K)$. Since f_p is continuous in the compact set K, in the ground model M, it is uniformly continuous in M. Therefore it is uniformly continuous in $M[G]$, because uniform continuity is absolute for transitive models. Hence, in $M[G]$ we may extend f_p continuously to a function $\widetilde{f}_n : \overline{K} \to [0,1]$ by defining $\widetilde{f}_p(x) = \lim_{n \in \omega} f_p(x_n)$ for $x_n \in K$ such that $x_n \xrightarrow{n} x$. Let $\dot{f}_p$ be an $R(K)$-name for $\widetilde{f}_p$.

We recall the definition of limit for directed systems: if F is a filter over a partial order P, $(x_p)_{p \in F} \subseteq \mathbb{R}$ and $x \in \mathbb{R}$, we say that $\lim_{p \in F} x_n = x$ if for all $\varepsilon > 0$ there exists $p \in F$ such that $|x_q - x| < \varepsilon$ for all $q \leq p$. Completeness of $\mathbb{R}$ implies that a directed system $(x_p)_{p \in F}$ converges to some $x \in \mathbb{R}$ iff for all $\varepsilon > 0$ there exists $p \in F$ such that $|x_q - x_r| < \varepsilon$ for all $q, r \leq p$.

In $M[G]$ define $\Omega_G = \bigcup_{p \in G} \Omega_p$ and let $f_G : \Omega_G \cap \overline{K} \to [0,1]$ be given by

$$f_G(x) = \lim_{p \in G} \widetilde{f}_p(x).$$

It follows from Lemma 4.3 and the genericity of G that f_G is well-defined. Let K_G be the closure of the graph of f_G. Let $\dot{f}_G$ and $\dot{K}_G$ be the $R(K)$-names for f_G and K_G, respectively.

LEMMA 4.4. *Suppose that $K \subseteq [0,1]^\alpha$ is compact with no isolated points in the ground model M, and let G be an $R(K)$-generic over M. Then the* *following statements hold in* $M[G]$:

(a) $\Omega_G \cap \overline{K}$ *is dense in* $\overline{K}$;

(b) f_G *is continuous*;

(c) *if* K *is connected in* M, *then* K_G *is connected in* $M[G]$.

Proof. Working in M, we will show that given $x \in K$, an open neighbourhood V of x belonging to $\mathcal{B}_\alpha$ and $p \in P$, there exists $q \leq p$ such that $V \cap \Omega_q \neq \emptyset$. This is enough to prove (a), using, in $M[G]$, the genericity of G and the fact that K is dense in $\overline{K}$.

If $x \in \overline{\Omega}_p$, take q to be p except that $\Omega_q \supseteq \overline{\Omega}_p$ preserving the condition $\mu(\Omega_q) < \varepsilon_p$ for all $\mu \in M_p$ (this is possible by the regularity of μ), and use Lemma 4.2 to obtain Δ_q. If $x \notin \overline{\Omega}_p$, pick an open neighbourhood $W \in \mathcal{B}_\alpha$ of x included in V and disjoint from $\overline{\Omega}_p$. Since $V \cap K$ is a non-empty open set in a compact space with no isolated points, it is uncountable. So there exists $y \in V$ such that $\mu(\{y\}) = 0$ for all $\mu \in M_p$. Fix $\gamma = \min\{\varepsilon_p - \mu(\Omega_p) : \mu \in M_p\}$. Select an open neighbourhood $U \in \mathcal{B}_\alpha$ of y included in W such that $\mu(U) < \gamma$ for all $\mu \in M_p$. Pick $U' \in \mathcal{B}_\alpha$ such that $x \in U' \subseteq \overline{U'} \subseteq U$. Define $q = (f_p, \Omega_p \cup U', M_p, \varepsilon_p, \Delta_q)$, where Δ_q is obtained by Lemma 4.2.

Let us prove (b). Working in $M[G]$, fix $x \in \Omega_G \cap \overline{K}$ and $\varepsilon > 0$. Choose $p \in G$ such that $x \in \Omega_p$ and $|\widetilde{f}_{p_1}(y) - \widetilde{f}_{p_2}(y)| < \varepsilon/3$ for all $y \in \Omega_p \cap \overline{K}$ and $p_1, p_2 \leq p$ (using Lemma 4.3). By the continuity of $\widetilde{f}_p$, let $V \subseteq \Omega_p$ be an open neighbourhood of x such that $|\widetilde{f}_p(y) - \widetilde{f}_p(x)| < \varepsilon/3$ for all $y \in V \cap \overline{K}$. Thus for all $q \leq p$,

$$|\widetilde{f}_q(y) - \widetilde{f}_q(x)| \leq |\widetilde{f}_q(y) - \widetilde{f}_p(y)| + |\widetilde{f}_p(y) - \widetilde{f}_p(x)| + |\widetilde{f}_p(x) - \widetilde{f}_q(x)| < \varepsilon,$$

implying that $|f_G(y) - f_G(x)| \leq \varepsilon$ (because we can take $q \leq p$ such that $|f_G(y) - f_q(y)|$ and $|f_G(x) - f_q(x)|$ are sufficiently small), proving the continuity of f_G at x.

To prove (c) we will first see that if K is connected in M then $\overline{K}$ is connected in $M[G]$. If $\overline{K}$ is not connected, there exist open sets U and V in $[0,1]^\alpha$ such that $\overline{K} \cap U \cap V = \emptyset$, $\overline{K} \subseteq U \cup V$, $\overline{K} \cap U \neq \emptyset$ and $\overline{K} \cap V \neq \emptyset$. Therefore $K \cap U \cap V = \emptyset$, $K \subseteq U \cup V$, $K \cap U \neq \emptyset$ and $K \cap V \neq \emptyset$. By the compactness of $\overline{K}$ we may assume that $U, V \in \mathcal{B}_\alpha$. Since elements of $\mathcal{B}_\alpha$ are determined by finite rational coordinates, $K \cap U^M \cap V^M = \emptyset$, $K \subseteq U^M \cup V^M$, $K \cap U^M \neq \emptyset$ and $K \cap V^M \neq \emptyset$, contradicting the connectedness of K in M.

Having proved that $\overline{K}$ is connected, we now show that for every $x \in \overline{K} \setminus \Omega_G$, every open neighbourhood V of x, $r \in [0,1] \cap \mathbb{Q}$ and $n \in \omega$ there exists $y \in V \cap K \cap \Omega_G$ such that $|f_G(y) - r| < 1/n$. This will imply that $\pi^{-1}_{K_G,K}(x) = \{x\} \times [0,1]$. We may assume that $x \in K$, taking some $x' \in V \cap K \setminus \Omega_G$ instead of x. We note that $V \cap K \setminus \Omega_G \neq \emptyset$, because if $V \cap K \subseteq \Omega_G$, we may assume that $V^M \subseteq \Omega_p$ for some $p \in G$, which implies that $x \in \Omega_G$.

Working in M, given $x \in K$, a neighbourhood $V \in \mathcal{B}_\alpha$ of x, $r \in [0,1] \cap \mathbb{Q}$ and $p \in R(K)$, we will show that there exists $q \le p$ such that $\operatorname{diam}(\Delta_q) \le 1/n$ and either $x \in \Omega_q$ or there exists $y \in V \cap \Omega_q$ such that $|f_q(y) - r| < 1/n$. If $x \in \overline{\Omega}_p$, by the regularity of the measures we find an open set W such that $\overline{\Omega}_p \subseteq W$ and $\mu(W) < \varepsilon_p$ for all $\mu \in M_p$. Select $\Omega_q \in \mathcal{B}_\alpha$ such that $\overline{\Omega}_p \subseteq \Omega_q \subseteq \overline{\Omega}_q \subseteq W$, $M_q = M_p$, $\varepsilon_q = \varepsilon_p$ and $f_q = f_p$ and use Lemmas 4.2 and 4.1 to obtain Δ_q such that $q \in R(K)$ and $\operatorname{diam}(\Delta_q) < 1/n$. If $x \notin \overline{\Omega}_p$ we choose an open neighbourhood $W \subseteq V$ of x disjoint from $\overline{\Omega}_p$. Consider $y \in W$ such that $\mu(\{y\}) = 0$ for all $\mu \in M_p$. Let U be an open neighbourhood of y included in W such that $\mu(U) < \varepsilon_p - \mu(\Omega_p)$ for all $\mu \in M_p$. Define $\Omega_q = \Omega_p \cup U$. By the Tietze theorem we find $f_q : \overline{K} \to [0,1]$ with support in Ω_q such that $f_q|_{\Omega_p} = f_p$ and $f_q(y) = r$. Define $\varepsilon_q = \varepsilon_p$ and $M_q = M_p$ and use Lemmas 4.1 and 4.2 to obtain Δ_q such that $\operatorname{diam}(\Delta_q) \le 1/n$.

We conclude that, in $M[G]$, if $x \in \overline{K} \setminus \Omega_G$ then $\pi^{-1}_{K_G,K}(x) = \{x\} \times [0,1]$. By items (a) and (b), for $x \in \Omega_G$ we have $\pi^{-1}_{K_G,K}(x) = \{(x, f_G(x))\}$. Therefore $\pi^{-1}_{K_G,K}(x)$ is connected for all $x \in K$, which easily implies that K_G is connected. ∎

From Lemma 4.4(a) we conclude that $\pi_\alpha[K_G] = \overline{K}$.

Lemma 4.5. *For any $\alpha < \omega_1$ and any compact set $K \subseteq [0,1]^\alpha$ with no isolated points, the forcing $R(K)$ is c.c.c.*

Proof. Let $(p_\xi : \xi < \omega_1)$ be an uncountable family of conditions of $R(K)$. We may assume that Ω_{p_ξ}, ε_{p_ξ} and Δ_{p_ξ} are constant with respect to ξ; we will denote them, respectively, by Ω, ε and Δ. Given $\xi, \eta < \omega_1$ define $p = (f_{p_\xi}, \Omega, M_{p_\xi} \cup M_{p_\eta}, \varepsilon, \Delta)$. It is easy verify that $p \in R(K)$ and $p \le p_\xi, p_\eta$. ∎

Lemma 4.6. *Let $\alpha < \omega_1$, $K \subseteq [0,1]^\alpha$ be a compact set with no isolated points, $\varepsilon > 0$, μ a positive measure and $p \in R(K)$ such that $\varepsilon_p \le \varepsilon$ and $\mu \in M_p$. Let $(\mu_n)_{n\in\omega} \subseteq M([0,1]^\alpha)$ and $(x_n)_{n\in\omega} \subseteq K$ be such that $\mu_n(\{x_n\}) \xrightarrow{n} 0$, $(|\mu_n|)_{n\in\omega}$ weak* converges to μ, $\{\mu_n : n \in \omega\}$ is not 5ε-weakly relatively compact, and there exists a pairwise disjoint sequence $(A_n)_{n\in\omega} \subseteq \mathcal{B}_\alpha$ such that $\|\mu_n\| - |\mu_n|(A_n) < \frac{\varepsilon}{18}\|\mu_n\|$. Then there exist $\delta_1 > \delta_2 > 0$ such that for all $k \in \omega$ there exist $q \le p$ and $n_1, n_2 > k$ such that*

(i) $|\int_K f_q \, d\mu_{n_1}| > \delta_1$;
(ii) $|\int_K f_q \, d\mu_{n_2}| < \delta_2$;
(iii) $|\mu_{n_1}|(K \setminus \Omega_q) < (\delta_1 - \delta_2)/3$;
(iv) $|\mu_{n_2}|(K \setminus \Omega_q) < (\delta_1 - \delta_2)/3$;
(v) $x_{n_1}, x_{n_2} \in \Omega_q$, $|f_q(x_{n_1}) - f_q(x_{n_2}| < 1/k$;
(vi) $\operatorname{diam}(\Delta_q) \le 1/k$.

Proof. By Definition 2.1, passing to a subsequence we may assume that there exists a pairwise disjoint sequence $(W_n)_{n\in\omega} \subseteq \mathcal{B}_\alpha$ such that $|\mu_n(W_n)|$

$> 5\varepsilon$ for all $n \in \omega$. Since $(\mu_n)_{n\in\omega}$ is not weakly convergent (by Theorem 2.2), $\|\mu_n\|$ does not converge to 0. Passing to a subsequence we assume that $\|\mu_n\|$ converges to $r > 0$.

Define $\delta_1 = 5\varepsilon/3r$ and $\delta_2 = 3\varepsilon r/2$. To simplify the notation we will assume that $r = 1$, substituting μ_n by μ_n/r for all n, and μ by μ/r. Passing to a subsequence and using the hypothesis we assume that $|\mu_n|(A_n) > 1-\varepsilon/18$ for all n.

Fix $\delta > 0$ such that $\delta < (\varepsilon_p - \nu(\Omega_p))/6$ for all $\nu \in M_p$. Such a δ exists by the definition of $R(K)$. Using the Rosenthal lemma ([Di, p. 82]) and the hypothesis that $\mu_n(\{x_n\})$ converges to 0, passing to a subsequence we assume that $|\mu_n(\{x_m\})| < \delta$ for all $n, m \in \omega$.

Fix $k \in \omega$. Since $|\mu_n|$ converges weak* to μ, and $\mu(\Omega_p) < \varepsilon$, we choose $k_0 \geq k$ such that $|\mu_n|(\Omega_p) < \varepsilon$ for all $n \geq k_0$, and therefore

$$\Big|\int f_p \, d\mu_n\Big| < \varepsilon.$$

Setting $U_n = W_n \setminus \Omega_p$ we have, for all $n > k_0$,

$$|\mu_n(U_n)| > 5\varepsilon - \varepsilon = 4\varepsilon.$$

But

$$|\mu_n|(K \setminus A_n) < \varepsilon/18,$$

which implies that $|\mu_n(U_n \cap A_n)| > 3\varepsilon$. Define $B_n = U_n \cap A_n$.

Since A_n's are pairwise disjoint, we can find $k_1 \geq k_0$ such that $\mu(A_n) < \delta$ and $\nu(A_n) < \delta$ for all $n > k_1$ and $\nu \in M_p$. Thus, noting that $\nu(\Omega_p) \leq \varepsilon_p - 4\delta$, for all $\nu \in M_p$, we have $\nu(\Omega_p \cup A_n \cup A_j) < \varepsilon_p - 2\delta$ for all $n, j > k_1$ and $\nu \in M_p$.

Now we will take care of item (v). Passing to a subsequence, we assume that x_n converges to $x \in K$. If $x \in \overline{\Omega}_p$ we may assume that $x \in \Omega_p$, extending p to p' such that $\overline{\Omega}_p \subseteq \Omega_{p'}$ (using regularity of measures). In this case, passing to a subsequence, we assume that $x_n \in \Omega_p$ for all n, and define $C_n = \emptyset$. If $x \notin \overline{\Omega}_p$, we assume that $x_n \notin \overline{\Omega}_p$ for all n and $|\nu(\{x_n\})| < \delta$ for all $\nu \in M_p$. Select $C_n \in \mathcal{B}_\alpha$ disjoint from $\overline{\Omega}_p$ such that $x_n \in C_n$, $\nu(C_n) < \delta$ for all $\nu \in M_p$, and $|\mu_n|(C_m) < \delta$ for all $n, m \in \omega$. It is possible to choose such C_n because we have assumed that $\mu_n(\{x_m\}) < \delta$ for all n, m.

By the continuity of f_p we find distinct integers $n_1, n_2 > k_1$ such that $|f_p(x_{n_1}) - f_p(x_{n_2})| < 1/k$. Define

$$\Omega_q = \Omega_p \cup A_{n_1} \cup A_{n_2} \cup C_{n_1} \cup C_{n_2}.$$

Since $|\mu_n(B_n)| > 3\varepsilon$, by the regularity of μ_n we find a closed set $F_n \subseteq B_n$ such that $|\mu_n(F_n)| > 3\varepsilon$.

Let $f : K \to [0,1]$ be a continuous function such that $f|_{F_{n_1}} = 1$ and $f|_{K\setminus B_{n_1}} = f_p|_{K\setminus B_{n_1}}$, and $g : K \to [0,1]$ be a continuous function such that $g|_{K\setminus(C_{n_1}\cup C_{n_2})} = 1$ and $g(x_{n_1}) = g(x_{n_2}) = 0$ if $C_n \neq \emptyset$ for all n, and

$g = 1$ otherwise. Define $f_q = f \cdot g$, $M_q = M_p$, $\varepsilon_q = \varepsilon_p$ and define Δ_q as in Lemma 4.2. By Lemma 4.1 we assume that $\operatorname{diam}(\Delta_q) < 1/k$. It is easy to see that $q \in R(K)$ and $f_q|_{\Omega_p} = f_p|_{\Omega_p}$, because $B_n \cap \Omega_p = \emptyset$. Hence $q \leq p$. Then

$$\left|\int f_q d\mu_{n_1}\right| \geq |\mu_{n_1}(F_{n_1})| - |\mu_{n_1}|(\Omega_p) - |\mu_{n_1}|(C_{n_1} \cup C_{n_2}) > 3\varepsilon - \varepsilon - 2\delta \geq 5\varepsilon/3$$

and

$$\begin{aligned}\left|\int f_q \, d\mu_{n_2}\right| &\leq |\mu_{n_2}(B_{n_1})| + \left|\int f_p \, d\mu_{n_2}\right| + |\mu_{n_2}|(C_{n_1} \cup C_{n_2}) \\ &< \varepsilon/18 + \varepsilon + 2\delta < 3\varepsilon/2.\end{aligned}$$

Since $A_{n_1} \cup A_{n_2} \subseteq \Omega_q$ we have

$$|\mu_n|(K \setminus \Omega_q) < \frac{\varepsilon}{18} = \frac{\delta_1 - \delta_2}{3}$$

for $n \in \{n_1, n_2\}$. Thus we have proved items (i) to (iv). Item (v) follows from the choice of k_1 and k_2, in the case $x \in \Omega_p$, because we will have $f_q(x_n) = f_p(x_n)$. In the case $x \notin \Omega_p$ we have $C_n \neq \emptyset$ and item (v) follows from $f_q(x_{n_1}) = f_q(x_{n_2}) = 0$. Item (vi) is imediate from the choice of Δ_q. ■

5. Iteration of the forcing. In this section we iterate forcings of the kind $R(K)$, as described below, to prove the main result of this paper: the consistent existence of an indecomposable $C(K)$ space with density smaller than continuum.

We will construct by induction forcings $(P_\alpha)_{\alpha \leq \omega_1}$ and P_α-names $(\dot{K}_\alpha)_{\alpha \leq \omega_1}$ such that $P_\alpha \Vdash$ "$\dot{K}_\alpha$ is compact of countable weight". Let P_0 be a trivial forcing and $K_0 = [0,1]^2$. Having defined P_α and $\dot{K}_\alpha$ we define

$$P_{\alpha+1} = P_\alpha * \dot{Q}_\alpha,$$

where $\dot{Q}_\alpha$ is a P_α-name such that

$$P_\alpha \Vdash \dot{Q}_\alpha = R(\dot{K}_\alpha),$$

and we define $\dot{K}_{\alpha+1}$ to be a $P_{\alpha+1}$-name such that $P_{\alpha+1} \Vdash \dot{K}_{\alpha+1} = (\dot{K}_\alpha)_{\dot{G}_\alpha}$. If $\alpha \leq \omega_1$ is a limit ordinal and $(P_\beta)_{\beta<\alpha}$ and $(\dot{K}_\beta)_{\beta<\alpha}$ are defined, we define P_α to be the iteration with finite supports of $(P_\beta)_{\beta<\alpha}$, and we define $\dot{K}_\alpha$ so that

$$P_\alpha \Vdash \dot{K}_\alpha = \varprojlim\, (\overline{\dot{K}_\beta})_{\beta<\alpha}.$$

Set $P = P_{\omega_1}$. In $M[G_\alpha]$ consider $K_\alpha = (\dot{K}_\alpha)_{G_\alpha}$.

By the definition we have $K_\alpha \subseteq [0,1]^\alpha$ for $\alpha \geq \omega$, and $K_\alpha \subseteq [0,1]^{\alpha+2}$ for $\alpha < \omega$. To get uniform notation, if $\alpha < \omega$ and $x \in K_\gamma$ with $\gamma > \alpha$, we denote $x|_{\alpha+2}$ by $x|_\alpha$.

Since P is an iteration with finite supports of c.c.c. forcings, P is also c.c.c. and hence it preserves cardinals.

In K_α we define $(q_n|\alpha)_{n\in\omega} \subseteq K_\alpha$ inductively, in M_α. In M, we fix an enumeration $\{q_n|0 : n \in \omega\}$ of the pairs of rationals in $[0,1]^2$. Having defined $\{q_n|\alpha : n\in\omega\}$ in M_α, in $M_{\alpha+1}$ we define $q_n|(\alpha+1) = (q_n|\alpha, f_G(q_n|\alpha))$ if $q_n|\alpha \in \Omega_{G_{\{\alpha\}}}$, and $q_n|(\alpha+1) = (q_n|\alpha, 0)$ otherwise. For α a limit ordinal we define $q_n|\alpha = \bigcup_{\beta<\alpha} q_n|\beta$. Let $\dot{q}_n|\alpha$ be a P_α-name for $q_n|\alpha$. Set $q_n = q_n|\omega_1$ in M_{ω_1}.

LEMMA 5.1. *In $M[G]$, the set $\{q_n : n \in \omega\}$ is dense in K_{ω_1}.*

Proof. By the hypothesis $\{q_n|0 : n \in \omega\}$ is dense in K_0. If $\{q_n|\beta : n \in \omega\}$ is dense in K_β for all $\beta < \alpha$ and α a limit ordinal, then, in M_α, $\{q_n|\alpha : n \in \omega\}$ is dense in K_α. In fact, if there exists a non-empty set $V \in \mathcal{B}_\alpha$ such that V is disjoint from $\{q_n|\alpha : n \in \omega\}$, then since V is a finite union of elementary open sets there exists $\beta < \alpha$ such that $\pi_\beta[V]$ is a non-empty open set in K_β, contradicting the assumption that $\{q_n|\beta : n \in \omega\}$ is dense in K_β.

Suppose that $\{q_n|\alpha : n \in \omega\}$ is dense in K_α. Let V be a non-empty open set of $K_{\alpha+1}$. Since $\mathrm{Gr}(f_{G_{\{\alpha\}}})$ is dense in $K_{\alpha+1}$, V intersects $\mathrm{Gr}(f_{G_{\{\alpha\}}}) = \pi^{-1}(\Omega_{G_{\{\alpha\}}})$, which is open in $K_{\alpha+1}$. Therefore, taking this intersection instead of V we assume that $V \subseteq \mathrm{Gr}(f_{G_{\{\alpha\}}})$. By Lemma 4.4, $f_{G_{\{\alpha\}}}$ is continuous in $\Omega_{G_{\{\alpha\}}}$, so $\pi^{-1}_{K_{\alpha+1},K_\alpha}$ is a homeomorphism when restricted to $\Omega_{G_{\{\alpha\}}}$, hence $\pi_{K_{\alpha+1},K_\alpha}[V]$ is open in K_α, and therefore it intersects $\{q_n|\alpha : n \in \omega\}$, implying that V intersects $\{q_n|(\alpha+1) : n \in \omega\}$. ∎

LEMMA 5.2. *In $M[G]$, the space K_{ω_1} is connected.*

Proof. We proceed by induction. For $\alpha = 0$, $K_0 = [0,1]^2$ is connected in M_0. If K_α is connected in M_α, then by Lemma 4.4, $K_{\alpha+1}$ is connected in $M_{\alpha+1}$. Let $\alpha \le \omega_1$ be a limit ordinal and suppose that, in M_β, K_β is connected, for all $\beta < \alpha$, but K_α is not connected. By the compactness of K_α there exist $U, V \in \mathcal{B}_\alpha$ such that $K_\alpha \cap U \cap V = \emptyset$, $K_\alpha \subseteq U \cup V$, $K_\alpha \cap U \neq \emptyset$ and $K_\alpha \cap V \neq \emptyset$. Since elements of $\mathcal{B}_\alpha$ are determined by finite coordinates below α, there exists $\beta < \alpha$ such that $\pi_\beta[U]$ and $\pi_\beta[V]$ are open sets such that $U = \pi_\beta^{-1}[\pi_\beta[U]]$ and $V = \pi_\beta^{-1}[\pi_\beta[V]]$, which implies that $\overline{K}_\beta$, and so K_β, is not connected. ∎

LEMMA 5.3. *Let $\dot{\mu}$ be a P-name for a measure on K_{ω_1} and let $\dot{z}$ be a P-name for an element of K_{ω_1}. Then the following sets are closed unbounded in ω_1:*

(a) $C_{\dot{\mu}} = \{\alpha < \omega_1 : P \Vdash \dot{\mu}_{|\mathcal{B}_\alpha} \in M_\alpha$ *and* $|\dot{\mu}|_{|\mathcal{B}_\alpha} = |\dot{\mu}_{|\mathcal{B}_\alpha}|\}$;
(b) $C_{\dot{z}} = \{\alpha < \omega_1 : P \Vdash \dot{z}|_\alpha \in M_\alpha\}$.

Proof. Fix $\alpha_0 \in \omega_1$. We will construct by induction an increasing sequence $(\alpha_n)_{n\in\omega}$ in ω_1 such that P forces the following statements:

$$(*) \qquad \dot{\mu}_{|\mathcal{B}_{\alpha_n}} \in M_{\alpha_{n+1}},$$
$$(**) \qquad \forall V \in \mathcal{B}_{\alpha_n}\ (|\dot{\mu}|(V) = |\dot{\mu}_{|\mathcal{B}_{\alpha_{n+1}}}|(V)).$$

Working in $M[G]$ for G a P-generic over a ground model M, given n we find $\alpha_{n+1} > \alpha_n$ such that $\dot{\mu}_G \in M_{\alpha_n}$, since $\mathcal{B}_{\alpha_n} \in M[G_\alpha]$ is countable and $\dot{\mu}_{G|\mathcal{B}_{\alpha_n}}$ may be identified as a countable sequence of countable sequences of integers. We may also choose α_{n+1} large enough to contain the elements of $\mathcal{B}_{\omega_1}$ which decide the value of $|\dot{\mu}_G|(V)$ for $V \in \mathcal{B}_\alpha$, since $|\dot{\mu}_G|(V)$ is the supremum of finite sums of measures of disjoint Borel sets and each Borel set in K_{ω_1} may be approximated by elementary open sets.

Therefore, for each $p \in P$ we find $q \le p$ which forces $(*)$ and $(**)$ for some α_{n+1}. This means that the set D consisting of all conditions $p \in P$ such that, for some $\beta < \omega_1$,

$$p \Vdash \beta = \min\{\beta < \omega_1 : \dot{\mu}_{|\mathcal{B}_{\alpha_n}} \in M_\beta \wedge \forall V \in \mathcal{B}_{\alpha_n} \ (|\dot{\mu}|(V) = |\dot{\mu}_{|\beta}|(V))\},$$

is dense in P. For each $p \in D$ we define β^p as the least β such that p forces $(*)$ and $(**)$ with β instead of α_{n+1}. Note that $\beta^p \ne \beta^q$ implies that p and q are incompatible. Since P is c.c.c., $\{\beta^p : p \in D\}$ is countable. Since D is dense in P, taking $\alpha_{n+1} = \sup\{\beta^p : p \in D\}$ we have $(*)$ and $(**)$ satisfied.

Setting $\alpha = \sup\{\alpha_n : n \in \omega\}$ we have $\alpha \in C_{\dot{\mu}}$, which proves that $C_{\dot{\mu}}$ is umbounded. The proof that $C_{\dot{\mu}}$ is closed uses the same argument. Thus we have handled item (a). Item (b) is analogous. ∎

For G an $R(K)$-generic over M, in $M[G]$ we define $f'_G : K_G \to [0,1]$ by $f'_G(x,t) = t$ for $(x,t) \in K_G \subseteq \overline{K} \times [0,1]$. In M we let $\dot{f}'_G$ be an $R(K)$-name for f'_G. We note that

$$f'_{G|K_G \cap \Omega_G \times [0,1]} = f_G \circ \pi, \tag{$***$}$$

since $\pi^{-1}_{K_G,\overline{K}}(x) = \{f_G(x)\}$ for $x \in \Omega_G$.

Let G be a P-generic over M and fix $\alpha < \omega_1$. In $M[G]$ we define $\widetilde{f}'_{G_{\{\alpha\}}}$ to be the continuous extension of $f'_{G_{\{\alpha\}}}$ in $\overline{K}_\alpha^{M[G]}$, which exists since $f'_{G_{\{\alpha\}}}$ is uniformly continuous in K_α. Let $\bar{f}'_{G_{\{\alpha\}}}$ be a P-name for $\widetilde{f}'_{G_{\{\alpha\}}}$.

LEMMA 5.4. *Let $\varepsilon > 0$ be rational, $\alpha < \omega_1$ and $p \in P$ be such that $\varepsilon_{p(\alpha)} \le \varepsilon$ and $\dot{\mu} \in M_p$, for $\dot{\mu}$ a P_α-name. Let $(\dot{\mu}_n)_{n\in\omega}$ and $(\dot{x}_n)_{n\in\omega}$ be sequences of P-names such that p forces:*

(i) $\dot{\mu}_n \in M(K_{\omega_1})$;
(ii) $\dot{x}_n \in K_{\omega_1}$;
(iii) $\mu_{n|\mathcal{B}_\alpha} \in M_\alpha$;
(iv) $\dot{x}_n|_\alpha \in M_\alpha$;
(v) $|\dot{\mu}_{n|\mathcal{B}_\alpha}| = |\dot{\mu}_n|_{|\mathcal{B}_\alpha}$;
(vi) $\dot{\mu}_{n|\mathcal{B}_\alpha}(\{\dot{x}_n|_\alpha\}) \xrightarrow{n} 0$;
(vii) $|\dot{\mu}_{n|\mathcal{B}_\alpha}|$ *weak* converges to* $\dot{\mu}$;
(viii) $\{\dot{\mu}_{n|\mathcal{B}_\alpha} : n \in \check{\omega}\}$ *is not* $5\check{\varepsilon}$*-weakly relatively compact*;

(ix) *there exists a pairwise disjoint sequence* $(A_n)_{n\in\omega} \subseteq \mathcal{B}_\alpha$ *such that* $\|\dot\mu_n\| - |\dot\mu_n|(A_n) < \frac{\varepsilon}{18}\|\dot\mu_n\|$ *for all* n.

Then there exist $\delta_1, \delta_2 > 0$ *such that* $p \Vdash \forall k \in \check\omega\ \exists n_1, n_2 > k$

$$\Big|\int\limits_K \bar f'_{G_{\{\alpha\}}} \circ \pi_{\alpha+1}\, d\dot\mu_{n_1}\Big| > \check\delta_1, \qquad \Big|\int\limits_K \bar f'_{G_{\{\alpha\}}} \circ \pi_{\alpha+1}\, d\dot\mu_{n_2}\Big| < \check\delta_2,$$

$$|\dot f_{G_{\{\alpha\}}}(\dot x_{n_1}|_\alpha) - \dot f_{G_{\{\alpha\}}}(\dot x_{n_2}|_\alpha)| < 2/k.$$

Proof. Let G be a P-generic over a ground model M such that $p \in G$. In $M[G]$, set $\mu_n = (\dot\mu_n)_G$ and $x_n = (\dot x_n)_G$. In $M[G_\alpha]$, take $\mu = \dot\mu_{G_{\{\alpha\}}}$.

By hypothesis, $\mu_{n|\mathcal{B}_\alpha} \in M[G_\alpha]$ and $x_n|_\alpha \in M[G_\alpha]$ for all n. Let us work in $M[G_\alpha]$. By Lemma 4.6 there exist $\delta'_1 > \delta'_2 > 0$ such that for all $p' \le p(\alpha) \in R(K_\alpha)$ and $k \in \omega$, there exist $n_1, n_2 > k$ and $q \le p'$ such that $|\int f_q\, d\mu_{n_1|\mathcal{B}_\alpha}| > \delta'_1$, $|\int f_q\, d\mu_{n_2|\mathcal{B}_\alpha}| < \delta'_2$, and $|\mu_{n_i|\mathcal{B}_\alpha}|(K_\alpha \setminus \Omega_q) < (\delta'_1 - \delta'_2)/3$ for $i = 1, 2$. By Lemma 4.1 we may assume that

$$\operatorname{diam}(\Delta_q) \le \min\left\{\frac{|\int f_q\, d\mu_{n_1|\mathcal{B}_\alpha}| - \delta'_1}{\|\mu_{n_1|\mathcal{B}_\alpha}\|}, \frac{\delta'_2 - |\int f_q\, d\mu_{n_2|\mathcal{B}_\alpha}|}{\|\mu_{n_2|\mathcal{B}_\alpha}\|}\right\}.$$

Therefore, for all $r \le q$ we have

$$\Big|\int\limits_{\Omega_q} f_r\, d\mu_{n_1|\mathcal{B}_\alpha}\Big| > \Big|\int f_q\, d\mu_{n_1|\mathcal{B}_\alpha}\Big| - \operatorname{diam}(\Delta_q)\cdot|\mu_{n_1|\mathcal{B}_\alpha}|(\Omega_q) \ge \delta'_1$$

and

$$\Big|\int\limits_{\Omega_q} f_r\, d\mu_{n_2|\mathcal{B}_\alpha}\Big| < \Big|\int f_q\, d\mu_{n_2|\mathcal{B}_\alpha}\Big| + \operatorname{diam}(\Delta_q)\cdot|\mu_{n_2|\mathcal{B}_\alpha}|(\Omega_q) \le \delta'_2.$$

Since $\mu_{n|\mathcal{B}_\alpha} \in M[G_\alpha]$, in $M[G]$ we have

$$\int\limits_{\Omega^{M[G]}\cap\overline{K}} \widetilde{f}_r\, d\mu_{n|\mathcal{B}_\alpha} = \int\limits_{\Omega^M\cap K} f_r\, d\mu_{n|\mathcal{B}_\alpha}$$

for all $\Omega \in \mathcal{B}_\alpha$, and so in $M[G]$,

$$\Big|\int\limits_{\Omega_r\cap\overline{K}_\alpha} \widetilde{f}_r\, d\mu_{n_1|\mathcal{B}_\alpha}\Big| > \delta'_1, \tag{1}$$

$$\Big|\int\limits_{\Omega_r\cap\overline{K}_\alpha} \widetilde{f}_r\, d\mu_{n_2|\mathcal{B}_\alpha}\Big| < \delta'_2 \tag{2}$$

and

$$|\mu_{n_i|\mathcal{B}_\alpha}|(\overline{K}_\alpha \setminus \Omega_r) < (\delta'_1 - \delta'_2)/3 \tag{3}$$

for $i \in \{1, 2\}$. We observe that at this moment $\widetilde{f}_r$ represents the continuous extension of f_r in $\overline{K}_\alpha^{M[G]}$, and not in $\overline{K}_\alpha^{M[G_{\alpha+1}]}$, as with the previous definition. In the same way we interpret $f_{G_{\{\alpha\}}}$ as a continuous function from $\Omega_{G_{\{\alpha\}}} \cap \overline{K}_{\alpha+1}^{M[G]}$ into $[0, 1]$.

The inequalities (1) and (2) also hold for $f_{G_{\{\alpha\}}}$ instead of $\widetilde{f}_r$, since, by the definition of $f_{G_{\{\alpha\}}}$, for all $x \in \Omega_{G_{\{\alpha\}}}$ and $\varepsilon' > 0$ there exists $r \le p(\alpha)$ such that for all $s \le r$ we have $|f_G(x) - f_s(x)| < \varepsilon'$.

From (3) and the hypothesis (v) we conclude that $|\mu_{n_i}|(K \setminus \Omega_r \times [0,1]^{\omega_1 \setminus \alpha}) < (\delta_1' - \delta_2')/3$, for $i \in \{1,2\}$, and from (1) and (2) for f_G and from (∗∗∗) we have

$$\Big|\int\limits_{K\cap\Omega_r\times[0,1]^{\omega_1\setminus\alpha}} f'_{G_{\{\alpha\}}} \circ \pi_{\alpha+1}\, d\mu_{n_1}\Big| > \delta_1',$$

$$\Big|\int\limits_{K\cap\Omega_r\times[0,1]^{\omega_1\setminus\alpha}} f'_{G_{\{\alpha\}}} \circ \pi_{\alpha+1}\, d\mu_{n_2}\Big| < \delta_2'.$$

Hence

$$\Big|\int\limits_K f'_{G_{\{\alpha\}}} \circ \pi_{\alpha+1}\, d\mu_{n_1}\Big| > \delta_1' - (\delta_1' - \delta_2')/3,$$

$$\Big|\int\limits_K f'_{G_{\{\alpha\}}} \circ \pi_{\alpha+1}\, d\mu_{n_2}\Big| < \delta_2' + (\delta_1' - \delta_2')/3.$$

Thus, taking $\delta_1 = \delta_1' - (\delta_1' - \delta_2')/3$ and $\delta_2 = \delta_2' + (\delta_1' - \delta_2')/3$ we see that $\delta_1 > \delta_2 > 0$ satisfy the assertion of the lemma. The last inequality of the assertion follows from items (v) and (vi) of Lemma 4.6. ∎

Lemma 5.5. *Let $a, b \in M_\alpha$ be disjoint subsets of ω such that*

$$\overline{\{q_n|\alpha : n \in a\}} \cap \overline{\{q_n|\alpha : n \in b\}} \ne \emptyset.$$

Then $\overline{\{q_n : n \in a\}} \cap \overline{\{q_n : n \in b\}} \ne \emptyset$ in M_{ω_1}.

Proof. If $\overline{\{q_n|\beta : n \in a\}} \cap \overline{\{q_n|\beta : n \in b\}} = \emptyset$ for β a limit ordinal, there exists $\gamma < \beta$ where the separation occurs, in the model M_β. Thus, in order to prove the lemma it is enough to show that

$$\overline{\{q_n|(\alpha+1) : n \in a\}} \cap \overline{\{q_n|(\alpha+1) : n \in b\}} \ne \emptyset$$

and use induction.

Fix $\alpha < \omega_1$ and $a, b \subseteq \omega$ as in the hypothesis. Working in M_α let us prove that

$$R(K_\alpha) \Vdash \overline{\{\dot q_n|(\alpha+1) : n \in \check a\}} \cap \overline{\{\dot q_n|(\alpha+1) : n \in \check b\}} \ne \emptyset \text{ in } \dot K_{\alpha+1}.$$

Fix $p \in R(K_\alpha)$. We will show that there exists $q \le p$ such that

$$q \Vdash \overline{\{\dot q_n|(\alpha+1) : n \in \check a\}} \cap \overline{\{\dot q_n|(\alpha+1) : n \in \check b\}} \ne \emptyset. \tag{4}$$

Using the fact that K_α is metrizable and passing to a subsequence, we assume that there exists $z \in K_\alpha$ such that $q_n|\alpha$ converges to z.

We will consider two cases. If there exists $q \le p$ such that $z \in \overline{\Omega}_q$, by the regularity of the measures in M_q we may assume that $z \in \Omega_q$. Since $f_{G_{\{\alpha\}}}$ is continuous in $\Omega_{G_{\{\alpha\}}}$ and $R(K_\alpha) \Vdash \dot K_{\alpha+1} = \overline{\mathrm{Gr}(\dot f_{G_{\{\alpha\}}})}$, we see that (4) holds.

In the second case, $z \notin \overline{\Omega}_q$ for all $q \le p$. Fix $q \le p$ and $k \in \omega$. Choose an open neighbourhood V of z disjoint from $\overline{\Omega}_q$. Since $\mu(\{q_n|\alpha\}) \xrightarrow{n} 0$ for all $\mu \in M_q$, we may find $n_1, n_2 > k$ and disjoint open sets $U_1, U_2 \subseteq V$ such that

- $n_1 \in a$, $n_2 \in b$;
- $q_{n_1}|\alpha \in U_1$, $q_{n_2}|\alpha \in U_2$;
- $\mu(\overline{U_1 \cup U_2}) < \varepsilon_q - \mu(\Omega_q)$ for all $\mu \in M_q$.

Define $\Omega_r = \Omega_q \cup U_1 \cup U_2$, $f_r = f_q$, $\varepsilon_r = \varepsilon_q$, $M_r = M_q$ and Δ_r such that $r \in R(K_\alpha)$ and $\operatorname{diam}(\Delta_r) \le 1/k$. By the above conditions we have $r \le q$ and, for all $i \in \{1,2\}$,

$$r \Vdash \check{q}_{n_i}|\alpha \in \Omega_{G_{\{\alpha\}}} \text{ and } \dot{f}_{G_{\{\alpha\}}}(\check{q}_{n_i}|\alpha) < 1/k,$$

proving that

$$p \Vdash (\check{z}, 0) \in \overline{\{\dot{q}_n|(\alpha+1) : n \in \check{a}\}} \cap \overline{\{\dot{q}_n|(\alpha+1) : n \in \check{b}\}}. \ \blacksquare$$

Lemma 5.6. *If U, V are disjoint open subsets of K_{ω_1} such that $\overline{U} \cap \overline{V} \neq \emptyset$, then $\overline{U} \cap \overline{V}$ has at least two elements.*

Proof. Let $\dot{U}$ and $\dot{V}$ be P-names for disjoint open subsets of K_{ω_1} such that

$$P \Vdash \overline{\dot{U}} \cap \overline{\dot{V}} \neq \emptyset.$$

For any $\alpha < \omega_1$ let $\dot{U}_\alpha$ and $\dot{V}_\alpha$ be P-names such that

$$P \Vdash \pi_\alpha[\dot{U}] = \dot{U}_\alpha,\ \pi_\alpha[\dot{V}] = \dot{V}_\alpha.$$

Since K_{ω_1} is separable, if V is an open set of K_{ω_1} then there exists a countable union V' of basic open sets such that $\overline{V} = \overline{V'}$. Thus, we may assume that, in $M[G]$, $(\dot{U})_G$ and $(\dot{V})_G$ are countable unions of basic open sets. Choose $\gamma < \omega_1$ and $p \in P$ such that

$$p \Vdash \forall \alpha \ge \gamma\ (\dot{U} = \pi_\alpha^{-1}[\dot{U}_\alpha] \text{ and } \dot{V} = \pi_\alpha^{-1}[\dot{V}_\alpha]).$$

Fix $q \le p$. Let $\dot{z}$ be a P-name for an element of K such that

$$q \Vdash \dot{z} \in \overline{\dot{U}} \cap \overline{\dot{V}}.$$

Pick $\alpha > \gamma, \operatorname{supp}(q)$ such that $q \Vdash \dot{z}|_\alpha \in M_\alpha$, which is possible by Lemma 5.3.

By Lemma 5.1 and since K_α is metrizable, there exist P-names $\dot{a}$ and $\dot{b}$ for disjoint subsets of ω such that $q \Vdash \dot{a}, \dot{b} \in M_\alpha$ and

$$q \Vdash \{\dot{q}_n : n \in \dot{a}\} \subseteq \dot{U},\ \{\dot{q}_n : n \in \dot{b}\} \subseteq \dot{V},\ \dot{q}_n|\alpha \xrightarrow{n \in \dot{a}} \dot{z}|_\alpha,\ \dot{q}_n|\alpha \xrightarrow{n \in \dot{b}} \dot{z}|_\alpha.$$

Define $r \le q$ as $r(\beta) = q(\beta)$ if $\beta \neq \alpha$, and take for $f_{r(\alpha)}$ the null function, $\Omega_{r(\alpha)} = \emptyset$, $M_{r(\alpha)} = \{\delta_{\dot{z}|_\alpha}\}$, $\Delta_{r(\alpha)} = \emptyset$ and $\varepsilon_{r(\alpha)} = 1$.

In $M[G_\alpha]$ we will denote $(\dot{z}|_\alpha)_{G_\alpha}$ by $z|_\alpha$.

In $R(K_\alpha)$, working in M_α, there is no $s \le r(\alpha)$ such that $z|_\alpha \in \Omega_s$, because this would imply $\delta_{z|_\alpha}(\Omega_s) = 1$. Thus, analogously to the proof of

Lemma 5.5, we conclude that, in M_α,

$$r(\alpha) \Vdash (\dot{z}|_\alpha, 0) \in \overline{\{\check{q}_n|(\alpha+1) : n \in \check{a}\}} \cap \overline{\{\check{q}_n|(\alpha+1) : n \in \check{b}\}},$$

where $a = \dot{a}_{G_\alpha}$ and $b = \dot{b}_{G_\alpha}$.

Recall the last paragraph of the proof of Lemma 5.5. We may modify it taking f_r such that $f_r(q_i|\alpha) = 1$ for $i \in \{n_1, n_2\}$ and $f_{r|K\setminus(U_1\cup U_2)} = f_{q|K\setminus(U_1\cup U_2)}$ instead of $f_r = f_q$. Defining $r(\alpha) \in R(K_\alpha)$ in this way in $M[G_\alpha]$ we will have

$$r(\alpha) \Vdash \check{q}_{n_i}|\alpha \in \Omega_{G_{\{\alpha\}}} \text{ and } |f_{G_{\{\alpha\}}}(\check{q}_{n_i}|\alpha) - 1| < 1/k,$$

and therefore

$$r(\alpha) \Vdash (\dot{z}|_\alpha, 1) \in \overline{\{\check{q}_n|(\alpha+1) : n \in \check{a}\}} \cap \overline{\{\check{q}_n|(\alpha+1) : n \in \check{b}\}}.$$

We have proved that

$$r \Vdash \pi_{\alpha+1}^{-1}[(\dot{z}|_\alpha, 0)] \subseteq \overline{\dot{U}} \cap \overline{\dot{V}},\ \pi_{\alpha+1}^{-1}[(\dot{z}|_\alpha, 1)] \subseteq \overline{\dot{U}} \cap \overline{\dot{V}}$$

and so $r \Vdash |\overline{\dot{U}} \cap \overline{\dot{V}}| \geq 2$. ∎

Lemma 5.7. *Every operator on $C(K_{\omega_1})$ is a weak multiplier.*

Proof. Fix $p \in P$ and a P-generic G over a ground model M. Let us work in $M[G]$. Let $K = K_{\omega_1}$ and $\mathcal{B} = \mathcal{B}_{\omega_1}$.

Suppose that there exists an operator $T : C(K) \to C(K)$ which is not a weak multiplier. By Lemmas 3.5 and 5.1 there exist distinct $x_n \in \{q_m : m \in \omega\}$ such that for every Borel bounded function $f : K \to \mathbb{R}$, the set $\{T^*(\delta_{x_n}) - f\delta_{x_n} : n \in \omega\}$ is not relatively weakly compact. Define $f : K \to \mathbb{R}$ by $f(x_n) = T^*(\delta_{x_n})(\{x_n\})$ for all $n \in \omega$, and $f(x) = 0$ at the other points of K. Passing to a subsequence we assume that $f(x_n)$ converges to $L \in \mathbb{R}$. By Corollary 2.5 there exist sequences of measures $(\mu_n)_{n\in\omega}$ and $(\lambda_n)_{n\in\omega}$ such that $(\mu_n)_{n\in\omega}$ are pairwise disjoint, $(\lambda_n)_{n\in\omega}$ converges weakly to $\lambda \in M(K)$ and

$$T^*(\delta_{x_n}) - f\delta_{x_n} = \mu_n + \lambda_n.$$

Let $\dot{\mu}_n$ be P-names for μ_n and $\dot{x}_n$ be P-names for x_n.

Set $C = \bigcap_{n\in\omega} C_{\dot{\mu}_n}$, where $C_{\dot{\mu}_n}$ is defined as in Lemma 5.3. Since countable intersections of closed unbounded subsets of ω_1 are closed unbounded (see [Ku, Ch. II, Lemma 6.8]), from Lemma 5.3 we deduce that C is closed unbounded in ω_1.

Choose $\alpha \in C$ such that $\|\mu_{n|\mathcal{B}_\alpha}\| = \|\mu_n\|$ for all $n \in \omega$. Let us prove that such an α exists. For any pair (n, m) of positive integers pick $a_{(\mu_n,m)} \in \mathcal{B}$ such that $|\mu_n(a_{(\mu_n,m)})| > \|\mu_n\| - 1/m$. Choose $\beta_{(\mu_n,m)}$ such that $a_{(\mu_n,m)} \in \mathcal{B}_{\beta_{(\mu_n,m)}}$. Letting β be the supremum of all $\beta_{(\mu_n,m)}$ we find $\alpha > \beta, \operatorname{supp}(p)$ belonging to C.

We may also assume that $\{\mu_{n|\mathcal{B}_\alpha} : n \in \omega\}$ is not weakly relatively compact in $M(\overline{K}_\alpha)$, by taking α sufficiently large to contain a sequence W_n in $\mathcal{B}$ such that $|\mu_n(W_n)| > \varrho$ for some $\varrho > 0$. Similarly, using Proposition 2.3 and the fact that $(\mu_n)_{n\in\omega}$ are pairwise disjoint, we assume that there exist $(A_n)_{n\in\omega} \subseteq \mathcal{B}_\alpha$ pairwise disjoint such that $\|\mu_n\| - |\mu_n|(A_n) < \frac{\varrho}{90}\|\mu_n\|$. We assume yet that

$$T^*(\delta_{x_n})_{|\mathcal{B}_\alpha}(\{x_n|_\alpha\}) = T^*(\delta_{x_n})(\{x_n\}),$$
$$f\delta_{x_n|\mathcal{B}_\alpha}(\{x_n|_\alpha\}) = f\delta_{x_n}(\{x_n\}),$$
$$\lambda_{n|\mathcal{B}_\alpha}(\{x_n|_\alpha\}) = \lambda_n(\{x_n\}).$$

Here we use regularity of the measures, taking $a_{n,m}, b_{n,m}, c_{n,m} \in \mathcal{B}$ containing x_n such that

$$|T^*(\delta_{x_n})(a_{n,m}) - T^*(\delta_{x_n})(\{x_n\})| < 1/m,$$
$$|f\delta_{x_n}(b_{n,m}) - f\delta_{x_n}(\{x_n\})| < 1/m,$$
$$|\lambda_n(c_{n,m}) - \lambda_n(\{x_n\})| < 1/m,$$

and choosing α large enough for $\mathcal{B}_\alpha$ to contain all $a_{n,m}$'s, $b_{n,m}$'s and $c_{n,m}$'s.

Note that $\mu_{n|\mathcal{B}_\alpha}(\{x_n|_\alpha\}) \xrightarrow{n} 0$ because

$$\begin{aligned}\mu_{n|\mathcal{B}_\alpha}(\{x_n|_\alpha\}) &= \mu_n(\{x_n\}) = T^*(\delta_{x_n})(\{x_n\}) - f\delta_{x_n}(\{x_n\}) + \lambda_n(\{x_n\}) \\ &= T^*(\delta_{x_n})(\{x_n\}) - \int\limits_{\{x_n\}} f\delta_{x_n} + \lambda_n(\{x_n\}) \\ &= T^*(\delta_{x_n})(\{x_n\}) - f(x_n) + \lambda_n(\{x_n\}) = \lambda_n(\{x_n\}),\end{aligned}$$

which converges to 0, since, by the Dieudonné–Grothendieck theorem, for every pairwise disjoint sequence $(V_n)_{n\in\omega}$ of open neighbourhoods of x_n, $\lambda_n(V_n)$ converges to 0.

Let us work in M_α. Since K_α has countable weight, the space $C(K_\alpha)$ is separable. Therefore $B_{C(K_\alpha)^*}$, the unit ball of $C(K_\alpha)^*$ with the weak* topology, is metrizable (see [Fa, Proposition 3.24]). Since, by the Alaoglu theorem, $B_{C(K_\alpha)^*}$ is compact in the weak* topology (see [Fa, Theorem 3.1]), we may assume, passing to a subsequence, that $|\mu_{n|\mathcal{B}_\alpha}|$ weak* converges in $M(K_\alpha)$. Let μ be the weak* limit of $|\mu_{n|\mathcal{B}_\alpha}|$.

Passing to a subsequence we may assume that $(x_n|_\alpha)_{n\in\omega}$ converges.

Set $\varepsilon = \varrho/5$. Define $q \in P$ by $q(\alpha) = (0, \emptyset, \{\mu\}, \varepsilon, \emptyset)$ and $q(\beta) = p(\beta)$ for $\beta \neq \alpha$. Select $r \leq q$ satisfying the hypothesis of Lemma 5.4. Define

$$f_\alpha = \widetilde{f'}_{G_{\{\alpha\}}} \circ \pi_{\alpha+1}$$

and let $\dot{f}_\alpha$ be a P-name for f_α. Since $f_\alpha(x_n) = f_{G_{\{\alpha\}}}(x_n|_\alpha)$ for all n, by Lemma 5.4 we get $\delta_1 > \delta_2 > 0$ such that

$$r \Vdash \forall k\ \exists n_1, n_2 > k\ \left|\int \dot{f}_\alpha d\,\dot{\mu}_{n_1}\right| > \delta_1 > \delta_2 > \left|\int \dot{f}_\alpha d\,\dot{\mu}_{n_2}\right|,$$
$$\dot{x}_{n_1}, \dot{x}_{n_2} \in \Omega_{G_{\{\alpha\}}},\ |\dot{f}_\alpha(\dot{x}_{n_1}) - \dot{f}_\alpha(\dot{x}_{n_2})| < 2/k.$$

Thus we find infinite disjoint subsets $a_\alpha, b_\alpha \in M_\alpha[G_\alpha]$ of ω such that $|\int f_\alpha\, d\mu_n| > \delta_1$ for $n \in a_\alpha$, $|\int f_\alpha\, d\mu_n| < \delta_2$ for $n \in b_\alpha$, and

$$\lim_{n \in a_\alpha} f_\alpha(x_n) = \lim_{n \in b_\alpha} f_\alpha(x_n).$$

We will call the above limit L'.

Since λ_n weakly converges to λ, $\int f_\alpha\, d\lambda_n \xrightarrow{n} \int f_\alpha\, d\lambda$. Refining a_α and b_α we may assume that for all $n \in a_\alpha \cup b_\alpha$,

$$\left|\int f_\alpha\, d\lambda_n - \int f_\alpha\, d\lambda\right| < (\delta_1 - \delta_2)/8.$$

Refining a_α and b_α again, we may assume that for all $n \in a_\alpha \cup b_\alpha$,

$$|f(x_n) - L| < (\delta_1 - \delta_2)/8.$$

We remark that L is the limit of $f(x_n)$. Since $(x_n|_{\alpha+1})_{n\in a_\alpha}$ and $(x_n|_{\alpha+1})_{n\in b_\alpha}$ converge to (z, L), where z is the limit of $(x_n|_\alpha)_{n\in\omega}$, we have

$$\overline{\{x_n|_{\alpha+1} : n \in a_\alpha\}} \cap \overline{\{x_n|_{\alpha+1} : n \in b_\alpha\}} \neq \emptyset$$

in $K_{\alpha+1}$. Since $a_\alpha, b_\alpha \in M_\alpha[G_\alpha] = M_{\alpha+1}$ and $\{x_n : n \in \omega\} \subseteq \{q_m : m \in \omega\}$, by Lemma 5.5 we have

$$\overline{\{x_n : n \in a_\alpha\}} \cap \overline{\{x_n : n \in b_\alpha\}} \neq \emptyset$$

in K.

On the other hand, since $T(f_\alpha)(x_n) = \int f_\alpha\, dT^*(\{\delta_{x_n}\})$ and $f\delta_{x_n}(f_\alpha) = f(x_n)$, setting

$$U = \left\{x \in \mathbb{R} : \left|x - L - \int f_\alpha\, d\lambda\right| > \delta_1 - (\delta_1 - \delta_2)/4\right\},$$
$$V = \left\{x \in \mathbb{R} : \left|x - L - \int f_\alpha\, d\lambda\right| < \delta_2 + (\delta_1 - \delta_2)/4\right\},$$

we see that U and V are open disjoint subsets of $\mathbb{R}$ and

$$T(f_\alpha)(x_n) = \int f_\alpha\, d\mu_n + f(x_n) + \int f_\alpha\, d\lambda_n \in U \quad \text{for } n \in a_\alpha,$$
$$T(f_\alpha)(x_n) = \int f_\alpha\, d\mu_n + f(x_n) + \int f_\alpha\, d\lambda_n \in V \quad \text{for } n \in b_\alpha,$$

which implies, by the continuity of $T(f_\alpha)$, that $\overline{\{x_n : n \in a_\alpha\}}$ and $\overline{\{x_n : n \in b_\alpha\}}$ are disjoint, giving a contradiction. ■

THEOREM 5.8. *It is relatively consistent with ZFC that there exists an indecomposable Banach space $C(K)$ of density $\omega_1 < 2^\omega$ such that every operator on $C(K)$ has the form $gI + S$ for some $g \in C(K)$ and S weakly compact.*

Proof. Suppose that CH does not hold in the ground model M. Since P is c.c.c., it preserves cardinals, and since $(2^\omega)^M \le (2^\omega)^{M[G]}$, $\neg$CH also holds in $M[G]$. In $M[G]$ the space K_{ω_1} has weight ω_1, because it is a subspace of $[0,1]^{\omega_1}$, and therefore the density of $C(K_{\omega_1})$ is $\omega_1 < 2^\omega$. From Lemmas 3.2, 5.6 and 5.7 and Theorem 3.3 it follows that every operator on $C(K_{\omega_1})$ has the form $gI + S$ for some $g \in C(K_{\omega_1})$ and S weakly compact. From Lemmas 5.2 and 3.4 we conclude that $C(K_{\omega_1})$ is indecomposable. ∎

References

[Br] C. Brech, *On the density of Banach spaces $C(K)$ with the Grothendieck property*, Proc. Amer. Math. Soc. 134 (2006), 3653–3663.

[Di] J. Diestel, *Sequences and Series in Banach Spaces*, Springer, New York, 1984.

[Fa] M. Fabian, P. Hájek, J. Pelant, P. Habala, V. Montesinos and V. Zizler, *Functional Analysis and Infinite-Dimensional Geometry*, Springer, New York, 2001.

[Fr] D. H. Fremlin, *Consequences of Martin's Axiom*, Cambridge Univ. Press, Cambridge, 1984.

[GM] W. T. Gowers and B. Maurey, *The unconditional basic sequence problem*, J. Amer. Math. Soc. 6 (1993), 851–874.

[KP] M. I. Kadec and A. Pełczyński, *Bases, lacunary sequences and complemented subspaces in the spaces L_p*, Studia Math. 21 (1962), 161–176.

[Ko1] P. Koszmider, *Forcing minimal extensions of Boolean algebras*, Trans. Amer. Math. Soc. 351 (1999), 3073–3117.

[Ko2] —, *Banach spaces of continuous functions with few operators*, Math. Ann. 330 (2004), 151–183.

[Ku] K. Kunen, *Set Theory. An Introduction to Independence Proofs*, North-Holland, 1980.

[Pe1] A. Pełczyński, *Banach spaces on which every unconditionally converging operation is weakly compact*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 10 (1962), 641–648.

[Pe2] —, *On strictly singular and strictly non-singular operators I*, ibid. 13 (1962), 31–41.

[Pl] G. Plebanek, *A construction of a Banach space $C(K)$ with few operators*, Topology Appl. 143 (2004), 217–239.

[Ru] W. Rudin, *Real and Complex Analysis*, 3rd ed., McGraw-Hill, New York, 1987.

[Sc] W. Schachermayer, *On some classical measure-theoretic theorems for non-sigma-complete Boolean algebras*, Dissertationes Math. 214 (1982).

[Ta] M. Talagrand, *Un nouveau $C(K)$ qui possède la propriété de Grothendieck*, Israel J. Math. 37 (1980), 181–191.

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