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Universal measure zero, large Hausdorff dimension, and nearly Lipschitz maps

by

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Abstract. We prove that each analytic set in $\mathbb{R}^n$ contains a universally null set of the same Hausdorff dimension and that each metric space contains a universally null set of Hausdorff dimension no less than the topological dimension of the space. Similar results also hold for universally meager sets.

An essential part of the construction involves an analysis of Lipschitz-like mappings of separable metric spaces onto Cantor cubes and self-similar sets.

1. Introduction. A separable metric space X is *universally null* (or has *universal measure zero*) if $\mu(X) = 0$ for each finite Borel measure μ on X that is diffused (i.e. vanishes on singletons). This is obviously equivalent to each Borel diffused measure μ on X being degenerate, in that for each Borel set $B \subseteq X$ one has either $\mu(B) = 0$ or $\mu(B) = \infty$. In particular, on a universally null space the s-dimensional Hausdorff measure is degenerate for all $s > 0$.

Can such a set have positive Hausdorff or even topological dimension? Fremlin [8, 439G] has an example of a subset E of the plane whose 1-dimensional Hausdorff measure is infinite and degenerate on E. In [29] it is proved that universally null sets with positive topological dimension exist if, and only if, there is a universally null set $E \subseteq \mathbb{R}$ with the cardinality of the continuum, and that there are universally null sets of arbitrary Hausdorff dimension.

The goal of the present paper is to get better results in this direction, namely, to prove that some metric spaces, including all analytic sets in Euclidean spaces, contain a universally null subset with the same Hausdorff dimension.

We outline the basic ideas of our construction. Let 2^ω be the usual Cantor space and provide it with the usual least difference metric given by

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$d(x,y) = 2^{-n}$, where n is the first integer for which $x(n) \neq y(n)$. It is well-known that the standard product probability measure on 2^ω is the one-dimensional Hausdorff measure and thus the Hausdorff dimension of 2^ω is 1. By a famous result of Edward Grzegorek [11] there are two sets $A, B \subseteq 2^\omega$ that have the same cardinality but are of different sizes from the measure-theoretic viewpoint: A has positive (outer) measure and B is universally null. A diagonal set $D \subseteq A \times B \subseteq 2^\omega \times 2^\omega$ (i.e. a set $D \subseteq A \times B$ with both projections one-to-one and onto) has positive one-dimensional Hausdorff measure, for the projection of D onto A is Lipschitz. It is also universally null: the projection onto B is one-to-one, and a continuous one-to-one preimage of a universally null set is universally null.

But how to get such a set in 2^ω? Can one copy this diagonal set into 2^ω in such a way that the Hausdorff dimension of the copy is still large? Yes indeed, there is a homeomorphism of 2^ω onto $2^\omega \times 2^\omega$ that is not quite Lipschitz, yet "Lipschitz enough" to give the preimage of the diagonal set D Hausdorff dimension 1. The preimage of D is universally null, because D is. So 2^ω contains a universally null set of full Hausdorff dimension. This method works for a class of generalized Cantor sets.

To get a universally null set of full Hausdorff dimension in $\mathbb{R}$ one can just approximate $\mathbb{R}$ from within by sufficiently thick Cantor sets. And it turns out that this approximation can be achieved in a wide class of metric spaces that are, in a sense, linearly ordered (so-called *monotone spaces*, cf. Section 4), and include e.g. all analytic subsets of $\mathbb{R}$ and all analytic ultrametric spaces.

Employing classical projection and intersection theorems one can extend the construction to Euclidean spaces. And using [29, Lemma 5.1] on topological dimension and Lipschitz maps yields, in any metric space X, a universally null set of Hausdorff dimension no less than the topological dimension of X.

The method can be applied in a more abstract setting (Section 5), yielding similar results for universally meager sets (Section 7) and some other classes of small sets (Section 8).

In this paper Cantor and self-similar sets, monotone spaces, and the ways Lipschitz and Lipschitz-like maps act between them (Sections 3 and 4) receive more attention than is strictly necessary for our purposes. It is because these notions seem to be interesting in their own right.

2. Preliminaries. In this section we recall Hausdorff measures and dimensions and their basic properties and introduce the notion of nearly Lipschitz map.

The set of all natural numbers including zero is denoted ω. The cardinality of the continuum is denoted $\mathfrak{c}$. The cardinality of a set A is denoted $|A|$. The power set of A is denoted $\mathcal{P}(A)$.

Metric spaces. Most metric spaces under consideration are separable. Recall that a metric space is *perfect* if it has no isolated points, *Polish* if it is separable and completely metrizable, and *analytic* if it is a continuous image of a Polish space.

In order to avoid unnecessary technicalities, all metric spaces are assumed to be bounded. This is no constraint, since any metric d can be redefined to be bounded e.g. by $d_1 = \min(d, 1)$, which leaves close distances unaffected.

Within a context of a metric space we use some self-explanatory notation. In particular, $\operatorname{diam} E$ denotes the diameter of a set E, and $\operatorname{dist}(A, B) = \inf\{d(x, y) : x \in A,\ y \in B\}$.

Hausdorff measure. Recall that given $s \geq 0$, the s-dimensional *Hausdorff measure* $\mathcal{H}^s$ on a metric space X is defined thus: For each $\delta > 0$ and $E \subseteq X$ set

$$\mathcal{H}^s_\delta(E) = \inf \sum_n (\operatorname{diam} E_n)^s, \tag{1}$$

where the infimum is taken over all finite or countable covers $\{E_n\}$ of E by sets of diameter at most δ, and put

$$\mathcal{H}^s(E) = \sup_{\delta > 0} \mathcal{H}^s_\delta(E).$$

The basic properties of $\mathcal{H}^s$ are well-known. It is an outer measure and its restriction to Borel sets is a G_δ-regular Borel measure in X. General references: [6, 15, 22]. We shall need the following theorem of Howroyd [12] that generalizes earlier results of Besicovitch [2] and Davies [3]. We shall call it the Davies Theorem.

THEOREM 2.1 (Davies Theorem). *Let X be an analytic metric space and $s > 0$. If $\mathcal{H}^s(X) > 0$, then there is a compact set $K \subseteq X$ such that $0 < \mathcal{H}^s(K) < \infty$.*

We shall also need the famous Frostman Lemma:

THEOREM 2.2 (Frostman Lemma [15, 8.19]). *Let X be an analytic metric space and $s > 0$. If $\mathcal{H}^s(X) > 0$, then there is a nontrivial finite Borel measure $\mu \leq \mathcal{H}^s$ on X such that $\mu B(x, r) \leq r^s$ for each $x \in X$, $r > 0$.*

Hausdorff dimension. The *Hausdorff dimension* of E is denoted and defined by

$$\dim_{\mathsf{H}} E = \sup\{s : \mathcal{H}^s(E) > 0\}.$$

Hausdorff dimension is an intrinsic value. It is monotone, and *countably stable* in the sense that $\dim_{\mathsf{H}} \bigcup_n E_n = \sup_n \dim_{\mathsf{H}} E_n$ for any countable family $\{E_n\}$ of sets. Hausdorff measures and dimensions of a metric space and its Lipschitz image are related as follows:

LEMMA 2.3 ([6, Lemma 6.1]). *If $f : X \to Y$ is Lipschitz and $\mathcal{H}^s(fX) > 0$, then $\mathcal{H}^s(X) > 0$, for each $s > 0$. In particular,* $\dim_{\mathsf{H}} X \geq \dim_{\mathsf{H}} f(X)$.

Upper Hausdorff dimension. We shall also consider the following variation of Hausdorff dimension. For a separable metric space E let E^* denote its completion and define the *upper Hausdorff dimension* of E by

$$\overline{\dim}_{\mathsf{H}} E = \inf\{\dim_{\mathsf{H}} K : E \subseteq K \subseteq E^*,\ K \text{ is } \sigma\text{-compact}\}.$$

Hence $\overline{\dim}_{\mathsf{H}}$ is an intrinsic value. It is easy to see that for E a subset of a complete metric space X one may alternatively define

$$\overline{\dim}_{\mathsf{H}} E = \inf\{\dim_{\mathsf{H}} K : E \subseteq K \subseteq X,\ K \text{ is } \sigma\text{-compact}\},$$

which shows that $\overline{\dim}_{\mathsf{H}}$ is countably stable. See [30] for more on upper Hausdorff dimension including an equivalent definition based on Hausdorff-like measures.

Obviously $\overline{\dim}_{\mathsf{H}} \geq \dim_{\mathsf{H}}$ and if E is σ-compact, then $\overline{\dim}_{\mathsf{H}} E = \dim_{\mathsf{H}} E$. On the other hand, there is a dense G_δ-set $E \subseteq \mathbb{R}$ such that $\dim_{\mathsf{H}} E = 0$, and any such set has $\overline{\dim}_{\mathsf{H}} E = 1$ by Lemma 2.4 below. This shows that $\dim_{\mathsf{H}} E = \overline{\dim}_{\mathsf{H}} E$ may easily fail. Note also that $\overline{\dim}_{\mathsf{H}}$ is estimated from above by, but is in general not equal to, the lower packing dimension, and *a fortiori* by packing dimension. These facts can be deduced from [7, (3.20)].

The following simple lemma links upper Hausdorff dimension to category.

LEMMA 2.4. *Let X be a separable metric space such that $\overline{\dim}_{\mathsf{H}} U \geq s$ for each nonempty open set U. If $E \subseteq X$ is nonmeager, then $\overline{\dim}_{\mathsf{H}} E \geq s$.*

Proof. Notice that if K is compact and $\dim_{\mathsf{H}} K < s$, then K is nowhere dense: otherwise it would contain a nonempty open set U and thus by assumption $\dim_{\mathsf{H}} K \geq \overline{\dim}_{\mathsf{H}} U \geq s$.

If $\overline{\dim}_{\mathsf{H}} E < s$, then there is a countable cover $\{K_n\}$ of E by compact sets such that $\dim_{\mathsf{H}} K_n < s$ for each n. By the above, the sets K_n are nowhere dense, whence E is meager. ∎

Nearly Lipschitz mappings. We shall make use of the following natural weakening of the notion of Lipschitz mapping.

DEFINITION 2.5. A mapping $f : (X, d) \to (Y, \rho)$ between metric spaces is termed *nearly Lipschitz* if for any $\varepsilon > 0$ there is $\delta > 0$ such that

$$\rho(f(x), f(y)) \leq d(x,y)^{1-\varepsilon} \quad \text{whenever} \quad d(x,y) < \delta. \tag{2}$$

Equivalently, f is nearly Lipschitz if there is a function $F : [0,\infty) \to [0,\infty)$ such that

$$\rho(f(x), f(y)) \leq F(d(x,y)), \quad x, y \in X,$$

satisfying $\liminf_{r\to 0} \log F(r)/\log r \geq 1$.

If f is β-Hölder for each $\beta < 1$, then, given $\varepsilon > 0$, there is a constant M such that $\rho(f(x), f(y)) \le Md(x,y)^{1-\varepsilon/2}$ for all $x, y \in X$. If $\delta > 0$ is small enough that $M \le \delta^{-\varepsilon/2}$, then (2) holds. On the other hand, if Y is bounded and (2) holds with $\varepsilon = 1-\beta$, then f is clearly β-Hölder with $M = \delta^{-\beta} \operatorname{diam} Y$. In summary, if Y is bounded, then f is nearly Lipschitz if and only if it is β-Hölder for each $\beta < 1$.

Every Lipschitz mapping is obviously nearly Lipschitz, but not vice versa. The growth condition for a nearly Lipschitz mapping is nevertheless good enough to control both Hausdorff dimensions the same way as Lipschitz mappings:

Lemma 2.6. *If $f : X \to Y$ is nearly Lipschitz, then* $\dim_{\mathsf{H}} X \ge \dim_{\mathsf{H}} f(X)$ *and* $\overline{\dim}_{\mathsf{H}} X \ge \overline{\dim}_{\mathsf{H}} f(X)$.

We shall use this easy lemma on several occasions. The first estimate follows easily e.g. from [22, Theorem 29], the second is similar.

3. Cantor cubes and self-similar sets. In order to get results about small sets of large Hausdorff dimension we need to investigate certain properties of self-similar sets. We also introduce a class of metric spaces, *Cantor cubes*, that is on one hand simple enough and on the other hand profoundly related to self-similar sets.

Given a set I, denote by I^ω the set of all I-ary sequences and by $I^{<\omega} = \bigcup_{n\in\omega} I^n$ the set of all finite I-ary sequences. Notice that for $p, q \in I^{<\omega}$, $p \subseteq q$ means that q is an extension of p. If $p \in I \cup I^{<\omega}$ and $q \in I \cup I^{<\omega} \cup I^\omega$, the symbol $p^\frown q$ denotes the usual concatenation. If $f \ne g$ are in I^ω, the symbol $f \wedge g$ denotes the maximal common initial segment of f and g, and $n(f,g) = |f \wedge g|$ the length of $f \wedge g$. In more detail,

$$n(f,g) = \min\{n \in \omega : f(n) \ne g(n)\}, \qquad f \wedge g = f{\restriction}n(f,g) = g{\restriction}n(f,g).$$

($f{\restriction}n$ is the restriction = truncation of f.)

Cantor cubes. Fix $k = \{0, 1, \dots, k-1\} \in \omega$. Consider the cube k^ω.

Let $\boldsymbol{r} = \langle r_0, r_1, \dots, r_{k-1}\rangle \in (0,1)^k$. For each $p \in k^{<\omega}$ put

$$\chi_{\boldsymbol{r}}(p) = \prod_{i<|p|} r_{p(i)}. \tag{3}$$

The following formula defines a metric on k^ω

$$\rho_{\boldsymbol{r}}(f,g) = \begin{cases} \chi_{\boldsymbol{r}}(f \wedge g) & \text{if } f \ne g, \\ 0 & \text{if } f = g, \end{cases}$$

that gives k^ω the standard product topology. The resulting metric space $(k^\omega, \rho_{\boldsymbol{r}})$ is denoted by $\mathbb{C}(\boldsymbol{r})$. We often drop the subscript writing ρ and χ for $\rho_{\boldsymbol{r}}$ and $\chi_{\boldsymbol{r}}$.

It is easy to show (and follows at once from the theory of self-similar sets, see below) that the Hausdorff dimension of $\mathbb{C}(\boldsymbol{r})$ is the solution s of *Moran's equation*

$$\sum_{i<k} r_i^s = 1 \tag{4}$$

and that $\mathcal{H}^s(\mathbb{C}(\boldsymbol{r})) = 1$.

Definition 3.1. A metric space $\mathbb{C}$ is called a *Cantor cube* if there is $k \in \omega$ and $\boldsymbol{r} \in (0,1)^k$ such that $\mathbb{C} = \mathbb{C}(\boldsymbol{r})$.

Note that each Cantor cube is homeomorphic to 2^ω ([14, Theorem 7.4]). The best known examples of Cantor cubes are $\mathbb{C}(1/3, 1/3)$, which is bi-Lipschitz equivalent to the Cantor ternary set via the standard mapping $x \mapsto \frac{2}{3}\sum_{n\in\omega} 3^{-n}x(n)$, and $\mathbb{C}(1/2, 1/2)$. The latter has Hausdorff dimension one and its one-dimensional Hausdorff measure coincides on Borel sets with the standard product (Haar) measure.

Self-similar sets. Cantor cubes are particular instances of self-similar sets. We recall basic notions, referring to [25] for more details. Let (X, d) be a complete metric space. A mapping $F : X \to X$ is called a *similarity* if there is a constant $r > 0$ such that $d(Fx, Fy) = r \cdot d(x, y)$ for all $x, y \in X$. The constant r is called the *similarity ratio of* F.

Let $\mathcal{F} = \langle F_0, F_1, \ldots, F_{k-1}\rangle$ be a finite sequence of similarities whose respective similarity ratios $\boldsymbol{r} = \langle r_0, r_1, \ldots, r_{k-1}\rangle$ are strictly less than 1. Such a sequence is termed an *Iterated Similarity System* (ISS). The ISS $\mathcal{F}$ induces a set mapping (the so-called *Hutchinson operator*) $S(A) = \bigcup_{F\in\mathcal{F}} F(A)$. It is well-known that there is a unique nonempty compact set K such that $S(K) = K$. This set is called the *attractor* of $\mathcal{F}$. Attractors of ISS's are often termed *self-similar sets.* The ISS $\mathcal{F}$ and its attractor K are said to satisfy

- SSP (*Strong Separation Property*) if $F_iK \cap F_jK = \emptyset$ whenever $i<j<k$,
- SOSC (*Strong Open Set Condition*) if there is a nonempty open set $U \subseteq X$ that meets K and is such that $S(U) \subseteq U$ and $F_iU \cap F_jU = \emptyset$ whenever $i < j < k$.

SSP is rather restrictive. Obviously SSP implies SOSC, but not vice versa. Nevertheless, Proposition 3.4 below says that, in a sense, SOSC sets can be approximated from within by SSP sets. Recall that dropping "that meets K" from the definition of SOSC one gets the famous *Open Set Condition* that is known to be, in Euclidean spaces, equivalent to SOSC (Schief [24]).

For each $p \in k^{<\omega}$ set $F_p = F_{p(0)} \circ F_{p(1)} \circ \cdots \circ F_{p(|p|-1)}$, $K_p = F_p(K)$. Note that $\operatorname{diam} K_p = \chi(p) \operatorname{diam} K$, where $\chi(p)$ is defined by (3). If $\mathbb{C}(\boldsymbol{r})$ is a Cantor cube, then the ISS given by $F_i(f) = i^\frown f$ turns $\mathbb{C}(\boldsymbol{r})$ into a SSP self-similar set.

LEMMA 3.2. *Every self-similar set is a Lipschitz image of the Cantor cube $\mathbb{C}(\boldsymbol{r})$, where $\boldsymbol{r}$ is the vector of similarity ratios of the underlying* ISS.

Proof. If $p \subseteq q \in k^{<\omega}$, then $K_p \supseteq K_q$. Hence for each $f \in k^\omega$ there is a unique $x_f \in \bigcap_{n\in\omega} K_{f\restriction n}$; and each $x \in K$ equals to some x_f. Hence the mapping defined by $\phi(f) = x_f$ maps $\mathbb{C}(\boldsymbol{r})$ onto K. It is Lipschitz, because

$$d(x_f, x_g) \le \operatorname{diam} K_{f\wedge g} = \chi_{\boldsymbol{r}}(f \wedge g) \operatorname{diam} K = \rho_{\boldsymbol{r}}(f,g) \operatorname{diam} K. \ \blacksquare$$

LEMMA 3.3. *Every self-similar set satisfying* SSP *is bi-Lipschitz equivalent to the Cantor cube $\mathbb{C}(\boldsymbol{r})$, where $\boldsymbol{r}$ is the vector of similarity ratios of the underlying* ISS.

Proof. If K satisfies SSP, then $\operatorname{dist}(K_{p^\frown i}, K_{p^\frown j}) = \chi_{\boldsymbol{r}}(p) \operatorname{dist}(F_i(K), F_j(K))$ for any $p \in k^{<\omega}$ and $i,j < k$. Hence there is a constant c such that $c\chi_{\boldsymbol{r}}(p) \le \operatorname{dist}(K_{p^\frown i}, K_{p^\frown j})$ for any $p \in k^{<\omega}$ and $i < j < k$. We use this fact to prove that the Lipschitz mapping $\phi : \mathbb{C}(\boldsymbol{r}) \to K$ constructed in the above proof is bi-Lipschitz. Let $f, g \in \mathbb{C}(\boldsymbol{r})$ be distinct and $p = f \wedge g$. There are distinct $i,j < k$ such that $p^\frown i \subseteq f$ and $p^\frown j \subseteq g$. Therefore $x_f \in K_{p^\frown i}$ and $x_g \in K_{p^\frown j}$ and

$$d(x_f, x_g) \ge \operatorname{dist}(K_{p^\frown i}, K_{p^\frown j}) \ge c\chi_{\boldsymbol{r}}(p) = c\rho_{\boldsymbol{r}}(f,g),$$

as required. $\blacksquare$

The solution s of Moran's equation (4) is called the *similarity dimension* of K and denoted $\dim_{\mathsf{S}} K$. Hausdorff and similarity dimensions are related as follows:

- $\mathcal{H}^{\dim_{\mathsf{S}} K}(K) < \infty$, in particular $\dim_{\mathsf{H}} K \le \dim_{\mathsf{S}} K$ (cf. Lemma 3.2).
- If K satisfies SOSC, then $\dim_{\mathsf{H}} K = \dim_{\mathsf{S}} K$ ([25, Theorem 2.6]).
- If K satisfies SSP, then $0 < \mathcal{H}^{\dim_{\mathsf{S}} K}(K) < \infty$ (cf. Lemma 3.3).

PROPOSITION 3.4. *Let K be a self-similar set satisfying* SOSC. *For each $\varepsilon > 0$ there is a* SSP *self-similar set $C \subseteq K$ such that $\dim_{\mathsf{H}} C > \dim_{\mathsf{H}} K - \varepsilon$.*

Proof. Though not explicitly stated, the required set C is actually constructed in the proof of [25, Theorem 2.6]. $\blacksquare$

Mapping a Cantor cube onto its square. The following proposition is one of the core tools for our investigation of small sets.

PROPOSITION 3.5. *For each Cantor cube $\mathbb{C}$ there are continuous mappings $\pi_1, \pi_2 : \mathbb{C} \to \mathbb{C}$ such that*

(i) *π_1 is nearly Lipschitz,*
(ii) *$h : f \mapsto (\pi_1(f), \pi_2(f))$ is a homeomorphism of $\mathbb{C}$ and $\mathbb{C} \times \mathbb{C}$.*

Proof. Let $A \subseteq \omega$ be an infinite set with density 0, i.e.

$$\lim_{n\to\infty} \frac{|A \cap n|}{n} = 0.$$

Let $\phi : \omega \to \omega \setminus A$ and $\psi : \omega \to A$ be the unique increasing bijections enumerating $\omega \setminus A$ and A. Consider the mappings

$$\pi_1 : \mathbb{C} \to \mathbb{C}, \quad \pi_1(f) = f \circ \phi,$$
$$\pi_2 : \mathbb{C} \to \mathbb{C}, \quad \pi_2(f) = f \circ \psi.$$

and define

$$h(f) = (\pi_1(f), \pi_2(f)) = (f \circ \phi, f \circ \psi), \quad f \in \mathbb{C}.$$

Both π_1 and π_2 are obviously continuous, and thus so is h.

If $f \neq g$, then there is $n \in \omega$ such that $f(n) \neq g(n)$. If $n \notin A$, then $n = \phi(i)$ for some $i \in \omega$ and thus $\pi_1(f)(i) = f(n) \neq g(n) = \pi_1(g)(n)$, whence $\pi_1(f) \neq \pi_1(g)$. If $n \in A$, then $\pi_2(f) \neq \pi_2(g)$ by the same argument. In either case, $h(f) \neq h(g)$. Thus h is one-to-one.

For any $(f, g) \in \mathbb{C} \times \mathbb{C}$ the function

$$n \mapsto \begin{cases} f \circ \phi^{-1}(n), & n \in \omega \setminus A, \\ g \circ \psi^{-1}(n), & n \in A, \end{cases}$$

is mapped by h on (f, g). Thus h is also onto. Hence h is a homeomorphism.

It remains to prove that π_1 is nearly Lipschitz. To that end we have to show that given any $\varepsilon > 0$ there is $\delta > 0$ such that

$$\rho(f \circ \phi, g \circ \phi) \leq \rho(f, g)^{1-\varepsilon} \quad \text{whenever} \quad \rho(f, g) < \delta. \tag{5}$$

Let $\boldsymbol{r} \in (0, 1)^k$ be such that $\mathbb{C} = \mathbb{C}(\boldsymbol{r})$. Denote $r_{\max} = \max_{i<k} r_i$ and $r_{\min} = \min_{i<k} r_i$. Choose

- $\eta < \infty$ such that $r_{\max}^{\eta} < r_{\min}$,
- $N \in \omega$ such that $\eta|A \cap n|/n < \varepsilon$ for all $n \geq N$,
- $\delta < r_{\min}^N$.

Let $f, g \in k^\omega$ be such that $\rho(f, g) < \delta$. Put $p = f \wedge g$ and $n = n(f, g) = |p|$. If $\phi(i) < n$, then obviously $f(\phi(i)) = g(\phi(i))$. Therefore

$$\rho(f \circ \phi, g \circ \phi) \leq \prod_{i<n, i\notin A} r_{p(i)} \leq \frac{\prod_{i<n} r_{p(i)}}{\prod_{i<n, i\in A} r_{p(i)}} \leq \frac{\chi(p)}{r_{\min}^{|A\cap n|}} \leq \frac{\chi(p)}{r_{\max}^{\eta|A\cap n|}}. \tag{6}$$

Since $r_{\min}^n \leqslant \rho(f, g) < \delta < r_{\min}^N$, we have $n > N$. Since $\chi(p) \leqslant r_{\max}^n$,

$$r_{\max}^{\eta|A\cap n|} \geqslant \chi(p)^{\eta|A\cap n|/n} \geqslant \chi(p)^{\varepsilon}$$

by the choice of N. Therefore (6) yields (5) and π_1 is nearly Lipschitz. ■

4. Monotone spaces and Cantor cubes. In this section we show that a substantial portion of an analytic metric space satisfying a certain linearity property can be mapped onto any Cantor cube of the same Hausdorff dimension by a nearly Lipschitz map. The property is tailored to fit the proof and is general enough to cover two important classes of spaces: subsets of the line and ultrametric spaces.

Monotone spaces. The following is taken from [21]. Linear orders are considered. We adhere to the usual interval notation $(\leftarrow, a] = \{x \in X : x \leq a\}$, $(a, b) = \{x \in X : a < x < b\}$ and likewise $[a, \rightarrow)$, $[a, b]$ etc.

Definition 4.1 ([21]). A metric space (X, d) is called 1-*monotone* if there is a linear order $<$ on X such that $d(x, y) \leq d(x, z)$ for all $x < y < z$ in X.

More generally, (X, d) is called *monotone* if there is a linear order $<$ on X and a constant c such that $d(x, y) \leq c\, d(x, z)$ for all $x < y < z$ in X.

X is termed σ-*monotone* if it is a countable union of monotone subsets.

Proposition 4.2 ([21]).

- (i) *A metric space is monotone if and only if it is bi-Lipschitz equivalent to a* 1*-monotone space.*
- (ii) *A subspace of a monotone metric space is monotone.*
- (iii) *If X is σ-monotone, then it is a countable union of closed monotone subspaces.*

Recall that a metric space (X, d) is termed *ultrametric* if the triangle inequality reads

$$d(x, z) \leq \max(d(x, y), d(y, z)).$$

Cantor cubes are obviously ultrametric spaces.

Proposition 4.3 ([21]). *Each ultrametric space is monotone. In particular, each Cantor cube and each self-similar set satisfying* SSP *is monotone.*

Corollary 4.4. *If X is a* SOSC *self-similar set and* $0 \leq s < \dim_{\mathsf{H}} X$, *then there is a closed monotone set $E \subseteq X$ such that* $\dim_{\mathsf{H}} E = s$.

Proof. By Theorem 3.4 there is a SSP set $C \subseteq X$ such that $\dim_{\mathsf{H}} C > s$. The Davies Theorem yields a compact set $E \subseteq C$ such that $\dim_{\mathsf{H}} E = s$. By Lemma 3.3 and the above proposition, E is the required set. ∎

Monotone spaces vs. Cantor cubes. We now show that an analytic σ-monotone space contains a nearly Lipschitz preimage of a Cantor cube.

Theorem 4.5. *Let X be an analytic σ-monotone metric space and $s > 0$ such that $\mathcal{H}^s(X) > 0$. If $\mathbb{C}$ is a Cantor cube with* $\dim_{\mathsf{H}} \mathbb{C} \leq s$, *then there is a compact set $C \subseteq X$ such that $\mathcal{H}^s(C) > 0$ and a nearly Lipschitz onto mapping $\phi : C \to \mathbb{C}$.*

Proof. Proposition 4.2 ensures that X may be assumed to be 1-monotone. Observe that if $E \subseteq X$ is compact, then $\max E$ and $\min E$ exist and belong to E. Indeed, the set $\bigcap_{x\in E} E \cap [x, \to)$, being an intersection of a directed family of compact sets, is nonempty. Its (unique) element is the maximum of E; and likewise for the minimum.

We may, and will, assume that $s = \dim_{\mathsf{H}} \mathbb{C}$.

According to the Davies Theorem we may also assume that X is compact. In particular, there are $a, b \in X$ such that $X = [a, b]$.

Frostman's Lemma yields a nontrivial finite Borel measure $\mu \le \mathcal{H}^s$ on X such that

$$\mu B(x, r) \le r^s \quad \text{for each } x \in X \text{ and } r > 0. \tag{7}$$

We show that μ is Darboux in the following sense: For each $E \subseteq X$ compact,

$$\textit{if } 0 \le t \le \mu(E), \textit{ then } \mu([a, x] \cap E) = t \textit{ for some } x \in X. \tag{8}$$

Notice that 1-monotonicity of X ensures that $[x, y] \subseteq B(x, d(x, y))$. Therefore the mapping $g(x) = \mu([a, x] \cap E)$ is continuous:

$$g(y) - g(x) \le \mu[x, y] \le \mu B(x, d(x, y)) \le d(x, y)^s.$$

It follows that the sets

$$X_0 = \{x \in X : \mu([a, x] \cap E) \le t\}, \quad X_1 = \{x \in X : \mu([a, x] \cap E) \ge t\}$$

are closed and hence compact. Let $x_0 = \max X_0$, $x_1 = \min X_1$. If $x_1 \le x_0$, then $t \le g(x_1) \le g(x_0) \le t$ and we can set $x = x_1$. If $x_1 > x_0$, then the interval (x_0, x_1) is empty and subadditivity of μ yields $g(x_1) \le g(x_0) + \mu(x_0, x_1] = g(x_0) \le t \le g(x_1)$ and we can set $x = x_1$, too. (8) is proved.

Let $\boldsymbol{r} = \langle r_0, r_1, \dots, r_{k-1}\rangle \in (0, 1)^k$ be the parameters of the Cantor cube $\mathbb{C} = \mathbb{C}(\boldsymbol{r})$. Set $r_{\max} = \max_{i<k} r_i$ and $r_{\min} = \min_{i<k} r_i$. Since $r^s_{\max} < 1$, there is a sequence $\gamma_n \searrow 0$ of positive numbers such that $\sum_{n=1}^{\infty} (r^s_{\max})^{n\gamma_n} < \mu(X)$. Put

$$\beta = \mu(X) - \sum_{n=1}^{\infty} (r^s_{\max})^{n\gamma_n},$$

so that $\beta > 0$. (By choosing $\{\gamma_n\}$ appropriately, we can make β as close to $\mu(X)$ as we wish.)

For each $p \in k^{<\omega}$ put

$$\varepsilon_p = \chi(p)^{1+\gamma_{|p|}}.$$

For each $p \in k^{<\omega}$, we construct inductively a compact set $E_p \subset X$ as follows. Put $E_\emptyset = X$. Now suppose E_p has been constructed. Recalling Moran's equation (4) and using (8), choose

$$a \le t_{p^\frown 0} < t_{p^\frown 1} < \dots < t_{p^\frown k} \le b \in X$$

so that

$$\mu(E_p \cap [t_{p^\frown i}, t_{p^\frown (i+1)}]) = r_i^s \, \mu E_p \quad \text{for } i < k$$

and define for $i < k$,

$$E_{p^\frown i} = \{x \in E_p \cap [t_{p^\frown i}, t_{p^\frown (i+1)}] : d(x, t_{p^\frown (i+1)}) \geq \varepsilon_{p^\frown i}\}.$$

Note that

$$E_{p^\frown i} \supseteq E_p \cap [t_{p^\frown i}, t_{p^\frown (i+1)}] \setminus B(t_{p^\frown (i+1)}, \varepsilon_{p^\frown i}) \tag{9}$$

and therefore (7) yields

$$\mu(E_{p^\frown i}) \geq r_i^s \mu E_p - \varepsilon_{p^\frown i}^s = r_i^s \mu E_p - \chi(p^\frown i)^{s(1+\gamma_{|p^\frown i|})}. \tag{10}$$

We prove by induction that

$$\mu(X)\chi(p)^s \geq \mu(E_p) \geq \chi(p)^s \Big(\mu(X) - \sum_{n=1}^{|p|} (r_{\max}^s)^{n\gamma_n}\Big) \tag{11}$$

for all $p \in k^{<\omega}$. The left inequality is obvious. We prove the right one. For $p = \emptyset$ it is trivial. Assume that it holds for p and let $i < k$. Put $m = |p|$. By (10),

$$\begin{aligned}
\mu E_{p^\frown i} &\geq r_i^s \mu E_p - \chi(p^\frown i)^{s(1+\gamma_{|p^\frown i|})} \\
&\geq r_i^s \Big(\chi(p)^s \mu X - \chi(p)^s \sum_{n=1}^{m} (r_{\max}^s)^{n\gamma_n}\Big) - \chi(p^\frown i)^s \chi(p^\frown i)^{s\gamma_{m+1}} \\
&\geq \chi(p^\frown i)^s \mu X - \chi(p^\frown i)^s \Big(\sum_{n=1}^{m} (r_{\max}^s)^{n\gamma_n} + r_{\max}^{(m+1)s\gamma_{m+1}}\Big) \\
&= \chi(p^\frown i)^s \Big(\mu X - \sum_{n=1}^{m+1} (r_{\max}^s)^{n\gamma_n}\Big).
\end{aligned}$$

The induction step is complete and (11) follows. The definition of β thus gives

$$\mu(X)\chi(p)^s \geq \mu(E_p) \geq \beta \chi(p)^s \tag{12}$$

for all $p \in k^{<\omega}$. Thus $\mu E_p > 0$, and in particular $E_p \neq \emptyset$. All E_p are obviously compact and $E_p \subseteq E_q$ whenever p extends q. Hence the set

$$C_f = \bigcap_{n\in\omega} E_{f\restriction n}$$

is nonempty for each $f \in k^\omega$. Put

$$C = \bigcup_{f \in k^\omega} C_f = \bigcap_{n\in\omega} \bigcup_{p \in k^n} E_p.$$

We estimate $\mu(C)$. Moran's equation for $\mathbb{C}$ gives $\sum_{p\in k^n} \chi(p)^s = 1$ for each $n \in \omega$. Hence (12) yields

$$\mu(C) = \lim_{n\to\infty} \sum_{p\in k^n} \mu(E_p) \geq \lim_{n\to\infty} \beta \sum_{p\in k^n} \chi(p)^s \geq \beta > 0$$

and $\mathcal{H}^s(C) > 0$ follows, since $\mu \leq \mathcal{H}^s$.

Define $\phi : C \to k^\omega$ by $\phi(x) = f$ iff $x \in C_f$. It is obviously surjective. Provide k^ω with the corresponding metric $\rho = \rho_{\boldsymbol{r}}$. It remains to show that $\phi : C \to (k^\omega, \rho)$ is nearly Lipschitz. Let $\varepsilon > 0$ be given. Since $\gamma_n \searrow 0$, there is $n \in \omega$ such that $1 - \varepsilon < 1/(1+\gamma_n)$. Put $\delta = \min_{|p|<n} \varepsilon_p$.

Assume that $x < y$ in C are such that $d(x,y) < \delta$. If $\phi(x) = \phi(y)$, there is nothing to prove. Otherwise there are $p \in k^{<\omega}$ and $i < j < k$ such that $x \in E_{p^\frown i}$ and $y \in E_{p^\frown j}$. Here the assumption that X is 1-monotone comes into play again: Since $i < j$, we have $x < t_{p^\frown(i+1)} \le y$ and $d(x, t_{p^\frown(i+1)}) \ge \varepsilon_{p^\frown i}$. Therefore $d(x,y) \ge \varepsilon_{p^\frown i}$ and thus $\delta > \varepsilon_{p^\frown i}$. The definition of δ hence yields $|p^\frown i| \ge n$. Therefore

$$d(x,y) \ge \varepsilon_{p^\frown i} \ge \chi(p^\frown i)^{1+\gamma_n} \ge \chi(p)^{1+\gamma_n} r_{\min}^{1+\gamma_n}. \tag{13}$$

Since $\rho(\phi x, \phi y) = \chi(p)$, this inequality gives

$$\rho(\phi x, \phi y) = \chi(p) \le \frac{1}{r_{\min}}\, d(x,y)^{1/(1+\gamma_n)} \le \frac{1}{r_{\min}}\, d(x,y)^{1-\varepsilon},$$

which is enough. ∎

Lemma 3.2 yields the following corollary.

Corollary 4.6. *Let X be an analytic σ-monotone metric space and K a self-similar set. If $\mathcal{H}^{\dim_{\mathsf{S}} K}(X) > 0$, then there is a compact set $C \subseteq X$ such that $\mathcal{H}^{\dim_{\mathsf{S}} K}(C) > 0$ and a nearly Lipschitz onto mapping $\phi : C \to K$.*

Replacing $\mathcal{H}^s(X) > 0$ with $\dim_{\mathsf{H}} \mathbb{C} < \dim_{\mathsf{H}} X$ yields a stronger conclusion:

Theorem 4.7. *Let X be an analytic σ-monotone metric space and $\mathbb{C}$ a Cantor cube. If $\dim_{\mathsf{H}} \mathbb{C} < \dim_{\mathsf{H}} X$, then there is a compact set $C \subseteq X$ and a Lipschitz onto mapping $\phi : C \to \mathbb{C}$.*

Proof. We briefly show how to modify the above proof. Let $\mathbb{C} = \mathbb{C}(\boldsymbol{r})$ and let $\dim_{\mathsf{H}} \mathbb{C} = u < s < \dim_{\mathsf{H}} X$. *Mutatis mutandis* we may assume that $X = [a,b]$ is compact and $\mathcal{H}^s(X) > 0$. Let μ be the measure satisfying (7) and (8). Put $R = r_{\max}^{s-u}$. Then $R < 1$ and therefore the series $\sum R^n$ is convergent. Thus we can choose $\eta > 0$ so that $\eta^s \sum_{n=1}^\infty R^n < \mu X$.

This time put $\varepsilon_p = \eta\chi(p)$ and construct E_p's so that (10) reads (note that Moran's equation is now $\sum r_i^u = 1$)

$$\mu(E_{p^\frown i}) \ge r_i^u \mu E_p - \eta^s \chi(p^\frown i)^s.$$

The replacement for the estimate (11) now reads

$$\mu(E_p) \ge \chi(p)^u \Big(\mu(X) - \eta^s \sum_{n=1}^{|p|} R^n\Big)$$

and the replacement for (13) is

$$d(x,y) \ge \eta\chi(p^\frown i) \ge \eta r_{\min}\chi(p) = \eta r_{\min}\rho(\phi(x), \phi(y)),$$

which shows that ϕ is Lipschitz. ∎

COROLLARY 4.8. *Let K_1, K_2 be self-similar sets. Let K_1 satisfy* SOSC. *If* $\dim_{\mathsf{S}} K_2 < \dim_{\mathsf{S}} K_1$, *then there is a compact set $C \subseteq K_1$ and a Lipschitz onto mapping $\phi : C \to K_2$.*

Proof. Since K_1 satisfies SOSC, $\dim_{\mathsf{S}} K_1 = \dim_{\mathsf{H}} K_1$ and by Corollary 4.4 there is a closed monotone subset $X \subseteq K_1$ such that $\dim_{\mathsf{H}} X > \dim_{\mathsf{S}} K_2$. Apply Lemma 3.2 and the above theorem. ∎

5. Small sets with large dimension. Our goal is to get small sets of large Hausdorff dimension. Examples of classes of small sets are e.g. sets of universal measure zero, Q-sets or universally meager sets (cf. Section 8), to name but a few. In this and the next section we deal with a notion of small set in a rather general setting:

Suppose $\boldsymbol{\mathcal{S}}$ is some class of separable metric spaces. Elements of $\boldsymbol{\mathcal{S}}$ are thought of as "small sets". We will consider the following closure properties of $\boldsymbol{\mathcal{S}}$.

- $\boldsymbol{\mathcal{S}}$ is *backwards closed*. That means that $\boldsymbol{\mathcal{S}}$ is closed under one-to-one continuous preimages, i.e. if $f : X \to Y$ is one-to-one continuous and $Y \in \boldsymbol{\mathcal{S}}$, then $X \in \boldsymbol{\mathcal{S}}$. Note that if $\boldsymbol{\mathcal{S}}$ is backwards closed, then it is closed under homeomorphisms and embeddings.
- $\boldsymbol{\mathcal{S}}$ is *strongly backwards closed*. That means that $\boldsymbol{\mathcal{S}}$ is closed under $\boldsymbol{\mathcal{S}}$-to-one continuous preimages, i.e. if $f : X \to Y$ is continuous, $f^{-1}(y) \in \boldsymbol{\mathcal{S}}$ for all $y \in Y$ and $Y \in \boldsymbol{\mathcal{S}}$, then $X \in \boldsymbol{\mathcal{S}}$.
- $\boldsymbol{\mathcal{S}}$ is *σ-additive*, i.e. if X is a union of countably many elements of $\boldsymbol{\mathcal{S}}$, then $X \in \boldsymbol{\mathcal{S}}$.

Cardinal invariants. The following are some of the most common ideals in topology, set theory and measure theory.

- Let μ denote the standard product measure on 2^ω, and $\mathcal{N}$ the σ-ideal of μ-negligible subsets of 2^ω.
- $\mathcal{M}$ is the σ-ideal of meager subsets of 2^ω.
- $\mathcal{E}$ is the σ-ideal generated by μ-negligible closed subsets of 2^ω.

For any ideal $\mathcal{J}$ denote $\mathsf{non}\,\mathcal{J} = \min\{|A| : A \notin \mathcal{J}\}$.

It is well-known that either of the three relations of $\mathsf{non}\,\mathcal{N}$ and $\mathsf{non}\,\mathcal{M}$ is consistent with ZFC. It is obvious that $\mathsf{non}\,\mathcal{E} \le \min(\mathsf{non}\,\mathcal{N}, \mathsf{non}\,\mathcal{M})$. It is also known that both $\mathsf{non}\,\mathcal{E} = \min(\mathsf{non}\,\mathcal{N}, \mathsf{non}\,\mathcal{M})$ and $\mathsf{non}\,\mathcal{E} < \min(\mathsf{non}\,\mathcal{N}, \mathsf{non}\,\mathcal{M})$ are consistent (see [1]).

The $\mathcal{N}$ part of the following fact is well-known and follows at once from the Isomorphism Theorem for Measures ([14, (17.41)]).

LEMMA 5.1. *Let ν be a nontrivial finite diffused Borel measure on 2^ω, and $\mathcal{N}(\nu)$, $\mathcal{E}(\nu)$, respectively, the σ-ideals generated by ν-negligible sets and ν-negligible closed sets. Then* $\mathsf{non}\,\mathcal{N}(\nu) = \mathsf{non}\,\mathcal{N}$ *and* $\mathsf{non}\,\mathcal{E}(\nu) = \mathsf{non}\,\mathcal{E}$.

Proof. We outline the proof of the $\mathcal{E}$ part. Assume without loss of generality that ν is a diffused probability Borel measure on a compact set $S \subseteq [0,1]$ that is strictly positive (i.e. $\nu(U) > 0$ for each nonempty set U open in S). For $x \in S$ set $g(x) = \nu([0,x])$. The mapping $g : S \to [0,1]$ is obviously continuous and onto. Denoting by λ the Lebesgue measure on $[0,1]$, it is also obvious that the equivalence $E \in \mathcal{E}(\nu) \Leftrightarrow g(E) \in \mathcal{E}(\lambda)$ holds for any set $E \subseteq S$. Since g is two-to-one, this is enough for $\mathsf{non}\,\mathcal{E}(\nu) = \mathsf{non}\,\mathcal{E}(\lambda)$. In particular, $\mathsf{non}\,\mathcal{E} = \mathsf{non}\,\mathcal{E}(\lambda)$, and $\mathsf{non}\,\mathcal{E}(\nu) = \mathsf{non}\,\mathcal{E}$ follows. ■

Small sets in σ-monotone spaces. The goal of this subsection is to show that if $\boldsymbol{\mathcal{S}}$ is backwards closed, then the existence of a set in $\boldsymbol{\mathcal{S}}$ with large cardinality implies that many sets in $\boldsymbol{\mathcal{S}}$ have large Hausdorff or upper Hausdorff dimension.

In what follows, we often employ cardinal hypotheses on $\boldsymbol{\mathcal{S}}$: Given a cardinal κ (in most cases $\kappa = \mathsf{non}\,\mathcal{N}$ or $\kappa = \mathsf{non}\,\mathcal{E}$), define

$$\kappa \prec \boldsymbol{\mathcal{S}} \ := \ \text{there is } S \in \boldsymbol{\mathcal{S}} \text{ such that } |S| = \kappa.$$

Note that if $\boldsymbol{\mathcal{S}}$ is backwards closed and $\kappa \prec \boldsymbol{\mathcal{S}}$, then the witnessing space can be actually assumed to be a subset of 2^ω, for any separable metric space embeds into $[0,1]^\omega$ and the latter is a continuous image of 2^ω.

We first prove the result for Cantor cubes.

LEMMA 5.2 (Cantor Cube Lemma). *Let $\boldsymbol{\mathcal{S}}$ be backwards closed and $\mathbb{C}$ a Cantor cube.*

(i) *If $\mathsf{non}\,\mathcal{N} \prec \boldsymbol{\mathcal{S}}$, then $\mathbb{C}$ contains a set $E \in \boldsymbol{\mathcal{S}}$ such that $\dim_{\mathsf{H}} E = \dim_{\mathsf{H}} \mathbb{C}$.*
(ii) *If $\mathsf{non}\,\mathcal{E} \prec \boldsymbol{\mathcal{S}}$, then $\mathbb{C}$ contains a set $E \in \boldsymbol{\mathcal{S}}$ such that $\overline{\dim}_{\mathsf{H}} E = \dim_{\mathsf{H}} \mathbb{C}$.*

Proof. We prove only (i), since (ii) is proved in the same manner. Let $s = \dim_{\mathsf{H}} \mathbb{C}$. Recall that $\mathcal{H}^s(\mathbb{C}) = 1$, so that $\mathcal{H}^s$ is a finite Borel measure on $\mathbb{C}$. Hence Lemma 5.1 yields a set $A \subseteq \mathbb{C}$ such that $\mathcal{H}^s(A) > 0$ and $|A| = \mathsf{non}\,\mathcal{N}$. Since $\boldsymbol{\mathcal{S}}$ is backwards closed and $\mathbb{C}$ is homeomorphic to 2^ω, there is $B \in \mathcal{P}(\mathbb{C}) \cap \boldsymbol{\mathcal{S}}$ such that $|B| = \mathsf{non}\,\mathcal{N}$. Enumerate the two sets $A = \{a_\alpha : \alpha < \mathsf{non}\,\mathcal{N}\}$, $B = \{b_\alpha : \alpha < \mathsf{non}\,\mathcal{N}\}$. Consider the diagonal set $D = \{(a_\alpha, b_\alpha) : \alpha < \mathsf{non}\,\mathcal{N}\}$ and the mappings π_1, π_2 and h of Proposition 3.5. For each $\alpha < \mathsf{non}\,\mathcal{N}$ set $e_\alpha = h^{-1}(a_\alpha, b_\alpha)$ and put

$$E = \{e_\alpha : \alpha < \mathsf{non}\,\mathcal{N}\} = h^{-1}(D).$$

Obviously $\pi_1(e_\alpha) = a_\alpha$ and $\pi_2(e_\alpha) = b_\alpha$. Therefore π_1 takes E onto A and π_2 takes E onto B. The mapping π_2 is one-to-one on E and $B \in \boldsymbol{\mathcal{S}}$. Hence $E \in \boldsymbol{\mathcal{S}}$ as well. On the other hand, since π_1 is nearly Lipschitz, Lemma 2.6 ensures that $\dim_{\mathsf{H}} E \geq \dim_{\mathsf{H}} \pi_1 E = \dim_{\mathsf{H}} A = s = \dim_{\mathsf{H}} \mathbb{C}$, as required. ■

The main theorem of this section follows from the Cantor Cube Lemma and Theorem 4.5 by use of the following easy lemma.

LEMMA 5.3 (Preimage Lemma). *Let $\boldsymbol{\mathcal{S}}$ be backwards closed. Let X, Y be metric spaces, $\phi : X \to Y$ a nearly Lipschitz mapping onto Y, and $D \in \boldsymbol{\mathcal{S}}$ a subset of Y. There is a subset $E \in \boldsymbol{\mathcal{S}}$ of X such that* $\dim_{\mathsf{H}} E \geq \dim_{\mathsf{H}} D$ *and* $\overline{\dim}_{\mathsf{H}} E \geq \overline{\dim}_{\mathsf{H}} D$.

Proof. For each $y \in D$, pick $\widetilde{y} \in \phi^{-1}(y)$ and set $E = \{\widetilde{y} : y \in D\}$. Then $\phi : E \to D$ is one-to-one, onto and nearly Lipschitz. Hence $E \in \boldsymbol{\mathcal{S}}$ and the remainder of the lemma follows directly from Lemma 2.6. ■

THEOREM 5.4. *Let $\boldsymbol{\mathcal{S}}$ be backwards closed. Let X be an analytic σ-monotone space, and $s > 0$ such that $\mathcal{H}^s(X) > 0$.*

(i) *If* $\mathsf{non}\,\mathcal{N} \prec \boldsymbol{\mathcal{S}}$, *then X contains a set $E \in \boldsymbol{\mathcal{S}}$ such that* $\dim_{\mathsf{H}} E = s$.
(ii) *If* $\mathsf{non}\,\mathcal{E} \prec \boldsymbol{\mathcal{S}}$, *then X contains a set $E \in \boldsymbol{\mathcal{S}}$ such that* $\overline{\dim}_{\mathsf{H}} E = s$.

Proof. We prove only (i), since (ii) is proved in the same manner. By Theorem 4.5 there is a compact set $C \subseteq X$ and a nearly Lipschitz map $\phi : C \to \mathbb{C}$ onto any Cantor cube for which $\dim_{\mathsf{H}} \mathbb{C} = s$. By the Cantor Cube Lemma there is a set $D \in \mathcal{P}(\mathbb{C}) \cap \boldsymbol{\mathcal{S}}$ such that $\dim_{\mathsf{H}} D = s$. Finally, by the Preimage Lemma there is a set $E \in \mathcal{P}(X) \cap \boldsymbol{\mathcal{S}}$ such that $\dim_{\mathsf{H}} D = s$. ■

COROLLARY 5.5. *Let $\boldsymbol{\mathcal{S}}$ be backwards closed and σ-additive. Let X be an analytic σ-monotone space.*

(i) *If* $\mathsf{non}\,\mathcal{N} \prec \boldsymbol{\mathcal{S}}$, *then X contains a set $E \in \boldsymbol{\mathcal{S}}$ such that* $\dim_{\mathsf{H}} E = \dim_{\mathsf{H}} X$.
(ii) *If* $\mathsf{non}\,\mathcal{E} \prec \boldsymbol{\mathcal{S}}$, *then X contains a set $E \in \boldsymbol{\mathcal{S}}$ such that* $\overline{\dim}_{\mathsf{H}} E = \dim_{\mathsf{H}} X$.

Proof. Choose a sequence $s_n \nearrow \dim_{\mathsf{H}} X$. Using Theorem 5.4 for each n, we can find $E_n \in \mathcal{P}(X) \cap \boldsymbol{\mathcal{S}}$ for which $\dim_{\mathsf{H}} E_n \geq s_n$ ($\overline{\dim}_{\mathsf{H}} E_n \geq s_n$, respectively). Now let $E = \bigcup_n E_n$ and appeal to σ-additivity of $\boldsymbol{\mathcal{S}}$ to conclude $E \in \boldsymbol{\mathcal{S}}$. ■

Varying the closure and cardinal hypotheses yields a number of corollaries. Here is a sample one that follows from Proposition 4.3 and Corollary 4.4.

COROLLARY 5.6. *Let $\boldsymbol{\mathcal{S}}$ be backwards closed and σ-additive and suppose* $\mathsf{non}\,\mathcal{N} \prec \boldsymbol{\mathcal{S}}$. *Then:*

(i) *Each analytic ultrametric space X contains a set $E \in \boldsymbol{\mathcal{S}}$ for which* $\dim_{\mathsf{H}} E = \dim_{\mathsf{H}} X$.
(ii) *If X is a* SOSC *self-similar set and* $0 \leq s \leq \dim_{\mathsf{H}} X$, *then X contains a set $E \in \boldsymbol{\mathcal{S}}$ for which* $\dim_{\mathsf{H}} E = s$.

Small sets in Euclidean spaces. The preceding results certainly apply to analytic subsets of the line. To extend them to higher dimensions, we use classical projection and intersection theorems. Let $n \in \omega$ and let V be a linear subspace of $\mathbb{R}^n$. Denote the orthogonal complement of V by $V^\perp$ and the orthogonal projection onto V by proj_V. If $x \in \mathbb{R}^n$, then $V + x$ denotes the shift of V by x, i.e. the unique affine copy of V parallel to V passing through x.

Let $m \in \omega$, $m < n$. The symbol $G(n,m)$ denotes the *Grassmann manifold*, which is the space of all m-dimensional linear subspaces of $\mathbb{R}^n$. The *Grassmann measure* is the unique uniform Borel probability measure on $G(n,m)$ and is denoted by $\gamma_{n,m}$. We refer to [15] for further details.

LEMMA 5.7. *Let $X \subseteq \mathbb{R}^n$ be compact and $m < s \le n$. If $0 < \mathcal{H}^s(X) < \infty$, then the set*

$$X_V = \{x \in V : \dim_{\mathsf{H}} X \cap (V^\perp + x) = s - m\}$$

is Borel and for $\gamma_{n,m}$-almost all $V \in G(n,m)$, $\mathcal{H}^m(X_V) > 0$.

Proof. This is [15, Theorem 10.10], except for "X_V is Borel". But that follows at once from [16, Theorem 6.1]. ∎

LEMMA 5.8 ([15, Corollary 9.4]). *Let $X \subseteq \mathbb{R}^n$ be Borel. If* $\dim_{\mathsf{H}} X \le m$, *then* $\dim_{\mathsf{H}} \operatorname{proj}_V(X) = \dim_{\mathsf{H}} X$ *for $\gamma_{n,m}$-almost all $V \in G(n,m)$.*

LEMMA 5.9 ([7, Corollary 7.12]). *Let $E \subseteq \mathbb{R}^n$ be arbitrary, $V \in G(n,m)$ and $D \subseteq \operatorname{proj}_V E$. Then*

$$\dim_{\mathsf{H}} E \ge \dim_{\mathsf{H}} D + \inf_{x \in D} \dim_{\mathsf{H}} E \cap (V^\perp + x).$$

We also need the upper Hausdorff dimension counterpart of this lemma.

LEMMA 5.10. *Let $E \subseteq \mathbb{R}^n$ be arbitrary, $V \in G(n,m)$ and $D \subseteq \operatorname{proj}_V E$. Then*

$$\overline{\dim}_{\mathsf{H}} E \ge \overline{\dim}_{\mathsf{H}} D + \inf_{x \in D} \overline{\dim}_{\mathsf{H}} E \cap (V^\perp + x).$$

Proof. For $x \in V$ and any set $A \subseteq \mathbb{R}^n$ write $A^x = A \cap (V^\perp + x)$. Let $c < \inf_{x \in D} \overline{\dim}_{\mathsf{H}} E^x$ and $d < \overline{\dim}_{\mathsf{H}} D$. It is enough to show that $\overline{\dim}_{\mathsf{H}} E \ge c + d$, so, aiming for a contradiction, assume the contrary. Then there is a countable family of compact sets $\{K_n\}$ such that $E \subseteq \bigcup_n K_n$ and

$$\dim_{\mathsf{H}} K_n < c + d, \quad n \in \omega. \tag{14}$$

Fix $x \in D$. Since $E^x \subseteq \bigcup_n K_n^x$, there is n such that $\dim_{\mathsf{H}} K_n^x > c$, which in turn yields $\delta > 0$ such that $\mathcal{H}^c_\delta(K_n^x) \ge 1$ (cf. (1)). Therefore

$$D = \bigcup_{n \in \omega,\, \delta > 0} \{x \in D : \mathcal{H}^c_\delta(K_n^x) \ge 1\}.$$

In particular there are n and $\delta > 0$ such that

$$\overline{\dim}_{\mathsf{H}}\{x \in D : \mathcal{H}^c_\delta(K^x_n) \geq 1\} > d. \tag{15}$$

The initial part of the proof of [16, Theorem 6.1] shows that the mapping $x \mapsto \mathcal{H}^c_\delta(K^x_n)$ is upper semicontinuous on V. Therefore

$$C = \{x \in V : \mathcal{H}^c_\delta(K^x_n) \geq 1\} \tag{16}$$

is closed. It is also bounded, for K_n is compact. Hence C is a compact set. Therefore (15) yields $\dim_{\mathsf{H}} C > d$, and (16) yields $\dim_{\mathsf{H}} K^x_n \geq c$ for all $x \in C$. Finally, apply Lemma 5.9 to get $\dim_{\mathsf{H}} K_n \geq c + d$, which contradicts (14). ∎

THEOREM 5.11. *Let $\boldsymbol{\mathcal{S}}$ be strongly backwards closed and σ-additive. Let $X \subseteq \mathbb{R}^n$ be analytic.*

- (i) *If* $\mathsf{non}\,\mathcal{N} \prec \boldsymbol{\mathcal{S}}$, *then X contains a set $E \in \boldsymbol{\mathcal{S}}$ such that* $\dim_{\mathsf{H}} E = \dim_{\mathsf{H}} X$.
- (ii) *If* $\mathsf{non}\,\mathcal{E} \prec \boldsymbol{\mathcal{S}}$, *then X contains a set $E \in \boldsymbol{\mathcal{S}}$ such that* $\overline{\dim}_{\mathsf{H}} E = \dim_{\mathsf{H}} X$.

Proof. (i) Let $n \in \omega$. For $n = 1$ the assertion follows at once from Corollary 5.5. We proceed by induction. Assume that $n > 1$ and that the assertion holds for $n-1$. Let $X \subseteq \mathbb{R}^n$ be an analytic set and let $s = \dim_{\mathsf{H}} X$.

If $s \leq n-1$, then there is, by Lemma 5.8, a hyperplane $V \in G(n, n-1)$ such that $\dim_{\mathsf{H}} \operatorname{proj}_V X = s$. The induction hypothesis yields a set $A \subseteq \operatorname{proj}_V X$, $A \in \boldsymbol{\mathcal{S}}$, such that $\dim_{\mathsf{H}} A = s$. Since proj_V is Lipschitz, the Preimage Lemma yields a set $E \in \mathcal{P}(X) \cap \boldsymbol{\mathcal{S}}$ such that $\dim_{\mathsf{H}} E = s$.

Now suppose $s > n-1$. Since $\boldsymbol{\mathcal{S}}$ is σ-additive, we may assume that $\mathcal{H}^s(X) > 0$. Passing to the subset whose existence is guaranteed by the Davies Theorem, we may assume that X is compact and $0 < \mathcal{H}^s(X) < \infty$. From Lemma 5.7 (with $m = 1$), there is a line $L \in G(n, 1)$ such that the set

$$X_L = \{x \in L : \dim_{\mathsf{H}} X \cap (L^\perp + x) = s - 1\}$$

is Borel and satisfies $\mathcal{H}^1(X_L) > 0$. Now apply the induction hypothesis to obtain, firstly, a set $D \subseteq X_L$, $D \in \boldsymbol{\mathcal{S}}$ with $\dim_{\mathsf{H}} D = 1$ and, secondly, for each $x \in D$, a set $E_x \subseteq X \cap (L^\perp + x)$ with $E_x \in \boldsymbol{\mathcal{S}}$ and $\dim_{\mathsf{H}} E_x = s - 1$. Put

$$E = \bigcup_{x \in D} E_x$$

so that the sections of E perpendicular to V are precisely the E_x's and thus have Hausdorff dimensions $s - 1$. By Lemma 5.9, the set E satisfies $\dim_{\mathsf{H}} E \geq (s-1) + \dim_{\mathsf{H}} D = s$. Consider the restriction $\pi = \operatorname{proj}_L \restriction E$ of the projection. It is Lipschitz and takes E onto a set $D \in \boldsymbol{\mathcal{S}}$. Moreover $\pi^{-1}(x) = E_x \in \boldsymbol{\mathcal{S}}$ for all $x \in D$. Consequently, $E \in \boldsymbol{\mathcal{S}}$, for $\boldsymbol{\mathcal{S}}$ is strongly backwards closed. The induction step is complete, and so is the proof of (i).

(ii) is proved exactly the same way, except that Lemma 5.9 has to be replaced by Lemma 5.10. ∎

I do not know if "strongly backwards closed" can be relaxed to "backwards closed" in the preceding theorem.

6. General metric spaces. The methods used to get a small subset of large Hausdorff dimension apply only to special classes of metric spaces. In particular we do not know the answer to the following question:

QUESTION 6.1. Let $\boldsymbol{\mathcal{S}}$ be backwards closed and σ-additive and $\mathsf{non}\,\mathcal{N} \prec \boldsymbol{\mathcal{S}}$. Given a compact metric space X, is there a set $E \subseteq X$ with $E \in \boldsymbol{\mathcal{S}}$ and $\dim_{\mathsf{H}} E = \dim_{\mathsf{H}} X$?

Note that by the Davies Theorem, relaxing "compact" to "analytic" does not affect the answer. Proposition 8.2 below shows that separability of X alone is not enough for an affirmative answer.

Our next goal is to establish a weaker conclusion that applies to *any* metric space: There is a small set of Hausdorff dimension no less than the *topological* dimension of the space. The topological dimension we consider is covering dimension and we denote it by $\dim X$. Recall that if X is a metric space, then $\dim X$ equals the large inductive dimension of X (the Katětov–Morita Theorem) and if, in addition, X is separable, then $\dim X$ equals the small inductive dimension of X (see [5] or [4]). We shall need the following theorem from [29] and a refinement of the Cantor Cube Lemma 5.2.

THEOREM 6.2 ([29, Lemma 5.1]). *Let X be a metric space. If $\dim X \geq n \in \omega$, then there is a countable family $\{f_i : i \in \omega\}$ of Lipschitz mappings $f_i : X \to [0,1]^n$ such that $\bigcup_{i\in\omega} f_i(X) = [0,1]^n$.*

LEMMA 6.3. *Let $\boldsymbol{\mathcal{S}}$ be backwards closed and $\mathbb{C}$ a Cantor cube. If $\mathsf{non}\,\mathcal{N} \prec \boldsymbol{\mathcal{S}}$, then $\mathbb{C}$ contains a set $E \in \boldsymbol{\mathcal{S}}$ such that for any countable cover $\{L_n : n \in \omega\}$ of $\mathbb{C}$ there is n such that $\dim_{\mathsf{H}}(E \cap L_n) = \dim_{\mathsf{H}} \mathbb{C}$. A similar statement holds for $\mathsf{non}\,\mathcal{E}$ and $\overline{\dim}_{\mathsf{H}}$.*

Proof. We use the notation of the proof of the Cantor Cube Lemma. For each n put $A_n = \{a_\alpha : (a_\alpha, b_\alpha) \in h(L_n)\}$. Since $\{L_n\}$ covers $\mathbb{C}$, the family $\{A_n\}$ covers A. Since $\mathcal{H}^s(A) > 0$, there is n for which $\mathcal{H}^s(A_n) > 0$ and *a fortiori* $\dim_{\mathsf{H}} A_n = s$. If $a_\alpha \in A_n$, then $(a_\alpha, b_\alpha) \in h(L_n) \cap D$. Consequently, $e_\alpha = h^{-1}(a_\alpha, b_\alpha) \in h^{-1}(h(L_n) \cap D) = L_n \cap E$ and thus $\pi_1(e_\alpha) = a_\alpha$. This proves that $A_n \subseteq \pi_1(L_n \cap E)$. Now apply Lemma 2.6 to get $\dim_{\mathsf{H}}(L_n \cap E) = s$. ∎

THEOREM 6.4. *Let $\boldsymbol{\mathcal{S}}$ be backwards closed and X a metric space.*

(i) *If $\mathsf{non}\,\mathcal{N} \prec \boldsymbol{\mathcal{S}}$, then X contains a set $E \in \boldsymbol{\mathcal{S}}$ such that $\dim_{\mathsf{H}} E \geq \dim X$.*

(ii) *If* $\mathsf{non}\,\mathcal{E} \prec \boldsymbol{\mathcal{S}}$, *then* X *contains a set* $E \in \boldsymbol{\mathcal{S}}$ *such that* $\overline{\dim}_{\mathsf{H}} E \geq \dim X$.

Proof. Let $\dim X \geq n \in \omega$. By Theorem 4.5, there is a compact set $C \subseteq [0,1]$ and a nearly Lipschitz onto mapping $\phi : C \to \mathbb{C}(1/2, 1/2)$. Provide the cartesian powers C^n and $(\mathbb{C}(1/2,1/2))^n$ with the maximum metrics. The mapping $\phi^n : C^n \to (\mathbb{C}(1/2,1/2))^n$ defined by $\phi^n(f_1,\dots,f_n) = (\phi(f_1),\dots,\phi(f_n))$ is thus nearly Lipschitz and onto. The maximum and Euclidean metric on C^n are clearly bi-Lipschitz equivalent and $(\mathbb{C}(1/2,1/2))^n$ is easily seen to be isometric to $\mathbb{C}(\underbrace{1/2,\dots,1/2}_{2^n \text{ times}})$. Thus, in summary, there is a compact set $K \subseteq [0,1]^n$, a Cantor cube $\mathbb{C}$ with $\dim_{\mathsf{H}} \mathbb{C} = n$ and a nearly Lipschitz onto mapping $\psi : K \to \mathbb{C}$.

Now consider the family $\{f_i : i \in \omega\}$ of Lipschitz mappings given by Theorem 6.2 and the corresponding cover $\{\psi(f_i(X) \cap K) : i \in \omega\}$ of $\mathbb{C}$. Then Lemma 6.3 yields an $i \in \omega$ such that $\psi(f_i(X) \cap K)$ contains a set $A \in \boldsymbol{\mathcal{S}}$ for which $\dim_{\mathsf{H}} A = n$. Notice that $\psi \circ f_i$ is nearly Lipschitz and apply the Preimage Lemma. ∎

For completeness we recall [29, Theorem 3.6], which in our notation reads:

THEOREM 6.5. *Let* $\boldsymbol{\mathcal{S}}$ *be backwards closed*, σ*-additive and suppose that* $\mathfrak{c} \prec \boldsymbol{\mathcal{S}}$. *Then each separable metric space* X *contains a set* $E \in \boldsymbol{\mathcal{S}}$ *such that* $\dim E \geq \dim X - 1$.

7. Universally null and universally meager sets. We now localize the results of the previous sections to two important classes of small sets that satisfy the hypotheses of the preceding theorems: universally null and universally meager sets.

Universally null sets. Recall that a metric space X is *universally null* (**UN**) if each finite diffused Borel measure μ on X is trivial, i.e. $\mu(X) = 0$. If $E \subseteq X$ is a subspace of X, then E is **UN** iff it is μ-negligible for each finite Borel diffused measure on X. This property is obviously countably additive.

Universally meager sets. Piotr Zakrzewski [26, 28] defines a set E in a metric space to be *universally meager* (**UM**) if, for every perfect Polish space Z, a subset $Y \subseteq Z$ and a Borel one-to-one mapping $f : Y \to E$, Y is meager in Z. This is a strengthening of the classical notion of a *perfectly meager* set, which is a set each of whose perfect subsets is meager in itself. In his papers, Zakrzewski gathers evidence that the notion of **UM** is a category counterpart of **UN**. As to the relation of **UN** and **UM**, it is not provable within ZFC that either of the two classes is included in the other, but to date there is no ZFC proof that the two classes differ.

The following lemma claims that the classes UN, UM and $\mathsf{UN}\cap\mathsf{UM}$ satisfy the closure hypotheses of all of the above theorems on small sets. Most of it is obvious or easy to prove. The only nontrivial part—strong backwards closedness of UM—follows at once from [28, Lemma 2.1].

Lemma 7.1. *The classes* UN *and* UM *are σ-additive and strongly backwards closed.*

A deep result of Edward Grzegorek ensures that the three classes satisfy also the cardinal hypotheses. As for UM, Grzegorek proves it for *absolutely first category* sets in place of UM, but Zakrzewski shows in [26, Theorem 2.1] that the two classes are equal.

Lemma 7.2 ([11]). $\mathsf{non}\,\mathcal{N}\prec\mathsf{UN}$ *and* $\mathsf{non}\,\mathcal{M}\prec\mathsf{UM}$.

The following folklore lemma follows at once from the Isomorphism Theorem for Measures ([14, (17.41)]) and the definitions.

Lemma 7.3. *Let $E\subseteq 2^\omega$. If $|E|<\mathsf{non}\,\mathcal{N}$, then E is* UN. *If $|E|<\mathsf{non}\,\mathcal{M}$, then E is* UM.

Corollary 7.4. $\min(\mathsf{non}\,\mathcal{N},\mathsf{non}\,\mathcal{M})\prec\mathsf{UN}\cap\mathsf{UM}$.

Proof. Suppose first that $\mathsf{non}\,\mathcal{M}<\mathsf{non}\,\mathcal{N}$. By Lemma 7.2 there is $A\subseteq 2^\omega$ UM such that $|A|=\mathsf{non}\,\mathcal{M}<\mathsf{non}\,\mathcal{N}$. By Lemma 7.3, A is UN. The same argument works if $\mathsf{non}\,\mathcal{M}>\mathsf{non}\,\mathcal{N}$.

Now suppose that $\mathsf{non}\,\mathcal{M}=\mathsf{non}\,\mathcal{N}$. Take a UN set $A=\{a_\alpha:\alpha<\mathsf{non}\,\mathcal{N}\}\subseteq 2^\omega$ and a UM set $B=\{b_\alpha:\alpha<\mathsf{non}\,\mathcal{N}\}\subseteq 2^\omega$ and consider the diagonal set $D=\{(a_\alpha,b_\alpha):\alpha<\mathsf{non}\,\mathcal{N}\}\subseteq 2^\omega\times 2^\omega$. Obviously $|D|=\mathsf{non}\,\mathcal{N}=\mathsf{non}\,\mathcal{M}$. Both projections $(a_\alpha,b_\alpha)\mapsto a_\alpha$, $(a_\alpha,b_\alpha)\mapsto b_\alpha$ are obviously continuous and one-to-one. The former takes D to a UN set and the latter takes D to a UM set. Hence D is, by Lemma 7.1, $\mathsf{UN}\cap\mathsf{UM}$. ∎

The following three theorems are now, in view of the above lemmas, trivial consequences of Theorems 5.4, 5.11 and 6.4.

Theorem 7.5. *Each analytic σ-monotone space X contains*

- *a* UN *set $E\subseteq X$ such that* $\dim_{\mathsf{H}} E=\dim_{\mathsf{H}} X$,
- *a* $\mathsf{UN}\cap\mathsf{UM}$ *set $E\subseteq X$ such that* $\overline{\dim}_{\mathsf{H}} E=\dim_{\mathsf{H}} X$.

Theorem 7.6. *Each analytic set $X\subseteq\mathbb{R}^n$ contains*

- *a* UN *set $E\subseteq X$ such that* $\dim_{\mathsf{H}} E=\dim_{\mathsf{H}} X$,
- *a* $\mathsf{UN}\cap\mathsf{UM}$ *set $E\subseteq X$ such that* $\overline{\dim}_{\mathsf{H}} E=\dim_{\mathsf{H}} X$.

Theorem 7.7. *Each metric space X contains*

- *a* UN *set $E\subseteq X$ such that* $\dim_{\mathsf{H}} E\geq\dim X$,
- *a* $\mathsf{UN}\cap\mathsf{UM}$ *set $E\subseteq X$ such that* $\overline{\dim}_{\mathsf{H}} E\geq\dim X$.

Notice that in general there is no hope that E can have positive $(\dim X)$-dimensional Hausdorff measure. For instance, if $E \subseteq \mathbb{R}^n$, then $\mathcal{H}^n(E)$ is σ-finite, so if $\mathcal{H}^n(E) > 0$, then the restriction of $\mathcal{H}^n$ to E is a measure witnessing that E is not UN.

8. Remarks. In this section we discuss various consequences of Martin's axiom and other extra set-theoretic assumptions. The reader is supposed to be familiar with basic set theory.

Improving $\overline{\dim}_{\mathsf{H}} E \geq \dim_{\mathsf{H}} X$ for UM. The existence of UN and UM sets of large dimension stems from Grzegorek's theorems that yield, in ZFC, a set of large cardinality ($\mathsf{non}\,\mathcal{N}$ or $\mathsf{non}\,\mathcal{M}$) in 2^ω that is topologically small (UN or UM). With some extra assumptions about the two cardinals one can get more, for instance, $\mathsf{non}\,\mathcal{N} \leq \mathsf{non}\,\mathcal{M}$ (implied by Martin's axiom or by $\mathsf{non}\,\mathcal{N} = \aleph_1$) enables us to strengthen all of the inequalities $\overline{\dim}_{\mathsf{H}} E \geq \dim_{\mathsf{H}} X$ of the previous section to $\dim_{\mathsf{H}} E = \dim_{\mathsf{H}} X$:

PROPOSITION 8.1. *Assume* $\mathsf{non}\,\mathcal{N} \leq \mathsf{non}\,\mathcal{M}$*.*

- (i) *Each analytic σ-monotone space X contains a $\mathsf{UN}\cap\mathsf{UM}$ set E such that* $\dim_{\mathsf{H}} E = \dim_{\mathsf{H}} X$*.*
- (ii) *Each analytic set $X \subseteq \mathbb{R}^n$ contains a $\mathsf{UN}\cap\mathsf{UM}$ set E such that* $\dim_{\mathsf{H}} E = \dim_{\mathsf{H}} X$*.*
- (iii) *Each metric space X contains a $\mathsf{UN}\cap\mathsf{UM}$ set E such that* $\dim_{\mathsf{H}} E \geq \dim X$*.*

Sets with no UN or UM subsets of positive dimension. It would be nice to have an example of a set with positive Hausdorff dimension such that all its UN subsets have Hausdorff dimension zero; and likewise for upper Hausdorff dimension and UM. Under Martin's axiom such examples are easy to construct.

PROPOSITION 8.2. *Assume Martin's axiom. Let X be an analytic metric space.*

- (i) *There is a set $S \subseteq X$ such that* $\dim_{\mathsf{H}} S = \dim_{\mathsf{H}} X$*, but* $\overline{\dim}_{\mathsf{H}} E = 0$ *for each UN set $E \subseteq S$.*
- (ii) *There is a set $L \subseteq X$ such that* $\overline{\dim}_{\mathsf{H}} L = \overline{\dim}_{\mathsf{H}} X$*, but* $\overline{\dim}_{\mathsf{H}} E = 0$ *for each UM set $E \subseteq L$.*

Proof. (i) *Mutatis mutandis* we may assume that X is compact and $0 < \mathcal{H}^s(X) < \infty$, where $s = \dim_{\mathsf{H}} X$, so that $\mathcal{H}^s$ is a nontrivial finite diffused Borel measure on X.

It is well-known that Martin's axiom yields a $\mathfrak{c}$-*Sierpiński set*, a set $S \subseteq X$ such that $|S| = \mathfrak{c}$ and if $E \subseteq S$ and $\mathcal{H}^s(E) = 0$, then $|E| < \mathfrak{c}$. In particular $\mathcal{H}^s(S) > 0$ and thus $\dim_{\mathsf{H}} S = s$. On the other hand, if $E \subseteq S$ is UN, then

$\mathcal{H}^s(E) = 0$ and thus $|E| < \mathfrak{c}$. As proved in [10, Lemma 1.1], under Martin's axiom, $|E| < \mathfrak{c}$ implies that E is a γ-set; and by [30, Proposition 7.7], if E is a γ-set, then $\overline{\dim}_{\mathsf{H}} E = 0$.

(ii) It is enough to find, for each $s < \overline{\dim}_{\mathsf{H}} X$, a set $L \subseteq X$ such that $\overline{\dim}_{\mathsf{H}} L \geq s$ and $\overline{\dim}_{\mathsf{H}} E = 0$ for each **UM** set $E \subseteq L$. Put $G = \bigcup\{U \subseteq X : U \text{ open}, \overline{\dim}_{\mathsf{H}} U \leq s\}$. Then $\overline{\dim}_{\mathsf{H}} G \leq s$ and thus each open nonempty subset U of $X \setminus G$ satisfies $\overline{\dim}_{\mathsf{H}} U \geq s$.

Martin's axiom yields a $\mathfrak{c}$-*Luzin set*, a set $L \subseteq X \setminus G$ such that $|L| = \mathfrak{c}$ and if $E \subseteq L$ is meager, then $|E| < \mathfrak{c}$. In particular L is not meager and consequently, by Lemma 2.4, $\overline{\dim}_{\mathsf{H}} L \geq s$. If $E \subseteq L$ is **UM**, then it is meager and thus $|E| < \mathfrak{c}$, whence $\overline{\dim}_{\mathsf{H}} E = 0$ by [10] and [30] again. ∎

Other small sets. With extra set-theoretic assumptions one can make some other small sets have positive (upper) Hausdorff dimension. We consider the following classes of separable metric spaces:

- X is *universally small* if there are no nontrivial Borel-based *ccc* σ-ideals on X.
- X is a λ*-set* if every countable subset of X is G_δ.
- $X \subseteq Y$ is a λ'*-set in* Y if $X \cup D$ is a λ-set for each countable set $D \subseteq Y$.
- X is a σ*-set* if each F_σ-set in X is G_δ in X.
- X is a Q*-set* if each subset of X is G_δ in X.

The figure shows various inclusions. See [18, 19] for general reference and [27] for universally small sets.

$$\begin{array}{ccccc}
\mathbf{UN} & \Longleftarrow & \text{universally small} & \Longrightarrow & \mathbf{UM} \\
\Uparrow & & & & \Uparrow \\
Q\text{-set} & \Longrightarrow & \sigma\text{-set} & \Longrightarrow & \lambda\text{-set} \\
 & & & & \Uparrow \\
 & & & & \lambda'\text{-set}
\end{array}$$

All of these classes except σ-sets are backwards closed ([18]). All of them may consistently have large Hausdorff dimension.

PROPOSITION 8.3. *It is relatively consistent with* ZFC *that each metric space X contains a set E with* $\dim_{\mathsf{H}} E \geq \dim X$ *and E is simultaneously Q, λ' and universally small.*

Proof. Judah and Shelah [13] have a model of "ZFC + there exists an uncountable Q-set + there exists a subset of reals of cardinality $\aleph_1$ which is not Lebesgue measurable". In other words, denoting by **Q** the class of Q-sets, $\mathsf{non}\,\mathcal{N} = \aleph_1 \prec \mathbf{Q}$ in the model. Let Q be an uncountable Q-set. Let S be an uncountable universally small set (there is one, see [27]). Let L be an uncountable λ'-set (there is one, see [18]). All of the three sets may be assumed to be of cardinality $\aleph_1$. Let $\boldsymbol{\mathcal{S}}$ be the class of sets that are simultaneously Q, λ' and universally small. Construct a diagonal set $E \subseteq Q \times S \times L$. Since all of the three classes are backwards closed, so is $\boldsymbol{\mathcal{S}}$ and $E \in \boldsymbol{\mathcal{S}}$. Hence E witnesses $\mathsf{non}\,\mathcal{N} \prec \boldsymbol{\mathcal{S}}$. Now apply Theorem 6.4. ∎

There are numerous variations. For instance: Since there is an uncountable universally small λ'-set ([23, 27]), if $\mathsf{non}\,\mathcal{N} = \aleph_1$, then for each analytic space there is a universally small λ'-set $E \subseteq X$ such that $\dim_{\mathsf{H}} E = \dim_{\mathsf{H}} X$; since there is a λ-set of cardinality $\mathfrak{b}$ ([23]), if $\mathsf{non}\,\mathcal{N} \leq \mathfrak{b}$, then there is, for each n, a λ-set $E \subseteq \mathbb{R}^n$ such that $\dim_{\mathsf{H}} E = n$. There are also counterparts for $\overline{\dim}_{\mathsf{H}}$ and $\mathsf{non}\,\mathcal{E}$.

Most of the classes under consideration may also consistently have Hausdorff dimension zero:

Proposition 8.4. *Each of the following is relatively consistent with* ZFC*:*

(i) $\overline{\dim}_{\mathsf{H}} X = 0$ *for each σ-set,*
(ii) $\overline{\dim}_{\mathsf{H}} X = 0$ *for each λ'-set $X \subseteq \mathbb{R}$,*
(iii) $\dim_{\mathsf{H}} X = 0$ *for each λ-set.*

Proof. Since σ-sets are consistently countable ([17, Theorem 22]), (i) is obvious.

(ii) By [20, Theorem 1.1], it is consistent that every λ'-set is a γ-set and in [30, Proposition 7.7] it is proved that $\overline{\dim}_{\mathsf{H}} X = 0$ for every γ-set X.

(iii) By [19, Theorem 22], if ω_2 Cohen reals are added to a model of CH, then each λ-set in the extension is of cardinality $\aleph_1$ or less. Also $\mathsf{cov}\,\mathcal{M} = \aleph_2 = \mathfrak{c}$ in the model. In particular, $|X| < \mathsf{cov}\,\mathcal{M}$ for each λ-set. But if X is a separable metric space and $|X| < \mathsf{cov}\,\mathcal{M}$, then $\mathcal{H}^s(X) = 0$ for each $s > 0$ (see e.g. [9, 534B(c)]), and therefore $\dim_{\mathsf{H}} X = 0$. ∎

Proposition 8.5. *There is a λ-set X such that* $\overline{\dim}_{\mathsf{H}} X = \infty$.

Proof. Provide the set of irrationals ω^ω with the (variation of) the least difference metric $\rho(f,g) = 2^{-n(f,g)}$ (cf. the first paragraph of Section 3). Let $X \subseteq \omega^\omega$ be an unbounded set of cardinality $\mathfrak{b}$. Rothberger [23] (or see [19, Theorem 21]) shows that X is a λ-set. If $\overline{\dim}_{\mathsf{H}} X < \infty$, then it is contained in a σ-compact set. But that is not possible, since X is unbounded. ∎

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