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Embedding theorems for spaces of $\mathbb{R}$-places of rational function fields and their products

by

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Abstract. We study spaces $M(R(y))$ of $\mathbb{R}$-places of rational function fields $R(y)$ in one variable. For extensions $F|R$ of formally real fields, with R real closed and satisfying a natural condition, we find embeddings of $M(R(y))$ in $M(F(y))$ and prove uniqueness results. Further, we study embeddings of products of spaces of the form $M(F(y))$ in spaces of $\mathbb{R}$-places of rational function fields in several variables. Our results uncover rather unexpected obstacles to a positive solution of the open question whether the torus can be realized as a space of $\mathbb{R}$-places.

1. Introduction. For any field K, the set of all orderings on K, given by their positive cones P, is denoted by $\mathcal{X}(K)$. This set is non-empty if and only if K is formally real. The *Harrison topology* on $\mathcal{X}(K)$ is defined by taking as a subbasis the *Harrison sets*

$$H(a) := \{P \in \mathcal{X}(K) \mid a \in P\}, \qquad a \in K \setminus \{0\}.$$

With this topology, $\mathcal{X}(K)$ is a boolean space, i.e., it is compact, Hausdorff and totally disconnected (see [L2, p. 32]).

Associated with every ordering P on K is an $\mathbb{R}$-*place* $\lambda(P)$ of K, that is, a place of K with image contained in $\mathbb{R} \cup \{\infty\}$, which is compatible with the ordering in the sense that non-negative elements are sent to non-negative elements or ∞. The set of all $\mathbb{R}$-places of K will be denoted by $M(K)$. The Baer–Krull Theorem (see [L1, Theorem 3.10]) shows that the mapping

$$\lambda : \mathcal{X}(K) \to M(K)$$

(which we will also denote by λ_K) is surjective. Through λ, we equip $M(K)$ with the quotient topology inherited from $\mathcal{X}(K)$, making it a compact Hausdorff space (see [L1, p. 74 and Cor. 9.9]), and λ a continuous closed mapping.

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According to [L1, Theorem 9.11] the subbasis for the quotient topology on $M(K)$ is given by the family of open sets of the form

$$U(a) = \{\zeta \in M(K) \mid \zeta(a) > 0\}$$

where a is in the *real holomorphy ring* of K, i.e., $\zeta(a) \neq \infty$ for all $\zeta \in M(K)$. Since for every $b \in K$ the element $b/(1+b^2)$ is in the real holomorphy ring of K (see [L1, Lemma 9.5]), we see that

$$H'(b) := \{\zeta \in M(K) \mid \infty \neq \zeta(b) > 0\} = U\left(\frac{b}{1+b^2}\right)$$

is a subbasic set for every $b \in K$. So we can assume that the topology on $M(K)$ is given by the subbasic sets $H'(b)$, $b \in K$.

Throughout this paper, $R(y)$ will always denote the rational function field in one variable over the field R. For the case of real closed R, we gave in [KMO] a handy criterion for two orderings on $R(y)$ to be sent to the same $\mathbb{R}$-place by λ:

THEOREM 1.1. *Take a real closed field R and two distinct orderings P_1, P_2 of $R(y)$. Then $\lambda(P_1) = \lambda(P_2)$ if and only if the cuts induced by y with respect to P_1 and P_2 in R are the upper and the lower edge of a ball in R.*

See Section 2 for the notions in this theorem and for more details.

If R is any real closed field, each ordering P on $R(y)$ is uniquely determined by the cut (D, E) in R where $D = \{d \in R \mid y - d \in P\}$ and $E = R \setminus D$ (cf. [G]). Hence, if $\mathcal{C}(R)$ is the set of all cuts in R, then we have a bijection

$$\chi : \mathcal{C}(R) \to \mathcal{X}(R(y))$$

(which we will also denote by χ_R). With respect to the interval topology on $\mathcal{C}(R)$ and the Harrison topology on $\mathcal{X}(R(y))$, χ_R is in fact a homeomorphism (see Proposition 3.7). Theorem 1.1 can be reformulated as: *Two distinct cuts in R are mapped by $\lambda \circ \chi$ to the same place in $M(R(y))$ if and only if they are the upper and the lower edge of a ball in R.*

In the present paper, we put this result to work in order to find, for given formally real extensions F of a real closed field R, continuous embeddings ι of $M(R(y))$ in $M(F(y))$, by finding suitable embeddings of $\mathcal{C}(R)$ in $\mathcal{C}(F)$.

For any field extension $L|K$, the *restriction*

$$\mathrm{res} = \mathrm{res}_{L|K} : M(L) \ni \zeta \mapsto \zeta|_K \in M(K)$$

is continuous (see [D, 7.2]). An embedding $\iota : M(K) \to M(L)$ will be called *compatible with restriction* if $\mathrm{res} \circ \iota$ is the identity.

In order to determine when such embeddings of $M(R(y))$ in $M(F(y))$ exist, we have to look at the canonical valuations of the ordered fields R and F. The *canonical valuation* v of an ordered field is the valuation corresponding to its associated $\mathbb{R}$-place. If v is the canonical valuation of the ordered field F, then its restriction to R is the canonical valuation of the field R ordered by the restriction of the ordering of F, and we will denote it again by v. Recall that the ordering and canonical valuation of a real closed field are uniquely determined. By vF and vR we denote the respective value groups. Then $vF|vR$ is an extension of ordered abelian groups. Note that $vR = \{0\}$ if and only if R is archimedean ordered. In Section 5, we will prove:

THEOREM 1.2. *Take a real closed field R and a formally real extension field F of R. A continuous embedding ι of $M(R(y))$ in $M(F(y))$ compatible with restriction exists if and only if vR is a convex subgroup of vF, for some ordering of F. In particular, such an embedding always exists when R is archimedean ordered. If F is real closed, then there is at most one such embedding.*

For the case of F not being real closed, we prove a partial uniqueness result (Theorem 5.2).

Let us point out a somewhat surprising consequence of Theorem 1.2. If R is a non-archimedean real closed field and F is an elementary extension (e.g., ultrapower) of R of high enough saturation, then vR will not be a convex subgroup of vF and there will be no such embedding ι.

In Section 6 we consider the special case where R is archimedean ordered and give a more explicit construction of ι and a more explicit proof of the uniqueness. The construction we give is of interest also when other spaces of places are considered (e.g., spaces of all places, together with the Zariski topology).

It is well known that for an archimedean real closed field R, $M(R(y))$ is homeomorphic to the circle (over $\mathbb{R}$, with the usual interval topology). In fact, this is an easy consequence of Theorem 1.1. Hence our embedding result shows that each $M(F(y))$ contains the circle as a closed subspace.

While spaces of orderings are well understood, this is not the case for spaces of $\mathbb{R}$-places. Some important insight has been gained (see for instance [BG], [B1], [B2], [EO], [GM], [KMO], [MMO], [S]), but several essential questions have remained unanswered. For example, it is still an open problem which compact Hausdorff spaces are realized as $M(F)$ for some F. It is therefore important to determine operations on topological spaces (like passage to closed subspaces, taking finite disjoint unions, taking finite products) under which the class of realizable spaces is closed. It has been shown in [EO] that closed subspaces and finite disjoint unions of realizable spaces are again realizable, as well as products of a realizable space with any boolean space.

It has remained an open question whether the product of two realizable spaces is realizable. A test case is the torus; it is not known whether the torus (or any other subspace of $\mathbb{R}^n$ of dimension > 1) is realizable.

As $M(\mathbb{R}(y))$ is the circle, $M(\mathbb{R}(x)) \times M(\mathbb{R}(y))$ is the torus. In Section 7 we generalize our construction given in Section 6 to obtain a natural embedding of $M(\mathbb{R}(x)) \times M(\mathbb{R}(y))$ in $M(\mathbb{R}(x, y))$. In view of the above-mentioned negative result, this embedding cannot be continuous with an image that is closed in $M(\mathbb{R}(x, y))$, because otherwise it would follow from the realizability of closed subspaces that the torus is realizable. We show an even stronger negative assertion: the image of the embedding is dense in, while not being equal to, $M(\mathbb{R}(x, y))$. Hence, the image is not closed, and the embedding is not continuous.

In the final Section 9 we will show that for an arbitrary extension $L|K$, there is a continuous embedding of $M(K)$ in $M(L)$ compatible with restriction as soon as L admits a *K-rational place*, that is, a place trivial on K with image $K \cup \{\infty\}$. In particular, this applies when L is a rational function field over K.

2. Cuts, balls and $\mathbb{R}$-places. Take any totally ordered set T and $D, E \subseteq T$. We will write $D < E$ if $d < e$ for all $d \in D$ and $e \in E$. Note that $\emptyset < T$ and $T < \emptyset$. For $c \in T$, we will write $c > D$ if $c > d$ for all $d \in D$, and $c < E$ if $c < e$ for all $e \in E$.

A pair $C = (D, E)$ is called a *cut in* T if $D < E$ and $D \cup E = T$. In this case, D is an *initial segment of* T, that is, if $d \in D$ and $d > c \in T$, then $c \in D$; similarly, E is a *final segment of* T, that is, if $e \in E$ and $e < c \in T$, then $c \in E$.

We include the cuts $C_{-\infty} = (\emptyset, T)$ and $C_{\infty} = (T, \emptyset)$; the empty set is understood to be both an initial and a final segment of T. If $C_1 = (D_1, E_1)$ and $C_2 = (D_2, E_2)$ are two cuts, then we will write $C_1 < C_2$ if $D_1 \subsetneq D_2$.

Take any non-empty subset A of T. By A^+ we will denote the cut $(D, T \setminus D)$ for which D is the smallest initial segment of T which contains A. Similarly, by A^- we will denote the cut $(T \setminus E, E)$ for which E is the smallest final segment of T which contains A.

A cut (D, E) is called *principal* if D has a last element or E has a first element. In the first case, the cut is equal to $\{d\}^+$, where d is the last element of D; in this case we will denote it by d^+. In the second case, the cut is equal to $\{e\}^-$, where e is the first element of E; in this case we will denote it by e^-.

We will need the following fact:

LEMMA 2.1. *If C_1, C_2 are cuts in T such that $C_1 < C_2$, then $C_1 \leq a^- < a^+ \leq C_2$ for some $a \in T$.*

Proof. Write $C_1 = (D_1, E_1)$ and $C_2 = (D_2, E_2)$. If $C_1 < C_2$, then there is some $a \in D_2 \setminus D_1$. Then $C_1 \leq a^- < a^+ \leq C_2$. ∎

For any pair (D, E) such that $D < E$, we define the *between set*

$$\mathrm{Betw}_T(D, E) := \{c \in T \mid D < c < E\}.$$

Now consider any ordered field F with its canonical valuation v. If D, E are any subsets of F, we set

$$v(E - D) := \{v(e - d) \mid e \in E,\, d \in D\} \subseteq vF \cup \{\infty\}.$$

The following observation is easy to prove.

LEMMA 2.2. *Assume that D is an initial segment or E is a final segment of F. Then $v(E - D)$ is an initial segment of $vF \cup \{\infty\}$.*

A subset $B \subseteq F$ is called a *ball* in F (with respect to the valuation v) if it is of the form

$$B \;=\; B_S(a, F) \;:=\; \{b \in F \mid v(a - b) \in S \cup \{\infty\}\}$$

where $a \in F$ and S is a final segment of vF. We consider $S = \emptyset$ as a final segment of vF; we have $B_\emptyset(a, F) = \{a\}$.

The notion of "ball" does not refer to some space over F, but to the ultrametric underlying the natural valuation of F. Note that because of the ultrametric triangle law, every element of a ball is a center, that is, if $b \in B_S(a, F)$ then $B_S(a, F) = B_S(b, F)$. Therefore, $v(b - c) \in S$ for all $b, c \in B_S(a, F)$. A subset B of F is a ball if and only if for any choice of $a, b \in B$ and $c \in F$ such that $v(a - c) \geq v(a - b)$ it follows that $c \in B$.

If $0 \in B_S(a, F)$, then $B_S(a, F) = B_S(0, F)$ is a convex subgroup of the ordered additive group of F. Every ball in F is in fact a coset of a convex subgroup: $B_S(a, F) = a + B_S(0, F)$.

By a *ball complement* for the ball $B = B_S(a, F)$ we will mean a pair (D, E) of subsets of F such that $D < B < E$ and $F = D \cup B \cup E$. In this case again, D is an initial segment and E is a final segment of F.

LEMMA 2.3. *If (D, E) is a ball complement for $B = B_S(a, F)$, then*

$$v(E - D) = v(E - B) = v(B - D) = vF \setminus S.$$

Proof. First, we show that $v(E - D) = vF \setminus S$. For $d \in D$ and $e \in E$, we have $v(a - d) < S$ and $v(e - a) < S$ because $d, e \notin B$. From $d < a < e$ it then follows that $v(e - d) = \min\{v(e - a), v(a - d)\} < S$. This proves that $v(E - D) < S$.

Now take $\alpha \in vF$, $\alpha < S$. Choose $0 < c \in F$ such that $vc = \alpha$. Then $v(a - (a - c)) = vc = \alpha$, whence $a - c \notin B$ and therefore $d := a - c \in D$. Similarly, $a + c \notin B$ and therefore $e := a + c \in E$. Since $d < a < e$, we find $\alpha = v(2c) = v(e - d) \in v(E - D)$. Since $v(E - D)$ is an initial segment of $vF \cup \{\infty\}$ by Lemma 2.2, and S is a final segment, we can now conclude that $v(E - D) = vF \setminus S$.

Again by Lemma 2.2, also $v(E-B)$ and $v(B-D)$ are initial segments of $vF \cup \{\infty\}$. If $d \in D$, $e \in E$ and $b \in B$, then $d < b < e$, whence $v(b-d) \geq v(e-d)$ and $v(e-b) \geq v(e-d)$. Consequently, $v(E-D)$ is contained in $v(E-B)$ and $v(B-D)$. On the other hand, $d, e \notin B$ implies that $v(b-d), v(e-b) < S$. So by what we have proved earlier, $v(b-d), v(e-b) \in v(E-D)$. This shows that all three sets are equal. ∎

We will say that a cut is the *lower edge* of the ball $B = B_S(a,F)$ if it is the cut B^-; similarly, a cut is said to be the *upper edge* of the ball B if it is the cut B^+. Two cuts will be called *equivalent* if either they are equal, or one is the lower edge B^- and the other is the upper edge B^+ of a ball B.

A cut of the form B^+ or B^- for B a ball will be called a *ball cut.* Principal cuts in F are ball cuts: $a^+ = \{a\}^+ = B_\emptyset(a,F)^+$ and $a^- = \{a\}^- = B_\emptyset(a,F)^-$.

If a cut is neither the lower nor the upper edge of a ball, then we call it a *non-ball cut.* The equivalence class of a non-ball cut is a singleton. As the following lemma shows, the equivalence class of a ball cut consists of two distinct cuts.

Lemma 2.4. *If a cut is the upper or the lower edge of a ball in F, then the ball is uniquely determined. In particular, $B_1^+ = B_2^-$ for two balls B_1 and B_2 is impossible. Therefore, equivalence classes of balls contain at most two cuts.*

Proof. We show the assertion for a cut $B^+ = B_S(a,F)^+$; the lower edge case is similar.

Take any $d \in F$ and some final segment T of vF. Suppose that $B^+ = B_T(d,F)^+$. Since the balls $B_S(a,F)$ and $B_T(d,F)$ are final segments of the left cut set of B^+, their intersection is non-empty. So one of them is contained in the other. If they were not equal, the bigger one would contain an element which is bigger than all elements in the smaller ball, but that is impossible.

Now suppose that $B^+ = B_T(d,F)^-$. Then $d > B_S(a,F)$, so $v(a-d) < S$. Similarly, $a < B_T(d,F)$, so $v(a-d) < T$. Set $d' := (d+a)/2$; then $d < d' < a$ and $v(a-d') = v(a-d) = v(d'-d)$. Consequently, $d' > B_S(a,F)$ and $d' < B_T(d,F)$, a contradiction. ∎

In combination with Theorem 1.1, this lemma shows that the mapping λ will glue no more than two orderings into one $\mathbb{R}$-place. Another, quite different way of proof is by an application of the Baer–Krull Theorem

Proposition 2.5. *Take a real closed field F. Then for every $\zeta \in M(F(y))$, the preimage $\lambda^{-1}(\zeta)$ consists of at most two orderings.*

Let us add the following observation:

PROPOSITION 2.6. *For every formally real field F, the mapping $\lambda : \mathcal{X}(F) \to M(F)$ induces* continuous glueings, *that is, if $P_1, P_2 \in \mathcal{X}(F)$ are such that for every pair of open neighborhoods U_1 of P_1 and U_2 of P_2 there are $Q_1 \in U_1$ and $Q_2 \in U_2$ with $\lambda(Q_1) = \lambda(Q_2)$, then $\lambda(P_1) = \lambda(P_2)$.*

Proof. Take two orderings $P_1, P_2 \in \mathcal{X}(F)$ such that $\lambda(P_1) \neq \lambda(P_2)$. Since $M(F)$ is Hausdorff, there are disjoint open neighborhoods U_1' of $\lambda(P_1)$ and U_2' of $\lambda(P_2)$. Their preimages $U_1 := \lambda^{-1}(U_1')$ and $U_2 := \lambda^{-1}(U_2')$ are open neighborhoods of P_1 and P_2, respectively. Since $U_1 \cap U_2 = \emptyset$, there cannot exist any orderings $Q_1 \in U_1$ and $Q_2 \in U_2$ such that $\lambda(Q_1) = \lambda(Q_2)$. ∎

3. Topologies on $\mathcal{C}(F)$ and $\mathcal{X}(F)$. Take any ordered field F. We have already defined the ordering on $\mathcal{C}(F)$. Intervals are defined as in any other linearly ordered set. Note that the linear order of $\mathcal{C}(F)$ has endpoints C_∞ and $C_{-\infty}$.

The *interval topology* on $\mathcal{C}(F)$ (like on every other linearly ordered set with endpoints) is defined by taking as basic open sets all intervals of the form $(C_1, C_2) = \{C \in \mathcal{C}(F) \mid C_1 < C < C_2\}$ for any two cuts $C_1, C_2 \in \mathcal{C}(F)$, together with $(C_1, C_\infty]$ if $C_1 \neq C_\infty$, and $[C_{-\infty}, C_2)$ if $C_{-\infty} \neq C_2$.

Note that in the interval topology on $\mathcal{C}(F)$, an open interval may have a first or a last element different from $C_\infty, C_{-\infty}$. Indeed, if $C = a^+$ is a principal cut and $C_1 < a^-$, then (C_1, a^+) has last element C. Similarly, if $C = a^-$ and $a^+ < C_2$, then (a^-, C_2) has first element C. However, this is the only way in which first and last elements will arise in open intervals:

LEMMA 3.1. *Take an interval I that is open in the interval topology. If C is the first element of I, then $C = a^+$ for some $a \in F$. If C is the last element of I, then $C = a^-$ for some $a \in F$.*

Proof. Any finite intersection or arbitrary union of intervals of the form (C_1, C_2) will only have a first or a last element if that is already true for one of the intervals. Suppose that C is the first element of I; the case of C being the last element is similar. Then C is the first element of an interval (C_1, C_2), which means that there is no cut properly between C_1 and C. Therefore, our assertion follows from Lemma 2.1. ∎

Let us also note that Lemma 2.1 implies:

LEMMA 3.2. *The principal cuts lie dense in $\mathcal{C}(F)$.*

A subset of $\mathcal{C}(F)$ will be called *full* if it is closed under equivalence. We define the *full topology* on $\mathcal{C}(F)$ to consist of all full sets that are open in the interval topology. This topology is always strictly coarser than the interval topology because in the latter there are always open sets containing C_∞ without containing $C_{-\infty}$. Hence it is not Hausdorff, but it is quasi-compact.

Proposition 3.3. *For every ball B in F, the intervals $[B^-, B^+]$, (B^-, B^+) and their complements are full in $\mathcal{C}(F)$.*

Proof. Take any ball B_1 in F. If $B_1 \cap B = \emptyset$, then both B_1^+ and B_1^- lie in the complement of (B^-, B^+), and by Lemma 2.4, also in the complement of $[B^-, B^+]$.

If $B_1 \cap B \neq \emptyset$, then $B_1 \subseteq B$ or $B \subsetneq B_1$. In the latter case, again both B_1^+ and B_1^- lie in the complements of $[B^-, B^+]$ and (B^-, B^+). If $B_1 \subsetneq B$, then both B_1^+ and B_1^- lie in $[B^-, B^+]$ and in (B^-, B^+). Finally, if $B_1 = B$, then both B_1^+ and B_1^- lie in $[B^-, B^+]$ and in the complement of (B^-, B^+). ∎

Let us also observe:

Lemma 3.4. *If $F|R$ is an extension of ordered fields, then the restriction mapping* $\mathrm{res} : \mathcal{C}(F) \to \mathcal{C}(R)$ *preserves $\leq$ and equivalence and is continuous in both the interval and the full topology. The preimage of every full subset of $\mathcal{C}(R)$ under* res *is again full.*

Proof. It is clear that res preserves $\leq$. Hence, the preimage of every convex set in $\mathcal{C}(R)$ is convex in $\mathcal{C}(F)$. Therefore, if I is an open interval in $\mathcal{C}(R)$, then its preimage I' is convex, and if it has no smallest and no largest element, then it is open. If it has a smallest element C', then $\mathrm{res}(C')$ is the smallest element of I, hence equal to $C_{-\infty}$ in $\mathcal{C}(R)$. Therefore, I' contains the cut $C_{-\infty}$ of $\mathcal{C}(F)$, whence $C' = C_{-\infty}$. Similarly, a largest element of I' can only be equal to C_∞ in $\mathcal{C}(R)$. It follows that I' is open. We have proved that res is continuous with respect to the interval topology.

Suppose that B is a ball in F. Then $B_0 = B \cap R$ is either empty or a ball in R. In the first case, $\mathrm{res}\, B^- = \mathrm{res}\, B^+$, and in the second case, $\mathrm{res}\, B^- = B_0^-$ and $\mathrm{res}\, B^+ = B_0^+$. This proves that res preserves equivalence. This implies that the preimage U' of a full set U is again full: if $C_1 \in U'$ is equivalent to C_2, then $\mathrm{res}(C_1) \in U$ and $\mathrm{res}(C_2)$ are equivalent, whence $\mathrm{res}(C_2) \in U$ and $C_2 \in U'$. From this and the continuity shown above it follows that res is continuous with respect to the full topology. ∎

Take any ordered field L. The notion of "full" was introduced in [H] for $\mathcal{X}(L)$, but only for the Harrison sets. We generalize the definition to arbitrary subsets Y of $\mathcal{X}(L)$ by calling Y *full* if $\lambda_L^{-1}(\lambda_L(Y)) = Y$. We will call two orderings $P_1, P_2 \in \mathcal{X}(L)$ *equivalent* if $\lambda(P_1) = \lambda(P_2)$. Hence, Y is full if and only if it is closed under equivalence.

Note that the intersection of finitely many full sets is again a full set and the union of any family of full sets is also a full set. We define the *full topology* on $\mathcal{X}(L)$ by taking as open sets all full sets that are open in the Harrison topology. In general, this topology is strictly coarser than the Harrison topology and hence not Hausdorff, but it is always quasi-compact.

Remark 3.5. 1) If Y is a full open (or closed) subset of $\mathcal{X}(L)$, then $\lambda(Y)$ is an open (or closed, respectively) subset of $M(L)$.

2) For any $U \subset M(L)$, $\lambda^{-1}(U)$ is a full subset of $\mathcal{X}(L)$.

3) Take any extension $L|K$ of ordered fields. Then in the diagram

$$\begin{array}{ccc} \mathcal{X}(L) & \xrightarrow{\lambda_L} & M(L) \\ {\scriptstyle \text{res}}\big\downarrow & & {\scriptstyle \text{res}}\big\downarrow \\ \mathcal{X}(K) & \xrightarrow{\lambda_K} & M(K) \end{array}$$

the restriction mappings are continuous, and the diagram commutes (see [D, 7.2]). Being continuous mappings from compact spaces to Hausdorff spaces, the restriction mappings are also closed and proper.

The analogue of Lemma 3.4 is:

Lemma 3.6. *If $L|K$ is an extension of ordered fields, then the restriction mapping* res : $\mathcal{X}(L) \to \mathcal{X}(K)$ *preserves equivalence and is continuous with respect to both the Harrison and the full topology. The preimage of every full set in $\mathcal{X}(R)$ under* res *is again full.*

Proof. The continuity in the Harrison topology has just been stated. The fact that res preserves equivalence follows from the commutativity of the above diagram. As in the proof of Lemma 3.4, this implies the last assertion, and it follows that res is also continuous with respect to the full topology. ∎

If R is any real closed field, each ordering P on $R(y)$ is uniquely determined by the cut (D, E) in R where $D = \{d \in R \mid y - d \in P\}$ and $E = R \setminus D$ (cf. [G]). Hence, we have a bijection

$$\chi : \mathcal{C}(R) \to \mathcal{X}(R(y)),$$

which we will also denote by χ_R.

Proposition 3.7. *With respect to the interval topology on $\mathcal{C}(R)$ and the Harrison topology on $\mathcal{X}(R(y))$, χ is a homeomorphism. The same holds with respect to the full topologies. For $C_1, C_2 \in \mathcal{C}(R)$, C_1 is equivalent to C_2 if and only if $\chi(C_1)$ is equivalent to $\chi(C_2)$.*

Proof. The first assertion is a consequence of [KMO, Prop. 2.1]. For the proof of the second assertion, we first prove the third. By definition, $\chi(C_1)$ is equivalent to $\chi(C_2)$ if and only if $\lambda(\chi(C_1)) = \lambda(\chi(C_2))$. But by Theorem 1.1, this holds if and only if C_1 and C_2 are equivalent. It follows that the image of a full subset of $\mathcal{C}(R)$ under χ is again full, and the preimage of a full subset of $\mathcal{X}(R(y))$ under χ is again full. Now the second assertion follows from the first. ∎

This proposition, together with Theorem 1.1, gives us a description of $M(R(y))$ as the quotient space of $\mathcal{C}(R)$ with respect to the equivalence relation for cuts:

Proposition 3.8. *Via the mapping $\lambda\circ\chi$, the space $M(R(y))$ with the topology induced by the Harrison topology is the quotient space of $\mathcal{C}(R)$ with the full topology, where the quotient is taken modulo the equivalence of cuts. The full topology is the coarsest topology on $\mathcal{C}(R)$ for which $\lambda\circ\chi$ is continuous. The image of a full open set in $\mathcal{C}(R)$ under $\lambda\circ\chi$ is open.*

A place in $M(R(y))$ is called *principal* if it is the image under $\lambda\circ\chi$ of a principal cut in $\mathcal{C}(R)$. From Proposition 3.8 and Lemma 3.2 we obtain:

Lemma 3.9. *The principal cuts lie dense in $M(R(y))$.*

We will also need:

Proposition 3.10. *The restriction mappings in the following diagram are continuous* (*with respect to the interval topology and the Harrison topology as well as with respect to the full topologies*), *and the following diagram commutes:*

$$\begin{array}{ccccc}
\mathcal{C}(F) & \xrightarrow{\chi_F} & \mathcal{X}(F(y)) & \xrightarrow{\lambda_{F(y)}} & M(F(y)) \\
{\scriptstyle\text{res}}\big\downarrow & & {\scriptstyle\text{res}}\big\downarrow & & {\scriptstyle\text{res}}\big\downarrow \\
\mathcal{C}(R) & \xrightarrow{\chi_R} & \mathcal{X}(R(y)) & \xrightarrow{\lambda_{R(y)}} & M(R(y))
\end{array}$$

Proof. In view of Lemmas 3.4 and 3.6 and part 3) of Remark 3.5, it just remains to prove that the square on the left hand side of the diagram commutes. This follows from the fact that the cut induced by y in R under the restriction of some ordering from $F(y)$ is simply the restriction of the cut induced by y in F under this ordering. ∎

We note the following fact, which is straightforward to prove:

Lemma 3.11. *If ι is an embedding of $\mathcal{C}(R)$ in $\mathcal{C}(F)$, or of $\mathcal{X}(K)$ in $\mathcal{X}(L)$, or of $M(K)$ in $M(L)$, compatible with restriction, then the preimage of a set U under ι is equal to its image under restriction.*

4. Embeddings of $\mathcal{C}(R)$ in $\mathcal{C}(F)$. We consider an extension $F|R$ of ordered fields. Our goal is to construct an embedding ι of $M(R(y))$ in $M(F(y))$ under suitable assumptions on the extension; this will be done in Section 5. In view of Proposition 3.8, we first define an order preserving embedding of $\mathcal{C}(R)$ in $\mathcal{C}(F)$.

From now on we will frequently have to compare cuts in R with cuts in F. If $C = (D, E)$ is a cut in R, then we will say that the element $a \in F$ *fills* C if $D < a < E$ holds in F. Since a subset A of R is also a subset of F, we will have to distinguish whether A^+ and A^- are taken in R or in F. If this is not clear from the context, we will write A^{+_R} and A^{-_R} for the former, and A^{+_F} and A^{-_F} for the latter.

To find a suitable embedding of $\mathcal{C}(R)$ in $\mathcal{C}(F)$, we need to study the set of all elements in F that fill a cut in R. More generally, we have to consider the following situation.

LEMMA 4.1. *Take two non-empty sets $D < E$ in R. Assume that (D, E) is either a non-ball cut in R with* $\mathrm{Betw}_F(D, E) \neq \emptyset$, *or a ball complement in R. Then*

$$\mathrm{Betw}_F(D, E) = B_S(a, F)$$

for each $a \in$ $\mathrm{Betw}_F(D, E)$, *where S is the largest final segment of vF disjoint from $v(E - D)$ (or equivalently, the largest subset of vF such that $S > v(E - D)$).*

Proof. First, we show that $\mathcal{B} := \mathrm{Betw}_F(D, E)$ is contained in $B_S(a, F)$. Take any $d \in D$, $e \in E$ and $b \in \mathcal{B}$. As $d < a < e$ and $|a - b| < e - d$, we see that $v(a - b) \geq v(e - d)$. We show that we must have $v(a - b) > v(e - d)$, which yields $b \in B_S(a, F)$.

Suppose that $v(a-b) = v(e-d)$. We assume that $b < a$; the case of $b > a$ is symmetrical. Then it follows that $v(a-d) = v(e-d)$ and $v(b-d) \geq v(e-d)$, so that $v\big(\frac{a-b}{e-d}\big) = 0$, $v\big(\frac{a-d}{e-d}\big) = 0$ and $v\big(\frac{b-d}{e-d}\big) \geq 0$. We consider the residues under v, which are real numbers. Firstly, $v\big(\frac{a-b}{e-d}\big) = 0$ and $\frac{a-b}{e-d} > 0$ imply that $\big(\frac{a-b}{e-d}\big)v > 0$, and $v\big(\frac{b-d}{e-d}\big) \geq 0$ and $\frac{b-d}{e-d} > 0$ imply that $\big(\frac{b-d}{e-d}\big)v \geq 0$. Secondly, we have

$$0 \leq \left(\frac{b-d}{e-d}\right)v < \left(\frac{a-d}{e-d}\right)v,$$

where the last inequality holds because $\big(\frac{a-d}{e-d}\big)v - \big(\frac{b-d}{e-d}\big)v = \big(\frac{a-d}{e-d} - \frac{b-d}{e-d}\big)v = \big(\frac{a-b}{e-d}\big)v > 0$. So there are rational numbers $q_1, q_2 > 0$ such that

$$\left(\frac{b-d}{e-d}\right)v < q_1 < q_2 < \left(\frac{a-d}{e-d}\right)v,$$

which yields

$$b - d < q_1(e-d) < q_2(e-d) < a - d,$$

whence

$$b < d + q_1(e-d) < d + q_2(e-d) < a.$$

Consequently, $d + q_1(e-d)$, $d + q_2(e-d) \in \mathrm{Betw}_R(D, E)$, which can only happen in the ball complement case. In this case, $\mathrm{Betw}_R(D, E)$ is a ball

$B_{S_0}(a_0, R)$ in R, with $D < a_0 < E$. By Lemma 2.3, $S_0 = vR \setminus v(E-D)$. But

$$v(d + q_2(e-d) \;-\; (d + q_1(e-d))) = v((q_2 - q_1)(e-d)) = v(e-d) < S_0,$$

in contradiction to $d + q_1(e-d)$, $d + q_2(e-d) \in B_{S_0}(a_0, R)$. We have now proved that $\mathcal{B}$ is contained in $B_S(a, F)$.

It remains to show that $B_S(a, F)$ is contained in $\mathcal{B}$. If this were not the case, then for some $b \in B_S(a, F)$ there would exist some $d \in D$ with $b \leq d$, or some $e \in E$ with $b \geq e$. We will assume the first case and deduce a contradiction; the second case is symmetrical. Since $b \leq d < a$ and $B_S(a, F)$ is convex, we have $d \in B_S(a, F)$.

First, we consider the case of (D, E) being the complement of a ball $B_{S_0}(a_0, R)$ in R. Then $a_0 \in \mathcal{B} \subseteq B_S(a, F)$, so $B_S(a, F) = B_S(a_0, F)$. Further, we know from Lemma 2.3 that $v(E-D) = vR \setminus S_0$. By our choice of S, this implies that $S \cap vR = S_0$, and we obtain $d \in B_S(a_0, F) \cap R = B_{S_0}(a_0, R)$, a contradiction.

In the non-ball case, we use the relation $B_S(a, F) = B_S(d, F)$ to deduce that $B_S(a, F) \cap R = B_T(d, R)$, where $T := S \cap vR$. From $S > v(E-D)$ it follows that $B_T(d, R) < E$. In the present case, $\mathrm{Betw}_R(D, E) = \emptyset$, so we find that $B_T(d, R)$ is contained in D. Since $B_S(a, F)$ is convex and contains $a > D$, it follows that $B_T(d, R)$ is a final segment of D. But this contradicts our assumption that (D, E) is a non-ball cut. ∎

Remark 4.2. If (D, E) is the complement of a ball $B_{S_0}(a_0, R)$ in R, then we can choose $a = a_0$. Moreover, S is then equal to the largest final segment of vF disjoint from $vR \setminus S_0$ (or equivalently, the largest subset of vF such that $S > vR \setminus S_0$).

The next lemma tells us which cuts in F restrict to the same cut in R:

Lemma 4.3. *Take any cut C in R.*

(a) *If $C = (D, E)$, then the set of all cuts in F that restrict to C is $\{C' \in \mathcal{C}(F) \mid D^{+_F} \leq C' \leq E^{-_F}\}$. (If $D = \emptyset$, then D^{+_F} means the cut F^{-_F}, and if $E = \emptyset$, then E^{-_F} means the cut F^{+_F}.)*

(b) *Assume that $C = B_0^{+_R}$ or $C = B_0^{-_R}$ for a ball $B_0 = B_{S_0}(a_0, R) \neq R$ in R, and take the ball $B_S(a_0, F)$ as in Lemma 4.1. Then the set of all cuts in F that restrict to the cut C in R is $\{C' \in \mathcal{C}(F) \mid B_0^{+_F} \leq C' \leq B_S(a_0, F)^{+_F}\}$ for $C = B_0^{+_R}$, and $\{C' \in \mathcal{C}(F) \mid B_S(a_0, F)^{-_F} \leq C' \leq B_0^{-_F}\}$ for $C = B_0^{-_R}$. If vR is a convex subgroup of vF and C is not principal, then $B_0^{+_F} = B_S(a_0, F)^{+_F}$, $B_0^{-_F} = B_S(a_0, F)^{-_F}$, and the above sets are singletons.*

Proof. The proof of (a) is straightforward. Now assume the hypotheses of (b). We prove the assertions for $C = B_0^{+_R}$. For $C = B_0^{-_R}$, the proof is symmetrical. If (D, E) is the ball complement of B_0 in R, then $C = (D \cup B_0, E)$. By Lemma 4.1, $\mathrm{Betw}_F(D, E) = B_S(a, F)$, which implies that $\mathrm{Betw}_F(D \cup B_0, E) = \{b \in B_S(a, F) \mid b > B_0\}$. This implies the first assertion of (b).

For the proof of the second assertion, assume that vR is a convex subgroup of vF and that C is not principal. Then S_0 is a non-empty final segment of vR, and $S_0 \neq vR$ since $B_{S_0}(a_0, R) \neq R$ by assumption. We wish to show that S_0 is an initial segment of S. Since S_0 is a final segment of vR and vR is convex in vF, also S_0 is convex in vF. Hence if S_0 were not an initial segment of S, then there would be an element $\gamma \in S$ such that $\gamma < S_0$. On the other hand, $S > vR \setminus S_0$, whence $S_0 > \gamma > vR \setminus S_0 \neq \emptyset$. But this contradicts the convexity of vR in vF.

Since S_0 is an initial segment of S, the ball $B_{S_0}(a_0, R)$ is coinitial and cofinal in the ball $B_S(a_0, F)$. This implies that $B_0^{+F} = B_S(a_0, F)^{+F}$ and $B_0^{-F} = B_S(a_0, F)^{-F}$. ■

We define an order preserving embedding $\tilde{\iota}$ of $\mathcal{C}(R)$ in $\mathcal{C}(F)$ as follows. Take a cut C in R. If $C = (D, E)$ is a non-ball cut in R, then we set $\tilde{\iota}(C) = D^{+F}$ or $\tilde{\iota}(C) = E^{-F}$. If C is the lower or upper edge of a ball $B_0 \neq R$ in R and (D, E) is the ball complement of B_0, then we set $\tilde{\iota}(C) = D^{+F}$ if $C = B_0^{-R}$ is the lower edge, and $\tilde{\iota}(C) = E^{-F}$ if $C = B_0^{+R}$ is the upper edge. Finally, we set $\tilde{\iota}(R^{-R}) = R^{-F}$ and $\tilde{\iota}(R^{+R}) = R^{+F}$. Note that $\tilde{\iota}$ is uniquely determined by this definition if and only if no non-ball cut (D, E) in R is filled in F because then $D^+ = E^-$ will still hold in F.

REMARK 4.4. For a cut C in R, its image $\tilde{\iota}(C)$ is a non-ball cut in F if and only if C is a non-ball cut in R that is not filled in F. Hence if $\tilde{\iota}(C)$ is a non-ball cut in F then it is the only cut in F that restricts to C.

Indeed, if C is a ball cut in R, then by our definition of $\tilde{\iota}$, also $\tilde{\iota}(C)$ is a ball cut. If $C = (D, E)$ is a non-ball cut in R that is filled in F, then by Lemma 4.1, $D^{+F} = B^{-F}$ and $E^{-F} = B^{+F}$ for a ball $B = B_S(a, F)$ in F, so $\tilde{\iota}(C)$ is again a ball cut. But if the non-ball cut $C = (D, E)$ is not filled in F, then it is also a non-ball cut in F, as the restriction to R of a ball cofinal in the left or coinitial in the right cut set in F would be a ball in R cofinal in D or coinitial in E.

The embedding $\tilde{\iota}$ is order preserving since if $(D_1, E_1) < (D_2, E_2)$ are two cuts in R, then $E_1 \cap D_2 \neq \emptyset$ and therefore $D_1^{+F} \leq E_1^{-F} < D_2^{+F} \leq E_2^{-F}$.

If $B_{S_0}(a_0, R) \neq R$ is a ball in R, and if we take S as defined in Lemma 4.1, then by our definition,

$$\tilde{\iota}(B_{S_0}(a_0, R)^{-R}) = B_S(a_0, R)^{-F} \quad \text{and} \quad \tilde{\iota}(B_{S_0}(a_0, R)^{+R}) = B_S(a_0, R)^{+F}.$$

This together with $\tilde{\iota}(R^{-R}) = R^{-F}$ and $\tilde{\iota}(R^{+R}) = R^{+F}$ shows:

Lemma 4.5. *The embedding* $\tilde{\iota}$ *sends equivalent cuts to equivalent cuts. Hence the preimage of a full set is full.*

Let us also note:

Proposition 4.6. *If* vR *is cofinal in* vF (*which implies that there is no* $f \in F$ *such that* $f > R$), *then* $\tilde{\iota}$ *sends principal cuts to principal cuts. Otherwise, no principal cut is sent to a principal cut.*

Proof. A principal cut in R is the upper or lower edge of a ball $B_\emptyset(a_0, R)$. Take the ball $B_S(a_0, F)$ as in Lemma 4.1. By definition, $\tilde{\iota}(B_\emptyset(a_0, R)^{-_R}) = B_S(a_0, F)^{-_F}$ and $\tilde{\iota}(B_\emptyset(a_0, R)^{+_R}) = B_S(a_0, F)^{+_F}$. The latter cuts are principal if and only if $S = \emptyset$. By Remark 4.2, $S = \emptyset$ if and only if there is no $\gamma \in vF$ such that $\gamma > vR$, that is, if and only if vR is cofinal in vF. ∎

If there is at least one non-ball cut in R that is filled in F, then the embedding $\tilde{\iota}$ will *not* be continuous with respect to the interval topology. Even worse:

Proposition 4.7. *Take any extension* $F|R$ *of ordered fields. If there is at least one non-ball cut in* R *that is filled in* F, *then there exists no embedding of* $\mathcal{C}(R)$ *in* $\mathcal{C}(F)$ *that is continuous with respect to the interval topology and compatible with restriction.*

Proof. Take $C = (D, E)$ to be a non-ball cut in R that is filled in F. Then Lemma 4.1 shows that $\mathrm{Betw}_F(D, E)$ is equal to a ball B in F. In order to be compatible with restriction, an embedding has to send C to a cut C' in F which is equal to B^{+_F}, B^{-_F}, or a proper cut in B. Suppose that $C' \neq B^{+_F}$. Take any cut $C_1 < B^{-_F}$ and consider the open interval $I = (C_1, B^{+_F})$ in $\mathcal{C}(F)$. Then the restriction of I to $\mathcal{C}(R)$ is an interval in $\mathcal{C}(R)$ with last element C. This shows that the preimage of I under any embedding compatible with restriction is not open, as follows from Lemma 3.1 since C is not a principal cut.

In the case of $C' = B^{+_F}$, choose $C_2 \in \mathcal{C}(F)$ such that $B^{+_F} < C_2$ and consider the open interval $I = (B^{-_F}, C_2)$ in $\mathcal{C}(F)$. Its restriction to $\mathcal{C}(R)$ is an interval with first element C, hence again not open. ∎

The problem is that an open interval in $\mathcal{C}(F)$ can end in a set that fills a cut from R, in which case its preimage in $\mathcal{C}(R)$ will include an endpoint. However, a full open set will have to enter the between set from both sides, and so we obtain the following positive result if we switch from the interval topology to the full topology:

Proposition 4.8. *Assume that* vR *is a convex subgroup of* vF. *Then the embeddings* $\tilde{\iota} : \mathcal{C}(R) \to \mathcal{C}(F)$ *constructed above are exactly the embeddings that are continuous with respect to the full topology and compatible with restriction.*

Proof. Take an embedding $\tilde{\iota} : \mathcal{C}(R) \to \mathcal{C}(F)$ as constructed above. In view of Lemma 4.3, $\tilde{\iota}$ is compatible with restriction.

By virtue of Lemma 4.5, in order to show that $\tilde{\iota}$ is continuous with respect to the full topology, it suffices to show that the preimage of any full open set U is open in the interval topology of $\mathcal{C}(R)$. Take $C \in \mathcal{C}(R)$ with $\tilde{\iota}(C) \in U$. Since U is open in the interval topology of $\mathcal{C}(F)$, there is an open interval $I \subseteq U$ which contains $\tilde{\iota}(C)$. The preimage of I under $\tilde{\iota}$ is again an interval, and if C is not an endpoint of it, then C lies in some open subinterval of this preimage.

Now suppose that C is an endpoint of the preimage of I. Then either all cuts in I on the left side of $\tilde{\iota}(C)$ restrict to C, or all cuts in I on the right side of $\tilde{\iota}(C)$ restrict to C. In both cases, we see that more than one cut in F restricts to C. Since we have assumed vR to be a convex subgroup of vF, Lemma 4.3 shows that we are in one of the following cases:

(a) C is a non-ball cut,
(b) C is a principal cut,
(c) $C = R^{-_R}$ or $C = R^{+_R}$.

In all three cases, by our construction of $\tilde{\iota}$, we find that $\tilde{\iota}(C) = B^{-_F}$ or $\tilde{\iota}(C) = B^{+_F}$ for some ball B in F. Denote the restriction of B^{-_F} to R by C_1, and the restriction of B^{+_F} to R by C_2. Then $C = C_1$ or $C = C_2$.

Since U is assumed to be full, $B^{-_F}, B^{+_F} \in U$; and since U is open, $B^{-_F} \in I_1$ and $B^{+_F} \in I_2$ for some open intervals I_1 and I_2 contained in U.

We first deal with cases (a) and (b). In both cases, B^{-_F} is the smallest cut that reduces to C_1 and B^{+_F} is the largest cut that reduces to C_2. The open interval I_1 contains a cut on the left of B^{-_F}, which consequently restricts to a cut $C_1' < C_1$. Similarly, I_2 contains a cut on the right of B^{+_F}, which consequently restricts to a cut $C_2' > C_2$. For every $C' \in (C_2, C_2')$ we have $\tilde{\iota}(C_2) < \tilde{\iota}(C') < \tilde{\iota}(C_2')$, hence $\tilde{\iota}(C') \in I_2$. This shows that $[C_2, C_2')$ is contained in the preimage of I_2. Similarly, it is shown that $(C_1', C_1]$ is contained in the preimage of I_1.

In case (a), both B^{-_F} and B^{+_F} restrict to C, so we have $C = C_1 = C_2$. In case (b), where $C = a^{-_R}$ or $C = a^{+_R}$ for some $a \in R$, B^{-_F} restricts to a^{-_R} and B^{+_F} restricts to a^{+_R}. In both cases, $(C_1', C_1] \cup [C_2, C_2') = (C_1', C_2')$. It follows that C has the open neighborhood (C_1', C_2') which is contained in the preimage of U.

Now we consider case (c). In this case, $\tilde{\iota}(C) = R^{-_F}$, the largest cut that restricts to $C_1 = R^{-_R}$, or $\tilde{\iota}(C) = R^{+_F}$, the smallest cut that restricts to $C_2 = R^{+_R}$. The open interval I_1 contains a cut on the right of R^{-_F}, which consequently restricts to a cut $C_1' > R^{-_R}$. Similarly, I_2 contains a cut on the left of R^{+_F}, which consequently restricts to a cut $C_2' < R^{+_R}$. For every $C' \in (C_2', R^{+_R})$ we have $\tilde{\iota}(C_2') < \tilde{\iota}(C') < \tilde{\iota}(R^{+_R})$, hence $\tilde{\iota}(C') \in I_2$.

This shows that $(C_2', R^{+_R}]$ is contained in the preimage of I_2. Similarly, it is shown that $[R^{-_R}, C_1')$ is contained in the preimage of I_1. Now one of these two intervals is an open neighborhood of C.

It follows in all three cases that C has an open neighborhood which is contained in the preimage of U. This proves that the restriction of U is open.

Now assume that $\tilde{\iota}'$ is an embedding of $\mathcal{C}(R)$ in $\mathcal{C}(F)$, compatible with restriction. Suppose that there is a cut C in $\mathcal{C}(R)$ such that its image $\tilde{\iota}'(C)$ is not in accordance with our above construction.

First, we consider the case of $C = (D, E)$ being a non-ball cut. Then our assumption and the compatibility with restriction imply that $D^{+_F} < \tilde{\iota}'(C) < E^{-_F}$ in $\mathcal{C}(F)$. If the ball $B_S(a, F)$ is chosen as in Lemma 4.1, then $D^{+_F} = B_S(a, F)^{-_F}$ and $E^{-_F} = B_S(a, F)^{+_F}$. Therefore, the open interval (D^{+_F}, E^{-_F}) in $\mathcal{C}(F)$ is full by Proposition 3.3. But the preimage of this interval is the singleton $\{C\}$, hence not open.

Now we consider the case of $C = B_0^{+_R}$ for some ball B_0 in R; the case of $C = B_0^{-_R}$ is symmetrical. If (D, E) is the ball complement of B_0 in R, then our assumption and the compatibility with restriction imply that $D^{+_F} < B_0^{+_F} \le \tilde{\iota}'(C) < E^{-_F}$ in $\mathcal{C}(F)$. The same argument as before shows that (D^{+_F}, E^{-_F}) is a full open interval in $\mathcal{C}(F)$. Its preimage in $\mathcal{C}(R)$ has C as its last element. Since C is the upper edge of a ball not equal to R, it follows that this interval is not open.

Finally, we consider the case of $C = R^{+_R}$; the case of $C = R^{-_R}$ is symmetrical. Then our assumption and the compatibility with restriction show that $R^{+_F} < \tilde{\iota}'(C)$ in $\mathcal{C}(F)$. The open set $[C_{-\infty}, R^{-_F}) \cup (R^{+_F}, C_\infty]$ in $\mathcal{C}(F)$ is full by Proposition 3.3. But the preimage of it is either $\{R^{+_R}\}$ or $\{R^{-_R}, R^{+_R}\}$, hence not open. ∎

Our positive result is contrasted by the following negative result:

PROPOSITION 4.9. *Assume that vR is not a convex subgroup of vF. Then there are no embeddings $\tilde{\iota} : \mathcal{C}(R) \to \mathcal{C}(F)$ that are continuous with respect to the full topology and compatible with restriction.*

Proof. If vR is not a convex subgroup of vF, then there are $\alpha, \beta \in vR$ and $\gamma \in vF \setminus vR$ such that $\alpha < \gamma < \beta$. Take $S_0 := \{\delta \in vR \mid \gamma < \delta\}$ and $B_0 := B_{S_0}(0, R)$. Note that $B_0 \neq R$ because $\alpha \notin S_0$, and that B_0 is not a singleton because $\beta \in S_0$.

Now if $B_S(0, F)$ is as in Lemma 4.1, then it follows from Remark 4.2 that $\gamma \in S \setminus S_0$. This implies that $B_{S_0}(0, R)$ is not cofinal in $B_S(0, F)$, whence $B_0^{+_F} < B_S(0, F)^{+_F}$. Now assume that $\tilde{\iota} : \mathcal{C}(R) \to \mathcal{C}(F)$ is an embedding compatible with restriction. Then by Lemma 4.3, $B_0^{+_F} \le \tilde{\iota}(B_0^{+_R}) \le B_S(0, F)^{+_F}$. Suppose first that $B_0^{+_F} < \tilde{\iota}(B_0^{+_R})$. By Proposition 3.3, the open neighborhood $U := [C_{-\infty}, B_0^{-_F}) \cup (B_0^{+_F}, C_\infty]$ of $\tilde{\iota}(B_0^{+_R})$ in $\mathcal{C}(F)$ is full. But $\tilde{\iota}^{-1}(U) = [C_{-\infty}, B_0^{-_R}) \cup [B_0^{+_R}, C_\infty]$ or $\tilde{\iota}^{-1}(U) = [C_{-\infty}, B_0^{-_R}] \cup [B_0^{+_R}, C_\infty]$ in

$\mathcal{C}(R)$, both of which are not open since B_0 is not a singleton and therefore $B_0^{+_R}$ is not the immediate successor of $B_0^{-_R}$.

Suppose now that $B_0^{+_F} = \tilde{\iota}(B_0^{+_R})$. Again by Proposition 3.3, the open neighborhood $U := (B_S(0,F)^{-_F}, B_S(0,F)^{+_F})$ of $\tilde{\iota}(B_0^{+_R})$ in $\mathcal{C}(F)$ is full. But $\tilde{\iota}^{-1}(U) = (B_0^{-_R}, B_0^{+_R}]$ or $\tilde{\iota}^{-1}(U) = [B_0^{-_R}, B_0^{+_R}]$ in $\mathcal{C}(R)$, both of which are not open since $B_0 \neq R$. ∎

5. Embeddings of $M(R(y))$ in $M(F(y))$. We will now consider an extension of formally real fields $F|R$, with R real closed, but not necessarily archimedean. We will first consider the case where also F is real closed.

We assume that vR is convex in vF and start from one of the embeddings $\tilde{\iota} : \mathcal{C}(R) \to \mathcal{C}(F)$ constructed in the previous section (cf. Proposition 4.8). We define an embedding

$$\iota : M(R(y)) \to M(F(y))$$

in the following way. If $M(R(y)) \ni \zeta = \lambda_{R(y)} \circ \chi_R(C)$ for a cut C in R, then we set

$$\iota(\zeta) := \lambda_{F(y)} \circ \chi_F(\tilde{\iota}(C)).$$

Since $\tilde{\iota}$ is compatible with the equivalence of cuts, the embedding ι is well defined and the diagram

$$\begin{tikzcd} \mathcal{C}(F) \arrow[r, "\lambda_{F(y)}\circ\chi_F"] & M(F(y)) \\ \mathcal{C}(R) \arrow[u, "\tilde{\iota}"] \arrow[r, "\lambda_{R(y)}\circ\chi_R"] & M(R(y)) \arrow[u, "\iota"'] \end{tikzcd}$$

commutes.

THEOREM 5.1. *Take an extension $F|R$ of real closed fields. If vR is convex in vF, then the embedding ι as defined above does not depend on the particular choice of $\tilde{\iota}$ and is continuous and compatible with restriction.*

Conversely, if $\iota : M(R(y)) \to M(F(y))$ is continuous and compatible with restriction, then it induces an embedding $\tilde{\iota} : \mathcal{C}(R) \to \mathcal{C}(F)$ continuous with respect to the full topology and compatible with restriction, such that the above diagram commutes, and vR is convex in vF.

Proof. Take $\tilde{\iota}$ as constructed in the previous section. We show that ι is continuous. Take any open set U in $M(F(y))$. By Proposition 3.8, its preimage U_1 in $\mathcal{C}(F)$ is a full open set. Then by Proposition 4.8, the preimage U_2 of U in $\mathcal{C}(R)$ is a full open set. Again by Proposition 3.8, the image U_3 of U_2 in $M(R(y))$ is open. From Lemma 3.11 we know that $\mathrm{res}(U)$ is the preimage of U under ι. But from the commutativity of the diagram in Proposition 3.10 we know that

$$\mathrm{res}(U) = \mathrm{res} \circ \lambda_{F(y)} \circ \chi_F(U_1) = \lambda_{R(y)} \circ \chi_R \circ \mathrm{res}(U_1) = U_3.$$

So the preimage of U under ι is open. This proves the continuity of ι.

In the construction of $\tilde{\iota}$ in the previous section the only freedom we had was to choose either the upper or the lower edge of the ball which fills a non-ball cut in R; but these cuts correspond to the same $\mathbb{R}$-place in $M(F(y))$. This shows that all embeddings $\tilde{\iota}$ constructed in the previous section determine the same embedding ι.

We will now prove the second assertion. Take ι as in the assumption. For each $C \in \mathcal{C}(R)$, we wish to define $\tilde{\iota}(C)$ such that

$$\lambda_{F(y)} \circ \chi_F \circ \tilde{\iota}\,(C) = \iota \circ \lambda_{R(y)} \circ \chi_R\,(C).$$

Set $\xi := \lambda_{R(y)} \circ \chi_R(C) \in M(R(y))$ and $\xi' := \iota(\xi)$. Since ι is compatible with restriction, ξ is the restriction of ξ' to $R(y)$. By the commutativity of the diagram in Proposition 3.10, we find that if $C' \in \mathcal{C}(F)$ is sent to ξ' by $\lambda_{F(y)} \circ \chi_F$, then $\mathrm{res}(C')$ must be sent to ξ by $\lambda_{R(y)} \circ \chi_R$.

If C is a non-ball cut, then choose any $C' \in \mathcal{C}(F)$ such that $\lambda_{F(y)} \circ \chi_F(C'_1) = \xi'$ and define $\tilde{\iota}(C) := C'$. Since C is the only cut in R that is sent to ξ by $\lambda_{R(y)} \circ \chi_R$, it follows that $\mathrm{res}(C') = C$.

If C is a ball cut, that is, $C = B_0^{-R}$ or $C = B_0^{+R}$ for some ball B_0 in R, then we have to find the images for both B_0^{-R} and B_0^{+R}. We claim that the continuity of ι implies that the preimage of ξ' under $\lambda_{F(y)} \circ \chi_F$ is $\{B^{-F}, B^{+F}\}$ for some ball B in F with $\mathrm{res}(B^{-F}) = B_0^{-R}$ and $\mathrm{res}(B^{+F}) = B_0^{+R}$. We treat the case of $B_0 \neq R$ and leave the case of $B_0 = R$ to the reader.

We write $B_0 = B_{S_0}(a_0, R)$, take S as in Lemma 4.1, and set $B := B_S(a_0, F)$. Suppose the preimage of ξ' is not $\{B^{-F}, B^{+F}\}$. Take C' in the preimage. Then by what we have shown above, C' restricts to B_0^{-R} or B_0^{+R}. We assume the latter case; the former is symmetrical. Then $B_0^{+F} \leq C' < B^{+F}$. By Proposition 3.3, the open interval (B^{-F}, B^{+F}) is full, so $U := \lambda_{F(y)} \circ \chi_F((B^{-F}, B^{+F}))$ is open in $M(F(y))$ and contains ξ'. The restriction I of (B^{-F}, B^{+F}) to $\mathcal{C}(R)$ has $B_0^{+R} = \mathrm{res}(C')$ as its largest element, hence it is not open. The same argument as in the first part of this proof shows that the preimage U' of U under ι is equal to $\lambda_{F(y)} \circ \chi_F \circ \mathrm{res}\,((B^{-F}, B^{+F})) = \lambda_{F(y)} \circ \chi_F(I)$, which is not open. But this contradicts the continuity of ι. We see that the preimage of ξ' must be $\{B^{-F}, B^{+F}\}$. So we set $\tilde{\iota}(B_0^{-R}) = B^{-F}$ and $\tilde{\iota}(B_0^{+R}) = B^{+F}$ and note that $\mathrm{res}(B^{-F}) = B_0^{-R}$ and $\mathrm{res}(B^{+F}) = B_0^{+R}$.

We have now defined a mapping $\tilde{\iota} : \mathcal{C}(R) \to \mathcal{C}(F)$ which is compatible with restriction. Therefore, $\tilde{\iota}$ must be injective, and since res preserves $\leq$ by Lemma 3.4, $\tilde{\iota}$ must preserve $<$. By definition, $\tilde{\iota}$ also preserves equivalence.

It remains to show that $\tilde{\iota}$ is continuous with respect to the full topology. Take a full open set U in $\mathcal{C}(F)$. By Proposition 3.8, $U_1 := \lambda_{F(y)} \circ \chi_F(U)$ is open. By Lemma 3.11, $U_2 := \mathrm{res}(U_1)$ is the preimage of U_1 under ι, hence open since ι is continuous. By the commutativity of the diagram in

Proposition 3.10,

$$U_2 = \mathrm{res} \circ \lambda_{F(y)} \circ \chi_F(U) = \lambda_{R(y)} \circ \chi_R \circ \mathrm{res}(U).$$

Thus, the full set $\mathrm{res}(U)$ in $\mathcal{C}(R)$ is the preimage of U_2, hence open by Proposition 3.8. Again by Lemma 3.11, the full open set $\mathrm{res}(U)$ is the preimage of U under $\tilde{\iota}$. This proves the continuity of $\tilde{\iota}$. ∎

Now we will consider the case of F not being real closed. We choose a real closure R' of F and take $\iota' : M(R(y)) \to M(R'(y))$ to be the embedding constructed above. Since $\mathrm{res}_{R'(y)|F(y)}$ is continuous (cf. Remark 3.5, part 3))

$$\iota := \mathrm{res}_{R'(y)|F(y)} \circ \iota'$$

is a continuous mapping from $M(R(y))$ to $M(F(y))$. Since ι' is compatible with the restriction

$$\mathrm{res}_{R'(y)|R(y)} = \mathrm{res}_{F(y)|R(y)} \circ \mathrm{res}_{R'(y)|F(y)},$$

we see that ι is compatible with the restriction. For this reason, it is also injective.

As the real closure R' can be taken with respect to any ordering on F, we may lose the uniqueness of ι. However, we are able to show the following partial uniqueness result:

THEOREM 5.2. *Take two orderings P_1 and P_2 of F which induce the same $\mathbb{R}$-place, R'_1 and R'_2 the respective real closures of F, and $\iota'_i : M(R(y)) \to M(R'_i(y))$, $i = 1, 2$, the unique continuous embeddings compatible with restriction. Consider the following commuting diagram:*

$$\begin{tikzcd}
 & M(R'_1(y)) \arrow[dr, "\mathrm{res}_1"] & \\
M(R(y)) \arrow[ur, "\iota'_1"] \arrow[dr, "\iota'_2"'] & & M(F(y)) \arrow[ll, "\mathrm{res}"'] \\
 & M(R'_2(y)) \arrow[ur, "\mathrm{res}_2"'] &
\end{tikzcd}$$

Then

$$\mathrm{res}_1 \circ \iota'_1 = \mathrm{res}_2 \circ \iota'_2.$$

Proof. We will first show that the mappings coincide on all $\mathbb{R}$-places of $R(y)$ determined by the principal cuts.

Suppose that $\zeta = \lambda \circ \chi(a^+) = \lambda \circ \chi(a^-)$, where $a \in R$. Note that for the corresponding valuation v_ζ on $R(y)$, we have $vR < v_\zeta(a - y)$. Let $\zeta_i := \iota'_i(\zeta)$ for $i = 1, 2$. By the definition of the embedding ι'_i, we see that ζ_i is determined by the upper and lower edge of the ball $B_{S_i}(a, R'_i)$ where $S_i = \{\alpha \in vR'_i \mid \alpha > vR\}$. Then for the corresponding valuation v_{ζ_i} on

$R_i'(y)$ we have $vR < v_{\zeta_i}(a-y) < S_i$ in $v_{\zeta_i}R_i'$. Since these value groups are divisible (by [EP, Theorem 4.3.7], R_i' being real closed fields), the values $v_{\zeta_i}(a-y)$ are rationally independent over these value groups. Therefore, the valuations v_{ζ_i} are uniquely determined by the natural valuations on R_i' and the values $v_{\zeta_i}(a-y)$. The same remains true when we restrict to $F(y)$. There, by our assumption, the restrictions of the natural valuations on R_i' coincide, so the restrictions of the valuations v_{ζ_i} to $F(y)$ must coincide, too. Further, the residue fields of v_{ζ_i} on $F(y)$ are equal to the residue field of F because $v_{\zeta_i}(a-y)$ is rationally independent over vF. Since the restrictions to F of ζ_1 and ζ_2 coincide, the restrictions to $F(y)$ of these $\mathbb{R}$-places coincide as well. Therefore, $\mathrm{res}_1 \circ \iota_1'(\zeta) = \mathrm{res}_2 \circ \iota_2'(\zeta)$.

Now take $\zeta_1 = \mathrm{res}_1 \circ \iota_1'(\zeta)$ and $\zeta_2 = \mathrm{res}_2 \circ \iota_2'(\zeta)$ for some $\zeta \in M(R(y))$ and suppose they are distinct. Since $M(F(y))$ is Hausdorff, there are disjoint open neighborhoods $U_1 \ni \zeta_1$ and $U_2 \ni \zeta_2$. The preimages of U_1 and U_2 in $M(R(y))$ are open, and ζ lies in their intersection. So this intersection is not empty, and by the density of the principal places in $M(R(y))$ (cf. Lemma 3.9), there is a principal place ζ_0 in this intersection. But the images of ζ_0 under the two embeddings are equal and hence must lie in $U_1 \cap U_2$, a contradiction. ∎

6. Embeddings of $M(R(y))$ in $M(F(y))$ for archimedean R. In this section we will consider an extension of formally real fields $F|R$ in the special case where R is archimedean real closed. The general case has been treated in the previous section. Here, we wish to give a different, more explicit construction of a continuous embedding ι of $M(R(y))$ in $M(F(y))$ which is compatible with restriction.

We choose any real place ξ of F. Then $\overline{F} := \xi(F) \subseteq \mathbb{R}$. Since R is archimedean, we can assume that $\xi|_R = \mathrm{id}_R$ and that $\overline{F}|R$ is an extension of archimedean ordered fields. By ξ_y we denote the *constant extension* of ξ to $F(y)$, i.e., the unique extension of ξ which is trivial on $R(y)$. Its valuation ring is the smallest subring of $F(y)$ containing both the valuation ring of ξ and $R(y)$. The valuation associated with ξ_y is the *Gauß* or *functorial valuation* on $F(y)$ extending the valuation associated with ξ on F. On polynomials in $F[y]$ with coefficients in the valuation ring of ξ, ξ_y acts by applying ξ to the coefficients. Therefore, the residue field of ξ_y is $\xi(F)(y)$.

For every $\zeta \in M(R(y))$ we define the constant extension $\zeta_{\overline{F}}$ of ζ to $\overline{F}(y)$ as follows. As ζ is trivial on the archimedean field R, it is determined by an irreducible polynomial $p(y) \in R[y]$ (or by $1/y$). Since R is real closed and $\overline{F}$ is formally real, such a polynomial p remains irreducible over $\overline{F}$, and thus p (or $1/y$, respectively) determines a unique extension of ζ to $\overline{F}(y)$ which is trivial on $\overline{F}$. We set $\iota_{\overline{F}|R}(\zeta) := \zeta_{\overline{F}}$.

LEMMA 6.1. *The mapping* $\iota_{\overline{F}|R} : M(R(y)) \to M(\overline{F}(y))$ *is a continuous embedding compatible with the restriction. If* $\overline{F}$ *is real closed, then it is a homeomorphism.*

Proof. Since $\overline{F}|R$ is an extension of archimedean ordered fields, R lies dense in $\overline{F}$. It follows from [KMO, Theorem 3.2] that the restriction mapping from $M(\overline{F}(y))$ to $M(R(y))$ is a homeomorphism if $\overline{F}$ is real closed. Hence in this case, $\iota_{\overline{F}|R}$ is a homeomorphism.

If $\overline{F}$ is not real closed, then we consider a real closure R' of $\overline{F}$. By what we have shown already, $\iota_{R'|R}$ is a homeomorphism. Since $\mathrm{res}_{R'(y)|R(y)}$ is continuous, the same holds for $\iota_{\overline{F}|R} = \mathrm{res}_{R'(y)|\overline{F}(y)} \circ \iota_{R'|R}$. ∎

Now we define

$$\iota(\zeta) := \zeta_{\overline{F}} \circ \xi_y. \tag{6.1}$$

THEOREM 6.2. *The mapping* $\iota : M(R(y)) \to M(F(y))$ *is a continuous embedding.*

Proof. Take $a \in F(y)$. We have to show that the preimage of a subbasis set $H'(a)$ under ι is open in $M(R(y))$. If $\xi_y(a)$ is 0 or ∞, then the same holds for $\zeta_{\overline{F}} \circ \xi_y$ for every $\zeta \in M(R(y))$. In this case, $H'(a)$ is empty and we are done.

Assume now that $\xi_y(a) \neq 0, \infty$. Then $\xi_y(a)$ is a non-zero rational function $g(y) \in \overline{F}(y)$. The preimage of $H'(a)$ is therefore the set of all real places $\zeta \in M(R(y))$ such that $\zeta_{\overline{F}}(g) > 0$. In the case of $\overline{F} = R$ (which for instance holds when $R = \mathbb{R}$), this is precisely $H'(g)$ in $M(R(y))$. For the general case, we apply Lemma 6.1 to conclude that the preimage of $H'(g)$ under the constant extension mapping $\zeta \mapsto \zeta_{\overline{F}}$, and hence the preimage of $H'(a)$ under ι, is open. ∎

From Theorem 5.2, where we take $F = R(x)$, we now obtain:

THEOREM 6.3. *The mapping* ι *defined in* (6.1) *is the unique continuous embedding of* $M(R(y))$ *in* $M(R(x,y))$ *that is compatible with restriction and such that all places in the image of* ι *have restriction* ξ *to* $R(x)$.

We have chosen to give a direct proof of Theorem 6.3 although it can be derived from the theorems of the last section. In order to do this, we have to show that the embedding defined in (6.1) coincides with the embedding we have constructed before. To this end, we consider an ordering P of $R(y)$ and the cut C it induces in the archimedean real closed field R. If $R = \mathbb{R}$, then the only possibilities are $C = C_\infty$, $C = C_{-\infty}$, or $C = r^+, r^-$ for $r \in \mathbb{R}$. If $R \neq \mathbb{R}$, then C can also be a cut induced in R by some $r \in \mathbb{R} \setminus R$.

If $C = C_\infty$ or $C = C_{-\infty}$, we have $y > F$ or $y < F$ under the corresponding orderings. In this case, $0 < vy^{-1} < vF^{>0}$, where $vF^{>0}$ denotes the set of positive elements of vF.

In the case of $C = r^+, r^-$, we see that $\iota(C)$ is the upper or lower edge of $B_{vF^{>0}}(r, F)$. This ball is $r + \mathcal{M}$ where $\mathcal{M}$ is the valuation ideal of infinitesimals in F. Since C is induced by y, we find that $0 < v(y - r) < vF^{>0}$.

In the final case, we have two subcases. If C is not filled in F, then $v(y - f) \leq 0$ for every $f \in F$. If C is filled by some element in F, then we can identify this element with the real number r that fills the cut C. In this case, we obtain the same result as previously.

In all three cases, we find that the constant extension ξ_y of ξ must be trivial on $R(y)$, which implies that $\iota(\zeta)$ must be of the form $\zeta_{\overline{F}} \circ \xi_y$.

In the case of $R = \mathbb{R}$, we can show the above more directly:

Proposition 6.4. *Take ι to be an embedding of $M(\mathbb{R}(y))$ in $M(\mathbb{R}(x, y))$, compatible with restriction and such that all places in the image of ι have the same restriction to $\mathbb{R}(x)$. If for some $\xi \in \mathrm{im}(\iota)$ such that $\xi(x) = a$ and $\xi(y) = b$ we have, for some $n \in \mathbb{N}$,*

$$0 < v_\xi(x - a) < nv_\xi(y - b),$$

then the embedding is not continuous. The same holds if $\xi(x) = \infty$ and $x - a$ is replaced by $1/x$ and/or $\xi(y) = \infty$ and $y - b$ is replaced by $1/y$.

Proof. Take

$$f(x, y) = \frac{x - a + (y - b)^n}{x - a}.$$

Then $H'(f) \cap \mathrm{im}(\iota)$ is the singleton $\{\xi\}$. Indeed, $\xi \in H'(f)$ since $\xi(f) = 1$. But if $\xi' = \iota(\zeta) \neq \xi$, then $\zeta(y) \neq b$, whence $\xi'(f) = \infty$. The cases of $\xi(x) = \infty$ and/or $\xi(y) = \infty$ are similar. ∎

It is possible to generalize the approach of this section to the general setting of the previous section by replacing the $\mathbb{R}$-place ξ of F by the finest coarsening ξ' whose residue field contains R. (The valuation ring of ξ' is the compositum of the valuation ring of ξ and the subfield R of F.) But we would need an analogue of Lemma 6.1 for the case of non-archimedean fields R and $\overline{F} = \xi'(F)$. We found that the tools developed to deal with this analogue can be directly applied to construct the embedding of $M(R(y))$ in $M(F(y))$ in the setting of the previous section.

7. Embeddings of $\prod_{i=1}^n M(\mathbb{R}(x_i))$ in $M(\mathbb{R}(x_1, \ldots, x_n))$. In order to study possible embeddings of the torus in spaces of real places, we wish to consider embeddings of $M(\mathbb{R}(x)) \times M(\mathbb{R}(y))$ in $M(\mathbb{R}(x, y))$. Initially, we will treat the more general case of n variables. We consider the projection mapping

$$\rho : M(\mathbb{R}(x_1, \ldots, x_n)) \ni \xi \mapsto (\xi|_{\mathbb{R}(x_1)}, \ldots, \xi|_{\mathbb{R}(x_n)}) \in \prod_{i=1}^{n} M(\mathbb{R}(x_i)).$$

LEMMA 7.1. *The mapping ρ is surjective.*

We describe a general construction that will prove the lemma. Take $\mathbb{R}$-places $\xi_i \in M(\mathbb{R}(x_i))$. We wish to associate to them an $\mathbb{R}$-place ξ of $\mathbb{R}(x_1, \ldots, x_n)$ whose restriction to $\mathbb{R}(x_i)$ is ξ_i. We may assume that $\xi_i(x_i) \neq \infty$; otherwise, we can replace x_i by $1/x_i$. For $1 \leq i < n$, let ξ'_i be the place of $\mathbb{R}(x_i, \ldots, x_n)$ which is trivial on $\mathbb{R}(x_{i+1}, \ldots, x_n)$ and such that $\xi'_i(x_i) = \xi_i(x_i)$. Its residue field is $\mathbb{R}(x_{i+1}, \ldots, x_n)$. Then the place

$$\xi = \xi_n \circ \xi'_{n-1} \circ \cdots \circ \xi'_1. \tag{7.1}$$

satisfies the above conditions. This construction can be replaced by the symmetric ones where the x_i are permuted.

REMARK 7.2. There are many more possibilities for choosing a common extension ξ of the ξ_i. Set $\xi_i(x_i) = a_i$. Choose any rationally independent elements $r_1, \ldots, r_n \in \mathbb{R}$. Then there is a (uniquely determined) $\mathbb{R}$-place ξ of $\mathbb{R}(x_1, \ldots, x_n)$ such that for the valuation v associated with ξ we have $v(x_i - a_i) = r_i$. The value group of ξ is generated by the values $r_1, \ldots, r_n$ and is thus archimedean. In contrast to this, the value group of the place in (7.1) has rank n and is thus not archimedean if $n > 1$.

The surjectivity shows that there exist embeddings

$$\iota : \prod_{i=1}^{n} M(\mathbb{R}(x_i)) \hookrightarrow M(\mathbb{R}(x_1, \ldots, x_n)). \tag{7.2}$$

Such an embedding will be called *compatible* if $\rho \circ \iota$ is the identity.

THEOREM 7.3. *The image of every compatible embedding ι as in* (7.2) *lies dense in $M(\mathbb{R}(x_1, \ldots, x_n))$. But for $n > 1$, every non-empty basic open subset of $M(\mathbb{R}(x_1, \ldots, x_n))$ contains infinitely many places that are not in the image of ι.*

Proof. Take non-zero elements $f_1, \ldots, f_m \in \mathbb{R}(x_1, \ldots, x_n)$ such that

$$U := H'(f_1) \cap \cdots \cap H'(f_m) \neq \emptyset.$$

Take $\zeta \in U$ and write $f_i(x_1, \ldots, x_n) = g_i(x_1, \ldots, x_n)/h_i(x_1, \ldots, x_n)$. Choose an ordering on $\mathbb{R}(x_1, \ldots, x_n)$ compatible with ζ. Then the existential sentence

$$\exists X_1 \ldots \exists X_n : \bigwedge_{1 \leq i \leq m} h_i(X_1, \ldots, X_n) \neq 0 \wedge \frac{g_i(X_1, \ldots, X_n)}{h_i(X_1, \ldots, X_n)} > 0$$

holds in $\mathbb{R}(x_1, \ldots, x_n)$ with this ordering. By Tarski's Transfer Principle, it also holds in $\mathbb{R}$ with the usual ordering. That is, there exist $a_1, \ldots, a_n \in \mathbb{R}$ such that $h_i(a_1, \ldots, a_n) \neq 0$ and $g_i(a_1, \ldots, a_n)/h_i(a_1, \ldots, a_n) > 0$ for $1 \leq$

$i \le m$. Hence for every $\mathbb{R}$-place $\zeta \in M(\mathbb{R}(x_1,\dots,x_n))$ such that $\zeta(x_i) = a_i$ we will have $\zeta(f_i) = g_i(a_1,\dots,a_n)/h_i(a_1,\dots,a_n) > 0$. Among all such ζ there is precisely one in $\mathrm{im}(\iota)$. For this ζ, we have $\zeta \in U \cap \mathrm{im}(\iota)$. This proves that $\mathrm{im}(\iota)$ lies dense in $M(\mathbb{R}(x_1,\dots,x_n))$.

For $n > 1$, Remark 7.2 shows that there are infinitely many $\mathbb{R}$-places $\zeta \in M(\mathbb{R}(x_1,\dots,x_n))$ such that $\zeta(x_i) = a_i$. As only one of them is in $\mathrm{im}(\iota)$, $U \setminus \mathrm{im}(\iota)$ is infinite. ∎

Corollary 7.4. *A compatible embedding ι as in* (7.2) *cannot be continuous with respect to the product topology on $\prod_{i=1}^n M(\mathbb{R}(x_i))$.*

Proof. Suppose we have a continuous compatible embedding. Under the product topology, the space $\prod_{i=1}^n M(\mathbb{R}(x_i))$ is compact. As the continuous image of a compact space in a Hausdorff space is again compact (cf. [K, Chapter 5, Theorem 8]), we find that the image is closed in $M(\mathbb{R}(x_1,\dots,x_n))$. As it is also dense in $M(\mathbb{R}(x_1,\dots,x_n))$ by Theorem 7.3, it must be equal to $M(\mathbb{R}(x_1,\dots,x_n))$. But this contradicts the second assertion of Theorem 7.3. Hence the embedding cannot be continuous. ∎

Remark 7.5. All of the above can be generalized to the case of infinitely many elements x_i, $i \in I$, that are algebraically independent over $\mathbb{R}$. After choosing some well-ordering on I, the construction of the embedding

$$\iota : \prod_{i\in I} M(\mathbb{R}(x_i)) \hookrightarrow M(\mathbb{R}(x_i \mid i \in I))$$

proceeds by (possibly transfinite) induction. The above theorem and corollary remain valid. The proof of the theorem still works, as in the finitely many polynomials $f_1,\dots,f_m$ only finitely many variables x_i can appear. For infinite I, it is no longer true that the choice of the elements $a_1,\dots,a_n$ determines a unique place in $\mathrm{im}(\iota)$. Still, an application of Remark 7.2 shows that $U \setminus \mathrm{im}(\iota)$ is infinite.

We will now reprove the result of the corollary in the case of $n = 2$ by looking more closely at the topologies that are involved here. Every embedding of $M(\mathbb{R}(x)) \times M(\mathbb{R}(y))$ in $M(\mathbb{R}(x,y))$ will induce a topology on $M(\mathbb{R}(x)) \times M(\mathbb{R}(y))$ whose open sets are the preimages of the intersections of the open sets of $M(\mathbb{R}(x,y))$ with the image of the embedding.

Theorem 7.6. *For every compatible embedding ι, the topology induced on $M(\mathbb{R}(x)) \times M(\mathbb{R}(y))$ is finer than the product topology.*

Proof. Take a basic open set in the product topology of $M(\mathbb{R}(x)) \times M(\mathbb{R}(y))$ which is the interior or exterior of a circle given by $(x-a)^2+(y-b)^2 = r^2$, where $a, b, r \in \mathbb{R}$. We set

$$f(x,y) = r^2 - (x-a)^2 - (y-b)^2.$$

Then the set $\mathrm{im}(\iota) \cap H'(f)$ is precisely the image of the interior of the circle, and $\mathrm{im}(\iota) \cap H'(-f)$ is precisely the image of the exterior of the circle. This proves that the induced topology is equal to or finer than the product topology.

It remains to present an induced open set in $M(\mathbb{R}(x)) \times M(\mathbb{R}(y))$ which is not open in the product topology. Take the unique ξ in $\mathrm{im}(\iota)$ such that $\xi(x) = 0$ and $\xi(y) = 0$. Let

$$f(x,y) = \begin{cases} 1 + x/y & \text{if } \xi(x/y) = 0, \\ 1 + y/x & \text{if } \xi(y/x) = 0, \\ y^2/x^2 & \text{otherwise.} \end{cases}$$

It follows in all three cases that $\xi \in H'(f)$. The preimage of ξ under ι is (ξ_1, ξ_2) where $\xi_1(x) = 0$ and $\xi_2(y) = 0$. If the subset U induced by $H'(f)$ in $M(\mathbb{R}(x)) \times M(\mathbb{R}(y))$ were open, then it would contain the interior of a circle $x^2 + y^2 = r^2$ for some $r > 0$. But this is impossible since whenever $(\xi_1, \xi_2) \in U$, then for the first choice of f, $\xi_2(y) = 0$ must imply $\xi_1(x) = 0$, and for the other two choices of f, $\xi_1(x) = 0$ must imply $\xi_2(y) = 0$. ∎

Open Problem. What is the induced topology? Is it one-dimensional or two-dimensional?

8. Embeddings of more general products. For simplicity, we will only consider the product of two spaces $M(F_1)$ and $M(F_2)$; a generalization to any finite products can be achieved along the lines of the last section. We will also assume that F_1 and F_2 both contain $\mathbb{R}$. Then we can assume they are embedded in some extension field of $\mathbb{R}$ such that F_1 and F_2 are linearly disjoint over $\mathbb{R}$. We denote by F the field compositum of F_1 and F_2, that is, the smallest subextension of the given extension of $\mathbb{R}$ that contains both F_1 and F_2.

As before, we consider the corresponding projection mapping

$$\rho : M(F) \ni \xi \mapsto (\xi|_{F_1}, \xi|_{F_2}) \in M(F_1) \times M(F_2).$$

We show that ρ is surjective. Take $(\xi_1, \xi_2) \in M(F_1) \times M(F_2)$. Then there is an extension ξ_1' of ξ_1 from F_1 to F such that the residue field of ξ_1' is F_2. Then take $\iota(\xi_1, \xi_2) = \xi_2 \circ \xi_1'$. Here again, one obtains a different place of F by interchanging F_1 and F_2, showing that ρ is not injective.

The surjectivity shows that there exist embeddings

$$\iota : M(F_1) \times M(F_2) \to M(F).$$

As before, ι will be called *compatible* if $\rho \circ \iota$ is the identity.

If $F_1|\mathbb{R}$ and $F_2|\mathbb{R}$ are function fields, we can again prove that the image of every compatible embedding ι lies dense in $M(F)$. We will need the following fact. For a proof, see the second half of the proof of the lemma on p. 190 of [KP].

Lemma 8.1. *Take a field k and a function field $K = k(x_1, \ldots, x_d, z)$ where $x_1, \ldots, x_d$ are algebraically independent over k and z is separable-algebraic over $k(x_1, \ldots, x_d)$. If $f \in k[x_1, \ldots, x_d, Z]$ is the irreducible polynomial of z over $k(x_1, \ldots, x_d)$ and if $a_1, \ldots, a_d, b \in k$ are such that*

$$f(a_1, \ldots, a_d, b) = 0 \quad \textit{and} \quad \frac{\partial f}{\partial Z}(a_1, \ldots, a_d, b) \neq 0,$$

then K admits a k-rational place ξ such that $\xi(x_i) = a_i$ for $1 \leq i \leq d$, and $\xi(z) = b$.

Theorem 8.2. *If $F_1|\mathbb{R}$ and $F_2|\mathbb{R}$ are function fields of transcendence degree ≥ 1, then the image of every compatible embedding ι lies dense in $M(F)$. But every non-empty basic open subset of $M(F)$ contains infinitely many places that are not in the image of ι.*

Proof. We write $F_1 = \mathbb{R}(x_1, \ldots, x_d, z_1)$ and $F_2 = \mathbb{R}(x_{d+1}, \ldots, x_{d+e}, z_2)$ with $x_1, \ldots, x_{d+e}$ algebraically independent over $\mathbb{R}$, z_1 separable-algebraic over $\mathbb{R}(x_1, \ldots, x_d)$, and z_2 separable-algebraic over $\mathbb{R}(x_{d+1}, \ldots, x_{d+e})$. Then $F = \mathbb{R}(x_1, \ldots, x_{d+e}, z_1, z_2)$. Let $G_1 \in k[x_1, \ldots, x_d, Z_1]$ be the irreducible polynomial of z_1 over $k(x_1, \ldots, x_d)$ and $G_2 \in k[x_{d+1}, \ldots, x_{d+e}, Z]$ be the irreducible polynomial of z_2 over $k(x_{d+1}, \ldots, x_{d+e})$.

Take non-zero elements $f_1, \ldots, f_m \in F$ such that $U := H'(f_1) \cap \ldots \cap H'(f_n) \neq \emptyset$. Take $\zeta \in U$ and write

$$f_i(x_1, \ldots, x_{d+e}, z_1, z_2) = \frac{g_i(x_1, \ldots, x_{d+e}, z_1, z_2)}{h_i(x_1, \ldots, x_{d+e})}$$

with polynomials $g_i \in \mathbb{R}[X_1, \ldots, X_{d+e}, Z_1, Z_2]$ and $h_i \in \mathbb{R}[X_1, \ldots, X_{d+e}]$. Choose an ordering on F compatible with ζ. Then the existential sentence

$$\begin{aligned}
&\exists X_1 \ldots \exists X_{d+e} \exists Z_1 \exists Z_2 : \\
&\qquad G_1(X_1, \ldots, X_d, Z_1) = 0 \wedge \frac{\partial G_1}{\partial Z_1}(X_1, \ldots, X_d, Z_1) \neq 0 \\
&\qquad \wedge\, G_2(X_{d+1}, \ldots, X_{d+e}, Z_2) = 0 \wedge \frac{\partial G_2}{\partial Z_2}(X_{d+1}, \ldots, X_{d+e}, Z_2) \neq 0 \\
&\qquad \wedge \bigwedge_{1 \leq i \leq m} h_i(X_1, \ldots, X_{d+e}) \neq 0 \wedge \frac{g_i(X_1, \ldots, X_{d+e}, Z_1, Z_2)}{h_i(X_1, \ldots, X_{d+e})} > 0
\end{aligned}$$

holds in F with this ordering. By Tarski's Transfer Principle, it also holds in $\mathbb{R}$ with the usual ordering. That is, there exist $a_1, \ldots, a_{d+r}, b_1, b_2 \in \mathbb{R}$

such that

$$(8.1)\qquad G_1(a_1,\dots,a_d,b_1)=0\wedge\frac{\partial G_1}{\partial Z_1}(a_1,\dots,a_d,b_1)\neq 0,$$

$$(8.2)\qquad G_2(a_{d+1},\dots,a_{d+e},b_2)=0\wedge\frac{\partial G_2}{\partial Z_2}(a_{d+1},\dots,a_{d+e},b_2)\neq 0,$$

$$(8.3)\qquad \bigwedge_{1\le i\le m} h_i(a_1,\dots,a_{d+e})\neq 0\wedge\frac{g_i(a_1,\dots,a_{d+e},b_1,b_2)}{h_i(a_1,\dots,a_{d+e})}>0.$$

Hence for every $\mathbb{R}$-place $\zeta\in M(F)$ such that $\zeta(x_i)=a_i$ and $\zeta(z_j)=b_j$ we will have $\zeta(f_i)>0$, $1\le i\le m$. By Lemma 8.1, (8.1) guarantees that there is $\zeta_1\in M(F_1)$ such that $\zeta_1(x_i)=a_i$, $1\le i\le d$, and $\zeta_1(z_1)=b_1$, and (8.2) guarantees that there is $\zeta_2\in M(F_2)$ such that $\zeta_2(x_i)=a_i$, $d+1\le i\le d+e$, and $\zeta_2(z_2)=b_2$. Consequently, there is $\zeta\in\operatorname{im}(\iota)$ with $\zeta(x_i)=a_i$ and $\zeta(z_j)=b_j$. It follows that $\zeta\in U\cap\operatorname{im}(\iota)$. This proves that the image of our construction lies dense in $M(F)$.

From Remark 7.2 it again follows that there are infinitely many $\mathbb{R}$-places ζ of $\mathbb{R}(x_1,\dots,x_{d+e})$ such that $\zeta(x_i)=a_i$. These places can be extended to F by setting $\zeta(z_j)=b_j$. All of them have archimedean value group. In contrast, all places in $\operatorname{im}(\iota)$ are compositions of two non-trivial places and therefore have non-archimedean value group. This shows that $U\setminus\operatorname{im}(\iota)$ is infinite. ∎

As before, one proves:

COROLLARY 8.3. *If $F_1|\mathbb{R}$ and $F_2|\mathbb{R}$ are function fields, then a compatible embedding cannot be continuous with respect to the product topology on $M(F_1)\times M(F_2)$.*

9. Raising the transcendence degree. In this final section, we show how to use previous constructions to embed $M(K)$ in $M(L)$, for an arbitrary field K and suitable transcendental extensions L of K.

THEOREM 9.1. *Assume that L admits a K-rational place ξ. Then*

$$\iota: M(K)\ni\zeta\mapsto\zeta\circ\xi\in M(L)$$

is a continuous embedding compatible with restriction.

Proof. It is clear that the embedding is compatible with restriction. For the continuity, take $f\in L$ and assume that $H'(f)\cap\operatorname{im}(\iota)\neq\emptyset$. Pick $\zeta\in M(K)$ such that $\zeta\circ\xi=\iota(\zeta)\in H'(f)$. It follows that $(\zeta\circ\xi)(f)\neq\infty$, and therefore $\infty\neq\xi(f)\in K$. For arbitrary $\zeta\in M(K)$, we see that $(\zeta\circ\xi)(f)=\zeta(\xi(f))$, so $\zeta\circ\xi\in H'(f)\Leftrightarrow\zeta\in H'(\xi(f))$. Hence, $\iota^{-1}(H'(f))=H'(\xi(f))$, which proves that ι is continuous. ∎

There are fields L of arbitrary transcendence degree over K which allow a unique K-rational place ξ. This fact has been used in [EO] to show that a given space of $\mathbb{R}$-places can be realized over arbitrarily large fields. The other extreme is:

Corollary 9.2. *Take a collection x_i, $i \in I$, of elements algebraically independent over K. Then there are at least $|K|^{|I|}$ many distinct continuous embeddings of $M(K)$ in $M(K(x_i \mid i \in I))$, all of them compatible with restriction and having mutually disjoint images.*

This follows from the fact that for every choice of elements $a_i \in K$ there is a K-rational place ξ of L such that $\xi(x_i) = a_i$.

Corollary 9.3. *There are at least $2^{\aleph_0}$ many continuous embeddings of $M(\mathbb{R}(x))$ in $M(\mathbb{R}(x, y))$, all of them compatible with restriction and having mutually disjoint images.*

It should be noted that Theorem 1.2 does not follow from Theorem 9.1. The condition that vR is a convex subgroup of vF does by no means imply that $F(y)$ admits an $R(y)$-rational place.

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