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How far is $C_0(\Gamma, X)$ with Γ discrete from $C_0(K, X)$ spaces?

by

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Abstract. For a locally compact Hausdorff space K and a Banach space X we denote by $C_0(K, X)$ the space of X-valued continuous functions on K which vanish at infinity, provided with the supremum norm. Let n be a positive integer, Γ an infinite set with the discrete topology, and X a Banach space having non-trivial cotype. We first prove that if the nth derived set of K is not empty, then the Banach–Mazur distance between $C_0(\Gamma, X)$ and $C_0(K, X)$ is greater than or equal to $2n + 1$. We also show that the Banach–Mazur distance between $C_0(\mathbb{N}, X)$ and $C([1, \omega^n k], X)$ is exactly $2n + 1$, for any positive integers n and k. These results extend and provide a vector-valued version of some 1970 Cambern theorems, concerning the cases where $n = 1$ and X is the scalar field.

1. Introduction. We follow the standard notation and terminology for topological spaces and Banach space theory that can be found in [11] and [14] respectively. When K is a compact Hausdorff space, the space $C_0(K, X)$ will be denoted by $C(K, X)$. If X is the scalar field, these spaces will also be denoted by $C_0(K)$ and $C(K)$ respectively. As usual, when K is the set $\mathbb{N}$ of natural numbers with the discrete topology or its Aleksandrov one-point compactification $\gamma\mathbb{N}$, we denote $C_0(\mathbb{N})$ by c_0 and $C(\gamma\mathbb{N})$ by c. If there is an isomorphism T from the Banach space X onto the Banach space Y we will write $X \sim Y$. Moreover, the Banach–Mazur distance $d(X, Y)$ between X and Y is defined by $\inf\{\|T\| \, \|T^{-1}\|\}$ where the infimum is taken over all isomorphisms T from X onto Y.

In this paper we are mainly interested in the Banach–Mazur distance between $C_0(\Gamma, X)$ spaces, where Γ are sets with the discrete topology, and $C_0(K, X)$ spaces. The origin of our research goes back to Banach. In 1932, he stated that $d(c_0, c) \leq 4$ [1, p. 181]. To prove this, he used the following isomorphism T_λ from c onto c_0:

$$T_\lambda(a_1, a_2, a_3, \ldots) = (\lambda a, a_1 - a, a_2 - a, \ldots), \tag{1.1}$$

where $\lambda = 1$ and $(a_n)_{n\in\mathbb{N}}$ converges to a. A better estimate for this distance

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can be obtained from (1.1) by taking $\lambda = 2$. Namely, $d(c_0, c) \le 3$. Finally, in 1970 Cambern [4] (see also [6] and [12]) calculated the exact value of this distance:

$$d(c_0, c) = 3. \tag{1.2}$$

Moreover, by the classical Mazurkiewicz and Sierpiński Theorem [15] (see also [18, Theorem 8.6.10, p. 155]) and the Bessaga and Pełczyński Theorem [3, Theorem 1] we deduce that if c_0 is isomorphic to a $C(K)$ space, then K is homeomorphic to an interval of ordinals $[1, \omega^n k]$ endowed with the order topology for some positive integers n and k, where ω denotes the first infinite ordinal. Thus, to determine the Banach–Mazur distance between c_0 and each of the $C(K)$ spaces, we are led to the following natural question:

Problem 1.1. *What are the values of* $d(c_0, C([1, \omega^n k])$ *for* $1 \le n, k < \omega$*?*

The purpose of the present paper is twofold: firstly, to provide a vector-valued extension of (1.2); secondly, to solve Problem 1.1 completely. To state our main results we recall that the derived set of a topological space K is the set $K^{(1)}$ of all accumulation points of K. If $1 \le n < \omega$, we define the consecutive derived sets by induction: $K^{(n+1)} = (K^{(n)})^{(1)}$, and $K^{(\omega)} = \bigcap_{1 \le n < \omega} K^{(n)}$. Moreover, a Banach space X has *non-trivial cotype* [8] if it has cotype q for some $2 \le q < \infty$. Recall that a Banach space $X \ne \{0\}$ is said to have *cotype* $2 \le q < \infty$ if there is a constant $\kappa > 0$ such that no matter how we select finitely many vectors $v_1, \ldots, v_n$ from X,

$$\Big(\sum_{i=1}^{n} \|v_i\|^q\Big)^{1/q} \le \kappa \Big(\int_0^1 \Big\|\sum_{i=1}^{n} r_i(t) v_i\Big\|^2 \, dt\Big)^{1/2},$$

where $r_i : [0,1] \to \mathbb{R}$ denote the *Rademacher functions*, defined by setting

$$r_i(t) = \operatorname{sign}(\sin 2^i \pi t).$$

We first prove the following lower bounds for the Banach–Mazur distances between certain $C_0(K, X)$ spaces. This is a generalization of the main result of [4], which concerned the case where $n = 1$ and X is the scalar field.

Theorem 1.2. *Let $1 \le n < \omega$, Γ an infinite set with the discrete topology, K a locally compact Hausdorff space and X a Banach space having non-trivial cotype. Then*

$$C_0(\Gamma, X) \sim C_0(K, X) \text{ and } K^{(n)} \ne \emptyset \ \Rightarrow\ d(C_0(\Gamma, X), C_0(K, X)) \ge 2n + 1.$$

To obtain some upper bounds for the distances mentioned in Problem 1.1 we prove:

Theorem 1.3. *Let $1 \le n, k < \omega$ and X a Banach space. Then*

$$d(C_0(\mathbb{N}, X), C([1, \omega^n k], X)) \le 2n + 1.$$

As an immediate consequence of Theorems 1.2 and 1.3 we get the following generalization of (1.2) which at the same time solves Problem 1.1.

COROLLARY 1.4. *Let* $1 \leq n, k < \omega$ *and let* X *be a Banach space having non-trivial cotype. Then*

$$d(C_0(\mathbb{N}, X), C([1, \omega^n k], X)) = 2n + 1.$$

We do not know whether the statement of Corollary 1.4 remains true without the hypothesis that X has non-trivial cotype. We also notice that Theorem 1.2 can be applied to obtain some generalizations of classical results on $C_0(\Gamma)$ spaces. For instance, it is well known that if a $C(K)$ space is isomorphic to some $C_0(\Gamma)$ space, where Γ is an infinite set with the discrete topology, then $K^{(\omega)} = \emptyset$ (see [2], [3] and [16]). As a consequence of Theorem 1.2 we give a simple proof of the following extension of this result.

COROLLARY 1.5. *Let* Γ *be an infinite set with the discrete topology,* K *a locally compact Hausdorff space and* X *a Banach space having non-trivial cotype. Then*

$$C_0(K, X) \sim C_0(\Gamma, X) \ \Rightarrow \ K^{(\omega)} = \emptyset.$$

Proof. Let T be an isomorphism from $C_0(K, X)$ onto $C_0(\Gamma, X)$. Take $1 \leq n < \omega$ such that $\|T\| \, \|T^{-1}\| < 2n+1$. Then by Theorem 1.2, $K^{(n)} = \emptyset$. ∎

Finally, the classical Milyutin Theorem [17, Theorem 21.5.10] shows that we cannot remove the non-trivial cotype hypothesis in Corollary 1.5. Indeed,

$$C_0(\mathbb{N}, C([0, 1])) \sim C([0, 1]) \sim C([0, 1], C[0, 1]),$$

nevertheless, $[0, 1]^{(\omega)} = [0, 1]$.

2. Preliminary results. In this section, we shall prove two propositions which play a central role in the proof of Theorem 1.2. We denote by S_X the unit sphere of a Banach space X. For a subset J of a topological space K we denote by $\mathring{J}$ the set of interior points of J. Recall that an isomorphism T of $C_0(K, X)$ into $C_0(\Gamma, X)$ is said to be *norm-increasing* if $\|f\| \leq \|T(f)\|$ for every $f \in C_0(K, X)$.

PROPOSITION 2.1. *Let* K *be a locally compact Hausdorff space such that* $K^{(n)} \neq \emptyset$ *for some* $1 \leq n < \omega$, Γ *be an infinite set with the discrete topology and* X *a Banach space having non-trivial cotype. Fix* $e \in S_X$ *and* $0 < \epsilon < 1$. *If* T *is a norm-increasing isomorphism from* $C_0(K, X)$ *into* $C_0(\Gamma, X)$ *then there are points* $x_1, \ldots, x_n \in K$, *compact subsets* $J_1, \ldots, J_n$ *of* K *and functions* $h_1, \ldots, h_n$ *in* $C_0(K)$ *satisfying:*

(a) $x_i \in \mathring{J}_i \cap K^{(n-i+1)}$ *for* $1 \leq i \leq n$.
(b) $J_i \subset \mathring{J}_{i-1}$ *for* $1 < i \leq n$.
(c) $0 \leq h_i \leq 1$, $h_i(x) = 1$ *if* $x \in J_i$ *for* $1 \leq i \leq n$, *and* $h_i(x) = 0$ *if* $x \notin \mathring{J}_{i-1}$ *for* $1 < i \leq n$.

(d) *The sets $G_i = \{y \in \varGamma : \|T(e \cdot h_i)(y)\| \geq \epsilon\}$, $1 \leq i \leq n$, are non-empty and mutually disjoint.*

Proof. We proceed by finite induction. Let $x_1 \in K^{(n)}$ and let J_1 be a compact neighborhood of x_1. By the Urysohn Lemma [11, Theorem 1.5.11, p. 41], we can find $h_1 \in C_0(K)$ with $0 \leq h_1 \leq 1$ and $h_1(x) = 1$ if $x \in J_1$. Moreover, since $0 < \epsilon < 1$ and T is norm-increasing, the set $G_1 = \{y \in \varGamma : \|T(e \cdot h_1)(y)\| \geq \epsilon\}$ is non-empty.

Given $1 \leq r < n$, suppose by induction that we have obtained points $x_1, \ldots, x_r$, compact sets $J_1, \ldots, J_r$, and functions $h_1, \ldots, h_r$ in $C_0(K)$ satisfying (a)–(d).

Since K is a locally compact Hausdorff space, it is possible to find points $a_1, a_2, \ldots$ in $(\mathring{J}_r \setminus \{x_r\}) \cap K^{(n-r)}$ and mutually disjoint compact subsets $L_1, L_2, \ldots$ satisfying

$$a_i \in \mathring{L}_i \subset L_i \subset \mathring{J}_r \quad \text{for every } 1 \leq i < \omega.$$

The Urysohn Lemma gives functions $f_1, f_2, \ldots \in C_0(K)$ such that, for every $1 \leq i < \omega$, $0 \leq f_i \leq 1$, $f_i(x) = 1$ if $x \in L_i$ and $f_i(x) = 0$ if $x \notin \mathring{J}_r$, and moreover $f_i \cdot f_j = 0$ if $i \neq j$.

Let $G = G_1 \cup \cdots \cup G_r$. We claim that there exists $1 \leq m < \omega$ such that

$$\{y \in \varGamma : \|T(e \cdot f_m)(y)\| \geq \epsilon\} \cap G = \emptyset. \tag{2.1}$$

Indeed, otherwise, assuming $G = \{y_1, \ldots, y_s\}$ and denoting

$$\varLambda_i = \{j \in [1, \omega[: \|T(e \cdot f_j)(y_i)\| \geq \epsilon\}$$

for each $1 \leq i \leq s$, we would obtain

$$[1, \omega[\subseteq \varLambda_1 \cup \cdots \cup \varLambda_s,$$

and we infer that $\varLambda_l$ must be infinite for some $1 \leq l \leq s$. Let $l_1, l_2, \ldots$ be distinct integers in $\varLambda_l$.

Since X has cotype q for some $2 \leq q < \infty$, there is a constant $Q > 0$ such that no matter how we select finitely many vectors $v_1, \ldots, v_p \in X$, if $0 < \eta \leq \|v_i\|$ for each $1 \leq i \leq p$, there are scalars $r_i = \pm 1$ such that

$$\Big\| \sum_{i=1}^{p} r_i v_i \Big\| \geq \eta Q \sqrt[q]{p}. \tag{2.2}$$

Pick $1 \leq m < \omega$ satisfying $\epsilon Q \sqrt[q]{m} > \|T\|$. Then according to (2.2) there exist scalars $r_i = \pm 1$ for $1 \leq i \leq m$ such that

$$\Big\| \sum_{i=1}^{m} r_i T(e \cdot f_{l_i})(y_l) \Big\| \geq \epsilon Q \sqrt[q]{m} > \|T\|.$$

Since $f_{l_i} \cdot f_{l_j} = 0$ if $i \neq j$, the function $A = \sum_{i=1}^{m} r_i (e \cdot f_{l_i}) \in C_0(K, X)$ is such that $\|A\| \le 1$. However,

$$\|T\| \ge \|T(A)\| \ge \Big\|T\Big(\sum_{i=1}^{m} r_i(e \cdot f_{l_i})\Big)(y_l)\Big\| > \|T\|,$$

a contradiction which establishes our claim.

Now take $1 \le m < \omega$ satisfying (2.1) and set $J_{r+1} = L_m$, $h_{r+1} = f_m$ and $G_{r+1} = \{y \in \Gamma : \|T(e \cdot f_m)(y)\| \ge \epsilon\}$. It is easy to check that conditions (a)–(d) hold for $r+1$, so we are done. ∎

To state the next proposition, we need to recall some notation and a classical representation theorem for the dual of $C_0(K, X)$ spaces. For an X-valued measure μ, $|\mu|$ denotes the variation of μ, and $rcabv(K, X)$ is the Banach space of all regular, countably additive, Borel, bounded variation measures, endowed with the variation norm. Throughout we will use the Singer Representation Theorem: there exists an isometric isomorphism between $C_0(K, X)^*$ and $rcabv(K, X^*)$ such that a linear functional φ and the corresponding measure μ are related by

$$\langle \varphi, f\rangle = \int f\, d\mu, \qquad f \in C_0(K, X),$$

where the integral is the *immediate integral* of Dinculeanu [9, p. 11]. When K is a compact Hausdorff space, this characterization can be found in [13]. The locally compact case can be derived from the compact one as explained in [5, p. 2].

The next proposition can be established by an argument similar to that used in the proof of [7, Lemma 2.1(a)]. For completeness, we give the whole argument.

PROPOSITION 2.2. *Let X be a Banach space having non-trivial cotype, K a locally compact Hausdorff space, Γ an infinite set with the discrete topology and T an isomorphism of $C_0(K, X)$ into $C_0(\Gamma, X)$. Then for every $y \in \Gamma$ and every $\eta > 0$ the set*

$$\{x \in K : |T^*(\varphi \cdot \delta_y)|(\{x\}) > \eta \text{ for some } \varphi \in S_{X^*}\}$$

is finite, where δ_y stands for the unit point mass at y.

Proof. Assume that, on the contrary, for some $\eta > 0$ the set

$$\{x \in K : \|T^*(\varphi \cdot \delta_y)(\{x\})\| > \eta \text{ for some } \varphi \in S_{X^*}\}$$

is infinite. Suppose that X has cotype q for some $2 \le q < \infty$, and let $Q > 0$ be as in the proof of Proposition 2.1. Pick $1 \le n < \omega$ satisfying $\eta Q \sqrt[q]{n} > 2\|T\|$. Fix also distinct points $x_1, \ldots, x_n \in K$ and $\varphi_1, \ldots, \varphi_n \in S_{X^*}$ such that

$$\|T^*(\varphi_i \cdot \delta_y)(\{x_i\})\| > \eta, \qquad 1 \le i \le n.$$

Thus, there are $v_1, \ldots, v_n$ in S_X such that

$$\langle T^*(\varphi_i \delta_y)(\{x_i\}), v_i\rangle > \eta, \qquad 1 \le i \le n. \tag{2.3}$$

Since $T^*(\varphi_i \cdot \delta_y)$ is regular for each $1 \le i \le n$, we can take mutually disjoint open neighborhoods $U_1, \ldots, U_n$ of $x_1, \ldots, x_n$, respectively, satisfying

$$|T^*(\varphi_i \cdot \delta_y)|(U_i \setminus \{x_i\}) \le \eta/2.$$

By the Urysohn Lemma, we can find $h_i \in C_0(K)$ with $0 \le h_i \le 1$, $h_i(x_i) = 1$ and $h_i(x) = 0$ if $x \in K \setminus U_i$. Define $f_i \in C_0(K,X)$ by $f_i = v_i \cdot h_i$. By (2.3) we have

$$\begin{aligned} \|(Tf_i)(y)\| &\ge |\langle \varphi_i, (Tf_i)(y)\rangle| = \Big|\int f_i \, dT^*(\varphi_i \cdot \delta_y)\Big| \\ &\ge |\langle T^*(\varphi_i \cdot \delta_y)(\{x_i\}), v_i\rangle| \\ &\quad - \Big|\int f_i \, dT^*(\varphi_i \cdot \delta_y) - \langle T^*(\varphi_i \cdot \delta_y)(\{x_i\}), v_i\rangle\Big| \\ &> \eta - |T^*(\varphi_i \cdot \delta_y)|(U_i \setminus \{x_i\}) \ge \eta/2. \end{aligned}$$

According to (2.2) there are scalars $r_i = \pm 1$ such that

$$\Big\|\sum_{i=1}^{n} r_i (Tf_i)(y)\Big\| \ge \eta Q \sqrt[q]{n}/2. \tag{2.4}$$

On the other hand, since $U_i \cap U_j = \emptyset$ if $i \ne j$ and $\|f_i\| \le 1$ for each $1 \le i \le n$, we have

$$\Big\|\sum_{i=1}^{n} r_i f_i\Big\| \le 1.$$

Therefore, by (2.4) and the choice of η we conclude

$$\|T\| \ge \Big\|T\Big(\sum_{i=1}^{n} r_i f_i\Big)\Big\| \ge \Big\|T\Big(\sum_{i=1}^{n} r_i f_i\Big)(y)\Big\| > \|T\|,$$

which is the required contradiction. ∎

Another basic ingredient in the proof of our main result is a Radon–Nikodým type vector measure theorem (see [10, Theorem 5, p. 269]).

Theorem 2.3. *Let X be a Banach space, K a locally compact Hausdorff space and $\mu \in rcabv(K, X^*)$. Then there exists a function $\gamma : K \to X^*$ such that:*

(a) $\|\gamma(x)\| = 1$ *for every* $x \in K$.
(b) *The map* $x \mapsto \langle \gamma(x), f(x)\rangle$ *is measurable and*

$$\int f \, d\mu = \int \langle \gamma(x), f(x)\rangle \, d|\mu|(x)$$

for every $f \in C_0(K, X)$.

3. Lower bounds on Banach–Mazur distances between $C_0(K,X)$ spaces. The aim of this section is to prove our main result, Theorem 1.2.

We will argue by contradiction in four steps.

STEP 1. Assuming the existence of an isomorphism T of $C_0(K, X)$ onto $C_0(\mathit{\Gamma}, X)$ such that $\|T\|\,\|T^{-1}\| < 2n+1$ we construct some special functions α and β in $C_0(\mathit{\Gamma})$.

Without loss of generality we may assume that T is norm-increasing and $\|T^{-1}\| = 1$, for otherwise we simply replace T by $\|T^{-1}\|T$.

Pick $0 < \epsilon < 1$ and $\eta > 0$ such that

$$\|T\| < (2n+1)\frac{1-\epsilon}{1+\epsilon}, \qquad \eta < \min\left\{\epsilon, \frac{(2n+1)(1-\epsilon) - \|T\|}{2}\right\}.$$

Fix $e \in S_X$. Since $K^{(n)} \neq \emptyset$ there are points $x_1, \ldots, x_n \in K$, compact subsets $J_1, \ldots, J_n \subset K$, functions $h_1, \ldots, h_n \in C_0(K)$, and subsets $G_1, \ldots, G_n \subset \mathit{\Gamma}$ satisfying the statements of Proposition 2.1. Define, for each $1 \leq i \leq n$,

$$f_i = e \cdot h_i \in C_0(K, X), \qquad g_i = \chi_{G_i} \cdot Tf_i,$$

where χ_{G_i} is the characteristic function of G_i. Denote by G the finite set $\bigcup_{i=1}^n G_i$. According to Proposition 2.2 the set

$$H = \bigcup_{y \in G} \{x \in K : |T^*(\varphi \cdot \delta_y)|(\{x\}) > \eta \text{ for some } \varphi \in S_{X^*}\}$$

is finite. Pick $z \in \mathring{J}_n \setminus H$ and $e^* \in S_{X^*}$ such that $\langle e^*, e\rangle = 1$, and define the vector measure

$$\mu = (T^{-1})^*(e^* \cdot \delta_z).$$

By Theorem 2.3 there exists a function $\gamma : \mathit{\Gamma} \to X^*$ satisfying the statements of that theorem.

Since $\|\gamma(y)\| = 1$ for every $y \in \mathit{\Gamma}$, we have $|T^*(\gamma(y) \cdot \delta_y)|(\{z\}) < \eta$ for each $y \in G$. Then, by regularity, we can find an open neighborhood $U \subset J_n$ of z such that

$$|T^*(\gamma(y) \cdot \delta_y)|(U) < \eta \quad \text{ for every } y \in G.$$

By the Urysohn Lemma, we can find $h_{n+1} \in C_0(K)$ such that $0 \leq h_{n+1} \leq 1$, $h_{n+1}(z) = 1$ and $h_{n+1}(x) = 0$ if $x \notin U$. Set $f_{n+1} = e \cdot h_{n+1}$ and define $\alpha, \beta \in C_0(\mathit{\Gamma})$ by setting, for every $y \in \mathit{\Gamma}$,

$$\alpha(y) = \langle \gamma(y), Tf_{n+1}(y)\rangle,$$

$$\beta(y) = \left\langle \gamma(y), g_1(y) + 2\sum_{i=2}^n g_i(y) + 2Tf_{n+1}(y)\right\rangle.$$

STEP 2. We prove that $\|\beta\| = \max\{2\|\alpha\|, |\beta(y)| : y \in G\}$.

In order to establish this, notice that for every $y \in G$,

$$\begin{aligned} (3.1) \qquad |\alpha(y)| &= |\langle \gamma(y), Tf_{n+1}(y)\rangle| = \left|\int Tf_{n+1}\, d\gamma(y) \cdot \delta_y\right| \\ &= \left|\int f_{n+1}\, dT^*(\gamma(y) \cdot \delta_y)\right| \leq |T^*(\gamma(y) \cdot \delta_y)|(U) < \eta < 1. \end{aligned}$$

On the other hand, take $y_0 \in \Gamma$ such that $\|\alpha\| = |\alpha(y_0)|$. Since γ satisfies item (b) of Theorem 2.3 and $|\mu|(\Gamma) = \|(T^{-1})^*(e^* \cdot \delta_z)\| \le 1$, we have

$$(3.2)\qquad \begin{aligned} |\alpha(y_0)| &= |\langle \gamma(y_0), Tf_{n+1}(y_0)\rangle| \ge \Big|\int \langle \gamma(y), Tf_{n+1}(y)\rangle d|\mu|(y)\Big| \\ &= \Big|\int Tf_{n+1}\, d\mu\Big| = \Big|\int Tf_{n+1}\, d(T^{-1})^*(e^* \cdot \delta_z)\Big| = \Big|\int f_{n+1}\, d(e^* \cdot \delta_z)\Big| \\ &= |\langle e^*, f_{n+1}(z)\rangle| = \langle e^*, e\rangle = 1. \end{aligned}$$

Hence $y_0 \in \Gamma \setminus G$. Moreover, since $\beta(y) = 2\alpha(y)$ for $y \in \Gamma \setminus G$, we are done.

STEP 3. We show that $\|\beta\| \ge (2n+1) - (2n-1)\epsilon$.

Fix y_0 such that $\|\beta\| = |\beta(y_0)|$. Once more, since γ satisfies item (b) of Theorem 2.3 and $|\mu|(\Gamma) = \|(T^{-1})^*(e^* \cdot \delta_z)\| \le 1$, we can write

$$\begin{aligned} |\beta(y_0)| &= \Big|\Big\langle \gamma(y_0), g_1(y_0) + 2\sum_{i=2}^{n} g_i(y_0) + 2Tf_{n+1}(y_0)\Big\rangle\Big| \\ &\ge \Big|\int \Big\langle \gamma(y), g_1(y) + 2\sum_{i=2}^{n} g_i(y) + 2Tf_{n+1}(y)\Big\rangle\, d|\mu|(y)\Big| \\ &= \Big|\int \Big(g_1 + 2\sum_{i=2}^{n} g_i + 2Tf_{n+1}\Big)\, d(T^{-1})^*(e^* \cdot \delta_z)\Big| \\ &= \Big|\Big\langle e^*, T^{-1}g_1(z) + 2\sum_{i=2}^{n} T^{-1}g_i(z) + 2f_{n+1}(z)\Big\rangle\Big| \\ &\ge \Big|\Big\langle e^*, f_1(z) + 2\sum_{i=2}^{n+1} f_i(z)\Big\rangle\Big| - |\langle e^*, f_1(z) - T^{-1}g_1(z)\rangle| \\ &\quad - 2\sum_{i=2}^{n} |\langle e^*, f_i(z) - T^{-1}g_i(z)\rangle|. \end{aligned}$$

Since T is norm-increasing, for every $x \in K$ and $1 \le i \le n$ we have

$$\begin{aligned} |\langle e^*, f_i(x) - T^{-1}g_i(x)\rangle| \le \|f_i - T^{-1}g_i\| &\le \|Tf_i - g_i\| \\ &= \|(1 - \chi_{G_i}) \cdot Tf_i\| \le \epsilon. \end{aligned}$$

Furthermore, by the definition of f_i,

$$\langle e^*, f_i(z)\rangle = \langle e^*, e\rangle = 1$$

for each $1 \le i \le n+1$. Therefore, we conclude that

$$\|\beta\| \ge (2n+1) - (2n-1)\epsilon.$$

STEP 4. As $\|\beta\| \ge (2n+1) - (2n-1)\epsilon$, according to Step 2 there are two possibilities:

(i) $2\|\alpha\| \ge (2n+1) - (2n-1)\epsilon$,

(ii) $|\beta(y)| \ge (2n+1) - (2n-1)\epsilon$ for some $y \in G$.

We will show that both lead to a contradiction.

Suppose first that (i) holds. Set $A = T^{-1}g_1 - 2f_{n+1}$. Since $0 \leq h_{n+1} \leq h_1 \leq 1$, for every $x \in K$ we have

$$\begin{aligned}\|T^{-1}(g_1)(x) - 2f_{n+1}(x)\| &\leq \|f_1(x) - 2f_{n+1}(x)\| + \|T^{-1}g_1(x) - f_1(x)\| \\ &\leq |h_1(x) - 2h_{n+1}(x)| + \epsilon \leq 1 + \epsilon.\end{aligned}$$

So $\|A\| \leq 1 + \epsilon$.

Recalling (3.1) and (3.2), we can fix $y_0 \in \Gamma \setminus G$ such that $\|\alpha\| = |\alpha(y_0)|$. It follows that

$$\begin{aligned}|\langle \gamma(y_0), T(A)(y_0)\rangle| &= 2|\langle \gamma(y_0), Tf_{n+1}(y_0)\rangle| = 2|\alpha(y_0)| \\ &\geq (2n+1) - (2n-1)\epsilon > (2n+1)(1-\epsilon).\end{aligned}$$

Consequently,

$$\|T\| \geq \left\| T\left(\frac{1}{1+\epsilon}A\right)\right\| > (2n+1)\frac{1-\epsilon}{1+\epsilon},$$

a contradiction to the choice of ϵ.

Next, assume that (ii) holds. We distinguish two cases.

Case 1: $\|\beta\| = |\beta(y_0)|$ for some $y_0 \in G_1$. In this case, since $G_1, \ldots, G_n$ are mutually disjoint we have

$$|\beta(y_0)| = |\langle \gamma(y_0), g_1(y_0) + 2Tf_{n+1}(y_0)\rangle| \geq (2n+1) - (2n-1)\epsilon.$$

Recalling (3.1), by the choice of η we deduce

$$\begin{aligned}|\langle \gamma(y_0), g_1(y_0)\rangle| &\geq (2n+1) - (2n-1)\epsilon - 2|\langle \gamma(y_0), Tf_{n+1}(y_0)\rangle| \\ &> (2n+1) - (2n-1)\epsilon - 2\eta > \|T\|.\end{aligned}$$

Therefore,

$$\|T\| \geq \|Tf_1\| \geq |\langle \gamma(y_0), Tf_1(y_0)\rangle| = |\langle \gamma(y_0), g_1(y_0)\rangle| > \|T\|,$$

which is a contradiction.

Case 2: $\|\beta\| = |\beta(y_0)|$ for some $y_0 \in G_i$, $i > 1$. Once again, since $G_1, \ldots, G_n$ are mutually disjoint we have

$$|\beta(y_0)| = |\langle \gamma(y_0), 2g_i(y_0) + 2Tf_{n+1}(y_0)\rangle| \geq (2n+1) - (2n-1)\epsilon.$$

Recalling that $\eta < \epsilon$, we infer

$$\begin{aligned}2|\langle \gamma(y_0), g_i(y_0)\rangle| &\geq (2n+1) - (2n-1)\epsilon - 2|\langle \gamma(y_0), Tf_{n+1}(y_0)\rangle| \\ &> (2n+1)(1-\epsilon).\end{aligned}$$

Next, set $B_i = T^{-1}g_1 - 2f_i$. Since $0 \leq h_i \leq h_1 \leq 1$, for every $x \in K$ we have

$$\begin{aligned}\|T^{-1}(g_1)(x) - 2f_i(x)\| &\leq \|f_1(x) - 2f_i(x)\| + \|T^{-1}g_1(x) - f_1(x)\| \\ &\leq |h_1(x) - 2h_i(x)| + \epsilon \leq 1 + \epsilon.\end{aligned}$$

It follows that $\|B_i\| \le 1+\epsilon$. Moreover

$$\begin{aligned}|\langle \gamma(y_0), T(B_i)(y_0)\rangle| &= 2|\langle \gamma(y_0), Tf_i(y_0)\rangle| \\ &= 2|\langle \gamma(y_0), g_i(y_0)\rangle| > (2n+1)(1-\epsilon).\end{aligned}$$

Thus,

$$\|T\| \ge \left\| T\left(\frac{1}{1+\epsilon}B_i\right)\right\| > (2n+1)\frac{1-\epsilon}{1+\epsilon},$$

which contradicts the choice of ϵ.

4. Upper bounds for $d(C_0(\mathbb{N},X), C([1,\omega^n k],X))$**.** In this section we show how to generalize the formula (1.1) of the introduction to obtain an upper bound for the Banach–Mazur distance between $C_0(\mathbb{N},X)$ and $C([1,\omega^n k],X)$, $1 \le k,n < \omega$, for arbitrary Banach spaces X. We start by proving the following crucial lemma.

LEMMA 4.1. *Let $1 \le n < \omega$ and X be a Banach space. For every $f \in C([1,\omega^n],X)$, define a sequence $(a_\xi)_{1\le\xi\le\omega^n}$ by*

$$a_{\omega^n} = 2f(\omega^n), \qquad a_{\omega^{n-1}i} = f(\omega^{n-1}i) - f(\omega^n) \quad \text{for } 1 \le i < \omega,$$

and if $n > 1$,

$$a_\xi = f(\omega^{n-1}i_1 + \cdots + \omega^{n-j}i_j) - f(\omega^{n-1}i_1 + \cdots + \omega^{n-(j-1)}(i_{j-1}+1))$$

whenever $\xi = \omega^{n-1}i_1 + \cdots + \omega^{n-j}i_j$ with $1 < j \le n$, $0 \le i_p < \omega$ for $1 \le p \le j-1$ and $1 \le i_j < \omega$. Then for every $\epsilon > 0$ there are only a finite number of ordinals $1 \le \xi \le \omega^n$ such that $\|a_\xi\| \ge \epsilon$.

Proof. First of all, each ordinal $1 \le \xi < \omega^n$ has a unique representation (the Cantor normal form [18, p. 153])

$$\xi = \omega^{n-1}i_1 + \cdots + \omega^{n-j}i_j$$

where $1 \le j \le n$, $0 \le i_p < \omega$ for $1 \le p \le j-1$ and $1 \le i_j < \omega$. Hence, for every $f \in C([1,\omega^n],X)$ the sequence $(a_\xi)_{1\le\xi\le\omega^n}$ is well defined.

We will argue by finite induction on n. Of course, the conclusion is true for $n=1$. Next, assume that it is true for $n-1$ with $n \ge 2$. Fix $f \in C([1,\omega^n],X)$ and consider the sequence $(a_\xi)_{1\le\xi\le\omega^n}$ defined as in the statement.

Pick $\epsilon > 0$. By the continuity of f there is $1 < m < \omega$ such that for every $\xi \in\,]\omega^{n-1}m, \omega^n]$, we have

$$\|f(\xi) - f(\omega^n)\| < \epsilon/2.$$

Therefore for every $\xi = \omega^{n-1}i_1 + \cdots + \omega^{n-j}i_j$ with $1 < j \le n$, $0 \le i_p < \omega$

for $1 \le p \le j-1$ and $1 \le i_j < \omega$ such that $\xi \in \,]\omega^{n-1}m, \omega^n[$ we deduce

$$\begin{aligned}\|a_\xi\| &= \|f(\omega^{n-1}i_1 + \cdots + \omega^{n-j}i_j) - f(\omega^{n-1}i_1 + \cdots + \omega^{n-(j-1)}(i_{j-1}+1))\| \\ &\le \|f(\xi) - f(\omega^n)\| + \|f(\omega^{n-1}i_1 + \cdots + \omega^{n-(j-1)}(i_{j-1}+1)) - f(\omega^n)\| \\ &< \epsilon.\end{aligned}$$

On the other hand, for every $1 \le r \le m$, consider $g_r \in C([1, \omega^{n-1}], X)$ given by $g_r(\xi) = f(\omega^{n-1}(r-1) + \xi)$. Moreover, for every $1 \le r \le m$, define a sequence $(a^r_\xi)_{1 \le \xi \le \omega^{n-1}}$ as follows:

$$a^r_{\omega^{n-1}} = 2g_r(\omega^{n-1}), \qquad a^r_{\omega^{n-2}i} = g_r(\omega^{n-2}i) - g_r(\omega^{n-1}) \quad \text{for } 1 \le i < \omega,$$

and if $n > 2$,

$$a^r_\xi = g_r(\omega^{n-2}i_1 + \cdots + \omega^{(n-1)-j}i_j) - g_r(\omega^{n-2}i_1 + \cdots + \omega^{n-j}(i_{j-1}+1))$$

whenever $\xi = \omega^{n-2}i_1 + \cdots + \omega^{(n-1)-j}i_j$ with $1 < j \le n-1$, $0 \le i_p < \omega$ for $1 \le p \le j-1$ and $1 \le i_j < \omega$. By the induction hypothesis, there are only a finite number of ordinals $1 \le \xi \le \omega^{n-1}$ such that $\|a^r_\xi\| \ge \epsilon$ for $1 \le r \le m$. Since

$$a^r_\xi = a_{\omega^{n-1}(r-1)+\xi}$$

for every $1 \le \xi < \omega^{n-1}$ and $1 \le r \le m$, we conclude that there are only a finite number of ordinals $\omega^{n-1}(r-1) + 1 \le \xi \le \omega^{n-1}r$ such that $\|a_\xi\| \ge \epsilon$. Since $[1, \omega^n]$ is the union of $[1, \omega^{n-1}], \ldots, [\omega^{n-1}(m-1), \omega^{n-1}m], [\omega^{n-1}m, \omega^n]$, we are done. ∎

Proof of Theorem 1.3. Observe that $C([1, \omega^n k], X)$ is isometrically isomorphic to the direct sum of k copies of $C([1, \omega^n], X)$, and $C_0(\mathbb{N}, X)$ is isometrically isomorphic to the direct sum of k copies of itself. So, it suffices to prove that

$$d(C_0(\mathbb{N}, X), C([1, \omega^n], X)) \le 2n + 1. \tag{4.1}$$

Denote by Γ_{ω^n} the interval of ordinals $[1, \omega^n]$ endowed with the discrete topology. We can replace $C_0(\mathbb{N}, X)$ in (4.1) by $C_0(\Gamma_{\omega^n}, X)$, because they are isometrically isomorphic.

For every $f \in C([1, \omega^n], X)$ define a map $T(f) : \Gamma_{\omega^n} \to X$ by

$$T(f)(\xi) = a_\xi \quad \text{for every } 1 \le \xi \le \omega^n,$$

where $(a_\xi)_{1 \le \xi \le \omega^n}$ is defined in Lemma 4.1. It follows directly from Lemma 4.1 that $T(f) \in C_0(\Gamma_{\omega^n}, X)$ for every $f \in C([1, \omega^n], X)$. Moreover, it is easy to check that $T : C([1, \omega^n], X) \to C_0(\Gamma_{\omega^n}, X)$ is a linear operator with $\|T\| \le 2$.

Conversely, for every sequence $g = (a_\xi)_{1 \le \xi \le \omega^n} \in C_0(\Gamma_{\omega^n}, X)$ define a map $S(g) : [1, \omega^n] \to X$ by setting

$$S(g)(\omega^n) = \frac{1}{2} a_{\omega^n}, \qquad S(g)(\omega^{n-1} i) = a_{\omega^{n-1} i} + \frac{1}{2} a_{\omega^n} \quad \text{for } 1 \le i < \omega,$$

and for every $\xi = \omega^{n-1} i_1 + \cdots + \omega^{n-j} i_j$ with $1 \le j \le n$, $0 \le i_p < \omega$ for $1 \le p \le j-1$ and $1 \le i_j < \omega$,

$$\begin{aligned} S(g)(\xi) = a_{\omega^{n-1} i_1 + \cdots + \omega^{n-j} i_j} &+ a_{\omega^{n-1} i_1 + \cdots + \omega^{n-(j-1)}(i_{j-1}+1)} \\ &+ \cdots + a_{\omega^{n-1} i_1 + \omega^{n-2}(i_2+1)} + a_{\omega^{n-1}(i_1+1)} + \frac{1}{2} a_{\omega^n}. \end{aligned}$$

We will prove that $S(g)$ is a continuous function for every $g \in C_0(\Gamma_{\omega^n}, X)$. To do this, fix $g = (a_\xi)_{1 \le \xi \le \omega^n} \in C_0(\Gamma_{\omega^n}, X)$. Given $\xi_0 \in [1, \omega^n]^{(1)}$ pick $\epsilon > 0$ and let Λ_ϵ be the finite set of all ordinals $1 \le \xi \le \omega^n$ such that $\|a_\xi\| \ge \epsilon/n$. We distinguish two cases.

Case 1: $\xi_0 = \omega^n$. Since Λ_ϵ is finite, there is $1 \le m < \omega$ such that

$$]\omega^{n-1} m, \omega^n[\, \cap \Lambda_\epsilon = \emptyset.$$

It follows from the definition of $S(g)$ that if $\xi \in \,]\omega^{n-1} m, \omega^n[$, then

$$\|S(g)(\xi) - S(g)(\xi_0)\| \le \|a_{\xi_1}\| + \cdots + \|a_{\xi_s}\|$$

for some $1 \le s \le n$ and $\xi = \xi_1 < \cdots < \xi_s < \xi_0$. Hence

$$\|S(g)(\xi) - S(g)(\xi_0)\| < \epsilon.$$

Case 2: $\xi_0 = \omega^{n-1} i_1 + \cdots + \omega^{n-j} i_j$ with $1 \le j < n$, $0 \le i_p < \omega$ for $1 \le p \le j-1$ and $1 \le i_j < \omega$. There is $1 \le m < \omega$ such that

$$]\omega^{n-1} i_1 + \cdots + \omega^{n-j}(i_j - 1) + \omega^{n-(j+1)} m, \xi_0[\, \cap \Lambda_\epsilon = \emptyset.$$

Once more, from the definition of $S(g)$, if $\xi \in \,]\omega^{n-1} i_1 + \cdots + \omega^{n-j}(i_j - 1) + \omega^{n-(j+1)} m, \xi_0[$, then

$$\|S(g)(\xi) - S(g)(\xi_0)\| \le \|a_{\xi_1}\| + \cdots + \|a_{\xi_s}\|$$

for some $1 \le s \le n - j$ and $\xi = \xi_1 < \cdots < \xi_s < \xi_0$. Consequently,

$$\|S(g)(\xi) - S(g)(\xi_0)\| < \epsilon.$$

Therefore, $S(g)$ is continuous at ξ_0.

Moreover, it is easy to check that $S : C_0(\Gamma_{\omega^n}, X) \to C([1, \omega^n], X)$ is a linear operator with

$$\|S\| \le \frac{2n+1}{2},$$

and the compositions $S \circ T$ and $T \circ S$ are, respectively, the identity operators in $C([1, \omega^n], X)$ and $C_0(\Gamma_{\omega^n}, X)$. This completes the proof of the theorem. ■

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