Rational Misiurewicz maps for which the Julia set is not the whole sphere

by

Magnus Aspenberg (Kiel)

Abstract. We show that Misiurewicz maps for which the Julia set is not the whole sphere are Lebesgue density points of hyperbolic maps.

1. Introduction. The notion of Misiurewicz maps goes back to the famous paper [14] by M. Misiurewicz. There have been some variations of the definition of complex Misiurewicz maps (see e.g. [9], [19]). Let us proceed with the following definition. First, let $J(f)$ be the Julia set of the function f and $F(f)$ its Fatou set. The set of critical points is denoted by $\mathrm{Crit}(f)$ and the omega limit set of x is denoted by $\omega(x)$.

DEFINITION 1.1. A rational non-hyperbolic map f is a *Misiurewicz map* if f has no parabolic periodic points and for every $c \in \mathrm{Crit}(f)$ we have $\omega(c) \cap \mathrm{Crit}(f) = \emptyset$.

In [17] Rivera-Letelier shows that Misiurewicz maps for unicritical polynomials of the form $f_c(z) = z^d + c$, $c \in \mathbb{C}$, are Lebesgue density points of hyperbolic maps. This paper extends this result to all Misiurewicz maps in the space of rational functions of a given degree $d \geq 2$, if the Julia set is not the whole sphere, i.e. every Misiurewicz map for which $J(f) \neq \hat{\mathbb{C}}$ is a Lebesgue density point of hyperbolic maps. The statement is false if the Julia set is the whole sphere (see e.g. [3]), because in this case, post-critically finite Misiurewicz maps are Lebesgue density points of Collet–Eckmann (CE) maps. In addition, these CE-maps have their Julia set equal to the whole sphere (see also [16]). It seems plausible that every Misiurewicz map for which $J(f) = \hat{\mathbb{C}}$ is a Lebesgue density point of CE-maps.

The following is the main result of this paper.

[41]

THEOREM A. *If f is a rational Misiurewicz map of degree $d \geq 2$, for which $J(f) \neq \hat{\mathbb{C}}$, then f is a Lebesgue density point of hyperbolic maps in the space of rational maps of degree d.*

The space of rational maps of degree d is a complex manifold of dimension $2d+1$. To prove Theorem A we will consider 1-dimensional balls around the starting map f. If $B(0, r)$ is a 1-dimensional ball in the parameter space of rational maps of degree $d \geq 2$, then we can associate a direction vector $v \in \mathbb{P}(\mathbb{C}^{2d})$ to $B(0, r)$, such that the plane in which $B(0, r)$ lies can be parameterized by $\{tv : t \in \mathbb{C}\}$. In this case we say that $B(0, r)$ *has direction* v.

Theorem A above follows directly from the following.

THEOREM B. *Let $r > 0$ and f_a, $a \in B(0, r)$, be a 1-dimensional family of rational functions of degree $d \geq 2$ and suppose that $f = f_0$ is Misiurewicz map for which $J(f) \neq \hat{\mathbb{C}}$. Then for almost all directions v of $B(0, r)$, f is a Lebesgue density point of hyperbolic maps in the ball $B(0, r)$.*

Combining [17] and Theorem A, every CE-map for which the Julia set is not the whole sphere can be approximated by a hyperbolic map. In particular, this holds for all polynomial CE-maps. In view of [17] and [3] it is a natural conjecture that almost every CE-map has its Julia set equal to the whole sphere.

REMARK 1.2. We note that any Misiurewicz map which is not a flexible Lattès map can be approximated by a hyperbolic map even in the case $J(f) = \hat{\mathbb{C}}$, see [1]. Later Gauthier ([8, Theorem 6.3]) proved this using simpler arguments with good families (see [2]) and non-normality. Similar arguments appear also in [5]. In fact, the main result in [8] is that Misiurewicz maps belong to the support of bifurcation currents.

In the last section we sketch how to approximate a Misiurewicz map (not being a flexible Lattès map) with a hyperbolic map combining Theorem A and [8].

For a survey of Lattès maps and the definition of flexible Lattès maps see e.g. [13].

2. Preliminary lemmas. We will use the following lemmas by R. Mañé.

THEOREM 2.1 (Mañé's Theorem I). *Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map and $\Lambda \subset J(f)$ a compact invariant set not containing critical points or parabolic points. Then either Λ is a hyperbolic set or $\Lambda \cap \omega(c) \neq \emptyset$ for some recurrent critical point c of f.*

THEOREM 2.2 (Mañé's Theorem II). *If $x \in J(f)$ is not a parabolic periodic point and does not intersect $\omega(c)$ for some recurrent critical point c, then for every $\varepsilon > 0$, there is a neighborhood U of x such that*

- *For all $n \geq 0$, every connected component of $f^{-n}(U)$ has diameter $\leq \varepsilon$.*
- *There exists $N > 0$ such that for all $n \geq 0$ and every connected component V of $f^{-n}(U)$, the degree of $f^n|_V$ is $\leq N$.*
- *For all $\varepsilon_1 > 0$ there exists $n_0 > 0$ such that every connected component of $f^{-n}(U)$ with $n \geq n_0$ has diameter $\leq \varepsilon_1$.*

An alternative proof of Mañé's Theorem by L. Tan and M. Shishikura can be found in [18]. Let us also note that a corollary of Mañé's Theorem II is that a Misiurewicz map cannot have any Siegel disks, Herman rings or Cremer points (see [11] or [18]).

For $k \geq 0$, define

$$P^k(f) = \overline{\bigcup_{n > k,\, c \in \mathrm{Crit}(f) \cap J(f))} f^n(c).}$$

Given a Misiurewicz map f, there is some $k \geq 0$ such that $P^k(f)$ is a compact, forward invariant subset of the Julia set which contains no critical points.

By Mañé's Theorem I, the set $\Lambda = P^k(f)$ is hyperbolic. It is then well-known that there is a holomorphic motion h on Λ:

$$h : \Lambda \times B(0, r) \to \mathbb{C}.$$

For each fixed $a \in B(0, r)$ the map $h = h(z, a) = h_a$ is an injection from Λ to $h_a(\Lambda) = \Lambda_a$ and for fixed $z \in \Lambda$ the map $h = h(z, a)$ is holomorphic in a.

Each critical point $c_j \in J(f)$ moves holomorphically, if it is non-degenerate (i.e. c_j is simple), by the Implicit Function Theorem. If it is degenerate, we have to use a new parameterisation to be able to view each critical point as an analytic function of the parameters. If the parameter space is 1-dimensional one can use the Puiseaux expansion (see e.g. [6, Theorem 1, p. 386]). By reparameterising using a simple base change of the form $a \mapsto a^q$ for some integer $q \geq 1$, the critical points then move holomorphically. In the multi-dimensional case, i.e. if we consider the whole $2d + 1$-dimensional ball $\mathbb{B}(0, r)$ in the parameter space, there is a corresponding result following from the standard theory of resolution of singularities (see e.g. [10] for a survey). A real analytic version of this result is stated in Lemma 9.4 in [15]. The complex analytic version is stated in [1] and can be formulated as follows: There is a proper, holomorphic map $\psi : U \to V$, where U and V are open sets in $\mathbb{C}^{2d+1}$ containing the origin, such that

$$f'(z, \psi(a)) = E(z - c_1(a)) \cdot \ldots \cdot (z - c_{2d-2}(a)),$$

where each $c_j(a)$ is a holomorphic function on U and E is holomorphic and non-vanishing. We therefore assume that all critical points c_j on the Julia set move holomorphically.

For some $k \geq 0$ we have $v_j := f^{k+1}(c_j) \in \Lambda$ for all $c_j \in \mathrm{Crit}(f) \cap J(f)$. Thus we can define the parameter functions

$$x_j(a) = v_j(a) - h_a(v_j(0)).$$

Let $\mathbb{B}(0, r)$ be a full dimensional ball in the parameter space of rational maps around $f = f_0$. We already know that Misiurewicz maps which are not flexible Lattès maps cannot carry an invariant line field on its Julia set (see [4]). This implies that not all the functions x_j can be identically zero in $\mathbb{B}(0, r)$.

LEMMA 2.3. *If f is a Misiurewicz map and not a flexible Lattès map, then at least one x_j is not identically zero in $\mathbb{B}(0, r)$.*

In fact, it follows a posteriori that no such x_j is identically zero. However, let us now assume that I is the set of indices j such that x_j is not identically zero in $\mathbb{B}(0, r)$. We know that $I \neq \emptyset$. In the end, we prove that in fact $I = \{1, \ldots, 2d - 2\}$.

Hence the sets $\{a : x_j(a) = 0\}$, $j \in I$, are all analytic sets of codimension 1. Hence for almost all directions v the functions x_j, $j \in I$, are not identically zero in the corresponding disk $B(0, r)$. From now on, fix such a disk $B(0, r)$ for some $r > 0$.

DEFINITION 2.4. Given $0 < k < 1$, a disk $D_0 = B(a_0, r_0) \subset B(0, r)$ is a *k-Whitney disk* if $|a_0|/r_0 = k$.

A *Whitney disk* is a *k*-Whitney disk for some $0 < k < 1$.

We will now use a distortion lemma from [4, Lemma 3.5]. In this lemma we put $\xi_n = \xi_{n,j}$ and

$$\xi_{n,j}(a) = f_a^n(c_j(a)),$$

where $a \in B(0, r)$. Moreover, choose some $\delta' > 0$ such that $\mathcal{N}$ is a fixed $10\delta'$-neighbourhood of Λ such that $\Lambda_a \subset \mathcal{N}$ for all $a \in B(0, r)$ and $\mathrm{dist}(\Lambda_a, \partial\mathcal{N}) \geq \delta'$. This $\delta' > 0$ shall be fixed throughout the paper and depends only on f.

LEMMA 2.5. *Let $\varepsilon > 0$. If $r > 0$ is sufficiently small, there exists a number $0 < k < 1$ only depending on the function x_j, and a number $S = S(\delta')$, such that the following holds for any k-Whitney disk $D_0 = B(a_0, r_0) \subset B(0, r)$: there is an $n > 0$ such that the set $\xi_n(D_0) \subset \mathcal{N}$ and has diameter at least S. Moreover,*

$$(1) \qquad \left| \frac{\xi_k'(a)}{\xi_k'(b)} - 1 \right| \leq \varepsilon$$

for all $a, b \in D_0$ and all $k \leq n$.

The difference from standard Koebe distortion lemmas is that in this case we have very small distortion also of the argument. Hence, if ε is small in the above lemma, we have good geometry control of the shape of $\xi_n(D_0)$

up to the large scale $S > 0$, i.e. it is almost round. We will use the fact that this holds for every x_j, $j \in I$.

3. Conclusion and proof of Theorem B. We recall the following folklore lemma. For proofs see e.g. [12] (see also [7] for the case of polynomials).

LEMMA 3.1. *Let f be a Misiurewicz map for which $J(f) \neq \hat{\mathbb{C}}$. Then the Lebesgue measure of $J(f)$ is zero.*

For each critical point $c_j = c_j(0) \in J(f)$, $j \in I$, put $D_j = \xi_{n_j,j}(D_0)$, where n_j is the number n in Lemma 2.5. Hence for every j, the diameter of D_j is at least S and we have good control of the geometry, if $\varepsilon > 0$ is small in Lemma 2.5.

Next we prove the following lemma.

LEMMA 3.2. *For each compact subset $K \subset F(f)$ there is a perturbation $r = r(K)$ such that $K \subset F(f_a)$ for all $a \in B(0,r)$.*

Proof. It follows from [18] and [11] that the only Fatou components for Misiurewicz maps are those corresponding to attracting cycles. Recall that $f = f_0$.

Given $K \subset F(f_0)$, there is some integer n and some small disk $B_j \subset F(f_0)$ around each attracting orbit such that $K \subset f_0^{-n}(D)$, where $D = \bigcup_j B_j$. Choose D such that $f_0(D) \subset D$. Since $f_a(D) \subset D$ for small perturbations $a \in B(0,r)$, we have $f_a^n(D) \subset D$ for all $n \geq 0$. Hence the family $\{f_a^n\}_{n=0}^{\infty}$ is normal on D and consequently $D \subset F(f_a)$ for any such parameter $a \in B(0,r)$. Moreover, $f_a^{-n}(D)$ moves continuously with the parameter in the Hausdorff distance topology. Therefore there is some $r > 0$ such that also $K \subset f_a^{-n}(D)$ for all $a \in B(0,r)$. The lemma is proved. ∎

Let $\delta > 0$. Define
$$E_\delta = \{z \in F(f_0) : \operatorname{dist}(z, J(f_0)) \geq \delta\}.$$
Now, there is some $\delta_0 > 0$ (depending only on $f = f_0$) such that for every $0 < \delta < \delta_0$ there exists an $r = r(\delta) > 0$ such that $E_\delta \subset F(f_a)$ for every $a \in B(0,r)$, by Lemma 3.2.

Clearly, $r(\delta) \to 0$ as $\delta \to 0$. Since the Lebesgue measure of $J(f_0)$ is zero, for every $\varepsilon_1 > 0$ there is some $\delta > 0$ such that the Lebesgue measure of $\{z : \operatorname{dist}(z, J(f_0)) \leq \delta\}$ is less than ε_1. Hence there exists some $\delta > 0$ such that for every disk D of diameter at least $S/2$ ($S > 0$ is the large scale from Lemma 2.5) we have
$$\frac{\mu(D \cap E_\delta)}{\mu(D)} \geq 1 - \varepsilon_1.$$

For this $\delta > 0$, there is some $r = r(\delta) > 0$ such that also $E_\delta \subset F(f_a)$ for all $a \in B(0, r)$. Since every D_j contains a disk of diameter $S/2$ (because of bounded distortion), we get

$$\frac{\mu(D_j \cap E_\delta)}{\mu(D_j)} \geq 1 - \varepsilon_1',$$

where $\varepsilon_1'(\varepsilon_1) \to 0$ as $\varepsilon_1 \to 0$. By Lemma 2.5,

$$\frac{\mu(\xi_{n_j,j}^{-1}(D_j \cap E_\delta))}{\mu(D_0)} \geq 1 - C\varepsilon_1'$$

for some constant $C > 0$ depending on the ε in Lemma 2.5. We have $C \to 1$ as $\varepsilon \to 0$. Now every parameter $a \in \xi_{n_j,j}^{-1}(D_j \cap E_\delta)$ has the property that $c_j(a) \in F(f_a)$. For every a in

$$A = \bigcap_j \xi_{n_j,j}^{-1}(D_j \cap E_\delta),$$

the critical point $c_j(a)$ is in $F(f_a)$. If $I \neq \{1, \ldots, 2d - 2\}$, then there is a small neighbourhood around a in the ball $\mathbb{B}(0, r)$ where all $c_j(a) \in F(f_a)$ for $j \in I$ and, by the definition of I (since $x_j \equiv 0$ for $j \neq I$), the other $c_j(a)$ still land at the hyperbolic set Λ_a. This means that f_a is a J-stable Misiurewicz map. But this contradicts [4]. Hence $I = \{1, \ldots, 2d - 2\}$, so no x_j is identically zero.

Consequently, for every $a \in A$, every $c_j(a)$ is in $F(f_a)$ and it follows that f_a is a hyperbolic map. Since $\varepsilon_1 > 0$ can be chosen arbitrarily small, the Lebesgue density of hyperbolic maps at $a = 0$ is equal to 1 and Theorem B follows.

4. Hyperbolic approximation of Misiurewicz maps when $J(f)$ $= \hat{\mathbb{C}}$. In this section we sketch how to perturb a Misiurewicz map (not being a flexible Lattès map) to a hyperbolic map in the case $J(f) = \hat{\mathbb{C}}$. Start with a Misiurewicz map $f = f_0$ with $J(f) = \hat{\mathbb{C}}$ such that f is not a flexible Lattès map. Using again the desingularisation process (see p. 3), we may assume that all critical points $c_1, \ldots, c_l$ move holomorphically. In [8] Gauthier pointed out that the set

$$G_0 = \bigcap_{j=1}^{l-1} \{x_j = 0\}$$

is a good family in the sense of [2]. In particular, it is a complex analytic set of dimension 1. Moreover, since f is a Misiurewicz map and not a flexible

Lattès map, the set

$$G_0' = \bigcap_{j=1}^{l} \{x_j = 0\}$$

reduces to a single point in the parameter space, because f cannot carry an invariant line field on its Julia set ([4, Lemma 2.3]). This implies that x_l is not identically zero in G_0. Now take a k-Whitney disk $D_0 \subset G_0 \cap \mathbb{B}(0, r)$ and use Lemma 2.5. We find that $\xi_{n,l}(D_0)$ grows to the large scale $S > 0$, where S only depends on the function x_l. Since $J(f) = \hat{\mathbb{C}}$, by the eventually onto-property (non-normality), after finitely many iterates, say $m = m(S)$, $\xi_{n+m,l}(D_0)$ has covered the whole Riemann sphere. We find thereby a solution to $\xi_{n+m,l}(a) = c_l(a)$ in G_0, for a arbitrarily close to 0. The function f_a is a new Misiurewicz map where one critical point lies in a super-attracting orbit. Hence $J(f_a) \neq \hat{\mathbb{C}}$ and we can apply Theorem A to get a hyperbolic map g arbitrarily close to f_a. Since a was arbitrarily close to 0, the proof is finished.

References

[1] M. Aspenberg, *Perturbations of rational Misiurewicz maps*, preprint, arXiv:0804.1106.
[2] —, *Rational Misiurewicz maps are rare*, preprint, math.DS/0701382.
[3] —, *The Collet–Eckmann condition for rational maps on the Riemann sphere*, Ph. D. thesis, Stockholm, 2004; http://www.diva-portal.org/kth/abstract.xsql?dbid=3788. To appear in Math. Z.
[4] —, *Rational Misiurewicz maps are rare II*, Comm. Math. Phys. 291 (2009), 645–658.
[5] M. Aspenberg and J. Graczyk, *Dimension and measure for semi-hyperbolic rational maps of degree 2*, C. R. Math. Acad. Sci. Paris 347 (2009), 395–400.
[6] E. Brieskorn and H. Knörrer, *Plane Algebraic Curves*, Birkhäuser, Basel, 1986.
[7] L. Carleson, P. W. Jones, and J.-C. Yoccoz, *Julia and John*, Bol. Soc. Brasil. Mat. (N.S.) 25 (1994), 1–30.
[8] T. Gauthier, *Fractions rationnelles Misiurewicz et courants de bifurcation*, preprint, Toulouse, 2009.
[9] J. Graczyk, J. Kotus, and G. Świątek, *Non-recurrent meromorphic functions*, Fund. Math. 182 (2004), 269–281.
[10] J. Kollár, *Lectures on Resolution of Singularities*, Ann. of Math. Stud. 166, Princeton Univ. Press, Princeton, NJ, 2007.
[11] R. Mañé, *On a theorem of Fatou*, Bol. Soc. Brasil. Mat. (N.S.) 24 (1993), 1–11.

[12] N. Mihalache, *Collet–Eckmann condition for recurrent critical orbits implies uniform hyperbolicity on periodic orbits*, Ergodic Theory Dynam. Systems 27 (2007) 1267–1286.

[13] J. Milnor, *On Lattès maps*, in: Dynamics on the Riemann Sphere. A Bodil Branner Festschrift, Eur. Math. Soc., 2006, 9–43.

[14] M. Misiurewicz, *Absolutely continuous invariant measures for certain maps of an interval*, Publ. Math. IHES 53 (1981), 17–51.

[15] D. Müller and F. Ricci, *Solvability for a class of doubly characteristic differential operators on 2-step nilpotent groups*, Ann. of Math. (2) 143 (1996), 1–49.

[16] M. Rees, *Positive measure sets of ergodic rational maps*, Ann. Sci. École Norm. Sup. (4) 19 (1986), 383–407.

[17] J. Rivera-Letelier, *On the continuity of Hausdorff dimension of Julia sets and similarity between the Mandelbrot set and Julia sets*, Fund. Math. 170 (2001), 287–317.

[18] M. Shishikura and T. Lei, *An alternative proof of Mañé's theorem on non-expanding Julia sets*, in: The Mandelbrot Set, Theme and Variations, London Math. Soc. Lecture Note Ser. 274, Cambridge Univ. Press, Cambridge, 2000, 265–279.

[19] S. van Strien, *Misiurewicz maps unfold generically (even if they are critically non-finite)*, Fund. Math. 163 (2000), 39–54.

Magnus Aspenberg
Mathematisches Seminar
Christian-Albrechts Universität zu Kiel
Ludewig-Meyn Str. 4
24 098 Kiel, Germany
E-mail: aspenberg@math.uni-kiel.de