FUNDAMENTA
MATHEMATICAE
206 (2009)

Embedding odometers in cellular automata

by

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To Michał Misiurewicz with admiration and affection

Abstract. We consider the problem of embedding odometers in one-dimensional cellular automata. We show that (1) every odometer can be embedded in a gliders-with-reflecting-walls cellular automaton, which one depending on the odometer, and (2) an odometer can be embedded in a cellular automaton with local rule $x_i \mapsto x_i + x_{i+1} \bmod n$ $(i \in \mathbb{Z})$, where n depends on the odometer, if and only if it is "finitary."

1. Introduction. An *odometer* is the "+1" map on a countable product of finite cyclic groups. A (one-dimensional) *cellular automaton* (X, T) is a dynamical system defined by a local rule on a closed, T-invariant subset of either $A^{\mathbb{N}}$ or $A^{\mathbb{Z}}$, where A is a finite alphabet. In [3] the authors and M. Pivato partially solved the "give me a cellular automaton and I will find an odometer that can be embedded in it" problem. In this paper we completely solve the converse problem: "give me an odometer and I will find a cellular automaton that it can be embedded in."

Theorem 1. *Every odometer can be embedded in a gliders-with-reflecting-walls cellular automaton.*

Although finitary odometers (defined in Theorem 2 below) can be embedded in a number of cellular automata [7], Theorem 1 identifies a (relatively small) class of cellular automata such that *every* odometer can be embedded in one of them.

Theorem 2. *Every finitary odometer* $(\mathbb{Z}(S), +1)$, *i.e. one such that the set of prime divisors of the members of S is finite*, *can be embedded in the one-dimensional*, *two-sided cellular automaton with local rule* $x_i \mapsto x_i + x_{i+1} \bmod n$ $(i \in \mathbb{Z})$, *defined on the space of all doubly infinite sequences with*

2010 *Mathematics Subject Classification*: Primary 37B10, 37B15.
Key words and phrases: odometer, embedded, cellular automaton.

DOI: 10.4064/fm206-1-8

entries from $\mathbb{Z}_n$, the ring of integers modulo n, where n is the product of the primes that divide infinitely members of S.

Conversely, only finitary odometers can be embedded in such cellular automata.

Definitions and background. Let $S = (s_1, s_2, \ldots)$ be a sequence of integers greater than 1. Define

$$\mathbb{Z}(S) := \prod_{k\geq 1} \mathbb{Z}/s_k\mathbb{Z} \quad \text{and} \quad \widetilde{\mathbb{Z}}(S) := \operatorname*{inv\,lim}_{k\to\infty} (\mathbb{Z}/s_1\cdots s_k\mathbb{Z}, \beta_k),$$

where the binding maps $\beta_k : s_1 \cdots s_{k+1}\mathbb{Z} \to s_1 \cdots s_k\mathbb{Z}$ are defined by

$$z \mapsto z \bmod s_1 \cdots s_k.$$

Addition in $\mathbb{Z}(S)$ is "with carrying," addition in $\widetilde{\mathbb{Z}}(S)$ is coordinatewise, i.e. without carrying. $\mathbb{Z}(S)$ and $\widetilde{\mathbb{Z}}(S)$ are isomorphic, compact, abelian, topological groups [4].

The $+1$ map on $\mathbb{Z}(S)$ is defined by

$$z \mapsto z + (1, 0, 0, \ldots)$$

and the $+\tilde{1}$ map on $\widetilde{\mathbb{Z}}(S)$ is defined by

$$z \mapsto z + (1, 1, \ldots).$$

$(\mathbb{Z}(S), +1)$ and $(\widetilde{\mathbb{Z}}(S), +\tilde{1})$ are topologically conjugate (any topological group isomorphism of $\mathbb{Z}(S)$ onto $\widetilde{\mathbb{Z}}(S)$ that takes 1 to $\tilde{1}$ is a topological conjugacy) and are called the *S-adic odometer*. When $S = (n, n, \ldots)$, $(\mathbb{Z}(S), +1)$ is the well-known *n-adic odometer*, denoted $(\mathbb{Z}(n), +1)$.

By Theorem 7.6 of [2], a complete topological conjugacy invariant of $(\mathbb{Z}(S), +1)$ is the *multiplicity function* $\mathrm{MULT}_S : \{\text{primes}\} \to \{0, 1, \ldots, \infty\}$, defined by

$$\mathrm{MULT}_S(p) := \sum_i \{\max j : p^j \text{ divides } s_i\}.$$

Thus $\mathrm{MULT}_S(p)$ is the total number of times that p divides members of S.

Throughout this paper a two-sided *cellular automaton* (X, T) will be a dynamical system defined on a closed, T-invariant subset of $A^{\mathbb{Z}}$, where A is a finite alphabet and T is given by a *local rule* $\tau : A^{2m+1} \to A$ for some $m \geq 0$ as follows: $[T(x)]_i = \tau(x_{i-m}, \ldots, x_{i+m})$ $(i \in \mathbb{Z})$. We note that T is continuous and commutes with the *shift* $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$, defined by $[\sigma(x)]_i = x_{i+1}$ $(i \in \mathbb{Z})$. When appropriate, we will write $x \in A^{\mathbb{Z}}$ as $x_L.x_R$, where the dot separates the negative indices from the non-negative ones. One-sided cellular automata are similarly defined.

When A has n elements, we may sometimes assume that $A = \mathbb{Z}_n$, the ring of integers modulo n. The cellular automaton defined on all doubly infinite sequences with entries from $\mathbb{Z}_n$ and local rule $x_i \mapsto x_i + x_{i+1} \bmod n$ $(i \in \mathbb{Z})$

will be denoted $(\mathbb{Z}_n^{\mathbb{Z}}, T_n)$. The maps T_n have no memory and so we define one-sided cellular automata $(T_n)_R : \mathbb{Z}_n^{\mathbb{N}_0} \to \mathbb{Z}_n^{\mathbb{N}_0}$ by the same local rule. Here $\mathbb{N} := \{1, 2, \ldots\}$ is the natural numbers and $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$.

A more geometric class of cellular automata is the class of *gliders-with-reflecting-walls* cellular automata [6, Example 6.5].

The alphabet for all these one-sided cellular automata is

$$\{W, L, R, \emptyset\},$$

where W is a stationary wall, L is a left-moving particle, R is a right-moving particle, and $\emptyset$ is an empty space.

The spaces $X \subseteq A^{\mathbb{N}}$ satisfy: for every $x \in X$, $x_1 = W$, $x_i = W$ for infinitely many i, and between any two consecutive W there is exactly one particle.

The local rule for these automata is as follows:

- Walls do not move.
- If the space immediately to the left of L is empty, then L and $\emptyset$ change places. If the space immediately to the left of L is W, then L becomes R but does not move.
- If the space immediately to the right of R is empty, then R and $\emptyset$ change places. If the space immediately to the right of R is W, then R becomes L but does not move.

For a dynamical system (X, f), where X is a subset of some $A^{\mathbb{Z}}$ or $A^{\mathbb{N}}$, the *space-time diagram* of (X, f) with *seed* x is the array whose (i, j)th entry is $[f^j(x)]_i$. It is a convenient way of visualizing the forward f-orbit of x, $\{f^j(x) : j \geq 0\}$. Here we think of "increasing time" as going down. Space-time diagrams for systems on one-sided sequences are similarly defined, and are convenient ways of visualizing odometers.

For dynamical systems (X, f) and $(\widehat{X}, \widehat{f})$, we say that (X, f) can be *embedded* in $(\widehat{X}, \widehat{f})$ if there is a closed, $\widehat{f}$-invariant subset $\widehat{X}'$ of $\widehat{X}$ such that (X, f) is topologically conjugate to $(\widehat{X}', \widehat{f}|_{\widehat{X}'})$.

Every odometer can be embedded in a gliders-with-reflecting-walls cellular automaton. Gliders-with-reflecting-walls cellular automata (X, T) are defined on one-sided infinite sequences with entries from $\{W, L, R, \emptyset\}$, with local rules defined in the preceding section.

THEOREM 1. *Every odometer can be embedded in a gliders-with-reflecting-walls cellular automaton.*

Proof. Let $S = (s_1, s_2, \ldots)$.

First assume that at least one s_i is even. Since the multiplicity function is a complete topological conjugacy invariant of $(\mathbb{Z}(S), +1)$, the order of the s_i is irrelevant, so we may assume that s_1 is even.

Consider the set X of all points in $\{W, L, R, \emptyset\}^{\mathbb{N}}$ of the form

$$W \leftarrow \tfrac{1}{2}s_1 \rightarrow W \leftarrow \tfrac{1}{2}s_1 s_2 \rightarrow W \leftarrow \cdots ,$$

where the gaps contain exactly one particle. The columns of gaps in the space-time diagram of a gliders-with-reflecting-walls cellular automaton with any such point as seed are periodic with least periods $s_1, s_1 s_2, \ldots$.

We show that this one-sided cellular automaton is topologically conjugate to $(\widetilde{\mathbb{Z}}(s_1, s_2, s_3, \ldots), +\tilde{1})$. Let $\widetilde{T}$ be the gliders-with-reflecting-walls cellular automaton map and label the gaps, left-to-right, $G_1, G_2, \ldots$. Consider the space-time diagram of $(X, \widetilde{T})$ with seed $\bar{x}$, defined by "R appears at the extreme left of each gap." For x in the forward $\widetilde{T}$-orbit-closure of $\bar{x}$, define

$$x \mapsto z = (z_1, z_2, \ldots) \in \prod_{k \ge 1} \mathbb{Z}/s_1 \cdots s_k \mathbb{Z}$$

as follows. For $i \ge 1$, let z_i, $0 \le z_i \le s_1 \cdots s_i - 1$, satisfy

$$x|_{G_i} = \widetilde{T}^{z_i}(\overbrace{R, \emptyset, \emptyset, \ldots, \emptyset}^{\longleftarrow \frac{1}{2}s_1 s_2 \cdots s_i \longrightarrow}),$$

i.e. $(\overbrace{R, \emptyset, \emptyset, \ldots, \emptyset}^{\longleftarrow \frac{1}{2}s_1 s_2 \cdots s_i \longrightarrow})$ appears in row z_i in this space-time diagram. This map is continuous, one-to-one, and commutes with the appropriate actions, and so is a topological conjugacy.

Now assume that all the s_i are odd. In this case the cellular automaton $(X, \widetilde{T}^2)$, defined on all points of $\{W, L, R, \emptyset\}^{\mathbb{N}}$ of the form

$$W \leftarrow s_1 \rightarrow W \leftarrow s_1 s_2 \rightarrow W \leftarrow \cdots ,$$

in the forward $\widetilde{T}^2$-orbit-closure of $\bar{x}$ (defined as above, R at the extreme left of each gap), is topologically conjugate to $(\widetilde{\mathbb{Z}}(S), +\tilde{1})$. ∎

An odometer can be embedded in a cellular automaton with local rule $x_0 + x_1$ if and only if it is "finitary". The word "finitary" in the title of this section refers to odometers $(\mathbb{Z}(S), +1)$ such that the set of prime divisors of the members of S is finite.

Throughout this section, $(\mathbb{Z}_n^{\mathbb{Z}}, T_n)$ will denote the two-sided cellular automaton with local rule $x_i \mapsto x_i + x_{i+1} \bmod n$ $(i \in \mathbb{Z})$. To avoid notational clutter, we may write T rather than T_n when n is clear.

LEMMA 1. *Let $\bar{x} = \ldots 000.100 \ldots$ and let X be the forward T_n-orbit-closure of $\bar{x}$.*

(1) *For any $n \ge 2$, $\bar{x}_R := 100\ldots$ is $(T_n)_R$-fixed.*
(2) *For any $x \in X$, if some column $[T_n^j(x)]_i$ $(j \ge 0)$ in the space-time diagram of (X, T_n) with seed x is periodic with least period m, then the column immediately to the left, $[T_n^j(x)]_{i-1}$ $(j \ge 0)$, is periodic with least period mn' for some factor n' of n ($n' = 1$ or n is possible).*

(3) *For any $n \geq 2$, $\bar{x}$ has an infinite forward T_n-orbit.*
(4) *For $n = p$ prime, there exist $1 = k_1 < k_2 < \cdots$ such that for every $i \geq 1$, the columns $[T_p^j(\bar{x})]_i$ $(j \geq 0)$, $i = -k_{i+1} + 1, \ldots, -k_i$, are periodic with least period p^i.*

Proof. Write T in place of T_n. (1) is clear.

(2) We prove this part for $n = p$ prime, leaving it to the reader to supply the details for the general case. Suppose that column i in the space-time diagram of (X, T) with seed x is periodic with least period m: $[T^j(x)]_i = [T^{j+m}(x)]_i$ $(j \geq 0)$. If

$$\sum_{j=0}^{m-1} [T^j(x)]_i \equiv 0 \bmod p,$$

then column $i - 1$ is periodic with least period m. If

$$\sum_{j=0}^{m-1} [T^j(x)]_i \not\equiv 0 \bmod p,$$

then column $i - 1$ is periodic with least period pm.

(3) If $\bar{x}$ has a finite forward T-orbit, then there exists $K \geq 0$ such that $x_{-k} = 0$ for every point x in this orbit and for every $k \geq K$. This contradicts (2).

(4) follows from (1), (2), and (3). ∎

We divide the "if and only if" statement of Theorem 2 into two separate theorems.

THEOREM 2A. *Every finitary odometer $(\mathbb{Z}(S), +1)$, i.e. one such that the set of prime divisors of the members of S is finite, can be embedded in the one-dimensional, two-sided cellular automaton $(\mathbb{Z}_n^{\mathbb{Z}}, T_n)$ with local rule*

$$x_i \mapsto x_i + x_{i+1} \bmod n \quad (i \in \mathbb{Z}),$$

where n is the product of the primes that divide infinitely many members of S.

Since the multiplicity function is a complete topological conjugacy invariant, every finitary odometer is topologically conjugate to one of the following two canonical forms:

(1) the n-adic odometer, $(\mathbb{Z}(n), +1) := (\mathbb{Z}(n, n, \ldots), +1)$, where n is the product of distinct primes,
(2) $(\mathbb{Z}(m, n, n, \ldots), +1)$, where m and n are relatively prime and n is the product of distinct primes.

Theorem 2A follows from Lemmas 2–7 below.

LEMMA 2. *For p prime and $m \geq 2$ such that p is not a factor of m, both $(\mathbb{Z}(p), +1)$ and $(\mathbb{Z}(m, p, p, \ldots), +1)$ can be embedded in $(\mathbb{Z}_p^{\mathbb{Z}}, T_p)$.*

Proof. Throughout this proof, we write T in place of T_p.

First we prove the lemma for $(\mathbb{Z}(p), +1)$. Consider the space-time diagram of $(\mathbb{Z}_p^{\mathbb{Z}}, T)$ with seed

$$\bar{x} = \ldots 000.1000 \ldots .$$

We show that T restricted to the forward T-orbit-closure of $\bar{x}$ is topologically conjugate to $(\mathbb{Z}(p), +1)$. Recall from Lemma 1(3), (1) that $\bar{x}$ has an infinite forward T-orbit and that $\bar{x}_R := 100\ldots$ is T_R-fixed.

Define a mapping $x \mapsto z$ from the forward T-orbit-closure of $\bar{x}$ to $\mathbb{Z}(p)$ as follows. For x in the forward T-orbit-closure of $\bar{x}$, let $z = (z_1, z_2, \ldots) \in \mathbb{Z}(p)$ be such that

$$T^{\sum_{i=1}^{k} z_i p^{i-1}}(\bar{x}) \to x \quad \text{as } k \to \infty.$$

That such a sequence exists follows from Lemma 1. (The partial sums of $\sum_{i=1}^{\infty} z_i p^{i-1}$ are the rows in the space-time diagram of $(\mathbb{Z}_p^{\mathbb{Z}}, T)$ with seed $\bar{x}$ at which the appropriate "right tails" of x appear, so z is well-defined.) This mapping is continuous, one-to-one, and commutes with the appropriate actions. Therefore it is a topological conjugacy.

The proof of the lemma for $(\mathbb{Z}(m, p, p, \ldots), +1)$ follows the proof for $(\mathbb{Z}(p), +1)$ provided we can find a seed $\bar{y} = \bar{y}_L.\bar{y}_R$ such that $\bar{y}$ has an infinite forward T-orbit and $\bar{y}_R$ is T_R-periodic with least period m. That we can do this is Lemma 4 below. ∎

Lemma 3. *$(\mathbb{Z}_p^{\mathbb{N}_0}, T_R)$ is topologically conjugate to the full one-sided shift $(\mathbb{Z}_p^{\mathbb{N}_0}, \sigma_L)$, where σ_L is the left-shift defined by $[\sigma_L(x)]_i := x_{i+1}$ $(i \geq 0)$.*

Proof. The topological conjugacy $x \mapsto y$ is given by $y_i := [T_R^i(x)]_0$ $(i \geq 0)$. For a more general result, see [1]. ∎

Lemma 4. *Let $m \geq 1$. There is a point $\bar{y} = \bar{y}_L.\bar{y}_R$ with an infinite forward T-orbit and such that $\bar{y}_R$ is T_R-periodic with least period m.*

Proof. By Lemma 3 there is a T_R-periodic point $\bar{y}_R = \bar{y}_0\bar{y}_1\ldots$ with least period m. It follows from Lemma 1(2) that column 0 in the space-time diagram of T (N.B. T, not T_R) with seed any left extension of $\bar{y}_R$ is periodic. So it suffices to show that $\bar{y}_R$ has a left extension such that the columns in the appropriate space-time diagram have arbitrarily large least periods.

By Lemma 3, for every $k \geq 1$ there are p^k T_R^k-fixed points. For $k = 1$ (since $p^2 > p^1$), there exist $\bar{y}_{-1}$ and $\bar{y}_{-2}$ such that $\bar{y}_{-2}\bar{y}_{-1}\bar{y}_0\ldots$ is not T_R-fixed. For $k = 2$, there exist $\bar{y}_{-3}$, $\bar{y}_{-4}$, and $\bar{y}_{-5}$ such that $\bar{y}_{-5}\bar{y}_{-4}\ldots$ is not T_R^2-fixed. Continue with $k = 3, 4, \ldots$. ∎

Lemma 5. *If (X, f) can be embedded in $(\widehat{X}, \widehat{f})$ and (Y, g) can be embedded in $(\widehat{Y}, \widehat{g})$, then $(X, f) \times (Y, g)$ can be embedded in $(\widehat{X} \times \widehat{Y}, \widehat{f} \times \widehat{g})$.* ∎

LEMMA 6. *Let $m, n \geq 2$ be relatively prime. Then $(\mathbb{Z}(mn), +1)$ is topologically conjugate to $(\mathbb{Z}(m) \times \mathbb{Z}(n), (+1, +1))$. If, in addition, $s \geq 2$ is relatively prime to both m and n, then $(\mathbb{Z}(s, mn, mn, \ldots), +1)$ is topologically conjugate to $(\mathbb{Z}(s, m, m, \ldots) \times \mathbb{Z}(n), (+1, +1))$.*

Proof. To prove the first statement it suffices to find a topological group isomorphism of $\mathbb{Z}(mn)$ onto $\mathbb{Z}(m) \times \mathbb{Z}(n)$ that takes $(1, 0, \ldots) \in \mathbb{Z}(mn)$ to $((1, 0, \ldots), (1, 0, \ldots)) \in \mathbb{Z}(m) \times \mathbb{Z}(n)$.

Map $\mathbb{Z}(mn)$ to $\mathbb{Z}(m) \times \mathbb{Z}(n)$ by

$$(z_0, z_1, \ldots) \mapsto ((z_0', z_1', \ldots), (z_0'', z_1'', \ldots)),$$

where for every $k \geq 0$, $\sum_{i=0}^{k} z_i' m^i$ is the beginning of the base m expansion of $\sum_{i=0}^{k} z_i (mn)^i$; similarly for z''. This map is well-defined, takes $(1, 0, \ldots)$ to $((1, 0, \ldots), (1, 0, \ldots))$, and satisfies all the conditions of topological group isomorphism, except possibly ontoness. To see that it maps $\mathbb{Z}(mn)$ onto $\mathbb{Z}(m) \times \mathbb{Z}(n)$, notice that it maps the set

$$\{k(1, 0, \ldots) \in \mathbb{Z}(mn) : k \geq 0\},$$

which is dense in $\mathbb{Z}(mn)$, onto the set

$$\{(k(1, 0, \ldots), k(1, 0, \ldots)) \in \mathbb{Z}(m) \times \mathbb{Z}(n) : k \geq 0\}.$$

The latter set is dense in $\mathbb{Z}(m) \times \mathbb{Z}(n)$ because m and n are relatively prime.

The proof of the second statement is similar. We omit the details. ∎

LEMMA 7. *Let $m, n \geq 2$ be relatively prime. Then $(\mathbb{Z}_{mn}^{\mathbb{Z}}, T_{mn})$ is topologically conjugate to $(\mathbb{Z}_m^{\mathbb{Z}} \times \mathbb{Z}_n^{\mathbb{Z}}, T_m \times T_n)$.*

Proof. Any ring isomorphism of $\mathbb{Z}_m \times \mathbb{Z}_n$ onto $\mathbb{Z}_{mn}$ is a topological conjugacy of $(\mathbb{Z}_m^{\mathbb{Z}} \times \mathbb{Z}_n^{\mathbb{Z}}, T_m \times T_n)$ onto $(\mathbb{Z}_{mn}^{\mathbb{Z}}, T_{mn})$. ∎

THEOREM 2B. *If an odometer $(\mathbb{Z}(S), +1)$ can be embedded in the one-dimensional, two-sided cellular automaton $(\mathbb{Z}_n^{\mathbb{Z}}, T_n)$ with local rule*

$$x_i \mapsto x_i + x_{i+1} \bmod n \qquad (i \in \mathbb{Z}),$$

then $(\mathbb{Z}(S), +1)$ is finitary, i.e. the set of prime divisors of the members of S is finite.

Proof. Suppose that $(\mathbb{Z}(S), +1)$ is topologically conjugate to (X, T_n), where X is a closed, T_n-invariant subset of $\mathbb{Z}_n^{\mathbb{Z}}$. Consider the space-time diagram of $(\mathbb{Z}(S), +1)$ with seed $(0, 0, \ldots)$. Every column is periodic and for p prime, p divides the least period of some column if and only if p divides some $s \in S$.

It follows from the uniform continuity of the topological conjugacy and its inverse that every column in any space-time diagram of (X, T_n) is periodic. For p prime, p divides the least period of some column in a space-time diagram of (X, T_n) if and only if p divides the least period of some column in the space-time diagram of $(\mathbb{Z}(S), +1)$ with seed $(0, 0, \ldots)$. The proof is completed by applying the lemma below. ■

LEMMA 8. *For any $n \geq 2$, the set of primes p such that p divides the least period of some column in a space-time diagram of (X, T_n) is finite.*

Proof. Since every column in the space-time diagram of $(\mathbb{Z}(S), +1)$ with seed $(0, 0, \ldots)$ is periodic, it follows from Lemma 1 that every column in any space-time diagram of (X, T_n) is periodic. Furthermore, if column i has least period m, then column $i - 1$ has least period $n'm$, where n' is a factor of n.

So if a column has least period m, then any prime that divides the least period of some column to its left also divides mn. Hence the set of all primes that divide the least period of any column is finite. ■

Acknowledgements. Some of this work was done while the first author was a guest of Trent University. He thanks it for its hospitality and support. The authors thank M. Pivato for introducing them to gliders with reflecting walls.

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Received 19 September 2008;
in revised form 16 April 2009