FUNDAMENTA
MATHEMATICAE
229 (2015)

# Borel completeness of some $\aleph_0$-stable theories

by

**Michael C. Laskowski** (College Park, MD) and
**Saharon Shelah** (Jerusalem and Piscataway, NJ)

**Abstract.** We study $\aleph_0$-stable theories, and prove that if $T$ either has eni-DOP or is eni-deep, then its class of countable models is Borel complete. We introduce the notion of $\lambda$-Borel completeness and prove that such theories are $\lambda$-Borel complete. Using this, we conclude that an $\aleph_0$-stable theory satisfies $I_{\infty,\aleph_0}(T,\lambda) = 2^\lambda$ for all cardinals $\lambda$ if and only if $T$ either has eni-DOP or is eni-deep.

**1. Introduction and preliminaries.** The main theme of the paper will be to produce many disparate models of an $\aleph_0$-stable theory, assuming some type of non-structure hypothesis. In all cases, to show the complexity of a model $M$, we concentrate on the regular types $p \in S(M)$ that have *finite dimension in $M$* i.e., for some (equivalently for every) finite $A \subseteq M$ on which the regular type is based and stationary, we have $\dim(p|A, M)$ finite. That is, there is no infinite, $A$-independent set of realizations of $p|A$ in $M$. Clearly, this notion is isomorphism invariant. If $f$ is an isomorphism between $M$ and $N$, then $p \in S(M)$ has finite dimension in $M$ if and only if $f(p)$ has finite dimension in $N$. This yields a criterion for two models to be non-isomorphic: If two models are isomorphic, their regular types of finite dimension must correspond.

In order to get our non-structure results, we need to identify and analyze both those regular types that are capable of having finite dimension in a model (which we term *eni*) as well as those regular types that are capable of 'supporting' an eni type. Lumped together, these regular types are called *eni-active* (see Definition 1.13) and we call a regular type *dull* if it is not eni-active. With Proposition 1.19, we see that this eni-active/dull partition of regular types has many equivalent descriptions. It is particularly useful that this dichotomy is preserved under the equivalence relation of non-orthogonality.

2010 *Mathematics Subject Classification*: Primary 03C45; Secondary 03E15.
*Key words and phrases*: Borel complete, Borel reducibility, $\aleph_0$-stable.

DOI: 10.4064/fm229-1-1  

The paper begins by stating some well-known results about models of $\aleph_0$-stable theories and then identifying various species of regular types. We close Section 1 by proving a structure result for dull types, Proposition 1.24, that indicates their name is apt. This result lays the foundation for Theorems 5.8 and 7.2.

In Section 2, we define the notion of having a 'DOP witness' and define many different variants of 'eni-DOP'. Fortunately, with Theorem 2.4, we see that an $\aleph_0$-stable theory admits one of these variants if and only if it admits them all. Thereafter, we choose the term 'eni-DOP' for its brevity. Theorem 2.4 also asserts that among $\aleph_0$-stable theories, eni-DOP is equivalent to the Omitting Types Order Property (OTOP), as well as to the existence of an independent triple of countable, saturated models over which the prime model is not saturated.

Our first major result, Theorem 4.12, proves that among $\aleph_0$-stable theories, those having eni-DOP are Borel complete. The existence of a *finite approximation* to a DOP witness (see Subsection 4.1) gives a procedure for constructing a model $M_G$ to code any bipartite graph $G$. In such a coding, the edge set of $G$ corresponds to the types of finite dimension in $M_G$. However, it is far from obvious how to recover the vertex set of $G$ from $M_G$. A weak attempt at this is given in Proposition 4.4, where given an isomorphism $f$ between two models $M_G$ and $M_{G'}$, there is a number $\ell$ (depending largely on $\mathrm{wt}(f(a)/a)$) so that the image of a complete graph of size $m > \ell$ is almost complete. As the number $\ell$ depends on the isomorphism and cannot be predicted in advance, we obtain our Borel completeness result by first coding an arbitrary tree $\mathcal{T}$ into a graph $G^*_{\mathcal{T}}$ in which each node $\eta \in \mathcal{T}$ corresponds to a sequence of finite, complete subgraphs of arbitrarily large size. Then, by composing this map with the coding of graphs into models described above, we obtain a $\lambda$-Borel embedding of subtrees of $\lambda^{<\omega}$ into models of our theory. It is noteworthy that had we been able to add finitely many constant symbols to the language, the proof of Borel completeness in the expanded language would have been much easier.

Once Theorem 4.12 has been established, for the remainder of the paper we assume that $T$ is $\aleph_0$-stable with eni-NDOP. In Section 5 we introduce and relate several notions of decompositions of a given model $M$. In Definition 5.4, decompositions are named [regular, eni, eni-active] according to the the species of $\mathrm{tp}(a_\nu/M_{\nu^-})$. With Theorems 5.7, 5.8, and 5.11 we measure the extent to which one can recover a model $M$ from a decomposition of it. Some of these results appear or are implicit in [11] and [3], but are included here to contrast the pros and cons of each species of decomposition.

In Section 6 we define an $\aleph_0$-stable theory $T$ to be *eni-deep* if it has eni-NDOP and some model $M$ has an eni-active decomposition with an infinite branch. With Theorem 6.9, we prove that any $\aleph_0$-stable, eni-deep theory is Borel complete. The proof uses a major result from [10] as a black box.

Finally, in Section 7, we collect our results into Theorem 7.2 that characterizes those $\aleph_0$-stable theories that have maximally large families of $L_{\infty,\aleph_0}$-inequivalent models of every cardinality.

We are grateful to the anonymous referee for mentioning that the class of eni types need not be closed under non-orthogonality and for insisting that the relationship between eni-active types and chains be described more precisely.

*For the whole of this paper, all theories are $\aleph_0$-stable.*

**1.1. Preliminary facts about $\aleph_0$-stable theories.** We begin by enumerating several well-known facts about models of $\aleph_0$-stable theories.

Definition 1.1. A non-algebraic, stationary type $p$ is *regular* if $p$ is orthogonal to every forking extension of itself. Further, $p$ is *strongly regular via* $\varphi$ if $\varphi \in p$ and for every strong type $q$ containing $\varphi$, $q$ is either orthogonal or parallel to $p$.

It is well known that the binary relation of non-orthogonality is an equivalence relation on the class of stationary, regular types.

Fact 1.2.

(1) *Over any set $A$, prime and atomic (indeed, constructible) models exist and are unique up to isomorphisms over $A$.*
(2) *If $M$ is a model and $p \not\perp M$, then there is a strongly regular $q \in S(M)$ non-orthogonal to $p$.*
(3) *Strongly regular types over models are* RK-minimal, *i.e., if $M \preceq N$, $q \in S(M)$ is strongly regular, and there is some $a \in N \setminus M$ such that* $\mathrm{tp}(a/M) \not\perp q$, *then $q$ is realized in $N$.*
(4) *Any pair $M \preceq N$ of models admits a* strongly regular resolution, *i.e., a continuous, elementary chain $\langle M_i : i \leq \alpha\rangle$ of elementary substructures of $N$ such that $M_0 = M$, $M_\alpha = N$, and $M_{i+1}$ is prime over $M_i \cup \{a_i\}$, where* $\mathrm{tp}(a_i/M_i)$ *is strongly regular.*
(5) *For any complete type $p \in S(M)$ over a model, there is a finite subset $A \subseteq M$ over which $p$ is based and stationary.*
(6) *A model is a-saturated (i.e., $\mathbf{F}^a_{\kappa(T)}$-saturated in the notation of* [6]) *if and only if it is $\aleph_0$-saturated.*

By combining Fact 1.2(2) and (3), we obtain the very useful '3-model lemma'.

Lemma 1.3. *Suppose $N_0 \preceq N_1 \preceq M$, $p \in S(N_1)$ is realized in $M$, and is non-orthogonal to $N_0$. Then there is a strongly regular $q \in S(N_0)$ non-orthogonal to $p$ that is realized in $M$ by an element $e$ satisfying $e \mathop{\smile\!\!\!\!\!\!|}_{N_0} N_1$.*

*Proof.* By Fact 1.2(2), choose a strongly regular $q \in S(N_0)$ non-orthogonal to $p$. Let $q'$ be the non-forking extension of $q$ to $S(N_1)$. As $p$ is realized in $M$, it follows from Fact 1.2(3) that $q'$ is realized in $M$ as well. But any $e$ realizing $q'$ satisfies $e \underset{N_0}{\perp} N_1$. ∎

The following notion is implicit in several proofs of atomicity in [11].

DEFINITION 1.4. A set $A$ is *essentially finite with respect to a strong type* $p$ if, for all finite sets $D$ on which $p$ is based and stationary, there is a finite $A_0 \subseteq A$ such that $p|DA_0 \vdash p|DA$.

LEMMA 1.5. *Fix a strong type $p$. If either of the following conditions holds:*

(1) *$p \perp A$ and $B$ is a (possibly empty) $A$-independent set of finite sets; or*

(2) *if $A$ is essentially finite with respect to $p$, $p \perp B$, and $A \underset{A\cap B}{\perp} B$,*

*then $A \cup B$ is essentially finite with respect to $p$.*

*Proof.* To establish (1), suppose $B = \{b_i : i \in I\}$ is $A$-independent. Choose any finite $D$ over which $p$ is based and stationary. Now, choose a finite $B_0 \subseteq B$ such that $D \underset{AB_0}{\perp} B$ and then choose a finite $A_0 \subseteq A$ such that $DB_0 \underset{A_0}{\perp} A$. We claim that $p|DA_0B_0 \vdash p|DAB$.

To see this, first note that since $p \perp A$, we have $p \perp A_0$, which coupled with $DA_0B_0 \underset{A_0}{\perp} A$ implies $p|DA_0B_0 \vdash p|DAB_0$. Since $DB_0 \underset{A}{\perp} (B \setminus B_0)$ we obtain $p|DAB_0 \vdash p|DAB$, proving (1).

To prove (2), write $E := A \cap B$. Choose a finite $D$ on which $p$ is based and stationary. Choose $B_0 \subseteq B$ finite such that $D \underset{B_0A}{\perp} B$. As $A \underset{E}{\perp} B$ we have $B \underset{EB_0}{\perp} DB_0A$ and $EB_0 \subseteq DB_0A \cap B$. Choose a finite $A_0 \subseteq A$ such that $DB_0 \underset{A_0}{\perp} A$. Finally, as $A$ is essentially finite with respect to $p$, choose $A_1 \subseteq A$ finite such that $A_0 \subseteq A_1$ and $p|DB_0A_1 \vdash p|DB_0A$. Put $D^* := DB_0A_1$. As $D^*A = DB_0A$, we have $B \underset{EB_0}{\perp} D^*A$ and $EB_0 \subseteq D^*A$.

To see that $p|D^* \vdash p|DAB$, first note that from the above, $p|D^* \vdash p|D^*A$. Also, since $p \perp B$, $p \perp EB_0$ and since $D^*A \underset{EB_0}{\perp} B$ we conclude that $p|D^*A \vdash p|DAB$. ∎

Next, we give a criterion for $\lambda$-saturation of a model of an $\aleph_0$-stable theory. For the moment, call a non-algebraic type $p \in S(M)$ *$\lambda$-full* if $\dim(p|A, M) \geq \lambda$ for some (every) finite set $A \subseteq M$ on which $p$ is based and stationary.

LEMMA 1.6. *For $\lambda$ any infinite cardinal, a model $M \models T$ is $\lambda$-saturated if and only if every strongly regular $p \in S(M)$ is $\lambda$-full.*

*Proof.* Left to right is clear, so fix an infinite cardinal $\lambda$ and a model $M$ in which every strongly regular type is $\lambda$-full. If $M$ is not $\lambda$-saturated, then there is a subset $A \subseteq M$, $|A| < \lambda$, and a type $q \in S(A)$ that is omitted in $M$. Among all possible choices, choose $q$ of least Morley rank. Let $q' \in S(M)$ denote the unique non-forking extension of $q$ to $M$, let $a$ be any realization of $q'$, and let $N = M[a]$ be any prime model over $M \cup \{a\}$. By Fact 1.2(2) there is an element $b \in N \setminus M$ such that $p = \mathrm{tp}(b/M)$ is strongly regular. Choose $B \subseteq M$, $|B| < \lambda$, such that $A \subseteq B$, $p$ is based and stationary over $B$, and $\mathrm{tp}(a/Bb)$ forks over $B$. Since $p$ is $\lambda$-full, there is $b^* \in M$ realizing $p|B$. Choose any $a^* \in \mathfrak{C}$ realizing $q|B$ with $\mathrm{tp}(a^*/Bb^*)$ forking over $B$. Now $a^* \notin M$, lest $q$ be realized in $M$. Thus, $r = \mathrm{tp}(a^*/M)$ is non-algebraic, yet $MR(r) < MR(q)$, hence $r|C$ is realized in $M$ for any $C \supseteq Bb^*$ on which $r$ is based and stationary and $|C| < \lambda$. However, any realization of $r|C$ is a realization of $q$, contradicting $q$ being omitted in $M$. ∎

The following corollary is immediate.

Corollary 1.7. *A countable model $M$ is saturated if and only if every strongly regular $q \in S(M)$ has infinite dimension.*

Given two sets $A, B$, we say that $A$ *has the Tarski–Vaught property in* $B$, written $A \subseteq_{\mathrm{TV}} B$, if $A \subseteq B$ and every $L(A)$-formula $\varphi(x, a)$ that is realized in $B$ is also realized in $A$.

Lemma 1.8.

(1) *If $B \subseteq_{\mathrm{TV}} B'$, then for every $a$, if* $\mathrm{tp}(a/B)$ *is isolated by the $L(B)$-formula $\varphi(x, b)$, then* $\mathrm{tp}(a/B')$ *is also isolated by $\varphi(x, b)$.*
(2) *Suppose that $B$ and $C$ are sets with $B$ containing a model $M$ and $B \underset{M}{\perp} C$. Then $B \subseteq_{\mathrm{TV}} BC$. Furthermore, if $A$ is atomic over $B$, then $AB \underset{M}{\perp} C$ and $A$ is atomic over $BC$ via the same $L(B)$-formulas.*
(3) *Suppose that $\langle A_i : i < \alpha\rangle$ and $\langle B_i : i < \alpha\rangle$ are both continuous, increasing subsets of a model such that each $A_i$ contains and is atomic over $B_i$, and $B_i \subseteq_{\mathrm{TV}} B_j$ whenever $i < j < \alpha$. Then:*
  (a) *$B_i \subseteq_{\mathrm{TV}} \bigcup B_i$;*
  (b) *$\bigcup A_i$ is atomic over $\bigcup B_i$; and*
  (c) *if, in addition, each $A_i$ was maximal atomic over $B_i$ inside $N$, then $A_i \preceq \bigcup A_i \preceq N$ for each $i$, and $\bigcup A_i$ is maximal atomic over $\bigcup B_i$.*

*Proof.* (1) is [6, XII, Lemma 1.12(3)], but we prove it here for convenience. Let $\psi(x, b_1, b')$ be any formula over $B'$ with $b_1$ from $B$ and $b'$ from $B'$. Let

$$\theta(y, z, w) := \forall x \forall x' \big([\varphi(x, y) \wedge \varphi(x', y)] \to (\psi(x, z, w) \leftrightarrow \psi(x, z, w))\big).$$

It suffices to show that $\theta(b, b_1, b')$ holds. However, if it failed, then since $b, b_1$ are from $B$ and $B \subseteq_{\mathrm{TV}} B'$, we would have $\neg\theta(b, b_1, b_2)$ for some $b_2$ from $B$. But this contradicts $\varphi(x, b)$ isolating $\mathrm{tp}(a/B)$.

(2) That $B \subseteq_{\mathrm{TV}} BC$ follows from the finite satisfiability of non-forking over models. That $AB \underset{M}{\perp} C$ is a restatement of isolated types being dominated over models, and the atomicity of $A$ over $BC$ follows from (1).

(3) Let $A^* := \bigcup_{i<\alpha} A_i$ and $B^* := \bigcup_{i<\alpha} B_i$. The preservation of the TV-property under continuous chains of sets is identical to the preservation of elementarity under continuous chains of models, so $B_i \subseteq_{\mathrm{TV}} B^*$ for each $i$. To see that $A^*$ is atomic over $B^*$, choose a finite subset $D \subseteq A^*$ and choose $i < \alpha$ such that $D \subseteq A_i$. If $\varphi(\overline{x}, b_i)$ isolates $\mathrm{tp}(D/B_i)$, then by iterating (1), the same formula isolates $\mathrm{tp}(D/B_j)$ for every $i < j < \alpha$, and hence also isolates $\mathrm{tp}(D/B^*)$.

To obtain (c), suppose that each $A_i$ is maximal atomic inside $N$ over $B_i$. As there is a prime model $N_i \preceq N$ containing each $A_i$, the maximality of $A_i$ implies that $A_i \preceq N$, so $A^* \preceq N$ as well. To demonstrate that $A^*$ is maximal, choose any $c \in N$ such that $A^*c$ is atomic over $B^*$. Choose $i < \alpha$ such that both $\mathrm{tp}(c/A^*)$ does not fork and is stationary over $A_i$ and $\mathrm{tp}(c/B_i)$ is isolated. We will show that $cA_i$ is atomic over $B_i$, which implies $c \in A_i$ by the maximality of $A_i$. To show this atomicity, first note that since $A_i$ is atomic over $B_i$ and $B_i \subseteq_{\mathrm{TV}} B^*$, it follows from (1) that $\mathrm{tp}(A_i/B_i) \vdash \mathrm{tp}(A_i/B^*)$, hence $A_i \underset{B_i}{\perp} B^*$. The transitivity of non-forking implies that $cA_i \underset{B_i}{\perp} B^*$. Since $cA_i$ is atomic over $B^*$, it follows from the open mapping theorem that $cA_i$ is atomic over $B_i$. ∎

Here is an example of a pair of sets with the Tarski–Vaught property. The latter is proved in [6, XII, Lemma 2.3(3)].

FACT 1.9. *Suppose that $M_0, M_1, M_2$ are models with $M_1 \underset{M_0}{\perp} M_2$, $N_0$ is a-saturated and independent of $M_1M_2$ over $M_0$, $N_1$ is a-prime over $N_0M_1$, and $N_2$ is a-prime over $N_0M_2$. Then $M_1M_2 \subseteq_{\mathrm{TV}} N_1N_2$.*

**1.2. Species of stationary regular types.** We begin this section by recalling a definition from [10].

DEFINITION 1.10. A stationary, regular type $q$ *lies directly above* $p$ if there is a non-forking extension $p' \in S(N)$ of $p$ with $N$ $\aleph_0$-saturated, a realization $c$ of $p'$, and an $\aleph_0$-prime model $N[c]$ over $N \cup \{c\}$ such that $q \not\perp N[c]$, but $q \perp N$. A regular type $q$ *lies above* $p$ if there is a sequence $p_0, \ldots, p_n$ of types such that $p_0 = p$, $p_n = q$, and $p_{i+1}$ lies directly above $p_i$ for each $i < n$. We say that $p$ *supports* $q$ if $q$ lies above $p$.

The following lemma gives a sufficient condition for supporting that does not mention $\aleph_0$-saturation.

LEMMA 1.11. *Suppose $p \in S(M)$ is regular, $a$ is any realization of $p$, and $M(a)$ is prime over $M \cup \{a\}$. If a stationary, regular $q$ satisfies $q \not\perp M(a)$, but $q \perp M$, then $q$ lies directly above $p$, hence $p$ supports $q$.*

*Proof.* Using Fact 1.2(2) and because 'lying directly above $p$' is closed under non-orthogonality, we may assume that $q \in S(M(a))$. Fix a finite $A \subseteq M(a)$ over which $q$ is based and stationary. As $M(a)$ is atomic over $M \cup \{a\}$, $\mathrm{tp}(A/Ma)$ is isolated. Choose any $\aleph_0$-saturated model $N \succeq M$ with $N \mathop{\smile\!\!\!\!\mid}_M a$. It follows by finite satisfiability that $Ma \subseteq_{\mathrm{TV}} Na$, so by Lemma 1.8(1), $\mathrm{tp}(A/Na)$ is isolated as well (in fact, by the same formula isolating $\mathrm{tp}(A/Ma)$). As $\mathrm{tp}(A/Na)$ is $\aleph_0$-isolated, we can choose an $\aleph_0$-prime model $N[a]$ over $N \cup \{a\}$ that contains $A$. Now, $p = \mathrm{tp}(a/N)$ is a non-forking extension of $p$ and $q \not\perp N[a]$. As $A$ is dominated by $a$ over $M$, $a \mathop{\smile\!\!\!\!\mid}_M N$ and $q \perp M$ we conclude that $q \perp N$. Thus, $q$ lies directly above $p$, so $p$ supports $q$ by definition. ■

DEFINITION 1.12. A stationary, regular type $p$ is *eni* (eventually non-isolated) if there is a finite set $A$ on which $p$ is based and stationary, but $p|A$ is non-isolated. Such a $p$ is ENI if it is both eni and strongly regular.

DEFINITION 1.13. The *ENI-active* types are the smallest class of stationary, regular types that contain the ENI types and are closed under automorphisms of the monster model, non-orthogonality, and supporting. Similarly, the *eni-active* types are the smallest class of stationary, regular types that are closed under automorphisms, non-orthogonality, and supporting.

With Proposition 1.19 we will see that every eni type is ENI-active, hence the classes of ENI-active and eni-active types coincide. One should note that whereas the class of eni types need not be closed under non-orthogonality, the class of eni-active types is.

DEFINITION 1.14. A stationary regular type $p$ is *dull* if it is not ENI-active.

Again, it follows from Proposition 1.19 below that a stationary regular type is dull if and only if it is not eni-active. Thus, in the notation of [10, Definition 3.7], if we take $\mathbf{P}$ to be *either* the class of ENI types *or* the closure of the class of eni types under non-orthogonality, then $\mathbf{P}^{\mathrm{active}}$ denotes the class of ENI-active types and $\mathbf{P}^{\mathrm{dull}}$ denotes the dull types. The remainder of this subsection is aimed at proving Proposition 1.19.

LEMMA 1.15. *Suppose that a model $M$ is prime over a finite set $A$, $c$ is a realization of a regular type $p \in S(M)$, and $M(c)$ is any prime model over $M \cup \{c\}$. If $p$ has infinite dimension in $M$, then $M(c)$ is also prime over $A$. In particular, $M$ and $M(c)$ are isomorphic over $A$.*

*Proof.* First, by increasing $A$ as necessary (while still keeping it finite) we may assume that $p$ is based and stationary on $A$. To prove the lemma, first note that it suffices to find a pair of models $N \preceq N' \preceq M$ such that $A \subseteq N$ and $N'$ is isomorphic over $A$ to any prime model $N(c)$ over $N \cup \{c\}$. Indeed, if we have such $N$ and $N'$, then as they are both countable and atomic over $A$, both are isomorphic to $M$ over $A$. Thus, $N(c)$ is isomorphic to both $M$ and $M(c)$ over $A$ and the lemma follows.

To produce the submodels $N$ and $N'$, first choose an infinite, $A$-independent set $J \subseteq M$ of realizations of $p|A$. Choose a partition $J = J_0 \cup J_1$ into disjoint, infinite sets. Next, choose $B \subseteq M$ to be maximal subject to the conditions that $AJ_0 \subseteq B$ and $B \underset{A}{\perp} J_1$. Let $N \preceq M$ be prime over $B$.

Claim. $N = B$, *hence* $N \underset{A}{\perp} J_1$.

*Proof.* Choose any $e \in N$. As $N$ is atomic over $B$, choose a finite set $C$, $A \subseteq C \subseteq B$, such that $\mathrm{tp}(e/C) \vdash \mathrm{tp}(e/B)$. As $J_0 \subseteq B$, it follows that $\mathrm{tp}(e/C) \vdash \mathrm{tp}(e/BJ_1)$. [Why? For $\bar{a}_1$ any tuple from $J_1$, a formula $\varphi(x, c, b, \bar{a}_1)$ belongs to $\mathrm{tp}(e/BJ_1)$ if and only if there is a cofinite $J_0' \subseteq J_0$ such that $\varphi(x, c, b, \bar{a}_0) \in \mathrm{tp}(e/B)$ for some $\bar{a}_0$ from $J_0'$.] In particular, $e \underset{B}{\perp} J_1$, which by transitivity implies $Be \underset{A}{\perp} J_1$. Thus, the maximality of $B$ implies that $e \in B$, proving the claim. ∎

Now choose $a \in J_1$ arbitrarily and choose any $N' \preceq M$ to be prime over $N \cup \{a\}$. As $\mathrm{tp}(a/A) = \mathrm{tp}(c/A)$ is stationary and both $a$ and $c$ are independent of $N$ over $A$, it follows that $\mathrm{tp}(a/N) = \mathrm{tp}(c/N)$. In particular, if we choose $N(c)$ to be any prime model over $N \cup \{c\}$, it will be isomorphic to $N'$ over $N$. Thus, $N$ and $N'$ are as desired, completing the proof of the lemma. ∎

Definition 1.16. A pair of models $M \preceq N$ is a *dull pair* if, for every $d \in N \setminus M$, $\mathrm{tp}(d/M)$ is dull whenever it is regular.

Lemma 1.17. *Suppose $M \preceq N$ is a dull pair, $c \in N \setminus M$ has $\mathrm{tp}(c/M)$ strongly regular, and $M(c) \preceq N$ is prime over $M \cup \{c\}$. Then $M(c) \preceq N$ is a dull pair.*

*Proof.* Choose any $d \in N \setminus M(c)$ such that $p := \mathrm{tp}(d/M(c))$ is regular. There are two cases. First, if $p \not\perp M$, then by Lemma 1.3 there is $e \in N \setminus M$ such that $q := \mathrm{tp}(e/M)$ is strongly regular and non-orthogonal to $p$. As $M \preceq N$ is a dull pair, $q$ and hence $p$ must be dull. On the other hand, suppose that $p \perp M$. If $p$ were not dull, it would be ENI-active, which by Lemma 1.11 would imply that $\mathrm{tp}(c/M)$ is ENI-active as well, again contradicting $M \preceq N$ being a dull pair. ∎

Lemma 1.18. *Suppose that $M \preceq N$ is a dull pair. Then for any $M'$ satisfying $M \preceq M' \preceq N$, both $M \preceq M'$ and $M' \preceq N$ are dull pairs.*

*Proof.* That $M \preceq M'$ is a dull pair is immediate. For the other pair, we argue by induction on $\alpha$ that

> For any $M'$ satisfying $M \preceq M' \preceq N$, if there is a strongly regular resolution $M = M_0 \preceq M_1 \preceq \cdots \preceq M_\alpha = M'$ then $M' \preceq N$ is a dull pair.

This would suffice by Fact 1.2(4), which asserts the existence of a strongly regular resolution of any $M'$. When $\alpha = 0$ there is nothing to prove. If $\alpha$ is a non-zero limit ordinal, then for any $d \in N \setminus M' = M_\alpha$ such that $p := \operatorname{tp}(d/M_\alpha)$ is regular, choose $\beta < \alpha$ such that $q := \operatorname{tp}(d/M_\beta)$ is parallel to $p$. By induction we see that $q$ is dull, hence $p$ is dull as well.

Finally, assume the inductive hypothesis holds for $\beta$ and suppose $M'$ has a strongly regular resolution of length $\alpha = \beta+1$. By the inductive hypothesis, $M_\beta \preceq N$ is a dull pair, $\operatorname{tp}(c_\beta/M_\beta)$ is strongly regular, and $M' = M_\alpha$ is prime over $M_\beta \cup \{c_\beta\}$. Thus, Lemma 1.17 implies that $M' \preceq N$ is a dull pair, and our induction is complete. ∎

PROPOSITION 1.19.

(1) *Every eni type is ENI-active.*
(2) *A type is eni-active if and only if it is ENI-active.*
(3) *A stationary, regular type is dull if and only if it is not eni-active.*

*Proof.* Once we have proved (1), clauses (2) and (3) follow immediately from the definitions. Fix an eni type $p$. Choose a finite set $A$ on which $p$ is based and stationary, yet $p|A$ is not isolated. Let $M$ be prime over $A$. As $M$ is atomic over $A$, it follows that $M$ omits $p|A$. Let $e$ be any realization of $p|M$ and let $M(e)$ be prime over $M \cup \{e\}$.

Assume that $p$ is not ENI-active, i.e., $p$ is dull. We will show that $M(e)$ is also prime over $A$, which is a contradiction since $M(e)$ visibly realizes $p|A$. To obtain this result, we begin with the following claim.

CLAIM. *There is a strongly regular resolution $M = M_0 \preceq \cdots \preceq M_n = M(e)$ of finite length $n$.*

*Proof.* First choose any maximal sequence $M = M'_0 \preceq M'_1 \preceq M'_n \preceq M(e)$ satisfying the conditions: (i) $M_{i+1}$ is prime over $M_i \cup \{d_i\}$ where $\operatorname{tp}(d_i/M'_i)$ is strongly regular, and (ii) $\operatorname{tp}(e/M'_{i+1})$ forks over $M'_i$. As the sequence of ordinals $\langle RM(e/M'_i) : i \leq n\rangle$ is strictly decreasing, such a sequence can have at most finite length. Also, for any such sequence, we must have $e \in M'_n$, because if not, then by Fact 1.2(2,3), there would be some strongly regular type $q \in S(M'_n)$ realized in $M(e) \setminus M'_n$ with $q \not\perp \operatorname{tp}(e/M'_n)$. However, if $d_n$ were any realization of $q$ in $M(e)$ and $M'_{n+1} \preceq M(e)$ were any prime model over $M'_n \cup \{d_n\}$, we would have $e$ forking with $M'_{n+1}$ over $M'_n$, which would contradict the maximality of the sequence.

Thus, any maximal sequence has $e \in M'_n$. It follows that $M'_n$ is prime over $M \cup \{e\}$, hence there is an isomorphism $f : M'_n \to M(e)$ fixing $M \cup \{e\}$ pointwise. Then the sequence $\langle f(M'_i) : i \leq n\rangle$ is a strongly regular resolution of $M(e)$, completing the proof of the claim. ∎

Fix such a strongly regular resolution $M = M_0 \preceq M_1 \preceq M_n = M(e)$ where $M_{i+1}$ is prime over $M_i \cup \{c_i\}$ and $\mathrm{tp}(c_i/M_i)$ is strongly regular. Next, note that $e$ forks over $M$ with any $d \in M(e) \setminus M$, so if $\mathrm{tp}(d/M)$ is regular it must be non-orthogonal to $p$ and hence dull. That is, $M \preceq M(e)$ is a dull pair. It follows from Lemma 1.18 that $M_i \preceq M(e)$ is also a dull pair whenever $i < n$.

Using this, we complete the proof by showing, by induction on $i \leq n$, that each $M_i$ is prime over $A$. When $i = 0$ this is immediate by hypothesis. So fix $i < n$ and assume that $M_i$ is prime over $A$. Let $q_i := \mathrm{tp}(c_i/M_i)$. Choose a finite set $B$, $A \subseteq B \subseteq M_i$, on which $q_i$ is based and stationary. Note that $M_i$ is prime over $B$ as well. As $M_i \preceq M(e)$ is a dull pair, $q_i$ is strongly regular and dull. In particular, $q_i$ is not ENI, hence $q_i$ has infinite dimension in $M_i$. Thus, $M_{i+1}$ is prime over $B$ by Lemma 1.15. However, as $\mathrm{tp}(B/A)$ is isolated, it follows that $M_{i+1}$ is prime over $A$ as well. ∎

**1.3. On dull types.** We begin by defining a strong notion of substructure.

Definition 1.20. A model $N$ is an $L_{\infty,\aleph_0}$*-substructure of* $M$ if $N \preceq M$ and for all finite $A \subseteq N$,

$$(N, a)_{a\in A} \equiv_{\infty,\aleph_0} (M, a)_{a\in A}.$$

The paradigm of this notion is when $M$ is atomic and $N \preceq M$. In this case, both $N$ and $M$ are atomic, hence back-and-forth equivalent, over every finite $A \subseteq N$. In Proposition 1.24, we will prove that this stronger notion of substructure holds for every dull pair $N \preceq M$. We begin with a lemma which gets its strength when coupled with Lemma 1.5.

Lemma 1.21. *Suppose $A \subseteq C$ is essentially finite with respect to a regular, stationary, but not eni type $p \in S(C)$. If $C$ is atomic over $A$, then so is $C \cup \{e\}$ for any realization $e$ of $p$.*

*Proof.* It suffices to show that $De$ is atomic over $A$ for every finite $D \subseteq C$. So fix any finite $D \subseteq C$. Choose a finite $D^*$, $D \subseteq D^* \subseteq C$, with $p$ based and stationary on $D^*$. As $A$ is essentially finite with respect to $p$, choose a finite $A_0 \subseteq A$ such that $p|D^*A_0 \vdash p|D^*A$. Since $p$ is not eni and $D^*A_0$ is finite, $\mathrm{tp}(e/D^*A_0)$ is isolated. Coupled with the fact that $D^*$ is atomic over $A$, this implies that $D^*e$ and hence $De$ is atomic over $A$, as required. ∎

Lemma 1.22. *Suppose that $N \preceq N(c)$, where $N(c)$ is prime over $N \cup \{c\}$ and $c$ realizes a dull type $p \in S(N)$. Then for every finite set $A$, there is $M \preceq N$ over which $p$ is based and an infinite Morley sequence $J \subseteq N$ in $p|M$ such that*

- *$A \subseteq M$;*
- *$N$ is atomic over $M \cup J$; and*
- *$N(c)$ is atomic over $M \cup J \cup \{c\}$.*

*Proof.* Without loss of generality, we may assume that $p$ is based and stationary on $A$. As $p$ is not eni, $p$ has infinite dimension in $N$, so we can find an infinite Morley sequence $J^* \subseteq N$ in $p|A$. Partition $J^*$ into two disjoint, infinite pieces $J^* = J_0 \cup J$. Arguing as in the proof of Lemma 1.15, choose $B \subseteq N$ maximal subject to the conditions (i) $AJ_0 \subseteq B$, and (ii) $B \underset{A}{\perp} J$. Just as in 1.15, $B$ is the universe of an elementary substructure, which we denote as $M$. Clearly, $A \subseteq M$.

Claim 1. *$M \preceq N$ is a dull pair.*

*Proof.* Choose any $e \in N$ such that $q = \mathrm{tp}(e/M)$ is regular. Any such $q$ must be non-orthogonal to $p$ and hence be dull: if this were not the case, then we would have $e \underset{M}{\perp} J$, which would contradict the maximality of $M$. ■

Claim 2. *$N$ is atomic over $M \cup J$.*

*Proof.* Choose $N_0 \preceq N$ to be maximal atomic over $M \cup J$. We argue that $N_0 = N$. If this were not the case, choose $e \in N$ such that $q := \mathrm{tp}(e/N_0)$ were regular. As $M \preceq N$ is a dull pair, it follows from Lemma 1.18 that $q$ is dull and hence not eni by Proposition 1.19. We argue by cases. First, if $q$ were non-orthogonal to $M$, then by Lemma 1.3 there would be $d \in N \setminus M$ such that $d \underset{M}{\perp} N_0$ which, since $J \subseteq N_0$, would contradict the maximality of $M$. On the other hand, if $q \perp M$, then by Lemmas 1.5(1) and 1.21 we would have $N_0 \cup \{e\}$ atomic over $M \cup J$, which contradicts the maximality of $N_0$. ■

Claim 3. *$M$ is maximal in $N(c)$ such that $M \underset{A}{\perp} Jc$.*

*Proof.* First, it is clear that $M \underset{A}{\perp} Jc$ by the defining property of $M$ and because $c \underset{A}{\perp} N$. The verification of the maximality of $M$ inside $N(c)$ is an exercise in non-forking. Namely, choose any $e \in N(c)$ such that $eM \underset{A}{\perp} Jc$. As $J \cup \{c\}$ is independent over $A$, we have $eMc \underset{A}{\perp} J$, hence $ec \underset{M}{\perp} J$. As $N$ is atomic over $M \cup J$ by Claim 2, we obtain $ec \underset{M}{\perp} N$. Combining this with the fact that $e \underset{M}{\perp} c$ yields $e \underset{M}{\perp} Nc$, hence $e \underset{M}{\perp} N(c)$. Thus, $e \in M$ as required. ■

We finish by using analogues of the proofs of Claims 1 and 2 (using $Jc$ in place of $J$) to show that $M \preceq N(c)$ is a dull pair and that $N(c)$ is atomic over $MJc$. ∎

LEMMA 1.23. *Suppose that $N \models T$, $\mathrm{tp}(c/N)$ is dull, and $N(c)$ is any prime model over $N \cup \{c\}$. Then $N$ is an $L_{\infty,\aleph_0}$-elementary substructure of $N(c)$.*

*Proof.* Given $N \preceq N(c)$ and a finite $A$, by enlarging $A$ slightly we may assume that $p = \mathrm{tp}(c/N)$ is based and stationary on $A$. Apply Lemma 1.22 to obtain $M \preceq N$ and $J$ such that $A \subseteq M$, $N$ is atomic over $MJ$, and $N(c)$ is atomic over $MJc$.

Let $g : MJ \to MJc$ be any elementary bijection that is the identity on $M$. Now, we show that $(N, a)_{a \in M} \equiv_{\infty,\aleph_0} (N(c), a)_{a\in M}$ by exhibiting the back-and-forth system

$$\mathcal{F} = \{\text{all finite partial functions } f : N \to N(c) \text{ such that } f \cup g \text{ is elementary}\}.$$

The verification that $\mathcal{F}$ is a back-and-forth system is akin to the verification that any two atomic models of a complete theory are back-and-forth equivalent. ∎

The following proposition follows by iterating Lemma 1.23:

PROPOSITION 1.24. *Suppose that $N \preceq M$ is a dull pair. Then $N$ is an $L_{\infty,\aleph_0}$-elementary substructure of $M$.*

*Proof.* Choose a strongly regular resolution $N = N_0 \preceq N_1 \preceq \cdots \preceq N_\alpha = M$ As $\mathrm{tp}(c_{i+1}/N_i)$ is dull for each $i < \alpha$, it follows from Lemma 1.23 that $N_i$ is an $L_{\infty,\aleph_0}$-elementary substructure of $N_{i+1}$. ∎

**2. eni-DOP and equivalent notions.** We begin with a central notion of [10] and contrast it with a slight strengthening.

DEFINITION 2.1. A stationary, regular type $p$ has a *DOP witness* if there is a quadruple $(M_0, M_1, M_2, M_3)$ of models, where $(M_0, M_1, M_2)$ form an independent triple of $a$-models, $M_3$ is a-prime over $M_1 \cup M_2$, $p$ is based on $M_3$, but $p \perp M_1$ and $p \perp M_2$. A *prime DOP witness* for $p$ is the same except that we require that $M_3$ be prime over $M_1 \cup M_2$ (as opposed to a-prime).

Visibly, among stationary, regular types, having either a DOP witness or a prime DOP witness is invariant under parallelism and automorphisms of the monster model $\mathfrak{C}$.

Recall that by Fact 1.2(6), an a-model is simply an $\aleph_0$-saturated model. As in [10], we are free to vary the amount of saturation of the models $(M_0, M_1, M_2)$.

LEMMA 2.2. *The following are equivalent for a stationary regular type $p$:*

(1) *$p$ has a prime DOP witness.*
(2) *There is a quadruple $(M_0, M_1, M_2, M_3)$ of models with $(M_0, M_1, M_2)$ forming an independent triple, $M_3$ prime over $M_1 \cup M_2$, $p$ based on $M_3$, but $p \perp M_1$ and $p \perp M_2$.*
(3) *Same as* (2), *but with* $\dim(M_1/M_0)$ *and* $\dim(M_2/M_0)$ *both finite.*
(4) *Same as* (2), *but with* $\dim(M_1/M_0) = \dim(M_2/M_0) = 1$.
(5) *$p$ has a prime DOP witness $(M_0, M_1, M_2, M_3)$ with* $\dim(M_1/M_0) = \dim(M_2/M_0) = 1$.

*Proof.* (1)⇒(2) is immediate.

(2)⇒(3). Let $(M_0, M_1, M_2, M_3)$ be any witness to (2). Choose a finite $d \subseteq M_3$ over which $p$ is based and stationary. As $M_3$ is prime over $M_1 \cup M_2$, choose finite $b \subseteq M_1$ and $c \subseteq M_2$ such that $\mathrm{tp}(d/M_1M_2)$ is isolated by a formula $\varphi(x, b, c)$. Let $M_1' \preceq M_1$ be prime over $M_0b$, let $M_2' \preceq M_2$ be prime over $M_0c$, and let $M_3' \preceq M_3$ be prime over $M_1' \cup M_2'$ with $d \subseteq M_3'$. Then $(M_0, M_1', M_2', M_3')$ are as required in (3).

(3)⇒(4). Assume that (3) holds. Among all possible quadruples of models witnessing (3), choose a triple $(M_0, M_1, M_2, M_3)$ with $\dim(M_1/M_0) + \dim(M_2/M_0)$ as small as possible. Clearly, we cannot have either $M_0 = M_1$ or $M_0 = M_2$, so $\dim(M_1/M_0)$ and $\dim(M_2/M_0)$ are each at least one. We argue that the minimum sum occurs when $\dim(M_1/M_0) = \dim(M_2/M_0) = 1$. Suppose this were not the case. Without loss of generality, assume that $\dim(M_1/M_0) \geq 2$. Choose an element $e \in M_1 \setminus M_0$ such that $\mathrm{tp}(e/M_0)$ is strongly regular and let $M_1' \preceq M_1$ be prime over $M_0 \cup \{e\}$. Let $M_3' \preceq M_3$ be prime over $M_1' \cup M_2$. There are two cases. On the one hand, if $p \not\perp M_3'$, then by e.g. [6, X, Claim 1.4], choose an automorphic copy $p'$ of $p$ that is based on $M_3'$ with $p \not\perp p'$. Then an automorphic copy of the quadruple $(M_0, M_1', M_2, M_3')$ contradicts the minimality of our choice. On the other hand, if $p \perp M_3'$, then the quadruple $(M_1', M_1, M_3', M_3)$ directly contradicts the minimality of our choice.

(4)⇒(5). Let $(M_0, M_1, M_2, M_3)$ be any witness to (4). Choose a finite $d \subseteq M_3$ over which $p$ is based and stationary. Now, choose an a-model $N_0$ satisfying $N_0 \underset{M_0}{\perp} M_3$, and choose a-prime models $N_1$ and $N_2$ over $N_0 \cup M_1$ and $N_0 \cup M_2$, respectively. As $M_3 \underset{M_\ell}{\perp} N_\ell$ for both $\ell = 1, 2$, it follows that $p \perp N_1$ and $p \perp N_2$. Also, as $M_1M_2 \subseteq_{\mathrm{TV}} N_1N_2$ by Fact 1.9, it follows from Lemma 1.8(1) that $\mathrm{tp}(d/N_1N_2)$ is isolated. Choose a prime model $N_3$ over $N_1 \cup N_2$ that contains $d$. Then $(N_0, N_1, N_2, N_3)$ is a prime DOP witness for $p$ with $\dim(N_1/N_0) = \dim(N_2/N_0) = 1$.

(5)⇒(1) is immediate. ■

Definition 2.3. A theory $T$ has *eni-DOP* if some eni type $p$ has a DOP witness. Similarly, $T$ has *ENI-DOP* (respectively, *eni-active DOP*) if some ENI-type (respectively, eni-active type) has a DOP witness.

It is fortunate, at least for the exposition, that $T$ having any of the three preceding notions are equivalent. In fact, this equivalence extends much further. Recall that a stable theory has the *Omitting Types Order Property* (*OTOP*) if there is a type $p(x,y,z)$ (where $x,y,z$ denote finite tuples of variables) such that for any cardinal $\kappa$ there is a model $M^*$ and a sequence $\langle (b_\alpha, c_\alpha) : \alpha < \kappa \rangle$ such that for all $\alpha, \beta < \kappa$,

$$M^* \text{ realizes } p(x, b_\alpha, c_\beta) \quad \text{if and only if} \quad \alpha < \beta.$$

Theorem 2.4. *The following are equivalent for an $\aleph_0$-stable theory $T$:*

1. *$T$ has eni-DOP.*
2. *$T$ has ENI-DOP.*
3. *$T$ has eni-active DOP.*
4. *There is an independent triple $(M_0, M_1, M_2)$ of countable, saturated models such that some (equivalently every) prime model over $M_1 \cup M_2$ is not saturated.*
5. *There is an independent triple $(N_0, N_1, N_2)$ of countable saturated models and strongly regular types $p, q \in S(N_0)$ such that $N_1$ is $\aleph_0$-prime over $N_0$ and a realization $b$ of $p$, $N_2$ is $\aleph_0$-prime over $N_0$ and a realization $c$ of $q$, and if $N_3$ is prime over $N_1N_2$, then there is a finite $d$ satisfying $\{b, c\} \subseteq d \subseteq N_3$ and an ENI type $r(x,d)$ that is omitted in $N_3$ and orthogonal to both $N_1$ and $N_2$.*
6. *$T$ has OTOP.*

*Proof.* If we let $\mathbf{P}$ denote any of eni, ENI, or eni-active, then it follows from Proposition 1.19 that $\mathbf{P}^{\text{active}}$ (which is the closure of $\mathbf{P}$ under automorphisms, non-orthogonality and 'supporting' within the class of stationary, regular types) would be the set of eni-active types. Thus, clauses (1), (2) and (3) are equivalent by way of [10, Corollary 3.9].

(1)$\Rightarrow$(4). Suppose that $(M_0, M_1, M_2, M_3)$ is a DOP witness for an eni type $p$ with each model countable and saturated. Let $N \preceq M_3$ be prime over $M_1 \cup M_2$, and by way of contradiction, assume that $N$ is saturated. Then as $N$ and $M_3$ are isomorphic over $M_1 \cup M_2$, by replacing $p$ by a conjugate type, we may assume that $p \in S(N)$. We will reach a contradiction to the saturation of $N$ by finding a finite subset $D^* \subseteq N$ on which $p$ is based and stationary, but $p|D^*$ is omitted in $N$.

First, since $p$ is eni and $N$ is saturated, choose a finite $D \subseteq N$ on which $p$ is based and stationary, but $p|D$ is not isolated. As $p \perp M_1$ and $p \perp M_2$, it follows from Lemma 1.5 that $M_1M_2$ is essentially finite with respect to $p$. Thus, there is a finite $D^* \subseteq DM_1M_2$ containing $D$ such that $p|D^* \vdash p|DM_1M_2$. As $p|D^*$ is a non-forking extension of $p|D$, it cannot be isolated. We argue that $p|D^*$ cannot be realized in $N$. Suppose $c \in N$ realized $p|D^*$. Then, as $cD^*$ is atomic over $M_1M_2$, we would have $\mathrm{tp}(c/D^*M_1M_2)$ isolated. However, since $\mathrm{tp}(c/D^*) \vdash \mathrm{tp}(c/D^*M_1M_2)$, we have $c \underset{D^*}{\perp} M_1M_2$. Thus, the Open Mapping Theorem would imply that $\mathrm{tp}(c/D^*)$ is isolated, which is a contradiction.

(4)$\Rightarrow$(5). Let $(M_0, M_1, M_2)$ exemplify (4), and fix a prime model $M_3$ over $M_1 \cup M_2$. As $M_3$ is not saturated, by Lemma 1.6 there is an ENI $r \in S(M_3)$ of finite dimension in $M_3$.

Claim. *The type $r$ is orthogonal to both $M_1$ and $M_2$.*

*Proof.* As the cases are symmetric, assume for a contradiction that $r \not\perp M_1$. By Fact 1.2(2) there is a strongly regular $p \in S(M_1)$ non-orthogonal to $r$. Choose a finite $A \subseteq M_3$ such that $r$ is based, stationary and strongly regular over $A$, and $r|A$ is omitted in $M_3$. Choose a finite $B \subseteq M_1$ over which $p$ is based, stationary and strongly regular, and let $r'$ and $p'$ be the unique non-forking extensions of $r|A$ and $p|B$ to $AB$. Since $M_1$ is saturated, $\dim(p|B, M_1)$ is infinite, hence $\dim(p', M_3)$ is infinite as well. Thus, so is $\dim(r', M_3)$, contradicting the fact that $r|A$ is omitted in $M_3$. ∎

Thus, $r$ has a prime DOP witness by Lemma 2.2(2), so Lemma 2.2(5) gives us the configuration we need.

(5)$\Rightarrow$(2). Given the triple $(M_0, M_1, M_2)$ and the type $r$ in (5), choose an a-prime model $M_3$ over $M_1 \cup M_2$. Then $(M_0, M_1, M_2, M_3)$ is a DOP witness for the ENI type $r$.

(5)$\Rightarrow$(6). Given the data from (5), let $w(x, u, y, z)$ be a type asserting that (i) $y$ and $z$ are $M_0$-independent solutions of $p$ and $q$, respectively; (ii) some $\varphi(u, y, z) \in w$ isolates $\mathrm{tp}(d/M_1M_2)$; and (iii) $r(x, u) \subseteq w$. We argue that the type $\exists u\, w(x, u, y, z)$ witnesses OTOP. To see this, fix any cardinal $\kappa$. Choose $\{b_i : i < \kappa\} \cup \{c_j : j < \kappa\}$ to be $M_0$-independent, where $\mathrm{tp}(b_i/M_0) = p$ and $\mathrm{tp}(c_j/M_0) = q$ for all $i, j \in \kappa$. For each $i, j$, let $M_1(b_i)$ be prime over $M_0 \cup \{b_i\}$, and $M_2(c_j)$ be prime over $M_0 \cup \{c_j\}$, and let $\overline{M}$ be prime over the union of these models. Now, for each pair $(i, j)$, choose a witness $d_{i,j}$ to $\varphi(u, b_i, c_j)$ from $\overline{M}$ and let $r_{i,j}$ be shorthand for $r(x, d_{i,j})$. It is easily checked that all of the types $r_{i,j}$ are orthogonal.

For each pair $(i, j)$ with $i \leq j$, choose a realization $e_{i,j}$ of $r_{i,j}$, and let $M^*$ be prime over $\overline{M} \cup \{e_{i,j} : i \leq j < \kappa\}$. Then, because of the orthogonality of the $r_{i,j}$, $M^*$ realizes $\exists u\, w(x, u, b_i, c_j)$ if and only if $i \leq j$.

(6)$\Rightarrow$(1). This is Corollary 5.12 below. (There is no circularity.) ∎

**3. $\lambda$-Borel completeness.** Throughout this section, we *fix a cardinal* $\lambda \geq \aleph_0$. We consider only models of size $\lambda$, typically those whose universe is the ordinal $\lambda$, in a language of size $\kappa \leq \lambda$. *For notational simplicity, we only consider relational languages.* Although it would be of interest to explore this notion in more generality, here we only study classes $\mathbf{K}$ of $L$-structures that are closed under $\equiv_{\infty,\aleph_0}$ and study the complexity of $\mathbf{K}/{\equiv_{\infty,\aleph_0}}$.

Definition 3.1. For any (relational) language $L$ with at most $\lambda$ symbols, let $L^{\pm} := L \cup \{\neg R : R \in L\}$, and let $S^\lambda_L$ denote the set of $L$-structures $M$ with universe $\lambda$. Let

$$L(\lambda) := \{R(\overline{\alpha}) : R \in L^{\pm},\ \overline{\alpha} \in {}^{\operatorname{arity}(R)}\lambda\}$$

and endow $S^\lambda_L$ with the topology formed by letting

$$\mathcal{B} := \{U_{R(\overline{\alpha})} : R(\overline{\alpha}) \in L(\lambda)\}$$

be a subbasis, where $U_{R(\overline{\alpha})} = \{M \in S^\lambda_L : M \models R(\overline{\alpha})\}$.

Definition 3.2. Given a language $L$ of size at most $\lambda$, a set $K \subseteq S^\lambda_L$ is $\lambda$-*Borel* if there is a $\lambda$-Boolean combination $\Psi$ of $L(\lambda)$-sentences (i.e., a propositional $L_{\lambda^+,\aleph_0}$-sentence of $L(\lambda)$) such that

$$K = \{M \in S^\lambda_L : M \models \Psi\}.$$

Given two languages $L_1$ and $L_2$, a function $f : S^\lambda_{L_1} \to S^\lambda_{L_2}$ is $\lambda$-*Borel* if the inverse image of every (basic) open set is $\lambda$-Borel.

That is, $f : S^\lambda_{L_1} \to S^\lambda_{L_2}$ is $\lambda$-Borel if and only if for every $R \in L_2$ and $\overline{\beta} \in {}^{\operatorname{arity}(R)}\lambda$, there is a $\lambda$-Boolean combination $\Psi_{R(\overline{\beta})}$ of $L_1(\lambda)$-sentences such that for every $M \in S^\lambda_{L_1}$, $f(M) \models R(\overline{\beta})$ if and only if $M \models \Psi_{R(\overline{\beta})}$.

As two countable structures are isomorphic if and only if they are $\equiv_{\infty,\aleph_0}$, a moment's thought tells us that when $\lambda = \aleph_0$, the notions of $\aleph_0$-Borel sets and functions defined above are equivalent to the usual notions of Borel sets and functions.

Definition 3.3. Suppose that $L_1, L_2$ are relational languages with at most $\lambda$ symbols, and for $\ell = 1, 2$, $K_\ell$ is a $\lambda$-Borel subset of $S^\lambda_{L_\ell}$ that is invariant under $\equiv_{\infty,\aleph_0}$. We say that $(K_1, \equiv_{\infty,\aleph_0})$ is $\lambda$-*Borel reducible to* $(K_2, \equiv_{\infty,\aleph_0})$, written

$$(K_1, \equiv_{\infty,\aleph_0}) \leq^B_\lambda (K_2, \equiv_{\infty,\aleph_0}),$$

if there is a $\lambda$-Borel function $f : S^\lambda_{L_1} \to S^\lambda_{L_2}$ such that $f(K_1) \subseteq K_2$ and, for all $M, N \in K_1$,

$$M \equiv_{\infty,\aleph_0} N \quad \text{if and only if} \quad f(M) \equiv_{\infty,\aleph_0} f(N).$$

Definition 3.4. A class $K$ is $\lambda$-*Borel complete for* $\equiv_{\infty,\aleph_0}$ if $(K, \equiv_{\infty,\aleph_0})$ is a maximum with respect to $\leq^B_\lambda$. We call a theory $T$ $\lambda$-*Borel complete* *for* $\equiv_{\infty,\aleph_0}$ if $\mathrm{Mod}_\lambda(T)$, the class of models of $T$ with universe $\lambda$, is $\lambda$-Borel complete for $\equiv_{\infty,\aleph_0}$.

To illustrate this notion, we prove a series of lemmas, culminating in a generalization of Friedman and Stanley's [2] result that subtrees of $\omega^{<\omega}$ are Borel complete. We make heavy use of the following characterizations of $\equiv_{\infty,\aleph_0}$-equivalence of structures of size $\lambda$.

FACT 3.5. *If $|L| \leq \lambda$, the following conditions are equivalent for $L$-structures $M$ and $N$ that are both of size $\lambda$:*

1. *$M \equiv_{\infty,\aleph_0} N$.*
2. *$M$ and $N$ satisfy the same $L_{\lambda^+,\aleph_0}$-sentences.*
3. *If $G$ is a generic filter of the Levy collapsing poset* $\mathrm{Lev}(\aleph_0,\lambda)$, *then in $V[G]$ there is an isomorphism $h : M \to N$ of countable structures.*

For all $\aleph_0 \leq \kappa \leq \lambda$, let $L_\kappa$ be the language consisting of the binary relation $\trianglelefteq$ and $\kappa$ unary predicate symbols $P_i(x)$. Let $\kappa\mathrm{CT}_\lambda$ denote the class of all $L_\kappa$-trees with universe $\lambda^{<\omega}$, colored by the predicates $P_i$.

LEMMA 3.6. *For any* (*relational*) *language $L$ satisfying $|L| \leq \kappa \leq \lambda$,*

$$(S_L^\lambda, \equiv_{\infty,\aleph_0}) \leq_\lambda^B (\kappa\mathrm{CT}_\lambda, \equiv_{\infty,\aleph_0}).$$

*Proof.* For each $n \in \omega$, let $\langle \varphi_{n,i}(\overline{x}) : i < \gamma(n) \leq \kappa \rangle$ be a maximal set of pairwise non-equivalent quantifier-free $L$-formulas with $\lg(\overline{x}) = n$. As well, fix a bijection $\Phi : \omega \times \kappa \to \kappa$.

Now, given any $L$-structure $M \in S_L^\lambda$, first note that since the universe of $M$ is $\lambda$, the finite sequences from $M$ naturally form a tree isomorphic to $\lambda^{<\omega}$ under initial segments.

So $f(M)$ will consist of this tree, with $\trianglelefteq$ interpreted as the initial segment relation. Furthermore, for each $j \in \kappa$, choose $(n,i) \in \omega \times \kappa$ such that $\Phi(n,i) = j$. If $i < \gamma(n)$, then put

$$P_j^{f(M)} := \{\overline{\alpha} \in \lambda^n : M \models \varphi_{n,i}(\overline{\alpha})\}$$

(if $i \geq \gamma(n)$, then for definiteness, say that $P_j$ always fails on $f(M)$).

Choose any $M, N \in S_L^\lambda$. It is apparent from the construction that if $M \equiv_{\infty,\aleph_0} N$, then $f(M) \equiv_{\infty,\aleph_0} f(N)$. The other direction is more interesting. Suppose that $f(M) \equiv_{\infty,\aleph_0} f(N)$. Consider the Levy collapsing forcing, $\mathrm{Lev}(\aleph_0,\lambda)$, that, for any generic filter $G$, $V[G]$ includes a bijection $g : \omega \to \lambda$. We work in $V[G]$. Note that both $f(M)$ and $f(N)$ are $\equiv_{\infty,\aleph_0}$-equivalent, countable structures. Thus, in $V[G]$, fix an $L_\kappa$-isomorphism $h : f(M) \to f(N)$. Using $h$, in $\omega$ steps we construct two branches $\eta, \nu \in \lambda^\omega$, where we think of $\eta$ as a branch through $f(M)$, while $\nu$ is a branch through $f(N)$, satisfying the following three conditions:

- for each $n \in \omega$, $h(\eta(n)) = \nu(n)$;
- $\{g(n) : n \in \omega\} \subseteq \mathrm{dom}(\eta)$; and
- $\{g(n) : n \in \omega\} \subseteq \mathrm{dom}(\nu)$.

Let $F = \{(\eta(n), \nu(n)) : n \in \omega\}$. As $\{g(n) : n \in \omega\}$ is all of $\lambda$, it follows that $\mathrm{dom}(F) = \lambda$ and $\mathrm{range}(F) = \lambda$. Furthermore, since $h(\eta(n)) = \nu(n)$, it follows that $P_j(\eta(n)) \leftrightarrow P_j(\nu(n))$ for each $j$. Thus, for each $n$, the $L$-quantifier free types of $\langle \eta(i) : i < n \rangle$ and $\langle \nu(i) : i < n \rangle$ are the same. In particular, it follows that $F$ is a bijection from $\lambda$ to $\lambda$ that preserves $L$-quantifier-free types. Thus, $F : M \to N$ is an isomorphism.

Of course, the isomorphism $F$ is in $V[G]$, but it follows easily by absoluteness that $M \equiv_{\infty,\aleph_0} N$ in $V$. ∎

Definition 3.7. Given any trees $T$ and $\{S_\eta : \eta \in T\}$, we form the tree $T^*(S_\eta : \eta \in T)$ that 'attaches $S_\eta$ to $T$ at $\eta$' as follows: The universe of $T^*(S_\eta : \eta \in T)$ (which, for simplicity, is writen as $T^*$ below) is the disjoint union of

$$T \sqcup \bigsqcup_{\eta \in T} S_\eta \setminus \{\langle\rangle\}$$

and, for $u, v \in T^*$, we say $u \leq_{T^*} v$ if and only if one of the following clauses holds:

- $u, v \in T$ and $u \trianglelefteq_T v$; or
- for some $\eta \in T$, $u, v \in \mathcal{S}_\eta \setminus \{\langle\rangle\}$ and $u \trianglelefteq_{S_\eta} v$; or
- $u \in T$, $v \in S_\eta \setminus \{\langle\rangle\}$ and $u \trianglelefteq_T \eta$.

Note that, in particular, elements from distinct $S_\eta$'s are incomparable, and that no element of any $S_\eta$ is 'below' any element of $T$. It is easily checked that if $T$ and each of the $S_\eta$'s is a subtree of $\lambda^{<\omega}$, then the attaching tree $T^*(S_\eta : \eta \in T)$ can also be construed as being a subtree of $\lambda^{<\omega}$.

Definition 3.8. A *subtree* of $\lambda^{<\omega}$ is simply a non-empty subset of $\lambda^{<\omega}$ that is closed under initial segments. Given a subtree $T$ of $\lambda^{<\omega}$, an element $\eta \in T$ is *contained in a branch* if there is some $\nu \in \lambda^\omega$ extending $\eta$ such that $\nu(n) \in T$ for every $n \in \omega$. A subtree $T$ of $\lambda^{<\omega}$ is *special* if, for every $\eta \in T$ that is contained in a branch, $\eta$ has no immediate successors that are leaves (i.e., every immediate successor of $\eta$ has a successor in $T$).

Lemma 3.9. $(\aleph_0\mathrm{CT}_\lambda, \equiv_{\infty,\aleph_0}) \leq^B_\lambda$ (*special subtrees of* $\lambda^{<\omega}, \equiv_{\infty,\aleph_0}$).

*Proof.* Fix a bijection $\Phi : \omega \times \omega \to \omega \setminus \{0, 1\}$. Let $T_0$ be the tree $\lambda^{<\omega}$.

Also, given any subset $V \subseteq \omega$, let $S_V$ be the rooted tree consisting of one copy of the tree $\omega^{\leq m}$ for each $m \in V$. Other than being joined at the root, the copies of $\omega^{\leq m}$ are disjoint.

Now, suppose we are given $M \in \aleph_0\mathrm{CT}_\lambda$, i.e., the tree $(\lambda^{<\omega}, \trianglelefteq)$, adjoined by countably many unary predicates $P_j(x)$. We construct a special tree $f(M)$

as follows: First, form the tree $T_0 = \lambda^{<\omega}$. For each $\eta \in T_0$, let

$$V(\eta) := \{\Phi(n,j) : M \models P_j(\eta)\}$$

where $n = \lg(\eta)$. Note that each $V(\eta) \subseteq \omega \setminus \{0,1\}$. Let $f(M)$ be the tree $T_0(S_{V(\eta)} : \eta \in T_0)$. By the remark above, as each of $T$, $T_0$ and each $S_V$ is a subtree of $\lambda^{<\omega}$, $f(M)$ is also a subtree of $\lambda^{<\omega}$. Furthermore, note that $T_0$ is recognizable in $f(M)$ as being precisely those elements of $f(M)$ that are contained in an infinite branch. Moreover, for every element $\eta \in f(M)$ that is not contained in an infinite branch, there is a uniform bound on the lengths of $\nu \in f(M)$ extending $\eta$. Combining this with the fact that $1 \notin V(\eta)$ for any $(\eta) \in T_0$, we conclude that $f(M)$ is special.

It is easily verified by the construction that if $M \equiv_{\infty,\aleph_0} N$, then $f(M) \equiv_{\infty,\aleph_0} f(N)$. Conversely, suppose that $M, N \in \aleph_0\,\mathrm{CT}_\lambda$ and that $f(M) \equiv_{\infty,\aleph_0} f(N)$. Choose any generic filter $G$ for the Levy collapse $\mathrm{Lev}(\aleph_0, \lambda)$. Then, in $V[G]$, there is a tree isomorphism $h : f(M) \to f(N)$ as both $f(M)$ and $f(N)$ are countable and back-and-forth equivalent. It suffices to prove that $M$ and $N$ are isomorphic in $V[G]$.

To see this, first note that since 'being part of an infinite branch' is an isomorphism invariant, the restriction of $h$ to $T_0$ is a tree isomorphism between the $T_0$ of $M$ and the $T_0$ of $N$. To finish, we need only show that for every $\eta \in T_0$ and $j \in \omega$, $M \models P_j(\eta)$ if and only if $N \models P_j(h(\eta))$. To see this, let $n = \lg(\eta)$ and $k = \Phi(n,j)$. Then $M \models P_j(\eta)$ if and only if there is an immediate successor $\nu$ of $\eta$ that is not part of an infinite branch, but has an extension $\mu$ of length $n+k$ that is a leaf. As this condition is also preserved by $h$, we conclude that $h|_{T_0}$ preserves each of the $\aleph_0$ colors as well. ∎

Corollary 3.10. *There are $\lambda$ pairwise $\equiv_{\infty,\aleph_0}$-inequivalent special subtrees of $\lambda^{<\omega}$.*

*Proof.* Let $L = \{R\}$ consist of a single, binary relation, and let DG be the class of all directed graphs (i.e., $R$-structures) with universe $\lambda$. It is well known that there are at least $\lambda$ pairwise $\equiv_{\infty,\aleph_0}$-inequivalent directed graphs. But, by composing the maps given in Lemmas 3.6 and 3.9, we get a $\lambda$-Borel embedding of $(\mathrm{DG}, \equiv_{\infty,\aleph_0})$ into (special subtrees of $\lambda^{<\omega}, \equiv_{\infty,\aleph_0}$) preserving $\equiv_{\infty,\aleph_0}$ in both directions. ∎

Theorem 3.11. *For any infinite cardinal $\lambda$, (subtrees of $\lambda^{<\omega}, \equiv_{\infty,\aleph_0}$) is $\lambda$-Borel complete.*

*Proof.* By Lemma 3.6, it suffices to show

$$(\lambda\mathrm{CT}_\lambda, \equiv_{\infty,\aleph_0}) \leq^B_\lambda (\text{subtrees of } \lambda^{<\omega}, \equiv_{\infty,\aleph_0}).$$

From the corollary above, fix a set $\{A_i : i \in \lambda\}$ of pairwise $\equiv_{\infty,\aleph_0}$-inequivalent special subtrees of $\lambda^{<\omega}$.

As notation, let $A_{\langle i\rangle}$ stand for the tree $A_i$, and let $A_{\langle\rangle}$ be the two-element tree $\{\langle\rangle, a\}$ satisfying $\langle\rangle \lhd a$. For each $u \subseteq \lambda$, let $T_u = \{\langle\rangle, a\} \cup \{\langle i\rangle : i \in u\}$ and $S_u = T_u(A_{\langle i\rangle} : i \in u)$. Note that for each $u \subseteq \lambda$, $S_u$ has a unique leaf $a$ attached to $\langle\rangle$, and the trees $S_u$ and $S_v$ are isomorphic if and only if $u = v$.

The proof now follows that of Lemma 3.9, using the trees $S_u$ to code the color of a node.

More formally, let $T_0 := \lambda^{<\omega}$ and fix an enumeration $\langle P_j(x) : j \in \lambda\rangle$ of the unary predicates. Given any $M \in \lambda\,\mathrm{CT}_\lambda$, for each node $\eta \in T_0$, let $V(\eta) := \{j \in \lambda : M \models V_j(\eta)\}$. Let $f(M)$ be the tree $T_0(S_{V(\eta)} : \eta \in T_0)$.

Note that as each of the $A_i$'s was special, $T_0$ is detectable in $f(M)$ as being the set of all nodes $\eta$ that are part of an infinite branch *and* have an immediate successor that is a leaf. The proof now mimicks Lemma 3.9. In particular, given an isomorphism $h : f(M) \to f(N)$ in $V[G]$, the restriction of $h$ to $T_0$ is an isomorphism of $M$ and $N$ as $\kappa\mathrm{CT}_\lambda$-structures. ∎

**4. The Borel completeness of $\aleph_0$-stable, eni-DOP theories.** This section is devoted to the proofs of Theorem 4.12 and Corollary 4.13. As the proof of the former is lengthy, the section is split into four subsections. The first describes two distinct types of eni-DOP witnesses. The second shows how one can encode bipartite graphs into models of $T$. However, Proposition 4.4, which gives a bit of positive information about the shapes of the bipartite graphs $G$ and $H$ whenever the associated models $M_G$ and $M_H$ are isomorphic, is rather weak. Thus, instead of trying to recover arbitrary bipartite graphs, in the third subsection we describe how to encode subtrees $\mathcal{T} \subseteq \lambda^{<\omega}$ into bipartite graphs $G^{[m]}_{\mathcal{T}}$, where the nodes of $\mathcal{T}$ correspond to complete, bipartite subgraphs of $G^{[m]}_{\mathcal{T}}$. Finally, in the fourth subsection we prove Theorem 4.12, with Corollary 4.13 following easily from it.

**4.1. Two types of eni-DOP witnesses.** Suppose that $T$ has eni-DOP. Call a 5-tuple $(M_0, M_1, M_2, M_3, r)$ an *eni-DOP witness* if it satisfies the assumptions of Theorem 2.4(5). A *finite approximation* $\mathcal{F}$ to an eni-DOP witness is a 5-tuple $(a, b, c, d, r_d)$, where $a, b, c, d$ are finite tuples from $(M_0, M_1, M_2, M_3)$, respectively; $\mathrm{tp}(b/a)$ and $\mathrm{tp}(c/a)$ are each stationary, regular types; each of $b, c$ contains $a$ and $\{b, c\}$ are independent over $a$; $r$ is based and stationary on $d$ with $r_d \in S(d)$ parallel to $r$; and $\mathrm{tp}(d/bc) \vdash \mathrm{tp}(d/M_1M_2)$. The last condition, coupled with the fact that $M_0, M_1, M_2$ are each a-models, yields the following *Extendability Condition*:

$$\mathrm{tp}(d/bc) \vdash \mathrm{tp}(d/b^*c^*)$$

for all $a^* \supseteq a$, $b^* \supseteq ba^*$, $c^* \supseteq ca^*$ such that $a^*$ is independent of $bc$ over $a$ and $b^*$ is independent of $c^*$ over $a^*$. As well, $r_d$ is ENI, $r_d \perp b$, and $r_d \perp c$.

For a fixed choice $\mathcal{F} = (a, b, c, d, r_d)$ of a finite approximation, the $\mathcal{F}$-*candidates over* $a$ consist of all 4-tuples $(b', c', d', r_{d'})$ such that $\mathrm{tp}(a, b, c, d) = \mathrm{tp}(a, b', c', d')$. There is a natural equivalence relation $\sim_{\mathcal{F}}$ on the $\mathcal{F}$-candidates over $a$ defined by

$$(b, c, d, r_d) \sim_{\mathcal{F}} (b', c', d', r_{d'}) \quad \text{if and only if} \quad r_d \not\perp r_{d'}.$$

LEMMA 4.1. *For any eni-DOP witness $(M_0, M_1, M_2, M_3, r)$, for any finite approximation $\mathcal{F}$, and for any pair $(b, c, d, r_d)$, $(b', c', d', r_{d'})$ of equivalent $\mathcal{F}$-candidates over $a$, every element of the set $\{b, c, b', c'\}$ depends on the other three over $a$.*

*Proof.* Everything is symmetric, so assume by way of contradiction that $b \underset{a}{\perp} cb'c'$. First, as $b'c' \underset{c}{\perp} b$, the Extendibility Condition implies that

$$\mathrm{tp}(d'/b'c') \vdash \mathrm{tp}(d'/b'c'bc).$$

In particular, $d' \underset{b'c'}{\perp} bc$, so $b \underset{c}{\perp} b'c'd'$ follows by the symmetry and transitivity of non-forking. Second, it follows from this and the Extendibility Condition that $\mathrm{tp}(d/bc) \vdash \mathrm{tp}(d/bcb'c'd')$, so $d \underset{bc}{\perp} b'c'd'$. Combining these two facts yields

$$d \underset{c}{\perp} b'c'd'.$$

But then, as $r_d \in S(d)$ is orthogonal to $c$, by e.g. [6, X, Claim 1.1], $r_d$ would be orthogonal to $b'c'd'$, which contradicts $r_d \not\perp r_{d'}$. ∎

It follows from the previous lemma that there are two types of behavior of a finite approximation $\mathcal{F}$. The following definition describes this dichotomy.

DEFINITION 4.2. Fix an eni-DOP witness $(M_0, M_1, M_2, M_3, r)$. A finite approximation $\mathcal{F} = (a, b, c, d, r_d)$ of it is *flexible* if there is an equivalent $\mathcal{F}$-candidate $(b', c', d', r_{d'})$ over $a$ for which some 3-element subset of $\{b, c, b', c'\}$ is independent over $a$. We say that the eni-DOP witness $(M_0, M_1, M_2, M_3, r)$ is of *flexible type* if it has a flexible finite approximation. A witness is *inflexible* if it is not flexible.

LEMMA 4.3. *Suppose that $(a, b, c, d, r_d)$ and $(a', b', c', d', r_{d'})$ are each finite approximations of an inflexible eni-DOP witness satisfying* $\mathrm{tp}(a) = \mathrm{tp}(a')$ *and $r_d \not\perp r_{d'}$. Then there is no finite set $A \supseteq aa'$ such that* $\mathrm{tp}(bc/A)$ *does not fork over $a$, exactly one element from $\{b', c'\}$ is in $A$, and the other element is independent of $A$ over $a'$.*

*Proof.* By way of contradiction, suppose that $A$ were such a set. For definiteness, suppose $b' \in A$ and $c' \underset{a'}{\perp} A$. Let $\mathcal{F}$ denote the finite approximation exemplified by $(A, bA, cA, dA, r_{dA})$. Fix an automorphism $\sigma \in \mathrm{Aut}(\mathfrak{C})$ fixing $Ac'$ pointwise such that $bcd \underset{Ac'}{\perp} \sigma(b)\sigma(c)\sigma(d)$. Then

$$(\sigma(b)A, \sigma(c)A, \sigma(d)A, r_{\sigma(d)A})$$

is an $\mathcal{F}$-candidate over $A$. Moreover, since $r_{dA} \not\perp r_d \not\perp r_{d'} \not\perp r_{\sigma(d)A}$ the transitivity of non-orthogonality of regular types implies that it is equivalent to $(bA, cA, dA, r_{dA})$. We will obtain a contradiction to the inflexibility of the eni-DOP witness by exhibiting a 3-element subset of $\{b, c, \sigma(b), \sigma(c)\}$ that is independent over $A$.

To do so, first note that since $b$ and $c$ are independent over $A$ and $\operatorname{tp}(c'/A)$ has weight 1, it follows that $c'$ cannot fork with both $b$ and $c$ over $A$. For definiteness, suppose that $b$ and $c'$ are independent over $A$. It follows that $\sigma(b)$ is also independent of $c'$ over $A$. These facts, together with the independence of $b$ and $\sigma(b)$ over $Ac'$, imply that the three-element set $\{b, \sigma(b), c'\}$ is independent over $A$.

We next claim that $\operatorname{tp}(bc/Ac')$ forks over $A$. If this were not the case, recalling that $b' \in A$, we would have $bc \mathop{\smile\!\!\!\!\!\!|}_{aa'} b'c'$. Then, by two applications of the Extendibility Condition, we would have $bcd \mathop{\smile\!\!\!\!\!\!|}_{aa'} b'c'd'$, which would contradict $r_d \not\perp r_{d'}$.

Now, the results in the previous two paragraphs, together with the fact that $\operatorname{tp}(c/Ab)$ has weight 1, imply that the set $\{b, \sigma(b), c\}$ is independent over $A$, contradicting the inflexibility of the eni-DOP witness. ∎

**4.2. Coding bipartite graphs into models.** In this subsection, we take a particular eni-DOP witness and show how we can embed an arbitrary bipartite graph $G$ into a model $M_G$. This mapping will be Borel, and isomorphic graphs will give rise to isomorphic models, but the converse is less clear. Proposition 4.4 demonstrates that the graphs $G$ and $H$ must be similar in some weak sense whenever $M_G$ and $M_H$ are isomorphic.

Fix an eni-DOP witness $(M_0, M_1, M_2, M_3, r)$ and a finite approximation $\mathcal{F} = (a, b, c, d, r_d)$ of it, choosing $\mathcal{F}$ to be flexible if the witness is. As notation, let $p = \operatorname{tp}(b/a)$ and $q = \operatorname{tp}(c/a)$.

We begin by describing how to code arbitrary bipartite graphs into models of $T$. Given a bipartite graph $G = (L_G, R_G, E_G)$, choose sets $\mathcal{B}_G := \{b_g : g \in L_G\}$ and $\mathcal{C}_G := \{c_h : h \in R_G\}$ such that $\mathcal{B}_G \cup \mathcal{C}_G$ is independent over $a$, $\operatorname{tp}(b_g/a) = p$ for each $b_g \in \mathcal{B}_G$, and $\operatorname{tp}(c_h/a) = q$ for each $c_h \in \mathcal{C}_G$. As well, for each $(g, h) \in L_G \times R_G$, choose an element $d_{g,h}$ such that $\operatorname{tp}(d_{g,h} b_g c_h/a) = \operatorname{tp}(dbc/a)$ and let $r_{g,h} \in S(d_{g,h})$ be conjugate to $r_d$. Note that $r_{g,h} \perp r_{g',h'}$ unless $(g, h) = (g', h')$. Let $\mathcal{D}_G = \{d_{g,h} : (g, h) \in E_G\}$ and $\mathcal{R}_G = \{r_{g,h} : (g, h) \in E_G\}$.

Inductively construct models $M_G^n$ of $T$ as follows: $M_G^0$ is any prime model over $\mathcal{B}_G \cup \mathcal{C}_G \cup \mathcal{D}_G$. Given $M_G^n$, let $\mathcal{P}_n = \{p \in S(M_G^n) : p \perp \mathcal{R}_G\}$. By the $\aleph_0$-stability of $T$, $\mathcal{P}_n$ is countable. Let $\mathcal{E}_n = \{e_s : s \in \mathcal{P}_n\}$ be independent over $M_G^n$ with each $e_s$ realizing $s$, and let $M_G^{n+1}$ be prime over $M_G^n \cup \mathcal{E}_n$. Finally, let $M_G = \bigcup_{n \in \omega} M_G^n$.

It is easily verified that if $G$ has universe $\lambda$, then the mapping $G \mapsto M_G$ is $\lambda$-Borel. Moreover, it is easy to see that for regular types $r \in S(M_G)$,

$$r \text{ has finite dimension in } M_G \text{ if and only if } r \not\perp r_{g,h} \text{ for some } (g,h) \in E_G.$$

Suppose that $f : M_G \to M_H$ were an isomorphism. Then $f$ maps the regular types in $S(M_G)$ of finite dimension onto the regular types in $S(M_H)$ of finite dimension. Thus, by construction of $M_G$ and $M_H$, this correspondence yields a bijection

$$\pi_f : E_G \to E_H.$$

Unfortunately, this identification need not extend to a bipartite graph isomorphism between $G$ and $H$. Specifically, there might be edges $e_1, e_2 \in E_G$ that share a vertex of $G$, while the corresponding edges $\pi_f(e_1), \pi_f(e_2) \in E_H$ do not have a common vertex. The bulk of our argument will be to show that images of sufficiently large, complete bipartite subgraphs of $G$ cannot be too wild.

To make this precise, for $X \subseteq E_G$, let $v_G(X)$ denote the smallest subset of the vertices of $G$ with $X \subseteq E_{v_G(X)}$. For $\ell$ very large, call a graph $G$ *almost $\ell$-complete bipartite* if it is $m_1 \times m_2$ bipartite with $0.99\ell \le m_i \le \ell$ for $i = 1, 2$ and each vertex has valence at least $0.9\ell$.

The proof of the following proposition is substantial, and occupies the remainder of this subsection.

Proposition 4.4. *For any bipartite graphs $G$ and $H$ and for any isomorphism $f : M_G \to M_H$, there is a number $\ell^*$, depending only on $f$, such that for all $\ell \ge \ell^*$, if $G_0 \subseteq G$ is any complete $\ell \times \ell$ bipartite subgraph, then $v_H(\pi_f(E_{G_0}))$ contains an almost $\ell$-complete bipartite subgraph.*

*Proof.* Fix bipartite graphs $G, H$, and an isomorphism $f : M_G \to M_H$. As notation, let $a' = f^{-1}(a)$, let $\mathcal{B}'_H = \{f^{-1}(b) : b \in \mathcal{B}_H\}$, and let $\mathcal{C}'_H = \{f^{-1}(c) : c \in \mathcal{C}_H\}$. Let further $X \subseteq \mathcal{B}_G \cup \mathcal{C}_G$ be minimal such that $\mathrm{tp}(a'/a\mathcal{B}_G \cup \mathcal{C}_G)$ does not fork over $Xa$, and let $X' \subseteq \mathcal{B}'_H \cup \mathcal{C}'_H$ be minimal such that $\mathrm{tp}(a/a'\mathcal{B}'_H\mathcal{C}'_H)$ does not fork over $X'a'$. Note that $|X| \le \mathrm{wt}(a'/a)$ and $|X'| \le \mathrm{wt}(a/a')$.

Let $\Lambda^*$ be the set of non-orthogonality classes of regular types in $S(M_G)$ of finite dimension in $M_G$. For each $S \in \Lambda^*$ let $(b_s, c_s)$ be the unique element of $\mathcal{B}_G \times \mathcal{C}_G$ such that there is a candidate $(a, b_s, c_s, d, r_d)$ over $a$ with $r_d \in S$ and let $(b'_s, c'_s)$ be the unique element of $\mathcal{B}'_H \times \mathcal{C}'_H$ such that there is a candidate $(a', b'_s, c'_s, d', r_{d'})$ over $a'$ with $r_{d'} \in S$.

For $\Lambda$ a finite subset of $\Lambda^*$, let $B(\Lambda) = \{b_s : S \in \Lambda\}$, $C(\Lambda) = \{c_s : S \in \Lambda\}$, and $v(\Lambda) = B(\Lambda) \cup C(\Lambda)$. Dually, define $B'(\Lambda)$, $C'(\Lambda)$, and $v'(\Lambda)$ using $(b'_s, c'_s)$ in place of $(b_s, c_s)$.

The proof splits into two cases depending on whether our eni-DOP witness is flexible or inflexible.

Case 1: *The eni-DOP witness is inflexible.* This case will be substantially easier than the other, and in fact, we prove that there is a number $e$ such that for all sufficiently large $\ell$, the image of any $\ell \times \ell$ bipartite graph contains an $(\ell - e) \times (\ell - e)$ complete, bipartite subgraph. The simplicity of this case is primarily due to the following claim.

Claim 1. *For any finite $\Lambda \subseteq \Lambda^*$ such that $v(\Lambda)$ is disjoint from $X$ and $v'(\Lambda)$ is disjoint from $X'$, we have $|v(\Lambda)| = |v'(\Lambda)|$.*

*Proof.* To see this, we again split into cases. First, if $p \perp q$, then we handle the two 'halves' separately. Note that for each $S \in \Lambda$, $\mathrm{tp}(b_s c_s/aa')$ does not fork over $a$, $\mathrm{tp}(b'_s, c'_s/aa')$ does not fork over $a'$, and by Lemma 4.3, each element of $\{b_s, c_s\}$ forks with $b'_s c'_s$ over $aa'$. Since $p \perp q$, this makes $\{b_s, b'_s\}$ fork over $aa'$. It follows that, working over $aa'$,

$$\mathrm{Cl}_p(B(\Lambda)) = \mathrm{Cl}_p(B'(\Lambda))$$

hence $|B(\Lambda)| = |B'(\Lambda)|$. It follows by a symmetric argument that $\mathrm{Cl}_q(C(\Lambda)) = \mathrm{Cl}_q(C'(\Lambda))$, so $|C(\Lambda)| = |C'(\Lambda)|$. It is now immediate that $|v(\Lambda)| = |v'(\Lambda)|$.

On the other hand, if $p \not\perp q$, then $\mathrm{Cl}_p$ is a closure relation on $p^*(\mathfrak{C}) \cup q^*(\mathfrak{C})$, where $p^*$ (resp. $q^*$) is the non-forking extension of $p$ (resp. $q$) to $aa'$. Furthermore, for each $S \in \Lambda$ we have $\mathrm{Cl}_p(b_s c_s) = \mathrm{Cl}_p(b'_s, c'_s)$. It follows that $\mathrm{Cl}_p(v(\Lambda)) = \mathrm{Cl}_p(v'(\Lambda))$. As each set is independent over $aa'$, we conclude that $|v(\Lambda)| = |v'(\Lambda)|$.

Let $w = \mathrm{wt}(a'/a)$ and $e = w + \mathrm{wt}(a/a')^2$. Suppose that $G_0 \subseteq G$ is an $\ell \times \ell$ complete, bipartite subgraph. Since $|X| \leq w$, there is an $(\ell - w) \times (\ell - w)$ complete subgraph $G_0^* \subseteq G_0$ such that $E_{G_0^*}$ is disjoint from $X$. By our choice of $e$ there is an $(\ell - e) \times (\ell - e)$ complete subgraph $G_1 \subseteq G_0^*$ such that $\pi_f(b, c)$ is not contained in $X'$ for all pairs $(b, c) \in E_{G_1}$. But then, by Lemma 4.3, we have $\pi_f(b, c)$ disjoint from $X'$ for all $(b, c) \in E_{G_1}$.

Now, $G_1$ is an $(\ell - e) \times (\ell - e)$ complete, bipartite subgraph of $G$. In particular, $G_1$ has $2(\ell - e)$ vertices and $(\ell - e)^2$ edges. Let $H_1$ be the subgraph of $H$ whose edges are $E_{H_1} := \pi_f(E_{G_1})$ and whose vertices are $v(H_1) := v_H(E_{H_1})$. Then $|E_{H_1}| = (\ell - e)^2$ since $\pi_f$ is a bijection and

$$|v(H_1)| = |v_H(E_{H_1})| = |v_G(E_{G_1})| = 2(\ell - e)$$

by Claim 1. By a classical optimal packing result, this is only possible when $H_1$ is itself a complete, $(\ell - e) \times (\ell - e)$ bipartite subgraph of $H$. ∎

Case 2: *The eni-DOP witness is flexible.* As we insisted that our finite approximation be flexible, it follows from Lemma 4.1 that $p \not\perp q$, so $p$-closure is a dependence relation on $p(\mathfrak{C}) \cup q(\mathfrak{C})$.

As well, for any candidate $(b, c, d, r_d)$ over $a$ and for any finite $A \supseteq a$, there is an equivalent candidate $(b', c', d', r_{d'})$ over $a$ such that $w_p(b'c'/A) = 1$.

Definition 4.5. For any finite subgraph $G_0 \subseteq G$, let $\Lambda(G_0)$ be the set of non-orthogonality classes $\{[r_{d_{g,h}}] : (g,h) \in E_{G_0}\}$. Note that $|\Lambda(G_0)| = |E_{G_0}|$ by the pairwise orthogonality of the types $r_{d_{g,h}}$.

A *manifestation* $\mathcal{M} = \mathcal{M}(\Lambda, a)$ *over* $a$ is a set $\{(b_s, c_s, d_s, r_{d_s}) : S \in \Lambda\}$ of candidates over $a$ with $r_{d_s} \in S$ for each $S \in \Lambda$. Associated to any manifestation $\mathcal{M}$ is a bipartite graph $G(\mathcal{M})$ with 'left nodes' $L(\mathcal{M}) = \{b_s : s \in \Lambda\}$, 'right nodes' $R(\mathcal{M}) = \{c_s : s \in \Lambda\}$, vertices $v(\mathcal{M}) = L(\mathcal{M}) \cup R(\mathcal{M})$, and edges $E(\mathcal{M}) = \{(b_s, c_s) : s \in \Lambda\}$.

If $G_0$ is a subgraph of $G$, then the *canonical manifestation of* $\Lambda(G_0)$ *over* $a$ *inside* $M_G$ is the set

$$\{(b_g, c_h, d_{g,h}, r_{g,h}) : (g,h) \in E_{G_0}\}.$$

A set $A$ *represents* $\Lambda$ *over* $a$ if $a \subseteq A$ and $v(\mathcal{M}) \subseteq A$ for some manifestation $\mathcal{M}$ of $\Lambda$ over $a$. A manifestation $\mathcal{M}'$ is $A$-*free* if $w_p(b'_s, c'_s/A) = 1$ for each $S \in \Lambda$ and $\{(b'_s, c'_s) : S \in \Lambda\}$ are independent over $A$.

Now, working in the monster model $\mathfrak{C}$, we define a measure of the complexity of $\Lambda$ over $a$. First, note that for any candidate $(b, c, d, r_d)$ over $a$, there is an equivalent candidate $(b', c', d', r_{d'})$ over $a$ with $w_p(b'c'/abc) = 1$. By choosing $b'c'$ to be independent over $abc$ of any given $A \supseteq abc$ we can insist that $w_p(b'c'/A) = 1$. It follows that $A$-free manifestations of $\Lambda$ exist over any set $A$ representing a finite $\Lambda$. Thus, the following definition makes sense.

Definition 4.6. The *maximal weight,* $\mathrm{mw}(\Lambda, a)$, is the largest integer $m$ such that for all finite $A$ representing $\Lambda$ over $a$, there is an $A$-free manifestation $\mathcal{M}'(\Lambda, a)$ over $a$ with $|v(\mathcal{M}')| = m + \Lambda$.

Lemma 4.7. *Suppose that $G$ is a bipartite graph, $G_0 \subseteq G$ is a connected subgraph of $G$, let $\mathcal{M}(\Lambda(G_0), a)$ be the canonical manifestation of $\Lambda(G_0)$ inside $M_G$, and let $\mathcal{M}'(\Lambda, a)$ be any other manifestation of $\Lambda(G_0)$. Then*

$$\mathrm{Cl}_p(v(\mathcal{M}') \cup \{v\}) = \mathrm{Cl}_p(v(\mathcal{M}') \cup v(G_0))$$

*for any $v \in v(G_0)$.*

*Proof.* Arguing by symmetry and induction, it suffices to show that for all non-empty $B \subseteq v(G_0)$ and every $c \in v(G_0) \setminus B$ such that $(b,c) \in E_{G_0}$ for some $b \in B$ we have $c \in \mathrm{Cl}_p(v(\mathcal{M}') \cup B)$. But this is immediate, since $\mathrm{Cl}_p(\{b', c', b, c\}) = \mathrm{Cl}_p(\{b', c', b\})$ for all equivalent candidates $(b, c, d, r_d)$ and $(b', c', d', r_{d'})$ over $a$. ∎

Lemma 4.8. *We have $k(G_0) \le \mathrm{mw}(\Lambda(G_0), a) \le |v(G_0)|$ for any bipartite graph $G$ and any finite $G_0 \subseteq G$.*

*Proof.* The upper bound is very soft. Let $A \supseteq a \cup v(G_0)$ be arbitrary and let $\mathcal{M}'$ be any other manifestation of $\Lambda(G_0)$ over $a$. Then

$$w_p(v(\mathcal{M}')/a) \leq w_p(v(\mathcal{M}')v(G_0)/a) = w_p(v(\mathcal{M}')/av(G_0)) + w_p(v(G_0)/a).$$

Since $w_p(b'_s c'_s/ab_s c_s) \leq 1$ for each $S \in \Lambda(G_0)$, we have $w_p(v(\mathcal{M}')/av(G_0)) \leq |\Lambda(G_0)|$. Also, by the independence of the nodes in $M_G$, $w_p(v(G_0)/a) = |v(G_0)|$. The upper bound on $\mathrm{mw}(\Lambda(G_0), a)$ follows immediately.

For the lower bound, again choose any $A \supseteq av(G_0)$ and let $C \subseteq v(G_0)$ consist of one vertex from every connected component of $G_0$. Clearly, $A$ represents $\Lambda(G_0)$ and $|C| = \mathrm{CC}(G_0)$. Let $\mathcal{M}'$ be any $A$-free manifestation of $\Lambda(G_0)$ over $a$. Then

$$w_p(v(\mathcal{M}')/a) \geq w_p(v(\mathcal{M}')C/a) - \mathrm{CC}(G_0) = w_p(v(\mathcal{M}')v(G_0)/a) - \mathrm{CC}(G_0)$$

with the second equality coming from Lemma 4.7. As before, for each $S$ in $\Lambda(G_0)$, we have $w_p(b'_s c'_s/ab_s c_s) \leq 1$, so $w_p(v(\mathcal{M}')/av(G_0)) \leq |\Lambda(G_0)|$. On the other hand, the $A$-freeness of $\mathcal{M}'$ implies that $w_p(v(\mathcal{M}')/A) = |\Lambda(G_0)|$, hence $w_p(v(\mathcal{M}')/av(G_0)) = |\Lambda(G_0)|$. Thus,

$$\begin{aligned} w_p(v(\mathcal{M}')v(G_0)/a) &= w_p(v(\mathcal{M}')/v(G_0)a) + w_p(v(G_0)/a) \\ &= |\Lambda(G_0)| + |v(G_0)|, \end{aligned}$$

from which the lower bound follows. ∎

Now, returning to our isomorphism $f : M_G \to M_H$, suppose that $G_0$ is any finite subgraph of $G$ that is disjoint from $X$, i.e., $\mathrm{tp}(G_0/aa')$ does not fork over $a$. Then:

Claim 2. *We have* $\mathrm{mw}(\Lambda(G_0), a') \leq |v(G_0)| + \mathrm{wt}(a/a')$.

*Proof.* Choose any finite $A$ containing $\{aa'\} \cup v(G_0) \cup v_H(\pi_f(E_{G_0}))$. So $A$ represents $\Lambda(G_0)$ over $a'$. Let $\mathcal{M}'$ be any $A$-free manifestation of $\Lambda(G_0)$ over $a'$. Now

$$w_p(v(\mathcal{M}')/a') \leq w_p(v(\mathcal{M}')aG_0/a') = w_p(v(\mathcal{M}')/aa'G_0) + w_p(aG_0/a').$$

But, as before, $w_p(b'_s c'_s/aa'b_s c_s) \leq 1$, so $w_p(v(\mathcal{M}')/aa'G_0) \leq |\Lambda(G_0)|$. Also,

$$w_p(aG_0/a') = w_p(G_0/aa') + \mathrm{wt}(a/a') = |v(G_0)| + w_p(a/a')$$

and the claim follows. ∎

Finally, choose a complete, bipartite subgraph $G_0 \subseteq G$, where $\ell$ is sufficiently large with respect to $W = \mathrm{wt}(a/a')$. Let $H_0$ be the bipartite graph with vertices $v_H(\pi_f(E_{G_0}))$ and edges $\pi_f(E_{G_0})$ and let $H_0^*$ be the subgraph of $H$ with the same vertex set as $H_0$. Note that $E_{H_0} \subseteq E_{H_0^*}$, but that equality need not hold.

As $G_0$ is $\ell \times \ell$ complete bipartite, $|v(G_0)| = 2\ell$ and $|\Lambda(G_0)| = \ell^2$. It follows immediately that $|E_{H_0}| = \ell^2$, and from Claim 2 and Lemma 4.8 that

$$k(H_0) \leq \mathrm{mw}(\Gamma(G_0), a') \leq 2\ell + W$$

where $W = \mathrm{wt}(a/a')$. So, by Corollary A.7 of the Appendix, $H_0$ contains an almost $\ell$-complete bipartite subgraph $H_1$. But then $H_1^*$, which is the subgraph of $H$ with the same vertex set as $H_1$, is almost $\ell$-complete as well. ∎

**4.3. Coding trees by complete, bipartite subgraphs.** As Proposition 4.4 is rather weak, we give up on coding arbitrary bipartite graphs into models of $T$. Rather, we seek to code subtrees of $\lambda^{<\omega}$ into bipartite graphs that have large, complete subgraphs.

Fix a sufficiently large integer $m$ and a tree $\mathcal{T} \subseteq \lambda^{<\omega}$. We will construct a bipartite graph $G_{\mathcal{T}}^{[m]}$ whose $7m \times 7m$ complete bipartite subgraphs $B_{\mathcal{T}}^m(\eta)$ code nodes $\eta \in \mathcal{T}$. Moreover, additional information about the level of $\eta$ and its set of immediate successors will be coded by the size of the intersection of $B_{\mathcal{T}}^m(\eta)$ and $B_{\mathcal{T}}^m(\nu)$ for other $\nu \in T$.

More precisely, fix a tree $(\mathcal{T}, \trianglelefteq)$ and a large integer $m$. We first define a bipartite graph $preG_{\mathcal{T}}^{[m]}$ to have universe $\mathcal{T} \times m \times 14$ with the edge relation

$$\big\{((\eta, i_1, n_1), (\eta, i_2, n_2)) : \eta \in \mathcal{T}, i_i, i_2 \in m, n_1 + n_2 \text{ is odd}\big\}.$$

So the 'left hand side' of $preG_{\mathcal{T}}^{[m]}$ is $L = \mathcal{T} \times m \times \{n \in 14 : n \text{ odd}\}$, the 'right hand side' is $R = \mathcal{T} \times m \times \{n \in 14 : n \text{ even}\}$, thereby associating a $7m \times 7m$ complete, bipartite graph to each node $\eta \in \mathcal{T}$.

Next, define a binary relation $E_0$ on $preG_{\mathcal{T}}^{[m]}$ by $(\eta_1, i_1, n_1)E_0(\eta_2, i_2, n_2)$ if and only if

- $\eta_2$ is an immediate successor of $\eta_1$, $i_1 = i_2$, $n_1 = n_2$, and
- either $\lg(\eta_1) = 0$ and $n_1 \in \{0, 1\}$ or $\lg(\eta_1) > 0$ and $n_1 \in \{10, 11, 12, 13\}$.

Let $E$ be the smallest equivalence relation containing $E_0$, i.e., the reflexive, symmetric and transitive closure of $E_0$.

Let $G_{\mathcal{T}}^{[m]} := preG_{\mathcal{T}}^{[m]}/E$ and, for each $\eta \in \mathcal{T}$, let $B_{\mathcal{T}}^m(\eta) = \{g \in G_{\mathcal{T}}^{[m]} : (\eta, i, n) \in g \text{ for some } i < m, n < 14\}$.

As notation, for each $\eta \in \mathcal{T}$, let $B_{\mathcal{T}}^m(\eta) = \{g \in G_{\mathcal{T}}^{[m]} : (\eta, i, n) \in g$ for some $i < m, n < 14\}$, let $\mathcal{S}_{\mathcal{T}}^m = \{B_{\mathcal{T}}^m(\eta) : \eta \in \mathcal{T}\}$, and let $g_{\mathcal{T}} : \mathcal{T} \to \mathcal{S}_{\mathcal{T}}^m$ be the bijection $\eta \mapsto B_{\mathcal{T}}^m(\eta)$. For all of these, when $\mathcal{T}$ and $m$ are clear, we delete reference to them. Finally, call an element $g \in G_{\mathcal{T}}^{[m]}$ a *singleton* if $g = \{(\eta, i, n)\}$ for a single element $(\eta, i, n) \in preG_{\mathcal{T}}^{[m]}$. All of the following facts are immediate:

Fact 4.9.

(1) *Every $B(\eta)$ is a $7m \times 7m$ complete, bipartite graph.*
(2) *If $g \in B(\eta)$ is a singleton and $E(g, h)$, then $h \in B(\eta)$.*
(3) *For all $\eta \in \mathcal{T}$, $i < m$, $(\eta, i, n)$ is a singleton for all $2 \leq n \leq 9$.*

(4) *If* $\lg(\nu) < \lg(\eta)$, *then* $B(\nu) \cap B(\eta) = \emptyset$ *if and only if* $\nu = \eta^-$. *Moreover, a non-empty intersection is a complete* $m \times m$ *bipartite graph if* $\eta^- = \langle\rangle_{\mathcal{T}}$, *and the intersection is* $2m \times 2m$ *complete, bipartite if* $\eta^- \neq \langle\rangle_{\mathcal{T}}$.

LEMMA 4.10. $\mathcal{S} = \{$*all* $7m \times 7m$ *complete, bipartite subgraphs of* $G_{\mathcal{T}}^{[m]}\}$.

*Proof.* That each $B(\eta) \in \mathcal{S}$ is a $7m \times 7m$ complete, bipartite subgraph is clear. Conversely, fix a $7m \times 7m$ complete, bipartite subgraph of $G_{\mathcal{T}}^{[m]}$. First, suppose that $X$ contains a singleton $a$. Without loss of generality, assume $a \in X \cap B(\eta) \cap L$. Then $E_X(a) = \{b \in X : E(a, b)\}$ has cardinality $7m$ and is contained in $B(\eta) \cap R$, hence $E_X(a) = B(\eta) \cap R$. But then, $X \cap R$ contains a singleton as well, so arguing similarly, $B(\eta) \cap L = X \cap L$, thus $X = B(\eta)$. It remains to show that $X$ contains a singleton. Choose $k$ maximal such that there is $\eta \in \mathcal{T}$, $\lg(\eta) = n$, and $X \cap B(\eta) \neq \emptyset$. Let $\nu = \eta^-$. If $X$ does not contain a singleton, then the maximality of $k$ implies that $X \cap (B(\eta') \setminus B(\nu)) = \emptyset$ for all $\eta' \in \mathrm{Succ}(\nu)$. Choose any $a \in X \cap B(\eta) \cap L$. Then $a \in B(\nu)$, and moreover, $E_X(a) \subseteq B(\nu) \cap R$. By counting, $E_X(a) = B(\nu) \cap R$, so $X$ contains a singleton, completing the proof. ■

For clarity, let $L_0 = \{R_1, R_2\}$ denote the language consisting of two binary relation symbols. Form an $L_0$-structure $(\mathcal{S}_{\mathcal{T}}^m, R_1, R_2)$ by declaring that $R_1(X, Y)$ holds if and only if $X \cap Y$ is an $m \times m$ complete, bipartite graph and $R_2(X, Y)$ holds if and only if $X \cap Y$ is a $2m \times 2m$ complete, bipartite graph.

LEMMA 4.11. *For any sufficiently large* $m$ *and trees* $(\mathcal{T}, \trianglelefteq), (\mathcal{T}', \trianglelefteq)$, *if there is an* $L_0$*-isomorphism* $\Phi : (\mathcal{S}_{\mathcal{T}}^m, R_1, R_2) \to (\mathcal{S}_{\mathcal{T}'}^m, R_1, R_2)$ *of the associated* $L_0$*-structures, then the composition* $h : (\mathcal{T}, \trianglelefteq) \to (\mathcal{T}', \trianglelefteq)$ *given by* $h = g_{\mathcal{T}'}^{-1} \circ \Phi \circ g_{\mathcal{T}}$ *is a tree isomorphism.*

*Proof.* For each $n \in \omega$, let $\mathcal{T}_n = \{\eta \in \mathcal{T} : \lg(\eta) < n\}$ and define $\mathcal{T}_n'$ analogously. Using Fact 4.9(4.9), one proves by induction on $n$ that $h|_{\mathcal{T}_n} : (\mathcal{T}_n, \trianglelefteq) \to (\mathcal{T}_n', \trianglelefteq)$ is a tree isomorphism. This suffices to prove the lemma. ■

### 4.4. $\aleph_0$-stable, eni-DOP theories are $\lambda$-Borel complete

THEOREM 4.12. *If* $T$ *is* $\aleph_0$*-stable with eni-DOP, then for any infinite cardinal* $\lambda$, *there is a* $\lambda$*-Borel embedding* $\mathcal{T} \mapsto M(\mathcal{T})$ *from subtrees of* $\lambda^{<\omega}$ *to* $\mathrm{Mod}_\lambda(T)$ *satisfying*

$$(\mathcal{T}_1, \trianglelefteq) \cong (\mathcal{T}_2, \trianglelefteq) \quad \textit{if and only if} \quad M(\mathcal{T}_1) \cong M(\mathcal{T}_2).$$

*Proof.* Fix any infinite cardinal $\lambda$. As in Subsection 4.2, fix an eni-DOP witness $(M_0, M_1, M_2, M_3, r)$ and a finite approximation $\mathcal{F} = (a, b, c, d, r_d)$ of it, choosing $\mathcal{F}$ to be flexible if the witness is. As notation, let $p = \mathrm{tp}(b/a)$

and $q = \mathrm{tp}(c/a)$. As well, for the whole of the proof, fix a recursive, fast growing sequence $\langle m_i : i \in \omega\rangle$ of integers, e.g., $m_0 = 10$ and $m_{i+1} = m_i!!$.

Given a subtree $\mathcal{T} \subseteq \lambda^{<\omega}$, let $G^*_{\mathcal{T}}$ be the bipartite graph which is the disjoint union $\bigcup_{i\in\omega} G^{[m_i]}_{\mathcal{T}}$, where the graphs $G^{[m_i]}_{\mathcal{T}}$ are constructed as in Subsection 4.3. Next, construct a model $M(\mathcal{T}) := M_{G^*_{\mathcal{T}}}$ from the bipartite graph $G^*_{\mathcal{T}}$ as in Subsection 4.2. Clearly, after some reasonable coding, we may assume that $M(\mathcal{T})$ has universe $\lambda$. It is routine to verify that both of the maps $\mathcal{T} \mapsto G^*_{\mathcal{T}}$ and $G^*_{\mathcal{T}} \mapsto M_{G^*_{\mathcal{T}}}$ (and hence their composition) are $\lambda$-Borel.

By looking at the constructions in Subsections 4.2 and 4.3, it is easily checked that isomorphic trees $\mathcal{T} \cong \mathcal{T}'$ give rise to isomorphic models $M(\mathcal{T}) \cong M(\mathcal{T}')$.

To establish the converse, suppose that $\mathcal{T}, \mathcal{T}'$ are subtrees such that $M(\mathcal{T}) \cong M(\mathcal{T}')$. Fix an isomorphism $f : M_{G^*_{\mathcal{T}}} \to M_{G^*_{\mathcal{T}'}}$ and choose $i$ so that $m_i \gg \ell^*$, where $\ell^*$ is the constant in the statement of Proposition 4.4.

For each $\eta \in \mathcal{T}$, by Fact 4.9(1), $B^m_{\mathcal{T}}(\eta)$ is a $7m_i \times 7m_i$ complete, bipartite subgraph of $G^{[m_i]}_{\mathcal{T}}$. Let $E(\eta)$ denote the edge set of $B^m_{\mathcal{T}}(\eta)$. Now $\pi_f(E(\eta))$ is a set of $(7m_i)^2$ edges in $G^*_{\mathcal{T}'}$. Let $v_{\mathcal{T}'}(\eta)$ be the smallest set of vertices in $G^*_{\mathcal{T}'}$ whose edge set contains $\pi_f(E(\eta))$.

By Proposition 4.4, the graph $J(\eta) := (v_{T'}(\eta), \pi_f(E(\eta)))$ has an almost $7m_i$-complete bipartite subgraph $K(\eta)$. Let $K^*(\eta)$ be the subgraph of $G^*_{\mathcal{T}'}$ whose vertex set is the same as $K(\eta)$. Note that the edge set of $K^*(\eta)$ contains the edge set of $K(\eta)$, so $K^*(\eta)$ is almost $7m_i$-complete as well.

As $K^*(\eta)$ is a connected subgraph of $G^*_{\mathcal{T}'}$, $K^*(\eta) \subseteq G^{[m_j]}_{\mathcal{T}'}$ for some $j$. As the valence of each vertex of $K^*(\eta)$ is $\sim 7m_i$ and $m_i \gg m_k$ for all $k < i$, we must have $j \geq i$.

Claim. $j = i$.

*Proof.* Choose $\nu \in \mathcal{T}'$ such that $K^*(\eta)$ and $B^{m_j}(\nu)$ share a connected subgraph $D$ with $e(D) \gg N_f$. Arguing as above, there is an almost $7m_j$-complete, bipartite subgraph $H^*(\nu)$ of $G^*_{\mathcal{T}}$ whose edge set (almost) contains $\pi_f^{-1}(E(\nu))$, where $E(\nu)$ is the edge set of $B^{m_j}_{\mathcal{T}'}(\nu)$. As before, $H^*(\nu) \subseteq G^{[m_k]}_{\mathcal{T}}$ for some $k$, and as the valence of every vertex is large, $k \geq j$. However, almost all of the edges of $D$ correspond to edges of $H^*(\nu)$. In particular, $H^*(\nu)$ contains edges from $B^{m_i}_{\mathcal{T}}(\eta)$. But, as $H^*(\eta)$ is connected, this implies $H^*(\eta) \subseteq G^{[m_i]}_{\mathcal{T}}$. Thus $k = j = i$. ∎

Thus, we have shown that for each $\eta \in \mathcal{T}$, $K^*(\eta)$ is an almost $7m_i$ complete bipartite subgraph of $G^{[m_i]}_{\mathcal{T}'}$. It follows as in the proof of Lemma 4.10 that for each $\eta \in \mathcal{T}'$, there is a unique $\nu \in \mathcal{T}'$ such that the subgraphs $K^*(\eta)$ and $B^{m_i}_{\mathcal{T}'}(\nu)$ have large intersection in $G^{[m_i]}_{\mathcal{T}'}$. Define

$$\Phi : \mathcal{S}^{m_i}_{\mathcal{T}} \to \mathcal{S}^{m_i}_{\mathcal{T}'}$$

by $\Phi(B^{m_i}(\eta)) = B^{m_i}_{\mathcal{T}'}(\nu)$ for this unique $\nu$. As the argument given above is reversible, $\Phi$ is a bijection. Furthermore, if $D \subseteq B^{m_i}(\eta)$ is either an $m \times m$ or a $2m \times 2m$ complete, bipartite subgraph, then applying Proposition 4.4 to $D$ yields a connected graph $K^*(D)$ whose number of edges satisfies

$$m_i^2 - N_f \le e(K(D)) \le m_i^2.$$

By taking $D = B^{m_i}(\eta_1) \cup B^{m_i}(\eta_2)$ for various $\eta_1, \eta_2 \in \mathcal{T}$, it follows that $\Phi$ is an $L_0$-isomorphism. Thus, by Lemma 4.11, $(\mathcal{T}, \trianglelefteq) \cong (\mathcal{T}', \trianglelefteq)$ as required. ∎

COROLLARY 4.13. *If $T$ is $\aleph_0$-stable with eni-DOP, then $T$ is Borel complete. Moreover, for every infinite cardinal $\lambda$, $T$ is $\lambda$-Borel complete for $\equiv_{\infty,\aleph_0}$.*

*Proof.* For both statements, by Theorem 3.11, it suffices to show that

$$(\text{subtrees of } \lambda^{<\omega}, \equiv_{\infty,\aleph_0}) \le^B_\lambda (\mathrm{Mod}_\lambda(T), \equiv_{\infty,\aleph_0})$$

for every $\lambda \ge \aleph_0$. So fix an infinite cardinal $\lambda$. The map $\mathcal{T} \mapsto M(T)$ given in Theorem 4.12 is $\lambda$-Borel. Choose any generic filter $G$ for the Levy collapsing poset $\mathrm{Lev}(\lambda, \aleph_0)$. By Fact 3.5, for any subtrees $\mathcal{T}_1, \mathcal{T}_2 \subseteq \lambda^{<\omega}$ in $V$, we have $\mathcal{T}_1 \equiv_{\infty,\aleph_0} \mathcal{T}_2$ in $V$ if and only if $\mathcal{T}_1 \cong \mathcal{T}_2$ in $V[G]$. As well, $M(\mathcal{T}_1) \equiv_{\infty,\aleph_0} M(\mathcal{T}_2)$ in $V$ if and only if $M(\mathcal{T}_1) \cong M(\mathcal{T}_2)$ in $V[G]$. Thus, since the mapping $\mathcal{T} \mapsto M(\mathcal{T})$ is visibly absolute between $V$ and $V[G]$, the result follows immediately from Theorem 4.12. ∎

**5. eni-NDOP and decompositions of models.** In this section, we assume throughout that $T$ is $\aleph_0$-stable with eni-NDOP. [In fact, the first few lemmas require only $\aleph_0$-stability.] We discuss three species of decompositions (regular, eni, and eni-active) of an arbitrary model $M$ and prove a theorem about each one. Theorem 5.7 asserts that in a regular decomposition $\mathfrak{d} = \langle M_\eta, a_\eta : \eta \in I\rangle$ of $M$, the model $M$ is atomic over $\bigcup_{\eta\in I} M_\eta$. This theorem plays a key role in Corollary 5.12.

Next, we discuss eni-active decompositions of a model $M$ and prove that for any $N \preceq M$ that contains $\bigcup_{\eta\in I} M_\eta$, the model $N$ is an $L_{\infty,\aleph_0}$-elementary substructure of $M$. In particular, Corollary 5.9 states that an eni-active decomposition determines a model up to $L_{\infty,\aleph_0}$-equivalence. This is extremely important when we compute $I_{\infty,\aleph_0}(T,\kappa)$ in Section 7.

Finally, we prove Theorem 5.11, which states that a model $M$ is atomic over $\bigcup_{\eta\in I} M_\eta$ for any eni decomposition of $M$ *provided that each of the models is maximal atomic* (see Definition 5.10). While the result sounds strong, it is of little use to us, as one has little control about what the maximal atomic submodels of an arbitrary model look like. This theorem was also proved by Koerwien [3], but is included here to contrast with Theorems 5.7 and 5.8.

Definition 5.1. An *independent tree of models* $\{M_\eta : \eta \in I\}$ satisfies

- $I$ is a subtree of $\mathrm{Ord}^{<\omega}$;
- $\eta \trianglelefteq \nu$ implies $M_\eta \preceq M_\nu$; and
- for each $\eta \in I$ and $\nu \in \mathrm{Succ}_I(\eta)$, $\bigcup_{\nu \trianglelefteq \gamma} M_\gamma \underset{M_\eta}{\perp} \bigcup_{\nu \not\trianglelefteq \delta} M_\delta$

In the decompositions that follow, our trees of models will have the additional property that $\mathrm{tp}(M_\nu/M_\eta) \perp M_{\eta^-}$ for every $\eta \neq \langle\rangle$ and every $\nu \in \mathrm{Succ}_I(\eta)$, but our early lemmas do not require this property.

Lemma 5.2. *Suppose $\{M_\eta : \eta \in I\}$ is any independent tree of models indexed by a finite tree $(I, \trianglelefteq)$. Then the set $\bigcup_{\eta\in I} M_\eta$ is essentially finite with respect to any strong type $p$ that is orthogonal to every $M_\eta$.*

*Proof.* We argue by induction on $|I|$. For $|I| = 1$, this is immediate by Lemma 1.5(1) (taking $A = M_{\langle\rangle}$ and $B = \emptyset$). So assume $\{M_\eta : \eta \in I\}$ is any independent tree of models with $|I| = n + 1$ and we have proved the lemma when $|I| = n$. Fix any strong type $p$ that is orthogonal to every $M_\eta$. Choose any leaf $\eta \in I$ and let $J \subseteq I$ be the subtree with universe $I \setminus \{\eta\}$. By the inductive hypothesis, $\bigcup_{\nu\in J} M_\nu$ is essentially finite with respect to $p$, so the result follows by Lemma 1.5(2), taking $A = \bigcup_{\nu\in J} M_\nu$ and $B = M_\eta$. ∎

Lemma 5.3. *Suppose $\{M_\eta : \eta \in I\}$ is any independent tree of models indexed by any tree $(I, \trianglelefteq)$ and let $N$ be any model that contains and is atomic over $\bigcup_{\eta\in I} M_\eta$. Let $p \in S(N)$ be any regular type that is not eni, but is orthogonal to every $M_\eta$, and let $c$ be any realization of $p$. Then $N \cup \{e\}$ is atomic over $\bigcup_{\eta\in I} M_\eta$.*

*Proof.* As notation, for $K \subseteq I$, we write $M_K$ for $\bigcup_{\nu\in K} M_\nu$. It suffices to show that $\mathrm{tp}(Dc/M_I)$ is isolated for any finite subset $D \subseteq N$ on which $p$ is based and stationary. To see this, fix such a set $D$. As $D$ is atomic over $M_I$, we can find a finite set $E \subseteq M_I$ such that $\mathrm{tp}(D/E)$ is isolated and $\mathrm{tp}(D/E) \vdash \mathrm{tp}(D/M_I)$. Choose a non-empty finite subtree $J \subseteq I$ such that $E \subseteq M_J$ and choose a prime model $N_J \preceq N$ over $M_J$ that contains $D$. By Lemmas 5.2 and 1.21 we see that $N_J \cup \{c\}$ is atomic over $M_J$. Choose a formula $\delta(x, h) \in \mathrm{tp}(Dc/M_J)$ that isolates the type. Now, let

$$\mathcal{F} = \{K,\ J \subseteq K \subseteq I : K \text{ is a subtree and } \mathrm{tp}(Dc/M_K) \text{ is isolated by } \delta(x, h)\}.$$

Clearly, $J \in \mathcal{F}$ and by Lemma 1.8, $\mathcal{F}$ is closed under unions of increasing chains. So choose a maximal element $K^* \in \mathcal{F}$ with respect to inclusion. To complete the proof of the lemma, it suffices to prove that $K^* = I$. If this were not the case, then choose a $\trianglelefteq$-minimal element $\eta \in I \setminus K^*$ and let $K' := K^* \cup \{\eta\}$. As $J$ was non-empty, $\eta \neq \langle\rangle$ and the independence of the tree yields $M_{K^*} \underset{M_{\eta^-}}{\perp} M_\eta$. But then, by Lemma 1.8(2), $\delta(x,h)$ isolates $\mathrm{tp}(Dc/M_{K'})$, contradicting the maximality of $K^*$. Thus, $K^* = I$. ∎

We define a plethora of decompositions.

Definition 5.4. Fix a model $M$. A [*regular*, *eni*, *eni-active*] decomposition $\mathfrak{d} = \langle M_\eta, a_\eta : \eta \in I\rangle$ inside $M$ consists of an independent tree $\{M_\eta : \eta \in I\}$ of elementary submodels of $M$ indexed by $(I, \trianglelefteq)$ satisfying the following conditions for each $\eta \in I$:

(1) Each $a_\eta \in M_\eta$ (but $a_{\langle\rangle}$ is meaningless).
(2) The set $C_\eta := \{a_\nu : \nu \in \mathrm{Succ}_I(\eta)\}$ is independent over $M_\eta$.
(3) For each $\nu \in \mathrm{Succ}_I(\eta)$ we have:
  (a) $\mathrm{tp}(a_\nu/M_{\nu^-})$ is [regular, eni, eni-active];
  (b) if $\eta \neq \langle\rangle$, then $\mathrm{tp}(a_\nu/M_\eta) \perp M_{\eta^-}$; and
  (c) $M_\nu$ is atomic over $M_\eta \cup \{a_\nu\}$.

A [*regular*, *eni*, *eni-active*] *decomposition of* $M$ is a [regular, eni, eni-active] decomposition inside $M$ with the additional property that for each $\eta \in I$, the set $C_\eta$ is a *maximal* $M_\eta$-independent set of realizations of [regular, eni, eni-active] types (that are orthogonal to $M_{\eta^-}$ when $\eta \neq \langle\rangle$).

We say that a decomposition (of any sort) is *prime* if $M_{\langle\rangle}$ is a prime submodel of $M$ and, for each $\nu \neq \langle\rangle$, $M_\nu$ is prime over $M_{\nu^-} \cup \{a_\nu\}$.

It is important to note that even though eni-NDOP implies eni-active NDOP, it is not the case that every eni-active decomposition is an eni decomposition. As well, note that if $\mathfrak{d} = \langle M_\eta, a_\eta : \eta \in I\rangle$ is a decomposition of $M$ (in any of the senses) and $N \preceq M$ contains $\bigcup_{\eta\in I} M_\eta$, then $\mathfrak{d}$ is also a decomposition of $N$. The following lemma requires no assumption beyond $\aleph_0$-stability.

Lemma 5.5. *For any $M$, prime* [*regular*, *eni*, *eni-active*] *decompositions of $M$ exist.*

*Proof.* Simply start with an arbitrary prime model $M_{\langle\rangle} \preceq M$, and given a node $M_\eta$, choose $C_\eta$ to be any maximal $M_\eta$-independent subset of $M$ of realizations of [regular, eni, eni-active] types (that are orthogonal to $M_{\eta^-}$ when $\eta \neq \langle\rangle$) and, for each $a_\nu \in C_\eta$, choose $M_\nu \preceq M$ to be prime over $M_\eta \cup \{a_\nu\}$. Any maximal construction of this sort will produce a prime [regular, eni, eni-active] decomposition of $M$. ∎

Of course, without any additional assumptions, such a decomposition may be of limited utility.

Lemma 5.6. ($T$ $\aleph_0$-stable with eni-NDOP) *Let $\mathfrak{d} = \langle M_\eta, a_\eta : \eta \in I\rangle$ be any regular decomposition inside $\mathfrak{C}$ and let $N$ be atomic over $\bigcup_{\eta\in I} M_\eta$. If an eni-active regular type $p$ satisfies $p \not\perp N$, then $p \not\perp M_\eta$ for some $\eta \in I$.*

*Proof.* Recall that eni-NDOP implies eni-active NDOP by Theorem 2.4. We first prove the lemma for all finite index trees $(I, \trianglelefteq)$ by induction on $|I|$. To begin, if $|I| = 1$, then we must have $N = M_{\langle\rangle}$ and there is nothing to prove. Assume the lemma holds for all trees of size $n$ and let $\mathfrak{d} = \langle M_\eta, a_\eta : \eta \in I\rangle$ be a decomposition inside $\mathfrak{C}$ indexed by $(I, \trianglelefteq)$ of size $n+1$. Let $N$ be atomic over $\bigcup_{\eta\in I} M_\eta$ and let $p$ be an eni-active type non-orthogonal to $N$. Choose a leaf $\eta \in I$ and let $J = I \setminus \{\eta\}$. If $(I, \trianglelefteq)$ were a linear order, then again $N = M_\eta$ and there is nothing to prove. If $(I, \trianglelefteq)$ is not a linear order, then choose any $N_J \preceq N$ to be prime over $\bigcup_{\nu\in J} M_\nu$. Then, by eni-active NDOP and Lemma 2.2(2), either $p \not\perp M_\eta$ or $p \not\perp N_J$. In the first case we are done, and in the second we finish by the inductive hypothesis since $|J| = n$. Thus, we have proved the lemma whenever the indexing tree $I$ is finite.

For the general case, fix a regular decomposition $\mathfrak{d} = \langle M_\eta, a_\eta : \eta \in I\rangle$ inside $\mathfrak{C}$, let $N$ be atomic over $\bigcup_{\eta\in I} M_\eta$ and choose an eni-active $p \not\perp N$. By employing Fact 1.2(2) and the observation that eni-active types are preserved under non-orthogonality, we may assume $p \in S(N)$. Choose a finite $D \subseteq N$ over which $p$ is based and stationary. As $D$ is finite and atomic over $\bigcup_{\eta\in I} M_\eta$, we can find a finite subtree $J \subseteq I$ such that $D$ is atomic over $\bigcup_{\eta\in J} M_\eta$. Fix such a $J$ and choose $M_J \preceq N$ such that $D \subseteq M_J$ and $M_J$ is prime over $\bigcup_{\eta\in J} M_\eta$. As $D \subseteq M_J$, $p \not\perp M_J$, so since $J$ is finite, the argument above implies that there is an $\eta \in J$ such that $p \not\perp M_\eta$. ∎

THEOREM 5.7. ($T$ $\aleph_0$-stable with eni-NDOP) *Suppose $\langle M_\eta, a_\eta : \eta \in I\rangle$ is a regular decomposition of $M$. Then $M$ is atomic over $\bigcup_{\eta\in I} M_\eta$.*

*Proof.* Choose $N \preceq M$ to be maximal atomic over $\bigcup_{\eta\in I} M_\eta$. We argue that $N = M$. If this were not the case, then choose $e \in M \setminus N$ such that $p := \mathrm{tp}(e/N)$ is regular. We obtain a contradiction in three steps.

CLAIM 1. *$p \perp M_\eta$ for all $\eta \in I$.*

*Proof.* Suppose this were not the case. Choose $\eta \in I$ $\triangleleft$-minimal such that $p \not\perp M_\eta$. Thus, either $\eta = \langle\rangle$ or $p \perp M_{\eta^-}$. By Lemma 1.3, there is an element $e \in M$ such that $\mathrm{tp}(e/M_\eta)$ is regular and non-orthogonal to $p$ (hence orthogonal to $M_{\eta^-}$ if $\eta \neq \langle\rangle$), but $e \underset{M_\eta}{\perp} N_\alpha$. This element $e$ contradicts the maximality of $C_\eta$ in Definition 5.4. ∎

CLAIM 2. *$p$ is dull.*

*Proof.* If $p$ were eni-active, then by Lemma 5.6 we would have $p \not\perp M_\eta$ for some $\eta \in I$, contradicting Claim 1. ∎

As $p$ is dull, it is not eni by Proposition 1.19. But this, coupled with Claim 1, implies that $N \cup \{e\}$ is atomic over $\bigcup_{\eta\in I} M_\eta$, which contradicts the maximality of $N$. Thus, $N = M$ and we finish. ∎

Theorem 5.8. ($T$ $\aleph_0$-stable with eni-NDOP) *Suppose* $\mathfrak{d} = \langle M_\eta, a_\eta : \eta \in I\rangle$ *is an eni-active decomposition of a model* $M$*. If* $N \preceq M$ *is atomic over* $\bigcup_{\eta\in I} M_\eta$*, then* $N \preceq M$ *is a dull pair. Thus, for every* $N'$ *satisfying* $N \preceq N' \preceq M$*, the model* $N$ *is an* $L_{\infty,\aleph_0}$*-elementary substructure of* $N'$ *and* $N'$ *is an* $L_{\infty,\aleph_0}$*-substructure of* $M$*.*

*Proof.* Given $M$ and $\mathfrak{d}$, choose any $N \preceq M$ atomic over $\bigcup_{\eta\in I} M_\eta$. To prove that $N \preceq M$ is dull, it suffices to show that there is no $e \in M \setminus N$ such that $\mathrm{tp}(e/N)$ is eni-active. So, by way of contradiction, assume that there were such an $e$. Let $p := \mathrm{tp}(e/N)$. By Lemma 5.6, we can choose an $\trianglelefteq$-minimal $\eta \in I$ such that $p \not\perp M_\eta$. By Lemma 1.3, there is $c \in M\setminus M_\eta$ such that $q := \mathrm{tp}(c/M_\eta)$ is non-orthogonal to $p$ and $c \underset{M_\eta}{\perp} N$. As $q$ is eni-active and orthogonal to $M_{\eta^-}$ (when $\eta \neq \langle\rangle$), the element $c$ contradicts the maximality of $C_\eta$ in Definition 5.4. Thus, $N \preceq M$ is a dull pair. The final sentence follows from Lemma 1.18 and Proposition 1.24. ∎

Corollary 5.9. ($T$ $\aleph_0$-stable with eni-NDOP) *Suppose* $\langle M_\eta, a_\eta : \eta \in I\rangle$ *is an eni-active decomposition of both* $M_1$ *and* $M_2$*. Then* $M_1 \equiv_{\infty,\aleph_0} M_2$*.*

*Proof.* Choose any $N_1 \preceq M_1$ to be prime over $\bigcup_{\eta\in I} M_\eta$. By Theorem 5.8, $N_1 \equiv_{\infty,\aleph_0} M_1$. By the uniqueness of prime models, there is $N_2 \preceq M_2$ that is both isomorphic to $N_1$ and prime over $\bigcup_{\eta\in I} M_\eta$. By Theorem 5.8 again, $N_2 \equiv_{\infty,\aleph_0} M_2$ and the result follows. ∎

The third theorem of this section involves eni decompositions of a model. Theorem 5.11 is of less interest to us, since when $M^*$ is uncountable, each of the component submodels $M_\eta$ may be uncountable as well.

Definition 5.10. A decomposition $\langle M_\eta, a_\eta : \eta \in I\rangle$ inside $M$ is *maximal atomic* if $M_{\langle\rangle}$ is a maximal atomic substructure of $M$ and, for each $\nu \neq \langle\rangle$, $M_\nu$ is maximal atomic over $M_{\nu^-} \cup \{a_\nu\}$.

Theorem 5.11. ($T$ $\aleph_0$-stable with eni-NDOP) *Every model* $M$ *is atomic over* $\bigcup_{\eta\in I} M_\eta$ *for every maximal atomic, eni decomposition* $\langle M_\eta : \eta \in I\rangle$ *of* $M$*.*

*Proof.* Given a maximal, atomic, eni decomposition $\{M_\eta : \eta \in I\}$ of a model $M$, choose an enumeration $\langle \eta_i : i < \alpha\rangle$ of $I$ such that $\eta_i \triangleleft \eta_j$ implies $i < j$. Note that $\eta_0 = \langle\rangle$. Next, define a continuous, elementary sequence $\langle N_i : i \leq \alpha\rangle$ of elementary substructures of $M$ satisfying:

- $N_0 = M_{\langle\rangle}$;
- $N_\beta = \bigcup_{i<\beta} N_i$ for every non-zero limit ordinal $\beta \leq \alpha$; and
- $N_{\beta+1} \preceq M$ is maximal atomic over $N_\beta \cup M_{\eta_\beta}$ whenever $\beta < \alpha$.

Using Lemma 1.8, it follows by induction on $\beta \leq \alpha$ that each model $N_\beta$ is atomic over $\bigcup_{i<\beta} M_{\eta_i}$. Thus, it suffices to prove that $N_\alpha = M$. Suppose that this were not the case. Choose $e \in M \setminus N_\alpha$ such that $p := \mathrm{tp}(e/N_\alpha)$ is regular. Choose $i \le \alpha$ least such that $p \not\perp N_i$. By superstability, either $i = 0$ or $i = \beta + 1$ for some $\beta < \alpha$. We argue by cases, arriving at a contradiction in each case.

CASE 1: *$p \not\perp M_\eta$ for some $\trianglelefteq$-least $\eta \in I$.* By Lemma 1.3, there is $c$ in $M \setminus M_\eta$ such that $q := \mathrm{tp}(c/M_\eta)$ is strongly regular, non-orthogonal to $p$, and $c \underset{M_\eta}{\perp} N_\alpha$. If $q$ were eni, then the element $c$ contradicts the maximality of $C_\eta$ in Definition 5.4. So assume $q$ is not eni. There are two subcases: First, if $\eta = \langle\rangle$, then by Lemma 1.21 (with $A = \emptyset$) we would have $M_{\langle\rangle} \cup \{c\}$ atomic, contradicting the maximality of $M_{\langle\rangle}$. On the other hand, if $\eta \neq \langle\rangle$, then $M_\eta$ would be atomic over $M_\nu \cup \{a_\eta\}$, where $\nu = \eta^-$. But then, by Lemma 1.5(1), we would have $M_\nu \cup \{a_\eta\}$ essentially finite with respect to $q$, hence again by Lemma 1.21 we would have $M_\eta \cup \{c\}$ atomic over $M_\nu \cup \{a_\eta\}$, contradicting the maximality of $M_\eta$.

CASE 2: *$p \perp M_\eta$ for every $\eta \in I$.* In this case, $p$ cannot be dull, because if it were, then by Lemma 5.3, $N_\alpha \cup \{e\}$ would be atomic over $\bigcup_{\eta\in I} M_\eta$. So assume $p$ is eni-active. As $N_0 = M_{\langle\rangle}$ and $p \not\perp N_i$, the conditions of the present case imply that $i = \beta+1$, so $N_i$ is atomic over $N_\beta \cup M_\beta$. Let $\nu = \eta_\beta^-$. As $N_\beta$ is atomic over $\bigcup_{\eta_j : j<\beta} M_j$ we have $N_\beta \underset{M_\nu}{\perp} M_{\eta_\beta}$. Since $p$ is eni-active, by eni-active NDOP we would have $p \not\perp N_\beta$ or $p \not\perp M_{\eta_\beta}$. The first possibility contradicts the minimality of $i$, while the second contradicts the conditions of the current case. ■

We close this section with an application of Theorem 5.7. The main point of the proof of Corollary 5.12 is that models that are atomic over an independent tree of countable models have a large number of partial automorphisms.

COROLLARY 5.12. *If $T$ is $\aleph_0$-stable and eni-NDOP, then $T$ cannot have OTOP.*

*Proof.* By way of contradiction suppose that there were a sufficiently large cardinal $\kappa$ and a model $M^*$ containing a sequence $\langle (b_\alpha, c_\alpha) : \alpha < \kappa\rangle$ and a type $p(x, y, z)$ such that for all $\alpha, \beta < \kappa$,

$$M^* \text{ realizes } p(x, b_\alpha, c_\beta) \quad \text{if and only if} \quad \alpha < \beta.$$

For each pair $\alpha < \beta$, fix a realization $a_{\alpha,\beta}$ of $p(x, b_\alpha, c_\beta)$. Choose a prime, regular decomposition $\langle M_\eta, a_\eta : \eta \in I\rangle$ of $M^*$. Note that each of the models $M_\eta$ is countable. By Theorem 5.7, $M^*$ is atomic over $\bigcup_{\eta\in I} M_\eta$, so for each pair $\alpha < \beta$ we can choose a finite $e_{\alpha,\beta}$ from $\bigcup_{\eta\in I} M_\eta$ such that $\mathrm{tp}(a_{\alpha,\beta}/b_\alpha, c_\beta \bigcup_{\eta\in I} M_\eta)$ is isolated by a formula $\theta(x, b_\alpha, c_\beta, e_{\alpha,\beta})$. We will eventually find a pair $\alpha < \beta$ and $e^*$ from $\bigcup_{\eta\in I} M_\eta$ such that

$$\mathrm{tp}(b_\beta, c_\alpha, e^*) = \mathrm{tp}(b_\alpha, c_\beta, e_{\alpha,\beta}).$$

This immediately leads to a contradiction, as $\theta(x, b_\beta, c_\alpha, e^*)$ would be realized in $M^*$ and any realization of it also realizes $p(x, b_\beta, c_\alpha)$, contrary to our initial assumptions.

We will obtain these $\alpha < \beta$ and $e^*$ by successively passing from our sequence to sufficiently long subsequences, each time adding some amount of homogeneity. First, for each $\alpha$, choose a finite subtree $J_\alpha \subseteq I$ such that $\mathrm{tp}(b_\alpha c_\alpha / \bigcup_{\eta\in I} M_\eta)$ does not fork and is as stationary as possible over $J_\alpha$. By an argument akin to the $\Delta$-system lemma, by passing to a subsequence we may assume that there is an $\eta^* \in I$ such that $J_\alpha \cap J_\beta = \{\nu : \nu \trianglelefteq \eta\}$ for all $\alpha \neq \beta$. For each $\alpha$, let $M^J_\alpha$ be the countable set $\bigcup_{\gamma\in J_\alpha} M_\gamma$. As well, let $\nu_\alpha$ be the (unique) immediate successor of $\eta^*$ contained in $J_\alpha$, let $H_\alpha = \{\gamma \in I : \nu_\alpha \trianglelefteq \gamma\}$, and let $M_\alpha = \bigcup_{\gamma\in H_\alpha} M_\gamma$. Note that the sets $H_\alpha$ are pairwise disjoint, and the independence of the tree implies that the sets $\{M_\alpha : \alpha \in \kappa\}$ are independent over $M_{\eta^*}$. By trimming further, we may additionally assume that the $J_\alpha$'s are tree isomorphic over $\eta^*$, and that the sets $M_\alpha$ are isomorphic over $M_{\eta^*}$.

Next, for each $\alpha < \beta$, partition each sequence $e_{\alpha,\beta}$ into three subsequences $r_{\alpha,\beta} \subseteq M_\alpha$, $s_{\alpha,\beta} \subseteq M_\beta$, and $t_{\alpha,\beta}$ disjoint from $M_\alpha \cup M_\beta$.

By the Erdős–Rado Theorem, we can pass to a subsequence such that for all $\alpha < \beta < \gamma$ we have:

- the partitions coincide, i.e., for each $i$, the $i$th coordinate of $e_{\alpha,\beta}$ is in $r_{\alpha,\beta}$ iff the $i$th coordinate of $e_{\beta,\gamma}$ is in $r_{\beta,\gamma}$;
- $\mathrm{tp}(t_{\alpha,\beta}/M_{\eta^*})$ is constant;
- $\mathrm{tp}(r_{\alpha,\beta}/M^J_\alpha)$ is constant; and
- $\mathrm{tp}(s_{\alpha,\beta}/M^J_\beta)$ is constant.

Additionally, by trimming the sequence still further, we may insist that for all pairs $\alpha < \beta$, there is $r^* \in H_\beta$ such that $\mathrm{tp}(r_{\alpha,\beta} M^J_\alpha / M_{\eta^*}) = \mathrm{tp}(r^* M^J_\beta / M_{\eta^*})$ and there is $s^* \in H_\alpha$ such that $\mathrm{tp}(s_{\alpha,\beta} M^J_\beta / M_{\eta^*}) = \mathrm{tp}(s^* M^J_\alpha / M_{\eta^*})$.

Finally, fix any such $\alpha < \beta$. By independence, we have

$$\mathrm{tp}(M^J_\alpha, M^J_\beta, r_{\alpha,\beta}, s_{\alpha,\beta}, t_{\alpha,\beta}) = \mathrm{tp}(M^J_\beta, M^J_\alpha, r^*, s^*, t_{\alpha,\beta}).$$

Let $e^*$ be the sequence formed from $r^* s^* t_{\alpha,\beta}$. As each of $b_\alpha$ and $c_\beta$ is dominated by $M^J_\alpha$ and $M^J_\beta$, respectively over $M_{\eta^*}$, it follows that

$$b_\alpha c_\beta \underset{M^J_\alpha M^J_\beta}{\perp} \bigcup_{\eta\in I} M_\eta$$

so $\mathrm{tp}(b_\alpha, c_\beta, e_{\alpha,\beta}) = \mathrm{tp}(b_\beta, c_\alpha, e^*)$, completing the proof. ■

**6. Borel completeness of eni-NDOP, eni-deep theories.** Throughout this section, we assume that $T$ is $\aleph_0$-stable with eni-NDOP, hence prime, eni-active decompositions exist for any model $N$ of $T$. We begin with a definition, which should be thought of as describing a potential 'branch' of an eni-active decomposition.

DEFINITION 6.1. An *eni-active chain* is a sequence $\langle M_i, a_i : i < \alpha\rangle$, where $2 \leq \alpha \leq \omega$ such that $a_i \in M_i$ for any $i$, and, for each $i$ such that $i+1 < \alpha$, $\mathrm{tp}(a_{i+1}/M_i)$ is eni-active, orthogonal to $M_{i-1}$ (when $i > 0$), and $M_{i+1}$ is prime over $M_i \cup \{a_{i+1}\}$. An eni-active chain is *finite* when $\alpha < \omega$. For $q$ a stationary, regular type, we say a finite chain is *$q$-topped* if $q \not\perp M_{\alpha-1}$, but $q \perp M_{\alpha-2}$. A finite chain is *ENI-topped* if it is $q$-topped for some ENI type $q$.

DEFINITION 6.2. An $\aleph_0$-stable, eni-NDOP theory is *eni-deep* if an eni-active $\omega$-chain exists.

LEMMA 6.3. *Given any model $M$ and regular type $p \in S(M)$, if some stationary, regular type $q$ lies directly over $p$, then there is a $q$-topped finite chain $\langle M_i, a_i : i < \alpha\rangle$ such that $M_0 = M$ and $\mathrm{tp}(a_1/M_0)$ realizes $p$.*

*Proof.* Choose an $\aleph_0$-saturated $N \succeq M$, $a$ realizing $p|N$, and an $\aleph_0$-prime model $N[a]$ over $N \cup \{a\}$ such that $q \not\perp N[a]$, while $q \perp N$. Choose a prime model $M(a) \preceq N[a]$ over $M \cup \{a\}$. As $q \perp N$, $q \perp M$. There are now two cases. First, if $q \not\perp M(a)$, then the two-element chain $\langle M, M(a)\rangle$ with $a_1 = a$ is as desired. Second, assume that $q \perp M(a)$. Choose an eni-active decomposition $\langle M_\eta : \eta \in I\rangle$ of $N[a]$ with $M_{\langle\rangle} = M(a)$ such that $M_\eta$ is prime over $M_{\eta^-} \cup \{a_\eta\}$ for every $\eta \in I \setminus \{\langle\rangle\}$. As $q \not\perp N[a]$ while $q \perp M(a)$, we can choose $\eta \neq \langle\rangle$ minimal such that $q \not\perp M_\eta$. As $q \perp M_{\eta^-}$,

$$M \preceq M_{\langle\rangle} \preceq M_{\eta|1} \preceq \cdots \preceq M_\eta$$

with $a_{\langle\rangle} = a$ and $a_{\ell+1} = a_{\eta|\ell}$ is a $q$-topped finite chain as required. ∎

Under the assumption of eni-NDOP, this leads to another characterization of the eni-active types.

PROPOSITION 6.4. ($T$ $\aleph_0$-stable, eni-NDOP) *A stationary, regular type $p$ is eni-active if and only if either $p$ is ENI or for every model $M$ such that $p \not\perp M$, there is a finite, ENI-topped chain $\langle M_i, a_i : i < \alpha\rangle$ such that $M_0 = M$ and $\mathrm{tp}(a_1/M_0) \not\perp p$.*

*Proof.* Let $\mathbf{P}$ denote the class of types satisfying the alleged characterization. It follows immediately from Lemma 1.11 that every type in $\mathbf{P}$ is eni-active. For the converse, $\mathbf{P}$ visibly contains the ENI types and is closed under non-orthogonality and automorphisms of the monster model. Thus, it suffices to show that if $q \in \mathbf{P}$ and $q$ lies directly over $p$, then $p \in \mathbf{P}$. To see this, choose any model $M$ such that $p \not\perp M$. Choose a regular $p' \in S(M)$ non-orthogonal to $p$. As $q$ lies directly over $p'$ as well, use Lemma 6.3 to find a $q$-topped finite chain $\langle M_i, a_i : i < \alpha\rangle$ with $M_0 = M$ and $\mathrm{tp}(a_1/M_0) \not\perp p$. Now, if $q$ is ENI, then this chain witnesses that $p \in \mathbf{P}$. On the other hand, if $q \in \mathbf{P}$ but is not ENI, then there is a finite ENI-topped chain $\langle N_j, b_j : j < \beta\rangle$ with $N_0 = M_{\alpha-1}$ and $\mathrm{tp}(b_1/N_0) \not\perp q$. The concatenation of these two finite chains is an ENI chain starting with $M_0 = M$ and $\mathrm{tp}(a_1/M_0) \not\perp p$. ∎

*Until the end of the proof of Theorem* 6.9, *fix an* $\aleph_0$*-stable, eni-NDOP theory that is eni-deep as witnessed by a specific eni-active* $\omega$*-chain* $\langle M_i, a_i : i \in \omega\rangle$.

Under these hypotheses, we aim to prove Theorem 6.9. By employing Proposition 6.4 for each $i \in \omega$, there is an integer $k = k(i) > i$ and an ENI-topped finite chain $\mathcal{C}_k = \langle N_j^k, b_j^k : j \le k\rangle$ such that for every $j \le i$, $N_j^k = M_j$ and $b_j^k = a_j$. As notation, using Fact 1.2(2), choose an ENI $q_k \in S(N_k^k)$ satisfying $q_k \perp N_{k-1}^k$.

We will use this configuration of ENI-topped chains to code arbitrary subtrees of $\mathcal{T} \subseteq \lambda^{<\omega}$ into models $M(\mathcal{T})$ preserving isomorphism in both directions. The 'reverse direction' i.e., showing that $M(\mathcal{T}_1) \cong M(\mathcal{T}_2)$ implying $(\mathcal{T}_1, \trianglelefteq) \cong (\mathcal{T}_2, \trianglelefteq)$ is quite involved and uses a 'black box' in the form of [10, Theorem 6.19]. We begin by recalling a number of definitions that appear in that paper. As we are concerned with eni-active decompositions, we take $\mathbf{P}$ to be the class of eni-active types. As eni-active types are regular, a $\mathbf{P}^r$-decomposition in the notation of [10] is precisely an eni-active decomposition.

Definition 6.5. Given a tree $I \subseteq \mathrm{Ord}^{<\omega}$, a *large subtree* of $I$ is a non-empty subtree $J \subseteq I$ such that for each $\eta \in J$, $\mathrm{Succ}_I(\eta) \setminus J$ is finite. We say that two trees $I_1$ and $I_2$ are almost isomorphic if there exist large subtrees $J_1 \subseteq I_1$ and $J_2 \subseteq I_2$ such that $(J_1, \trianglelefteq) \cong (J_2, \trianglelefteq)$.

A tree $I$ has *infinite branching* if, for every $\eta \in I$, $\mathrm{Succ}(\eta)$ is either infinite or empty. If a tree $I$ has infinite branching, for any integer $k$, we say a node $\eta \in I$ has *uniform depth* $k$ if every maximal branch of $\{\nu \in I : \eta \trianglelefteq \nu\}$ has length exactly $k$. A node $\eta$ *often has unbounded depth* if, for every large subtree $J \subseteq I$ with $\eta \in J$, there is an infinite branch in $J$ containing $\eta$.

Suppose $\eta \in I$ and $E_\eta$ is an equivalence relation on $\mathrm{Succ}(\eta)$. Then $\eta$ is an $(m,n)$*-cusp* if there are infinite sets $A_m, A_n, B \subseteq \mathrm{Succ}(\eta)$ such that

(1) the set $A_m \cup A_n$ is pairwise $E_\eta$-equivalent;
(2) each $\delta \in A_m$ has uniform depth $m$;
(3) each $\rho \in A_n$ has uniform depth $n$; and
(4) each $\gamma \in B$ is often unbounded.

A *cusp* is an $(m,n)$-cusp for some $m \neq n$.

Definition 6.6. Suppose $S \subseteq \mathbf{P}$ and $\mathfrak{d} = \langle M_\eta, a_\eta : \eta \in I \rangle$ is a $\mathbf{P}$-decomposition. We say $\mathfrak{d}$ *supports* $S$ if, for every $q \in S$, there is $\eta(q) \in \max(I) \setminus \{\langle\rangle\}$ such that $q \not\perp M_{\eta(q)}$, but $q \perp M_{\eta(q)^-}$. If $\mathfrak{d}$ supports $S$, then we let $\mathrm{Field}(S) := \{\eta(q) : q \in S\}$ and $I^S := \{\nu \trianglelefteq \eta : \eta \in \mathrm{Field}(S)\}$.

Definition 6.7. Fix a subset $S \subseteq \mathbf{P}$, a model $M$, and a function $\Phi : \omega \to \omega$. We say that an eni-active decomposition $\mathfrak{d} = \langle M_\eta, a_\eta : \eta \in I \rangle$ of $M$ is $\mathbf{P}$*-finitely saturated* if, for every finite $A \subseteq M$ and $p \in S(A) \cap \mathbf{P}$, there is $\eta \in I$ such that $\mathrm{tp}(a_\eta / M_{\eta^-}) \not\perp p$.

The decomposition $\mathfrak{d}$ is $(S, \Phi)$*-simple* if

(1) $\mathfrak{d}$ is $\mathbf{P}$-finitely saturated;
(2) $\mathfrak{d}$ supports $S$ (hence $I^S$ is defined);
(3) for $\mu \in I$, define $E_\mu$ by $E_\mu(\eta, \nu) \Leftrightarrow \mathrm{tp}(a_\eta / M_\mu) = \mathrm{tp}(a_\nu / M_\mu)$;
(4) for all $\eta, \nu \in I^S$,
   (a) if $\eta^- = \nu^- = \mu$, then $E_\mu(\eta, \nu)$,
   (b) $\mathrm{Succ}_{I^S}(\eta)$ is empty or infinite (hence $I^S$ has infinite branching),
   (c) $\eta$ is either of some finite uniform depth or is a cusp,
   (d) if $\eta$ is an $(m, n)$-cusp, then $\Phi(m - n) = \lg(\eta)$.

Theorem 6.19 from [10], which we take as a black box, states:

Theorem 6.8. *Suppose $S \subseteq \mathbf{P}$, a model $M$, and a function $\Phi : \omega \to \omega$ are given. If $\mathfrak{d}_1$ and $\mathfrak{d}_2$ are both $(S, \Phi)$-simple decompositions of $M$, then the trees $I_1^S$ and $I_2^S$ are almost isomorphic.*

With our eye on applying Theorem 6.8, we massage the data we were given at the top of this section.

Let $\mathcal{U} = \{k \in \omega : k = k(i) \text{ for some } i\}$. As $\mathcal{U}$ is infinite, by passing to an infinite subset, we may additionally assume that if $n < m$ are from $\mathcal{U}$, then $m > 2n$. It follows from this that for all pairs $n < m$, $n' < m'$ from $\mathcal{U}$,

$$m - n = m' - n' \quad \text{if and only if} \quad m = m' \text{ and } n = n'.$$

Next, it is routine to partition $\mathcal{U}$ into infinitely many infinite sets $V_i$ for which $k > i$ for every $k \in V_i$.

Fix an integer $i$. An '$i$-tree' is a subtree of $\omega^{<\omega}$ with a unique 'stem' $\{\langle 0^j \rangle : j < i\}$ of length $i$. As an example, for each $k \in V_i$, let

$$I_i(k) := \{\eta \in \omega^{\leq k} : \text{for all } j < i, \text{ if } \lg(\eta) > j, \text{ then } \eta(j) = 0\}.$$

If $I$ and $J$ are both $i$-trees (say with disjoint universes), then the *free join of $I$ and $J$ over $i$, written $I \oplus_i J$,* is the $i$-tree with universe $(I \cup J)/\!\sim$, where for each $j < i$, the (unique) nodes of $I$ and $J$ of length $j$ are identified, and any other $\sim$-class is a singleton. To set notation, for $n < m$ from $V_i$, let $I_i(n, m) := I_i(n) \oplus_i I_i(m)$. We associate an eni-active decomposition

$$\mathfrak{d}(n,m) := \langle N_\eta, b_\eta : \eta \in I_i(n,m)\rangle$$

satisfying:

- for $\lg(\eta) < i$, $N_\eta = M_i$ and $b_\eta = a_i$; and
- if $k(\eta) = n$ when $\eta \in I_i(n)$ and $k(\eta) = m$ when $\eta \in I_i(m)$, then $N_\eta \cong N^{k(\eta)}_{\lg(\eta)}$ and $\mathrm{tp}(b_\nu/N_{\nu^-}) = \mathrm{tp}(b^{k(\nu)}_{\lg(\nu)}/N^{k(\nu)}_{\lg(\nu^-)})$.

In particular, as $\mathfrak{d}(n,m)$ is a decomposition, $\{N_\eta : \eta \in I_i(n,m)\}$ form an independent tree of models.

Still with $i$ fixed, for every $\delta \in \omega^i$ choose from $V_i$ disjoint, 4-element sets $\{n(\delta^+), m(\delta^+), n(\delta^-), m(\delta^-)\}$ satisfying $n(\delta^+) < m(\delta^+)$ and $n(\delta^-) < m(\delta^-)$.

Now, for each $\delta \in \omega^{<\omega}$, let $\mathrm{diff}(\delta^+) = m(\delta^+) - n(\delta^+)$ and $\mathrm{diff}(\delta^-) = m(\delta^-) - n(\delta^-)$. It follows from our thinness conditions on $\mathcal{U}$ (and the disjointness of the sets $V_i$) that the set $D = \{\mathrm{diff}(\delta^+), \mathrm{diff}(\delta^-) : \delta \in \omega^{<\omega}\}$ is without repetition. Let $\Phi : \omega \to \omega$ be any function such that for every $\delta \in \omega^{<\omega}$,

$$\Phi(\mathrm{diff}(\delta^+)) = \Phi(\mathrm{diff}(\delta^-)) = \lg(\delta).$$

To ease notation, for each $\delta \in \omega^{<\omega}$, let $I(\delta^+) = I_i(n(\delta^+), m(\delta^+))$ and $\mathfrak{d}(\delta^+) = \mathfrak{d}(n(\delta^+), m(\delta^+))$, with analogous definitions for $I(\delta^-)$ and $\mathfrak{d}(\delta^-)$.

Next, let $I_0 := (\lambda \times \omega)^{<\omega}$. We denote elements of $I_0$ by pairs $(\eta, \delta)$. Note that $\lg(\eta) = \lg(\delta)$ for all $(\eta, \delta) \in I_0$. Let $\mathfrak{d}_0$ denote the eni-active decomposition $\langle M_{(\eta,\delta)}, a_{(\eta,\delta)} : (\eta,\delta) \in I_0\rangle$, where $M_{(\eta,\delta)} \cong M_{\lg(\eta)}$ via a map $f_{(\eta,\delta)}$, and $f_{(\eta,\delta)}(a_{(\eta,\delta)}) = a_{\lg(\eta)}$.

With all of the above as a preamble, we are now ready to code subtrees of $\lambda^{<\omega}$ into models of our theory.

Theorem 6.9. ($T$ $\aleph_0$-stable, eni-NDOP, eni-deep) *For any $\lambda \geq \aleph_0$, there is a $\lambda$-Borel embedding $\mathcal{T} \mapsto M(\mathcal{T})$ of subtrees of $\lambda^{<\omega}$ into models of size $\lambda$ satisfying*

$$(\mathcal{T}_1, \trianglelefteq) \cong (\mathcal{T}_2, \trianglelefteq) \quad \textit{if and only if} \quad M(\mathcal{T}_1) \cong M(\mathcal{T}_2).$$

*Proof.* Fix a cardinal $\lambda \geq \aleph_0$. We describe the map $\mathcal{T} \mapsto M(\mathcal{T})$. Fix a subtree $\mathcal{T} \subseteq \lambda^{<\omega}$. Begin by letting $\delta_0(\mathcal{T})$ be the eni-active decomposition formed by starting with the decomposition $\mathfrak{d}_0$ and simultaneously adjoining a copy of $\mathfrak{d}(\delta^+)$ to every node $(\eta,\delta) \in I_0$ for which $\eta \in \mathcal{T}$, as well as adjoining a copy of $\mathfrak{d}(\delta^-)$ to every node $(\eta,\delta) \in I_0$ for which $\eta \notin \mathcal{T}$. Let $I_0(\mathcal{T})$ denote the index tree of $\mathfrak{d}_0(\mathcal{T})$. Let $M_0(\mathcal{T})$ be prime over $\bigcup\{N_\nu : \nu \in I_0(\mathcal{T})\}$. For each $\nu \in \max(I_0(\mathcal{T}))$, let $q_\nu \in S(N_\nu)$ be the ENI type conjugate to $q_{\lg(\nu)} \in S(N^{\lg(\nu)}_{\lg(\nu)})$ and let $S = \{q_\nu : \nu \in \max(I_0(\mathcal{T}))\}$. Because of the independence of the tree and the fact that $M_0(\mathcal{T})$ is prime over the tree, each $q_\nu$ has finite dimension in $M_0(\mathcal{T})$.

Next, we recursively construct an elementary chain $\langle M_n(\mathcal{T}) : n \in \omega\rangle$ and a sequence $\langle \mathfrak{d}_n(\mathcal{T}) : n \in \omega\rangle$ as follows. We have already defined $M_0(\mathcal{T})$ and $\mathfrak{d}_0(\mathcal{T})$, so assume $M_n(\mathcal{T})$ is defined and $\mathfrak{d}_n(\mathcal{T})$ is an eni-active decomposition of $M_n(\mathcal{T})$ extending $\mathfrak{d}_0(\mathcal{T})$. Let $R_n$ consist of all $p \in S(M_n(\mathcal{T})) \cap \mathbf{P}$ satisfying $p \perp S$. Let $J_n := \{a_p : p \in R_n\}$ be a $M_n(\mathcal{T})$-independent set of realizations of each $p \in R_n$. For each $p \in R_n$, there is a $\lhd$-minimal $\eta(p) \in I_n(\mathcal{T})$ such that $p \not\perp N_{\eta(p)}$. Let $N_p$ be prime over $N_{\eta(p)} \cup \{a_p\}$. Let $\mathfrak{d}_{n+1}(\mathcal{T})$ be the natural extension of $\mathfrak{d}_n(\mathcal{T})$ formed by affixing each $N_p$ as an immediate successor of $N_{\eta(p)}$, and let $M_{n+1}(\mathcal{T})$ be prime over the independent tree of models in $\mathfrak{d}_{n+1}(\mathcal{T})$.

Finally, let $\mathfrak{d}(\mathcal{T}) := \bigcup_{n\in\omega} \mathfrak{d}_n(\mathcal{T})$ and let $M(\mathcal{T})$ be prime over $\mathfrak{d}(\mathcal{T})$. As notation, let $I(\mathcal{T})$ denote the index tree of $\mathfrak{d}(\mathcal{T})$.

The following facts are easily established:

(1) A type $p \in S(M(\mathcal{T})) \cap \mathbf{P}$ has finite dimension in $M(\mathcal{T})$ if and only if $p \not\perp S$.
(2) $\mathfrak{d}(\mathcal{T})$ is $\mathbf{P}$-finitely saturated.
(3) $\mathfrak{d}(\mathcal{T})$ supports $S$ and $I^S(\mathcal{T}) = I_0(\mathcal{T})$.
(4) $I^S(\mathcal{T})$ is infinitely branching.
(5) For $\nu \in I^S(\mathcal{T})$:
   - $\nu$ is a cusp if and only if $\nu \in I_0$; in particular, if $\nu = (\eta, \delta)$ and if $\eta \in \mathcal{T}$, then $\nu$ is an $(m(\delta^+), n(\delta^+))$-cusp, and if $\eta \notin \mathcal{T}$, then $\nu$ is an $(m(\delta^-), n(\delta^-))$-cusp;
   - if $\nu \in I_0(\mathcal{T}) \setminus I_0$, then $\nu$ is of uniform finite depth.

In particular, $\mathfrak{d}(\mathcal{T})$ is an $(S, \Phi)$-simple decomposition of $M(\mathcal{T})$.

Main Claim. *If $M(\mathcal{T}_1) \cong M(\mathcal{T}_2)$, then $(\mathcal{T}_1, \unlhd) \cong (\mathcal{T}_2, \unlhd)$.*

*Proof.* Suppose that $f : M(\mathcal{T}_1) \to M(\mathcal{T}_2)$ is an isomorphism. Then the image of $\mathfrak{d}(\mathcal{T}_1)$ under $f$ is a decomposition of $M(\mathcal{T}_2)$ with index tree $I(\mathcal{T}_1)$. As well, $\mathfrak{d}(\mathcal{T}_2)$ is also a decomposition of $M(\mathcal{T}_2)$ with index tree $I(\mathcal{T}_2)$. If, for $\ell = 1, 2$, we let $S_\ell$ denote the non-orthogonality classes of ENI types of finite dimension in $M(\mathcal{T}_\ell)$, then as isomorphisms preserve types of finite dimension, $f(S_1) = S_2$ setwise. It follows that both $f(\mathfrak{d}_1)$ and $\mathfrak{d}_2$ are $(S_2, \Phi)$-simple decompositions of $M(\mathcal{T}_2)$. Thus, by Theorem 6.8, the trees $I_0(\mathcal{T}_1)$ and $I_0(\mathcal{T}_2)$ are almost isomorphic.

Fix large subtrees $J_\ell \subseteq I_0(\mathcal{T}_\ell)$ and a tree isomorphism $h : J_1 \to J_2$. Note that for $\ell = 1, 2$, a node $\nu \in J_\ell$ has uniform depth $k$ in $J_\ell$ if and only if $\nu$ has uniform depth $k$ in $I_0(\mathcal{T}_\ell)$. It follows that $h$ maps cusps to cusps, and more precisely, $(m, n)$-cusps to $(m, n)$-cusps. Thus, the restriction $h'$ of $h$ to $J_1 \cap (\lambda \times \omega)^{<\omega}$ is a tree isomorphism mapping onto $J_2 \cap (\lambda \times \omega)^{<\omega}$ that sends $(m, n)$-cusps to $(m, n)$-cusps. However, as the pairs $(m, n)$ uniquely identify $\delta \in \omega^{<\omega}$ and even $\delta^+$ and $\delta^-$, it follows that $h'(\eta, \delta) = (\eta^*, \delta)$ for every

$(\eta, \delta) \in \mathrm{dom}(h')$. Also, if we let

$$P_\ell := \{(\eta, \delta) \in J_\ell \cap (\lambda \times \omega)^{<\omega} : (\eta, \delta) \text{ is a } \delta^+\text{-cusp}\}$$

then $h'$ maps $P_1$ onto $P_2$ as well. Recalling that from our construction, $(\eta, \delta) \in P_\ell$ if and only if $\eta \in \mathcal{T}_\ell$, we see that for every $(\eta, \delta) \in \mathrm{dom}(h')$,

$$\text{if } h'(\eta, \delta) = (\eta^*, \delta), \text{ then } \eta \in \mathcal{T}_1 \text{ if and only if } \eta^* \in \mathcal{T}_2.$$

To finish, we recursively construct maps $h^* : \lambda^{<\omega} \to \lambda^{<\omega}$ and $\delta^* : \lambda^{<\omega} \to \omega^{<\omega}$ satisfying:

(6) $(\eta, \delta^*(\eta)) \in J_1$;
(7) $h^*(\eta) = \eta^*$ if and only if $h'(\eta, \delta^*(\eta)) = (\eta^*, \delta^*(\eta))$;
(8) for all $\eta$ and all $\alpha, \alpha' \in \lambda$, $\delta^*(\eta\hat{\ }\langle\alpha\rangle) = \delta^*(\eta\hat{\ }\langle\alpha'\rangle)$; and
(9) for all $\eta \in \lambda^{<\omega}$ and $\alpha, \beta \in \lambda$,

$$(\eta\hat{\ }\langle\alpha\rangle, \delta^*(\eta\hat{\ }\langle\alpha\rangle)) \in J_1 \quad \text{and} \quad (h^*(\eta)\hat{\ }\langle\beta\rangle, \delta^*(h^*(\eta)\hat{\ }\langle\beta\rangle)) \in J_2.$$

To accomplish this, first let $\delta^*(\langle\rangle) = \langle\rangle$. Because $\delta^*(\eta)$ is defined, the definition of $h^*(\eta)$ is given by clause (7). As $(\eta, \delta^*(\eta)) \in J_1$ and since $J_\ell$ are large subtrees of $I_0(\mathcal{T}_\ell)$, it follows that there is $\delta' \in \mathrm{Succ}(\delta^*(\eta))$ such that clauses (8) and (9) hold for all $\alpha, \beta \in \lambda$. Define $\delta^*(\eta\hat{\ }\langle\alpha\rangle) = \delta'$ for every $\alpha$ and define $h^*(\eta\hat{\ }\langle\alpha\rangle)$ according to clause (7).

It is easily checked that $h^* : \lambda^{<\omega} \to \lambda^{<\omega}$ is a tree isomorphism. Additionally, as $h'$ mapped $P_1$ onto $P_2$, it follows that the restriction of $h^*$ to $\mathcal{T}_1$ is a tree isomorphism between $(\mathcal{T}_1, \trianglelefteq)$ and $(\mathcal{T}_2, \trianglelefteq)$. This completes the proof of both the Main Claim and Theorem 6.9. ∎

Corollary 6.10. *If $T$ is $\aleph_0$-stable with eni-NDOP and is eni-deep, then $T$ is Borel complete. Moreover, for every infinite cardinal $\lambda$, $T$ is $\lambda$-Borel complete for $\equiv_{\infty,\aleph_0}$.*

*Proof.* If $T$ has eni-DOP, then this is literally Corollary 4.13. If $T$ has eni-NDOP, then the proof is exactly like the proof of Corollary 4.13, using Theorem 6.9 in place of Theorem 4.12. ∎

**7. Main gap for models of $\aleph_0$-stable theories modulo $L_{\infty,\aleph_0}$-equivalence.** In this brief section, we combine our previous results to exhibit a dichotomy among $\aleph_0$-stable theories.

Definition 7.1. For $T$ any theory and $\lambda$ an infinite cardinal, let $\mathrm{Mod}_\lambda(T)$ denote the set of models of $T$ with universe $\lambda$.

For $T$ any theory and $\lambda$ any cardinal, $I_{\infty,\aleph_0}(T, \lambda)$ denotes the maximum cardinality of any pairwise non-$\equiv_{\infty,\aleph_0}$ collection from $\mathrm{Mod}_\lambda(T)$.

For any $M \models T$ of size $\lambda$, the *Scott height* $\mathrm{SH}(M)$ *of* $M$, is the least ordinal $\alpha < \lambda^+$ such that for any model $N$, $N \equiv_\alpha M$ implies $N \equiv_{\alpha+1} M$.

THEOREM 7.2. *The following conditions are equivalent for any $\aleph_0$-stable theory $T$:*

(1) *For all infinite cardinals $\lambda$, $I_{\infty,\aleph_0}(T,\lambda) = 2^\lambda$.*
(2) *For all infinite cardinals $\lambda$,* $\sup\{\mathrm{SH}(M) : M \in \mathrm{Mod}_\lambda(T)\} = \lambda^+$.
(3) *$T$ either has eni-DOP or is eni-deep.*

*Proof.* The equivalence (1)$\Leftrightarrow$(2) is the content of [7].

(3)$\Rightarrow$(1). Fix any infinite cardinal $\lambda$. If $T$ has either of the properties in (3), then by Corollary 4.13 or Corollary 6.10, $T$ is $\lambda$-Borel complete. However, it is well known (see e.g. [8]) that there is a family of $2^\lambda$ pairwise non-$\equiv_{\infty,\aleph_0}$ directed graphs with universe $\lambda$. It follows immediately that $I_{\infty,\aleph_0}(T,\lambda) = 2^\lambda$ in either case.

(1)$\Rightarrow$(3). Assume that $T$ is $\aleph_0$-stable, has eni-NDOP and is *eni-shallow* (i.e., not eni-deep). Then, by Corollary 5.9, models of $T$ are determined up to $\equiv_{\infty,\aleph_0}$-equivalence by their prime, eni-active decompositions. Thus, it suffices to count the number of prime, eni-active decompositions up to isomorphism ([1]).

To obtain this count, first note that if $T$ is eni-shallow, then as in [6, X, Theorem 4.4] (which builds on [6, VII, Section 5]), the depth of any index tree of an eni-active decomposition is an ordinal $\beta < \omega_1$. In any prime decomposition, each of the models $M_\eta$ is countable, hence there are at most $2^{\aleph_0}$ isomorphism types. So, as a weak upper bound, if $\lambda = \aleph_\alpha$, then the number of prime, eni-active decompositions of depth $\beta$ of a model of size $\lambda$ is bounded by $\beth_{(|\alpha|+|\beta|)^+}$. [Similar counting arguments appear in [6, X, Theorem 4.7].] From this, we conclude that for some cardinals $\lambda$, $I_{\infty,\aleph_0}(T,\lambda) < 2^\lambda$. ∎

**Appendix: Packing problems for bipartite graphs.** A *bipartite* graph $A$ consists of a set of vertices which are partitioned into two sets $L(A)$ and $R(A)$, together with a binary, irreflexive edge relation $E(A) \subseteq L(A) \times R(A)$. We say that $A$ is *complete bipartite* if the set of edges satisfies $E(A) = L(A) \times R(A)$. We call $A$ *balanced* if $\big||L(A)| - |R(A)|\big| \le 1$.

Define a function $e^* : \omega \to \omega$ by $e^*(2b) = b^2$ and $e^*(2b+1) = b(b+1)$ for all $b \in \omega$. A classical packing problem asserts:

FACT A.1. *A bipartite graph $A$ with at most $c \ge 2$ vertices has at most $e^*(c)$ edges, with equality holding if and only if $|A| = c$ and $A$ is complete and balanced.*

([1]) We say that two eni-active decompositions $\mathfrak{d}_1 = \langle M^1_\eta, a^1_\eta : \eta \in I_1\rangle$ and $\mathfrak{d}_2 = \langle M^2_\eta, a^2_\eta : \eta \in I_2\rangle$ are *isomorphic* if there is a tree isomorphism $f : (I_1, \trianglelefteq) \cong (I_2, \trianglelefteq)$ and an elementary bijection $f^* : \bigcup_{\eta\in I_1} M^1_\eta \to \bigcup_{\eta\in I_2} M^2_\eta$ such that, for each $\eta \in I_1$, $f^*|_{M^1_\eta}$ maps $M^1_\eta$ isomorphically onto $M^2_{f(\eta)}$.

For a bipartite graph $A$, let $v(A)$, $e(A)$, and $\mathrm{CC}(A)$ denote the number of vertices, edges, and connected components of $A$, respectively. Recall that in the discussion prior to the statement of Proposition 4.4, we defined an $m_1 \times m_2$ bipartite graph $A$ to be almost $\ell$-complete if $|m_i - \ell| \leq 0.01\ell$ for each $i = 1, 2$ and each vertex has valence at least $0.9\ell$. We aim to establish the following Fact.

Fact A.2. *Suppose that $N$ is a given integer and $\ell \gg N$ (explicit bounds on $\ell$ in terms of $N$ can be found from the proof). If $A$ is any bipartite graph with $v(A) \leq 2\ell + N$ and $e(A) \geq \ell^2 - N$, then $A$ is almost $\ell$-complete.*

The new statistic we investigate in this Appendix is $k(A)$, which we define to be $v(A) - \mathrm{CC}(A)$. Two special cases are that $v(A) = k(A) + 1$ for any connected bipartite graph $A$, and that any null bipartite graph $B$ has $k(B) = 0$.

We wish to find an analogue of Fact A.2 in which the upper bound on $v(A)$ is replaced by an upper bound on $k(A)$.

If $A$ and $B$ are each bipartite graphs with disjoint sets of vertices, then $A \amalg B$ denotes their *disjoint union.* It is the bipartite graph $C$ whose vertices are the union of the vertices of $A$ and $B$, and $E(C) = E(A) \cup E(B)$.

Note that all of our statistics are additive with respect to disjoint unions. For example, for $x \in \{n, e, \mathrm{CC}, k\}$, $x(A \amalg B) = x(A) + x(B)$. Thus, if $A$ is any bipartite graph and $B$ is null, then $k(A \amalg B) = k(A)$. The proof of the following lemma is routine.

Lemma A.3. *Suppose $A$ and $B$ are disjoint, and are each complete, balanced, bipartite graphs with $k(A) \geq k(B) \geq 1$. Let $A^+$ and $B^-$ be disjoint, complete, balanced bipartite graphs with $k(A^+) = k(A) + 1$ and $k(B^-) = k(B) - 1$. Then $k(A^+ \amalg B^-) = k(A \amalg B)$ and $e(A^+ \amalg B^-) \geq e(A \amalg B)$.*

One corollary is immediate by combining Fact A.1 with Lemma A.3.

Corollary A.4. *For all positive integers $a$ and all bipartite graphs $A$ with $k(A) \leq a$, we have $e(A) \leq e^*(a + 1)$, with equality holding if and only if $A = B \amalg C$, with $B$ complete and balanced, and $C$ null (possibly empty).*

Next, given a pair of integers $c, d$, let $f(c, d)$ be the least integer such that $e(A) \leq f(c, d)$ for all bipartite graphs of the form $A = B \amalg C$, where $k(B) \leq c$ and $k(C) \leq d$.

Lemma A.5.

(1) *For all $c, d \in \omega$, $f(c, d) = e^*(c + 1) + e^*(d + 1)$.*
(2) *If $1 \leq d \leq c$, then $f(c + 1, d - 1) \geq f(c, d)$.*

*Proof.* The first statement follows by applying Fact A.1 to each of $B$ and $C$, while the second follows from Lemma A.3. ∎

PROPOSITION A.6. *If $\ell > W^2/4$ and $A$ is a bipartite graph satisfying $k(A) \le 2\ell + W$ and $e(A) \ge \ell^2$, then $A$ contains a connected subgraph $B \subseteq A$ with at least $\ell^2 - W^2/4$ edges and at most $2\ell + W$ vertices.*

*Proof.* Let $\Phi$ be the set of all $A$ such that $k(A) \le 2\ell + W$ and $A$ does not have any connected component $B$ with $k(B) \ge 2\ell - 1$. Among all such $A$, choose $A^* \in \Phi$ so as to maximize the number of edges $e(A^*)$. By Lemma A.5,

$$\begin{aligned} e(A^*) &\le f(2\ell - 1, W + 1) = e^*(2\ell - 1) + e^*(W + 1) \\ &\le \ell(\ell - 1) + W^2/4 < \ell^2. \end{aligned}$$

Thus, our given graph $A$ is not in $\Phi$, so $A$ has a connected component $B$ with $k(B) \ge 2\ell - 1$. Now, if we decompose $A$ as $A = B \amalg C$, then $k(C) \le W + 1$, so by Lemma A.3, $e(C) \le e^*(W + 1) \le W^2/4$.

Since $\ell^2 \le e(A) = e(B) + e(C)$, this implies $e(B) \ge \ell^2 - W^2/4$. But, since $B$ is connected, $v(B) = k(B) + 1$ and $k(B) \le k(A) \le 2\ell + W$, so $B$ has at most $2\ell + W$ vertices. ∎

The final corollary follows by combining Proposition A.6 with Fact A.2.

COROLLARY A.7. *If $\ell \gg W$ and $A$ is a bipartite graph satisfying $k(A) \le 2\ell + W$ and $e(A) \ge \ell^2$, then $A$ has an almost $\ell$-complete subgraph.*

**Acknowledgments.** The first author was partially supported by NSF grants DMS-0600217 and 0901334.

The second author was partially supported by U.S.-Israel Binational Science Foundation Grant no. 2002323 and Israel Science Foundation Grant no. 242/03. Publication no. 1016.

Both authors were partially supported by NSF grants DMS-0600940 and 1101597.

## References

[1] E. Bouscaren and D. Lascar, *Countable models of nonmultidimensional $\aleph_0$-stable theories*, J. Symbolic Logic 48 (1983), 197–205.

[2] H. Friedman and L. Stanley, *A Borel reducibility theory for classes of countable structures*, J. Symbolic Logic 54 (1989), 894–914.

[3] M. Koerwien, *Comparing Borel reducibility and depth of an $\omega$-stable theory*, Notre Dame J. Formal Logic 50 (2009), 365–380.

[4] M. C. Laskowski, *An old friend revisited: Countable models of $\omega$-stable theories*, Notre Dame J. Formal Logic 48 (2007), 133–141.

[5] M. Makkai, *A survey of basic stability theory, with particular emphasis on orthogonality and regular types*, Israel J. Math. 49 (1984), 181–238.

[6] S. Shelah, *Classification Theory*, rev. ed., North-Holland, Amsterdam, 1990.

[7] S. Shelah, *On the number of non-almost isomorphic models of $T$ in a power*, Pacific J. Math. 36 (1971), 811–818.

[8] S. Shelah, *Existence of many $L_{\infty,\lambda}$-equivalent, nonisomorphic models of $T$ of power $\lambda$*, Ann. Pure Appl. Logic 34 (1987), 291–310.

[9] S. Shelah, *Characterizing an $\aleph_\epsilon$-saturated model of superstable NDOP theories by its $\mathbb{L}_{\infty,\aleph_\epsilon}$-theory*, Israel J. Math. 140 (2004), 61–111.

[10] S. Shelah and M. C. Laskowski, **P**-*NDOP and* **P**-*decompositions of $\aleph_\epsilon$-saturated models of superstable theories*, Fund. Math. 229 (2015), 47–81.

[11] S. Shelah, L. A. Harrington, and M. Makkai, *A proof of Vaught's conjecture for $\omega$-stable theories*, Israel J. Math. 49 (1984), 259–280.

Michael C. Laskowski
Department of Mathematics
University of Maryland
College Park, MD 20742, U.S.A.
E-mail: mcl@math.umd.edu

Saharon Shelah
Department of Mathematics
The Hebrew University of Jerusalem
Einstein Institute of Mathematics
Edmond J. Safra Campus, Givat Ram
Jerusalem, 91904, Israel
and
Department of Mathematics
Hill Center, Busch Campus
Rutgers, the State University of New Jersey
110 Frelinghuysen Road
Piscataway, NJ 08854-9019, U.S.A.
E-mail: shelah@math.huji.ac.il

*Received 2 November 2012;*
*in revised form 20 December 2013 and 8 May 2014*