P-NDOP and P-decompositions of
$\aleph_\epsilon$-saturated models of superstable theories

by

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Abstract. Given a complete, superstable theory, we distinguish a class $P$ of regular types, typically closed under automorphisms of $C$ and non-orthogonality. We define the notion of $P$-NDOP, which is a weakening of NDOP. For superstable theories with $P$-NDOP, we prove the existence of $P$-decompositions and derive an analog of the first author’s result in Israel J. Math. 140 (2004). In this context, we also find a sufficient condition on $P$-decompositions that implies non-isomorphic models. For this, we investigate natural structures on the types in $P \cap S(M)$ modulo non-orthogonality.

1. Introduction. Results by the first author, most notably Chapter X of [4] and the first half of [5], demonstrate that $\aleph_\epsilon$-saturated models of superstable theories with NDOP admit very desirable decompositions. In this paper, we generalize these results in three ways. First, we always assume that the theory $T$ is superstable, but we only have NDOP for a class $P$ of regular types. Second, we show that the tree structure of a decomposition of an $\aleph_\epsilon$-saturated model $M$ can be read off from the non-orthogonality classes of regular types in $S(M)$. Third, we show that these results for $\aleph_\epsilon$-saturated models give information about weak decompositions of arbitrary models of such theories.

In more detail, throughout the paper we assume we have a fixed, complete, superstable theory and we work within a monster model $C$. We fix a set $P$ of stationary, regular types over small subsets of $C$ that is closed under automorphisms of $C$ and the equivalence relation of non-orthogonality, and additionally assume that our theory satisfies $P$-NDOP. Typically, we fix a model $M$ that is at least $\aleph_\epsilon$-saturated (i.e., $M$ contains a realization of every strong type over every finite subset of $M$) and study $P$-decompositions inside $M$ of many varieties. Of primary interest are prime, $(\aleph_\epsilon, P)$-decompositions.

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\(d\) of \(M\) over \(\binom{B}{A}\) (see Definition 4.16) where \(A \subseteq B\) are ɛ-finite and every regular type \(p\) non-orthogonal to \(\text{stp}(B/A)\) is in \(P\). We associate a subset \(\mathcal{P}_P(d, M)\) of \(S(M) \cap P\) (see Definition 5.1) to such a pair. The main result of the paper, Theorem 5.12, asserts that this set of regular types depends only on \(\binom{B}{A}\). In particular, it is independent of the decomposition \(d\), and successive results show that these sets have a tree structure under inclusion.

In the final section of the paper, we show how this result, which holds only for \(\aleph_\varepsilon\)-saturated models, gives positive information for much weaker decompositions of models \(M_0\) without any saturation assumption.

2. Preliminaries. As mentioned above, we always work in a class of models of a complete, superstable, first-order theory \(T\). We fix a monster model \(\mathfrak{C}\), and all models and sets we discuss will be small subsets of \(\mathfrak{C}\). We assume that \(T\) eliminates quantifiers, so any model \(M\) will be an elementary submodel of \(\mathfrak{C}\), and we additionally assume that ‘\(T = T^{eq}\)’, so that every type over an algebraically closed set is stationary.

Definition 2.1. A set \(A\) is ɛ-finite if \(\text{acl}(A) = \text{acl}(a)\) for some \(a \in \mathfrak{C}^{eq}\).

Recall that as we are working in \(\mathfrak{C}^{eq}\), it would be equivalent to say that \(\text{acl}(A) = \text{acl}(\bar{a})\) for some finite tuple. It is easily seen that the union of two ɛ-finite sets is ɛ-finite. Furthermore, since \(T\) is superstable, any subset \(B \subseteq A\) of an ɛ-finite set is ɛ-finite. [Why? If \(B \subseteq A\) with \(\text{acl}(A) = \text{acl}(a)\), choose a finite \(\bar{b}\) from \(B\) such that \(B \upharpoonright \bar{a}\). Then \(\text{acl}(B) = \text{acl}(\bar{b})\).] Thus, the set of ɛ-finite subsets of \(\mathfrak{C}\) form an ideal.

Convention 2.2. \(\aleph_\varepsilon\) is a cardinal strictly between \(\aleph_0\) and \(\aleph_1\).

Thus, if we write ‘\(M\) is \(\lambda\)-saturated for some \(\lambda \geq \aleph_\varepsilon\)’, we mean that either \(M\) is \(\aleph_\varepsilon\)-saturated (i.e., realizes all types over ɛ-finite subsets) or \(M\) is \(\lambda\)-saturated for some \(\lambda \geq \aleph_1\). Recall that by e.g. [4 IV, 2.2(7)], for \(\lambda \geq \aleph_1\), \(M\) is \(\lambda\)-saturated if and only if for every subset \(A \subseteq M\) of size less than \(\lambda\), \(M\) realizes every type over \(\text{acl}(A)\).

We record several facts from [4] that will be used throughout this paper. The first is the Second Characterization Theorem, IV, 4.18, the second is X, Claim 1.6(5), the third is V, 1.12, and (4) follows easily from (2) and (3).

Fact 2.3. Suppose \(T\) is superstable and \(\lambda \geq \aleph_\varepsilon\).

(1) A model \(M\) is \(\lambda\)-prime over a set \(A\) if and only if (a) \(M \supseteq A\) and is \(\lambda\)-saturated; (b) \(M\) is \(\lambda\)-atomic over \(A\); and (c) every \(A\)-indiscernible sequence \(I \subseteq M\) has length at most \(\lambda\). (When \(\lambda = \aleph_\varepsilon\), the \(\lambda\) occurring in (c) should be replaced by \(\aleph_0\).)

(2) If \(M\) is \(\lambda\)-saturated, \(A \supseteq M\), and \(N\) is \(\lambda\)-prime over \(M \cup A\), then \(N\) is dominated by \(A\) over \(M\).
(3) If $M \preceq N$ are both $\lambda$-saturated, $p \in S(M)$ is regular, and there is some $c \in N \setminus M$ such that $tp(c/M) \not\perp p$, then $p$ is realized in $N$.

(4) If $M_0 \preceq M_1 \preceq M_2$ are all $\lambda$-saturated and there is $e \in M_2 \setminus M_1$ such that $tp(e/M_1)$ is regular and non-orthogonal to $M_0$, then there is $e^* \in M_2 \setminus M_1$ such that $e$ and $e^*$ are domination equivalent over $M_1$, with $e^* \perp M_1$.

3. **P-NDOP.** Our story begins by localizing the notion of DOP around a single parallelism class of stationary, regular types.

**Definition 3.1.** An independent triple of models $(M_0, M_1, M_2)$ are such that $M_0 = M_1 \cap M_2$ and $\{M_1, M_2\}$ are independent over $M_0$. For $\lambda \geq \aleph_\epsilon$, a $\lambda$-quadruple is a sequence $(M_0, M_1, M_2, M_3)$ of $\lambda$-saturated models, where $(M_0, M_1, M_2)$ form an independent triple, and $M_3$ is $\lambda$-prime over $M_1 \cup M_2$. A $\lambda$-DOP witness for a stationary, regular type $p$ is a $\lambda$-quadruple $(M_0, M_1, M_2, M_3)$ for which $Cb(p) \subseteq M_3$, but $p \perp M_1$ and $p \perp M_2$. We say that $p$ has a DOP witness if it has a $\lambda$-DOP witness for some $\lambda \geq \aleph_\epsilon$.

Visibly, whether a specific $\lambda$-quadruple is a $\lambda$-DOP witness for $p$ depends only on the parallelism class of $p$. To understand the consequences of this notion, we recall that a set $A$ is self-based on an independent triple $(M_0, M_1, M_2)$ of models if $A \not\perp_{M_3}^M M_i$ holds for each $i < 3$. The concept of self-basedness was defined explicitly in [1] and was used implicitly in the proof of [4, X, 2.2(c)]. The fact that, for any independent triple $(M_0, M_1, M_2)$, any finite set $A$ can be extended to a finite, self-based $B \subseteq AM_1M_2$ follows from [1, Lemma 2.4]. The Claim in [1] proof of Theorem 1.3 establishes the following fact.

**Fact 3.2.** If $A$ is self-based on the independent triple $(M_0, M_1, M_2)$, $p \in S(A)$ is stationary, $p \perp M_1$, and $p \perp M_2$, then $p \vdash p|AM_1M_2$.

Using this fact, an easy examination of the proof of [4, X, 2.2] yields:

**Fact 3.3.** Let $p$ be any stationary, regular type with a DOP witness. Then:

1. For every $\lambda \geq \aleph_\epsilon$, $p$ has a $\lambda$-DOP witness.
2. For every $\lambda$-DOP witness $(M_0, M_1, M_2, M_3)$ for $p$, there is an infinite, indiscernible set $I \subseteq M_3$ over $M_1 \cup M_2$ whose average type $Av(I, M_3)$ is parallel to $p$.
3. For every $\lambda$-DOP witness $(M_0, M_1, M_2, M_3)$ for $p$, there is a subset $A \subseteq M_3$, $|A| < \lambda$, over which $p$ is based and stationary, and a Morley sequence $\langle b_i : i < \lambda \rangle$ from $M_3$ in $p|AM_1M_2$.

We isolate one corollary from this that will be crucial for us later.
Corollary 3.4. For any $\lambda \geq \aleph_\epsilon$, if $(M_0, M_1, M_2, M_3)$ is a $\lambda$-DOP witness for a stationary, regular $p \in S(M_3)$, then for any realization $c$ of $p$, any $\lambda$-prime model $M_3[c]$ over $M_3 \cup \{c\}$ is isomorphic to $M_3$ over $M_1 \cup M_2$. In particular, $M_3[c]$ is $\lambda$-prime over $M_1 \cup M_2$.

Proof. By the uniqueness of $\lambda$-prime models, both statements will follow once we establish that $M_3 \cup \{c\}$ is the universe of a $\lambda$-construction sequence over $M_1 \cup M_2$. To see this, first fix a $\lambda$-construction sequence $(b_i : i < \delta)$ of $M_3$ over $M_1 \cup M_2$. For each $i < \delta$, write $B_i = M_1 \cup M_2 \cup \{b_j : j < i\}$ and fix a subset $X_i \subseteq B_i$, $|X_i| < \lambda$ such that $stp(b_i/X_i) \vdash stp(b_i/B_i)$.

Next, choose a subset $A \subseteq M_3$, $|A| < \lambda$, over which $p$ is based and stationary. By forming an increasing $\omega$-chain, we can increase $A$ slightly (still maintaining $|A| < \lambda$) so that $A$ is self-based on $(M_0, M_1, M_2)$ and $X_i \subseteq A$ whenever $b_i \in A$.

Let $(a_i : i < \gamma)$ be the enumeration of $A$ given by the ordering of the original construction. Easily, $(a_i : i < \gamma)$ is $\lambda$-constructible over $M_1 \cup M_2$.

Furthermore, it follows from Fact 3.2 that for any Morley sequence $I$ in $p|A$ with $|I| < \lambda$, we have $p|AI \vdash p|AIM_1M_2$. Using this, we have a $\lambda$-construction sequence $(a_i : i < \gamma)\langle c_j : j < \lambda \rangle$ over $M_1 \cup M_2$, where $(c_j : j < \lambda)$ is any Morley sequence in $p|A$ from $M_3$ (the existence of such a sequence follows from Fact 3.3(3)). From the uniqueness of $\lambda$-prime models and the fact that such models are $\lambda$-constructible we deduce that there is another $\lambda$-construction sequence of $M_3$ over $M_1 \cup M_2$ in which $(a_i : i < \gamma)\langle c_j : j < \lambda \rangle$ is an initial segment. Let $(b_k : k < \nu)$ be the tail of this sequence. For each $k < \nu$, let $B^*_k = M_1 \cup M_2 \cup A \cup \{c_j : j < \lambda\} \cup \{b_\ell : \ell < k\}$ and choose $Y_k \subseteq B^*_k$, $|Y_k| < \lambda$, such that $stp(b_k/Y_k) \vdash stp(b_k/B^*_k)$. Without loss of generality, we may assume $A \subseteq Y_k$ for each $k$. To complete the proof, it suffices to show that 

$$(a_i : i < \gamma)\langle c \rangle \langle c_i : i < \lambda \rangle \langle b_k : k < \nu \rangle$$

is a $\lambda$-construction sequence over $M_1 \cup M_2$.

We already know that $(a_i : i < \gamma)$ is a $\lambda$-construction sequence over $M_1 \cup M_2$. Using the first sentence of the previous paragraph, combined with the fact that $\{c\} \cup \{c_j : j < \lambda\}$ is independent over $A$, we inductively show that $(a_i : i < \gamma)\langle c \rangle \langle c_j : j < \lambda \rangle$ is also a $\lambda$-construction sequence over $M_1 \cup M_2$. Thus, it suffices to prove that $stp(b_k/Y_k) \vdash stp(b_k/B^*_k)$ for each $k < \nu$. For this, since both $tp(c/B^*_k)$ and $tp(b_k/B^*_k)$ do not fork over $Y_k$, it suffices to show that $tp(c/Y_k)$ is almost orthogonal to $stp(b_k/Y_k)$. To see this, choose $j < \lambda$ such that $tp(c_j/A)$ does not fork over $Y_k$. Now, $tp(c/Y_k) = tp(c_j/Y_k)$ and $tp(c_j/Y_k)$ is almost orthogonal to $stp(b_k/Y_k)$ since $stp(b_k/Y_k) \vdash stp(b_k/Y_kc_j)$, so we finish.

Next, we show additional closure properties of DOP witnesses.
**Definition 3.5.** A regular type $q$ lies directly above $p$ if there is a non-forking extension $p' \in S(M)$ of $p$ with $M \not\subseteq S(M)$, $\forall$-saturated, a realization $c$ of $p'$, and an $\aleph$-prime model $M[c]$ over $M \cup \{c\}$ such that $q \not\subseteq M[c]$, but $q \perp M$. A regular type $q$ lies above $p$ if there is a sequence $p_0, \ldots, p_n$ of types such that $p_0 = p$, $p_n = q$, and $p_{i+1}$ lies directly above $p_i$ for each $i < n$. (We allow $n = 0$, so in particular, any regular type lies above itself.) We say that $p$ supports $q$ if $q$ lies above $p$.

The nomenclature above is apt if one considers a branch of a decomposition tree. Suppose $M_0 \preceq \cdots \preceq M_n$ is a sequence of $\aleph$-saturated models such that for each $i < n$ there is $a_i \in M_{i+1}$ such that $tp(a_i/M_i)$ is regular (and orthogonal to $M_i$ when $i > 0$) and $M_{i+1}$ is $\aleph$-prime over $M_i \cup \{a_i\}$. Then any regular $q \not\subseteq M_n$ lies over any regular type $p$ non-orthogonal to $tp(a_0/M_0)$. Similarly, any such $p$ supports any such $q$.

**Proposition 3.6.** Fix a stationary, regular type $p$ with a DOP witness. Then:

1. Every type parallel to $p$ has a DOP witness.
2. Every automorphic image of $p$ has a DOP witness.
3. Every stationary, regular $q$ non-orthogonal to $p$ has a DOP witness.
4. Every stationary, regular $q$ lying above $p$ has a DOP witness.

**Proof.** (1) and (2) are immediate. For (3), choose $\lambda \geq \aleph$ and a $\lambda$-quadruple $(M_0, M_1, M_2, M_3)$ witnessing that $p$ has $\lambda$-DOP. Let $q$ be any stationary, regular type non-orthogonal to $p$. As $q$ is non-orthogonal to $M_3$, there is $q' \in S(M_3)$ non-orthogonal to $q$ (and hence to $p$) and conjugate to $q$. But now, $q' \perp M_1$ and $q' \perp M_2$, so $(M_0, M_1, M_2, M_3)$ witnesses that $q'$ has $\lambda$-DOP. Thus, $q$ has a DOP witness by (2).

(4) It suffices to prove this for $q$ lying directly above $p$. As both notions are parallelism invariant, we may assume that $p \in S(N)$, where $N$ is $\aleph$-saturated. Choose $c$ realizing $p$ and $N[c]$ $\aleph$-prime over $N \cup \{c\}$ such that $q \not\subseteq N[c]$, but $q \perp N$. Choose $q' \in S(N[c])$ non-orthogonal to $q$. Fix a cardinal $\lambda > |N|$, and choose a $\lambda$-DOP witness $(M_0, M_1, M_2, M_3)$ for $p$. Without loss of generality, we may assume that $N \preceq M_3$ and that $c \perp M_3$. Let $M^*$ be $\lambda$-prime over $N[c] \cup M_3$ and let $q^*$ be the non-forking extension of $q'$ to $M^*$. We argue that $(M_0, M_1, M_2, M^*)$ is a $\lambda$-DOP witness for $q^*$.

To see this, first note that $N[c]$ is $\aleph$-constructible over $N \cup \{c\}$, $N$ is $\aleph$-saturated, and $c \perp M_3$, so $N[c]$ is $\aleph$-constructible (hence $\lambda$-constructible) over $M_3 \cup \{c\}$. Since $M^*$ is $\lambda$-constructible over $N[c] \cup M_3$, it follows that $M^*$ is $\lambda$-constructible over $M_3 \cup \{c\}$, hence is $\lambda$-prime over $M_3 \cup \{c\}$. Thus, by Corollary 3.4, $M^*$ is $\lambda$-prime over $M_1 \cup M_2$. That is, $(M_0, M_1, M_2, M^*)$ is a $\lambda$-quadruple.
As well, $q' \in S(N[c])$ is orthogonal to $N$ and $N[c] \perp M_3$, so $q' \perp M_3$. As $M_1 \cup M_2 \subseteq M_3$, it follows immediately that $q^* \perp M_1$ and $q^* \perp M_2$. 

Throughout the remainder of this paper, we consider sets $\mathbf{P}$ of stationary, regular types over small subsets of the monster model $\mathfrak{C}$. We typically require $\mathbf{P}$ to be closed under automorphisms of $\mathfrak{C}$ and non-orthogonality.

**Definition 3.7.** Let $\mathbf{T}_{\text{reg}}$ denote the set of all stationary, regular types over small subsets of $\mathfrak{C}$ and fix a subset $\mathbf{P} \subseteq \mathbf{T}_{\text{reg}}$ that is closed under automorphisms and non-orthogonality.

As notation:

- A stationary type $q$ is orthogonal to $\mathbf{P}$, written $q \perp \mathbf{P}$, if $q$ is orthogonal to every $p \in \mathbf{P}$. We write $\mathbf{P}^{\perp} = \{ q \in \mathbf{T}_{\text{reg}} : q \perp \mathbf{P} \}$.
- $\mathbf{P}^{\text{active}}$ is the closure of $\mathbf{P}$ in $\mathbf{T}_{\text{reg}}$ under automorphisms, non-orthogonality, and supporting (i.e., if $p \in \mathbf{T}_{\text{reg}}$ supports some $q \in \mathbf{P}$, then $p \in \mathbf{P}^{\text{active}}$).
- $\mathbf{P}^{\text{dull}} = \mathbf{T}_{\text{reg}} \setminus \mathbf{P}^{\text{active}}$.

**Definition 3.8.** Let $\mathbf{P} \subseteq \mathbf{T}_{\text{reg}}$ be any set of regular types. A theory $T$ has $\mathbf{P}$-NDOP if no $p \in \mathbf{P}$ has a DOP witness.

The following corollary is merely a restatement of Proposition 3.6.

**Corollary 3.9.** For any $\mathbf{P} \subseteq \mathbf{T}_{\text{reg}}$, $T$ has $\mathbf{P}$-NDOP if and only if $T$ has $\mathbf{P}^{\text{active}}$-NDOP.

**Definition 3.10.** Given a class $\mathbf{P}$ of regular types, we define the $\mathbf{P}$-depth of a stationary, regular type $p$, $\text{dp}_\mathbf{P}(p) \in \text{ON} \cup \{-1\}$, by (1) $\text{dp}_\mathbf{P}(p) = -1$ if and only if $p \in \mathbf{P}^{\text{dull}}$; and (2) $\text{dp}_\mathbf{P}(p) \geq \alpha$ if and only if $p \in \mathbf{P}^{\text{active}}$ and for every $\beta \in \alpha$ there is a triple $(M,N,a)$ where $M$ is $\aleph_\epsilon$-saturated, $N$ is $\aleph_\epsilon$-prime over $M \cup \{a\}$, $p$ is parallel to $\text{tp}(a/M)$, and there is $q \in S(N)$ orthogonal to $M$ with $\text{dp}_\mathbf{P}(q) \geq \beta$.

As in Chapter X of [4], in the preceding definition it would be equivalent to replace ‘$\aleph_\epsilon$-saturation’ by ‘$\lambda$-saturation’ for any uncountable cardinal $\lambda$. The proof of the following lemma is identical to that of [4, X, Lemma 7.2].

**Lemma 3.11.** If $T$ has $\mathbf{P}$-NDOP, then any regular $p$ with $\text{dp}_\mathbf{P}(p) > 0$ is trivial, i.e., the set $p(\mathfrak{C})$ has a trivial pre-geometry with respect to the dependence relation of forking.

We close this section with two technical lemmas that will be used later. Note that a type $q$ (not necessarily regular) is orthogonal to $\mathbf{P}^{\text{dull}}$ if and only if every regular type non-orthogonal to $q$ is an element of $\mathbf{P}^{\text{active}}$.

**Lemma 3.12.** ($\mathbf{P}$-NDOP, $\lambda \geq \aleph_\epsilon$) Suppose that $M$ is $\lambda$-prime over an independent triple $(M_0,M_1,M_2)$ of $\lambda$-saturated models, $a$ is $\epsilon$-finite satisfying $\text{tp}(a/M) \perp \mathbf{P}^{\text{dull}}$ and $\text{tp}(a/M) \perp M_2$. Let $M[a]$ be any $\lambda$-prime model
over $M \cup \{a\}$. For any subset $N \subseteq M[a]$ that is maximal such that $N \perp_{M_1} M$ we have:

1. $N \preceq M[a]$, $N$ is $\lambda$-saturated, and $M[a]$ is $\lambda$-prime over $N \cup M$.
2. For any $a^* \subseteq N$ such that $N \perp_{M_1 a^*} a$, $N$ is $\lambda$-prime over $M_1 \cup \{a^*\}$.

Proof. To see that $N \preceq M[a]$ and $N$ is $\lambda$-saturated, choose $N^+ \preceq M[a]$ to be $\lambda$-prime over $N$. As $M_1$ is $\lambda$-saturated, it follows from Fact 2.3(2) that $N^+$ is dominated by $N$ over $M_1$, hence $N^+ \perp_{M_1} M$, so $N^+ = N$ by the maximality of $N$.

Next, choose $M^* \preceq M[a]$ to be maximal such that $M^*$ is $\lambda$-saturated and $\lambda$-atomic over $N \cup M$. (Since $T$ is superstable, the union of a continuous chain of $\lambda$-saturated models is $\lambda$-saturated, so $M^*$ exists.) Since $a$ is $\epsilon$-finite, any subset $I \subseteq M[a]$ that is indiscernible over $M$ has size at most $\lambda$ (when $\lambda = \aleph_\epsilon$, $I$ must be countable). It follows at once that every subset $I \subseteq M^*$ that is indiscernible over $N \cup M$ has size at most $\lambda$, so by Fact 2.3(1), $M^*$ is $\lambda$-prime over $N \cup M$. We complete the proof of (1) by showing that $M^* = M[a]$.

Suppose not. Choose $c \in M[a] \setminus M^*$ such that $q = \text{tp}(c/M^*)$ is regular. The argument splits into cases. First, if $q \perp N$ and $q \perp M$, then $(M_1, N, M, M^*)$ is a DOP witness for $q$, so by Corollary 3.4, any $\lambda$-prime model over $M^* \cup \{c\}$ is $\lambda$-prime over $N \cup M$, which contradicts the maximality of $M^*$. Second, if $q \not\perp N$, then choose a regular $r \in S(M^*)$ that does not fork over $N$ but $q \not\perp r$. Choose $d \in M[a] \setminus M^*$ realizing $r$. Then, by symmetry and transitivity of non-forking, $Nd \perp M$, which contradicts the maximality of $N$. Finally, suppose that $q \not\perp M$. As before, there is a regular $p \in S(M^*)$ that does not fork over $M$ but $q \not\perp p$, and an element $e \in M[a] \setminus M^*$ realizing $p$. As $p$ is regular, based on $M$, and non-orthogonal to $\text{tp}(a/M)$, we have $p \in P_{\text{active}}$ and $p \perp M_2$. So, by $P$-NDOP it must be that $p \not\perp M_1$. But then $p \not\perp N$, so arguing as above we contradict the maximality of $N$. This proves (1).

For (2), choose any such $a^*$. We show that $N$ is $\lambda$-prime over $M_1 \cup \{a^*\}$ via Fact 2.3(1). We already know that $N$ is $\lambda$-saturated. To see that $N$ is $\lambda$-atomic over $M_1 \cup \{a^*\}$, choose any finite set $c$ from $N$. As $N \subseteq M[a]$, $\text{tp}(c/Ma)$ is $\lambda$-isolated. But $c \perp Ma$, so $\text{tp}(c/M_1 a^*)$ is $\lambda$-isolated as well (see e.g. [4, IV, 4.1]). Finally, if $I \subseteq N$ is indiscernible over $M_1 \cup \{a^*\}$, then $I$ is indiscernible over $M_1$. But $N \perp_{M_1} M$, so $I$ is indiscernible over $N \cup M$. As $M[a]$ is $\lambda$-prime over $N \cup M$, it follows that $I$ has size at most $\lambda$, completing the proof of (2).

Lemma 3.13. ($P$-NDOP, $\lambda \geq \aleph_\epsilon$) Suppose that $M_1 \preceq M$ are both $\lambda$-saturated, $a$ is $\epsilon$-finite, $\text{tp}(a/M) \perp P_{\text{dull}}$, and either $\text{tp}(a/M)$ does not fork
over \(M_1\), or \(\tp(a/M)\) is regular and non-orthogonal to \(M_1\). Let \(M[a]\) be any \(\lambda\)-prime model over \(M \cup \{a\}\). For any subset \(N \subseteq M[a]\) that is maximal such that \(N \upharpoonright M_1\) we have:

1. \(N \preceq M[a]\), \(N\) is \(\lambda\)-saturated, and \(M[a]\) is \(\lambda\)-prime over \(N \cup M\).
2. For any \(a^* \subseteq N\) such that \(N \downarrow_{M_1} a\), \(N\) is \(\lambda\)-prime over \(M_1 \cup \{a^*\}\).

\[\text{Proof.}\] The proof is similar to that of Lemma \ref{lemma:3.12} only easier. The hypotheses on \(\tp(a/M)\) ensure that for any \(e \in M[a] \setminus M\), as \(e\) is dominated by \(a\) over \(M\), we have \(\tp(e/M) \not\models M_1\).

To see (1), take \(N^+ \preceq M[a]\) to be \(\lambda\)-prime over \(N\). As before, the maximality of \(N\) implies that \(N^+ = N\), so \(N \preceq M[a]\) and \(N\) is \(\lambda\)-saturated. As well, choose \(M^+ \preceq M[a]\) that is maximal such that \(M^+\) is \(\lambda\)-saturated and \(\lambda\)-atomic over \(N \cup M\). As before, indiscernible subsets of \(M^+\) over \(N \cup M\) have size at most \(\lambda\), so \(M^+\) is \(\lambda\)-prime over \(N \cup M\).

The verification that \(M^+ = M[a]\) is also similar. If not, choose \(c \in M[a] \setminus M^+\) such that \(q = \tp(c/M^+)\) is regular. If \(q \downarrow N\) and \(q \downarrow M\), then \((M_1, N, M, M^+)\) is a DOP witness for \(q\), which again contradicts the maximality of \(M^+\) by Corollary \ref{corollary:3.4} If \(q \not\models N\), then arguing as before there is a regular \(r \in S(M^+)\) that does not fork over \(N\), \(q \not\models r\), and a realization \(d\) of \(r\), which contradicts the maximality of \(N\). Finally, if \(q \not\models M\), then there is a regular \(p \in S(M^+)\) that does not fork over \(M\) but \(q \not\models p\), and a realization \(e\) of \(p\) in \(M[a]\). Our conditions on \(\tp(a/M)\) imply that \(\tp(e/M) \not\models M_1\), hence \(\tp(e/M) \not\models N\) and we argue as above, completing the reasoning for (1). The verification of (2) is identical to its counterpart in the proof of Lemma \ref{lemma:3.12}. \[\blacksquare\]

4. **P-decompositions.** Throughout this section, assume that \(T\) is superstable, and that \(P\) is a class of regular types, closed under automorphisms of \(C\) and non-orthogonality. We define a number of species of \(P\)-decompositions, along with a number of ways in which one \(P\)-decomposition can extend another.

**Definition 4.1.** Fix a model \(M\). A **weak \(P\)-decomposition inside \(M\)** is a sequence \(\bar{a} = \langle N_\eta, a_\eta : \eta \in I \rangle\) indexed by a tree \((I, \preceq)\) satisfying:

1. \(\{N_\eta : \eta \in I\}\) is an independent tree of elementary submodels of \(M\).
2. \(\eta \preceq \nu\) implies \(N_\eta \preceq N_\nu\).
3. For each \(\eta, a_\eta \in N_\eta\) (but \(a_\langle \rangle\) is meaningless).
4. For all \(\nu \in \Succ_I(\eta)\), \(N_\nu\) is dominated by \(a_\nu\) over \(N_\eta\).
5. If \(\eta \neq \langle \rangle\), then \(\tp(a_\nu/N_\eta) \downarrow N_\eta\) for each \(\nu \in \Succ_I(\eta)\).
6. For each \(\eta \in I\), \(\{a_\nu : \nu \in \Succ_I(\eta)\}\) is independent over \(N_\eta\), and \(\tp(a_\nu/N_\eta) \downarrow P\bot\) for each \(\nu \in \Succ_I(\eta)\).
Note that in the definition above, we do not require that $\text{tp}(a_\nu/N_\eta)$ be regular. However, the content of (6) is that any regular type $q \not\perp \text{tp}(a_\nu/N_\eta)$ is necessarily in $\mathbf{P}$.

**Lemma 4.2.** Suppose $\mathfrak{d} = \langle N_\eta, a_\eta : \eta \in I \rangle$ is a weak $\mathbf{P}$-decomposition inside $M$. Then:

1. If $I_1, I_2 \subseteq I$ are both downward closed and $I_0 = I_1 \cap I_2$, then
   \[
   \left( \bigcup_{\eta \in I_1} N_\eta \right) \cup \perp \left( \bigcup_{\eta \in I_2} N_\eta \right).
   \]

2. Suppose $\eta \in I$, $\nu = \eta^\ast \langle \alpha \rangle$, where $\alpha$ is least such that $\eta^\ast \langle \alpha \rangle \notin I$, the element $a_\nu \in M$ satisfies $\text{tp}(a_\nu/N_\eta) \perp \mathbf{P}^\perp$, if $\eta \neq \langle \rangle$ then $\text{tp}(a_\nu/N_\eta) \perp N_\eta^-$, and $a_\nu \perp \{a_\gamma : \gamma \in \text{Succ}_I(\eta)\}$, and $N_\nu \leq M$ is dominated by $a_\nu$ over $N_\eta$. Then $\mathfrak{d}^* = \mathfrak{d}^\ast \langle N_\nu, a_\nu \rangle$ is a weak $\mathbf{P}$-decomposition inside $M$.

There are two ways of defining when a weak $\mathbf{P}$-decomposition inside a model $M$ is ‘maximal’. Fortunately, at least when both $M$ and each of the submodels $N_\eta$ are $\aleph$-saturated, Lemma 4.4 below shows that the two notions are equivalent.

**Definition 4.3.** Suppose that $\mathfrak{d} = \langle N_\eta, a_\eta : \eta \in I \rangle$ of $M$ is a weak $\mathbf{P}$-decomposition inside $M$. For each $\eta \in I$, let
\[
C_\eta(M) = \{a \in M \setminus N_\eta : \text{tp}(a/N_\eta) \perp \mathbf{P}^\perp \text{ and } \perp N_\eta^- \text{ (when } \eta \neq \langle \rangle)\}.
\]
Then:

- **$\mathfrak{d}$ is a weak $\mathbf{P}$-decomposition of $M$** if, for every $\eta \in I$, $\{a_\nu : \nu \in \text{Succ}_I(\eta)\}$ is a maximal $N_\eta$-independent subset of $C_\eta(M)$.
- **$\mathfrak{d}$ $\mathbf{P}$-exhausts $M$** if, for every $\eta \in I$, every regular $p \in S(N_\eta) \cap \mathbf{P}$ orthogonal to $N_\eta^-$ (when $\eta \neq \langle \rangle$) and all $e \in p(\mathfrak{C})$, if $e \perp_{N_\eta} \{a_\nu : \nu \in \text{Succ}_I(\eta)\}$ then $e \perp_{N_\eta} M$.

**Lemma 4.4.** Suppose that $\mathfrak{d} = \langle N_\eta, a_\eta : \eta \in I \rangle$ is a weak $\mathbf{P}$-decomposition inside an $\aleph$-saturated model $M$ such that every $N_\eta$ is $\aleph$-saturated as well. Then $\mathfrak{d}$ is a weak $\mathbf{P}$-decomposition of $M$ if and only if $\mathfrak{d}$ $\mathbf{P}$-exhausts $M$.

**Proof.** For both directions, recall that if $h \in M \setminus N_\eta$, then there is a finite, $N_\eta$-independent set $\{b_i : i < n\} \subseteq M$ domination equivalent to $h$ over $N_\eta$ with $\text{tp}(b_i/N_\eta)$ regular for each $i < n$.

For the easy direction, suppose that $\mathfrak{d}$ is not a weak $\mathbf{P}$-decomposition of $M$. Choose $\eta \in I$ such that $A = \{a_\nu : \nu \in \text{Succ}_I(\eta)\}$ is not maximal in $C_\eta(M)$. Choose $h \in C_\eta(M)$ such that $h \perp A$, and from above, choose $\{b_i : i < n\} \subseteq M$ domination equivalent to $h$ over $N_\eta$. Then, for any $i < n$, the element $b_i$ and the type $\text{tp}(b_i/N_\eta)$ witness that $\mathfrak{d}$ does not $\mathbf{P}$-exhaust $M$.  


Conversely, suppose that $\mathfrak{d}$ is a weak $\mathbf{P}$-decomposition of $M$. Fix any
$\eta \in I$, any regular type $p \in S(N_\eta) \cap \mathbf{P}$ that is orthogonal to $N_\eta$ when
$\eta \neq \langle \rangle$, and any $e \in p(\mathfrak{C})$ with $\mathfrak{C}(\eta,M)$. We will show that $\mathfrak{C}_M$, where
$A = \{ a_\nu : \nu \in \text{Succ}_I(\eta) \}$.
To see this, using the notation above choose $n < \omega$ minimal such that
there are $h \in M \setminus N_\eta$ and $B = \{ b_i : i < n \} \subseteq M$ such that $\mathfrak{C}_M(h)$, and $h$ and
$B$ are domination equivalent over $N_\eta$ with $\text{tp}(b_i/N_\eta)$ regular for each $i$. It
follows from the minimality of $n$ that $\text{tp}(b_i/N_\eta)$ is non-orthogonal to $p$, hence
each $b_i$ is in $C_\eta(M)$. As $A$ is maximal in $C_\eta(M)$, we have $b_i \mathfrak{C}_M$ for each $i$
hence $\text{tp}(b_i/N_\eta A)$ is hereditarily orthogonal to $p$ (i.e., $\text{tp}(b_i/N_\eta A)$, as well as
every forking extension of it, is orthogonal to $p$). Thus, $\text{tp}(B/N_\eta A)$ is hered-
itarily orthogonal to $p$. This implies $\mathfrak{C}_M$. [Why? If not, then $\text{tp}(e/N_\eta A)$
would be parallel to $p$, so by orthogonality we would have $\mathfrak{C}_M$. This
would imply that $e$ and $B$ (and hence $e$ and $h$) are independent over $N_\eta$
which is a contradiction.] $\blacksquare$

For our next series of results, we insist that the model $M$ be sufficiently
saturated, and we additionally require that each submodel occurring in a
decomposition be sufficiently saturated as well. In most applications, $\aleph_\epsilon$
saturation would suffice, but it costs little to work in the more general context of $(\lambda, \mathbf{P})$-saturated models, which we now introduce.

Fix, for the remainder of this section, a pair $\langle \lambda, \mu \rangle$ of cardinals satisf-
ying $\lambda, \mu \geq \aleph_\epsilon$. Throughout this paper, if $\lambda = \mu = \aleph_\epsilon$, we write $(\aleph_\epsilon, \mathbf{P})$ in place of $(\aleph_\epsilon, \aleph_\epsilon, \mathbf{P})$.

**Definition 4.5.** We say that a model $M$ is $(\lambda, \mathbf{P})$-saturated if it is $\aleph_\epsilon$
saturated, and for each finite $A \subseteq M$ we have $\text{dim}(p,M) \geq \lambda$ for each
$p \in \mathbf{P} \cap S(A)$, and $\text{dim}(q,M) \geq \mu$ for all stationary, regular $q \in \mathbf{P} \perp \cap S(A)$.
(If either $\lambda$ or $\mu$ is $\aleph_\epsilon$, the associated dimension is at least $\aleph_0$.)

We say that a $(\lambda, \mathbf{P})$-saturated model $N$ is $(\lambda, \mathbf{P})$-prime over a set $X$ if
$N \supseteq X$ and $N$ embeds elementarily over $X$ into any $(\lambda, \mathbf{P})$-saturated model
containing $X$.

Note that our assumptions on $\lambda$ guarantee that any $(\lambda, \mathbf{P})$-saturated
model is $\aleph_\epsilon$-saturated, but we include this clause for emphasis. Also, if $\lambda = \mu$, then the $(\lambda, \mathbf{P})$-saturated models are precisely the $\lambda$-saturated models.
The standard facts about the existence of $(\lambda, \mathbf{P})$-prime models extend easily
to this context. To see this, call a type $\text{tp}(a/B)$ $(\lambda, \mathbf{P})$-isolated if any of
the three conditions hold: (1) $\text{tp}(a/B)$ is $\aleph_\epsilon$-isolated (i.e., $F_0^\mathbf{P}$-isolated), or
(2) $\text{tp}(a/B) \in \mathbf{P}$ and is $\lambda$-isolated, or (3) $\text{tp}(a/B) \in \mathbf{P} \perp$ and is $\mu$-isolated.
Next, call a set $B$ $(\lambda, \mathbf{P})$-primitive over $A$ if $B = A \cup \{ b_i : i < \alpha \}$, where
$\text{tp}(b_i/A \cup \{ b_j : j < i \})$ is $(\lambda, \mathbf{P})$-isolated for every $i$, and call a model $M$,
(\bar{\lambda},\mathbf{P})$-primary over $A$ if $M$ is \((\bar{\lambda},\mathbf{P})\)-saturated and its universe is \((\bar{\lambda},\mathbf{P})\)-primitive over $A$. This notion of isolation satisfies the same axioms as for $\text{F}^\alpha_{\lambda}$-isolation in [4 Chapter 4] and thus we obtain the same consequences. In particular:

- If $A \subseteq M^*$ with $M^*$ \((\bar{\lambda},\mathbf{P})\)$-saturated, then there is a $M \preceq M^*$ that is \((\bar{\lambda},\mathbf{P})\)$-primary over $A$.
- $M$ \((\bar{\lambda},\mathbf{P})\)$-primary over $A$ implies $M$ is \((\bar{\lambda},\mathbf{P})\)$-prime over $A$.
- If $M$ is \((\bar{\lambda},\mathbf{P})\)$-saturated, $M \subseteq A$, and $N$ is \((\bar{\lambda},\mathbf{P})\)$-prime over $A$, then $N$ is dominated by $A$ over $M$.

**Definition 4.6.** Suppose $M$ is \((\bar{\lambda},\mathbf{P})\)$-saturated. A weak \((\bar{\lambda},\mathbf{P})\)$-decomposition inside $M$ (of $M$) is a weak $\mathbf{P}$-decomposition inside $M$ (of $M$) for which each of the submodels $N_\eta$ is a \((\bar{\lambda},\mathbf{P})\)$-saturated elementary substructure of $M$.

A salient feature of weak \((\bar{\lambda},\mathbf{P})\)$-decompositions is that each of the submodels is itself $\aleph_\epsilon$-saturated. The proof of the following lemma is virtually identical to arguments in [4 Section X.3].

**Lemma 4.7.** \((\mathbf{P}\text{-NDOP})\) Suppose that $\langle N_\eta, a_\eta : \eta \in I \rangle$ is a weak \((\bar{\lambda},\mathbf{P})\)$-decomposition of a \((\bar{\lambda},\mathbf{P})\)$-saturated model $M$. Let $\overline{M} \preceq M$ be any $\aleph_\epsilon$-prime submodel of $M$ over $\bigcup_{\eta \in I} N_\eta$. Then if $p \in \mathbf{P}$ is non-orthogonal to $\overline{M}$, then there is a unique $\prec$-minimal $\eta \in I$ such that $p \not\prec N_\eta$.

**Proof.** We first show that $p \not\prec N_\eta$ for some $\eta \in I$. As $\overline{M}$ is $\aleph_\epsilon$-saturated, there is $q \in S(\overline{M})$ that is regular and non-orthogonal to $p$. As any such $q$ is in $\mathbf{P}$, we may assume that $p \in S(\overline{M})$ to begin with. Choose a finite $B \subseteq \overline{M}$ over which $p$ is based and stationary. As $B$ is $\aleph_\epsilon$-isolated over $\bigcup_{\eta \in I} N_\eta$, there is a finite subtree $I_0 \subseteq I$ such that $B$ is $\aleph_\epsilon$-isolated over $\bigcup_{\eta \in I_0} N_\eta$. Choose any $M_0 \preceq \overline{M}$ such that $B \subseteq M_0$ and $M_0$ is $\aleph_\epsilon$-prime over $\bigcup_{\eta \in I_0} N_\eta$. As there is some type $p' \in S(M_0)$ parallel to $p$, $\mathbf{P}$-$\text{NDOP}$ implies that $p \not\prec N_\eta$ for some $\eta \in I_0$.

Finally, using Lemma 4.2(1) it follows that there is a unique $\prec$-minimal $\eta \in I$ with $p \not\prec N_\eta$. □

The point of the following definition is that \((\bar{\lambda},\mathbf{P})\)$-decompositions have no control over types orthogonal to $\mathbf{P}$.

**Definition 4.8.** An $\aleph_\epsilon$-saturated model $N$ is $\mathbf{P}$-minimal over $X$ if $N \supseteq X$, but for any $\aleph_\epsilon$-saturated $N_0 \preceq N$ containing $X$, $\text{tp}(e/N_0) \perp \mathbf{P}$ for every $e \in N \setminus N_0$.

**Corollary 4.9.** \((\mathbf{P}\text{-NDOP})\) Suppose that $\langle N_\eta, a_\eta : \eta \in I \rangle$ is a weak \((\bar{\lambda},\mathbf{P})\)$-decomposition of a \((\bar{\lambda},\mathbf{P})\)$-saturated model $M$ and let $\overline{M} \preceq M$ be any $\aleph_\epsilon$-prime model over $\bigcup_{\eta \in I} N_\eta$. Then:
(1) Every $c \in M \setminus M$ satisfies $\tp(c/M) \perp P$.

(2) $M$ is $P$-minimal over $\bigcup_{\eta \in I} N_\eta$.

Proof. (1) Assume for a contradiction that there is $c \in M$ such that $\tp(c/M) \not\perp P$. As $P$ is closed under non-orthogonality and automorphisms of $\mathfrak{C}$, there is $p \in P \cap S(M)$ non-orthogonal to $\tp(c/M)$. Then, by Fact 2.3(3), there is $e \in M$ realizing $p$. So, by Lemma 4.7, $p \not\perp N_\eta$ for some $\eta \in I$. Thus, by Fact 2.3(4) there is $e^* \in M$ domination equivalent to $e$ over $M$ with $e^* \perp M$. As $\{a_\nu : \nu \in \text{Succ}_I(\eta)\} \subseteq M$, this contradicts the fact that $\langle N_\eta, a_\eta : \eta \in I \rangle$ is a weak $(\bar{x}, P)$-decomposition of $M$.

(2) Choose any $M_1 \preceq M$ that is $\aleph_\xi$-prime over $\bigcup_{\eta \in I} N_\eta$. Then (1) applies to $M_1$. That is, there is no $c \in M \setminus M_1$ such that $\tp(c/M_1) \not\perp P$. Thus, $M$ is $P$-minimal over $\bigcup_{\eta \in I} N_\eta$. ■

Next, we show that if we additionally assume that $P = P^{\text{active}}$, then we can extend the previous results to any $\aleph_\xi$-saturated submodel of $M$ containing the decomposition.

Proposition 4.10. ($P$-NDOP, $P = P^{\text{active}}$) Suppose that $\langle N_\eta, a_\eta : \eta \in I \rangle$ is a weak $(\bar{x}, P)$-decomposition of a $(\bar{x}, P)$-saturated model $M$. Let $M^* \preceq M$ be any $\aleph_\xi$-saturated model containing $\bigcup_{\eta \in I} N_\eta$. Then there is no $e \in M \setminus M^*$ such that $\tp(e/M^*) \not\perp P$.

Proof. As both $M^*$ and $M$ are $\aleph_\xi$-saturated, it suffices to prove that there is no $e \in M \setminus M^*$ such that $\tp(e/M^*) \in P$. Assume for a contradiction that there were such an $e$. Let $M_0 \preceq M^*$ be any $\aleph_\xi$-prime model over $\bigcup_{\eta \in I} N_\eta$. Next, form an increasing sequence $\langle M_\alpha : \alpha \leq \delta \rangle$ of $\aleph_\xi$-saturated models where $M_\delta = M^*$, $M_{\alpha+1}$ is $\aleph_\xi$-prime over $M_\alpha \cup \{b_\alpha\}$, with $\tp(b_\alpha/M_\alpha)$ regular, and for $\alpha < \beta$ a non-zero limit, $M_\alpha$ is $\aleph_\xi$-prime over $\bigcup_{\beta < \alpha} M_\beta$.

Choose $\alpha \leq \delta$ least such that there is some $e \in M \setminus M_\alpha$ for which $\tp(e/M_\alpha) \in P$. By superstability, $\alpha$ cannot be a non-zero limit ordinal. Now suppose $\alpha = \beta + 1$. On the one hand, if $p = \tp(e/M_\alpha) \in P$ were non-orthogonal to $M_\beta$, then by Fact 2.3(4), there would be $e^* \in M$ such that $q = \tp(e^*/M_\beta)$ is regular and non-orthogonal to $p$, contradicting the minimality of $\alpha$. On the other hand, if $p \perp M_\beta$, then as $P^{\text{active}} = P$, we have $r = \tp(b_\beta/M_\beta) \in P$, which again contradicts the minimality of $\alpha$.

Thus, $\alpha$ must equal zero, i.e., there is $e \in M \setminus M_0$ such that $p = \tp(e/M_0) \in P$. By Lemma 4.7, choose a $\preceq$-minimal $\eta \in I$ such that $p \not\perp N_\eta$.

Choose $q \in S(N_\eta)$ regular such that $p \not\perp q$ and let $q' \in S(M_0)$ be the non-forking extension of $q$ to $M_0$. As both $M_0$ and $M$ are $\aleph_\xi$-saturated, there is $c \in M \setminus M_0$ realizing $q'$. As $q' \in P$, we have $c \in C_\eta(M)$ in the notation of Definition 4.3, which contradicts the maximality of $\{a_\nu : \nu \in \text{Succ}_I(\eta)\}$. ■
Corollary 4.11. (P-NDOP, $P = P^{\text{active}}$) Suppose that $\langle N_\eta, a_\eta : \eta \in I \rangle$ is a weak $(\bar{X}, P)$-decomposition of $M$. Let $M^* \preceq M$ be any $\aleph_\epsilon$-saturated elementary submodel containing $\bigcup_{\eta \in I} N_\eta$. If $p \in P$ and $p \not\models M^*$, then $p \not\models M^*$.

Proof. As in the proof above, form an increasing sequence $\langle M_\alpha : \alpha \leq \delta \rangle$ of $\aleph_\epsilon$-saturated models, this time such that $M_0 = M^*$, $M_\delta = M$, $M_{\alpha+1}$ is $\aleph_\epsilon$-prime over $M_\alpha \cup \{b_\alpha\}$, where $tp(b_\alpha/M_\alpha)$ is regular, and for $\alpha < \delta$ a non-zero limit, $M_\alpha$ is $\aleph_\epsilon$-prime over $\bigcup_{\beta < \alpha} M_\beta$. Choose $\alpha \leq \delta$ least such that $p \not\models M_\alpha$. We will show that $\alpha = 0$. Clearly, $\alpha$ cannot be a non-zero limit by superstability. Assume for a contradiction that $\alpha = \beta + 1$. Then $p \not\models M_{\alpha}$, but $p \models M_{\beta}$. As before, this implies that $r = tp(b_\beta/M_\beta) \in P^{\text{active}} = P$. But now, $M_\beta$ is an $\aleph_\epsilon$-saturated model containing $\bigcup_{\eta \in I} N_\eta$, yet there is an element of $M \setminus M_\beta$ realizing $r \in P$, with contradicts Proposition 4.10. Thus, $\alpha = 0$, so $p \not\models M^*$. ■

Corollary 4.12. (P-NDOP, $P = P^{\text{active}}$) Suppose that $\langle N_\eta, a_\eta : \eta \in I \rangle$ is a weak $(\bar{X}, P)$-decomposition of a $(\bar{X}, P)$-saturated model $M$. If $p \in P$ and $p \not\models M$, then there is a unique $\prec$-minimal $\eta \in I$ such that $p \not\models N_\eta$.

Proof. Let $M^* \preceq M$ be any $\aleph_\epsilon$-prime model over $\bigcup_{\eta \in I} N_\eta$. By Corollary 4.11, $p \not\models M^*$, so by Lemma 4.7, $p \not\models N_\eta$ for some $\prec$-minimal $\eta \in I$. As in the proof of Lemma 4.7, the uniqueness follows from Lemma 4.2(1). ■

Until this point in our discussion, the submodels occurring in a decomposition could be very large, with an extreme case being that any model $M$ has a one-element decomposition $\langle M \rangle$. The next definition limits the size of the submodels, while retaining the fact that they are at least $\aleph_\epsilon$-saturated.

Definition 4.13. A prime $(\bar{X}, P)$-decomposition inside $M$ (of $M$) is a weak $(\bar{X}, P)$-decomposition inside $M$ (of $M$) in which $N_{\langle \rangle}$ is $(\bar{X}, P)$-prime over $\emptyset$ and, for each $\eta \in I \setminus \{\langle \rangle\}$, $N_\eta$ is $(\bar{X}, P)$-prime over $N_{\eta^-} \cup \{a_\eta\}$.

Definition 4.14. Fix a $(\bar{X}, P)$-saturated $M$. A prime $(\bar{X}, P)$-decomposition $\bar{d}_2 = \langle N^2_\eta, a^2_\eta : \eta \in J \rangle$ end extends the prime $(\bar{X}, P)$-decomposition $\bar{d}_1 = \langle N^1_\eta, a^1_\eta : \eta \in I \rangle$ if $I \subseteq J$ and, for each $\eta \in I$, $N^2_\eta = N^1_\eta$ and $a^2_\eta = a^1_\eta$.

We say $\bar{d}_2$ is a regular end extension of $\bar{d}_1$ if, in addition, $tp(a_\eta/N_{\eta^-})$ is regular for each $\eta \in J \setminus I$. Furthermore, $\bar{d}_2$ is a standardly regular end extension of $\bar{d}_1$ if $tp(a_\eta/N_{\eta^-}) = tp(a_\nu/N_{\nu^-})$ whenever $\eta, \nu \in J \setminus I$, $\eta^- = \nu^-$, and $tp(a_\eta/N_{\eta^-}) \not\models tp(a_\nu/N_{\nu^-})$.

The following lemma is straightforward, and relies on the fact that if $N \preceq M$ are both $(\bar{X}, P)$-saturated with $a \in M \setminus N$ satisfying $tp(a/N) \in P$, then there is $N[a] \preceq M$ that is $(\bar{X}, P)$-prime over $N \cup \{a\}$ and that $N[a]$ contains realizations of every regular type over $N$ non-orthogonal to $tp(a/N)$. Proofs of similar statements appear in [4, Section X.3].
Lemma 4.15. Suppose $\mathfrak{d} = \langle N_\eta, a_\eta : \eta \in I \rangle$ is a prime $(\mathfrak{X}, \mathfrak{P})$-decomposition inside a $(\mathfrak{X}, \mathfrak{P})$-saturated model $M$. Then:

1. $\mathfrak{d}$ is a prime $(\mathfrak{X}, \mathfrak{P})$-decomposition of $M$ if and only if it has no proper (standedly regular) end extension.
2. There is a prime $(\mathfrak{X}, \mathfrak{P})$-decomposition $\mathfrak{d}^*$ of $M$ that is a standardly regular end extension of $\mathfrak{d}$.

Following the main theme of [5], we wish to identify the class of $\mathfrak{P}$-decompositions that lie above a specific triple $(N, N', a)$, where $N \preceq N'$, with $\text{tp}(a/N') \perp \mathfrak{P} \perp$ and $\text{tp}(a/N') \perp N$, that is, triples where $a$ could play the role of $a_{\nu}$ in some $\mathfrak{P}$-decomposition with $N' = N_{\eta}$ and $N = N_{\eta^-}$. However, as in [5], this is too much data to record at once, so we seek an $\epsilon$-finite approximation of it.

Specifically, for $M$ any model, let

$$\Gamma(M) := \{ (A, B) : A \subseteq B \subseteq M \text{ are both } \epsilon\text{-finite} \}.$$ 

We frequently write $(\begin{smallmatrix} B \\ A \end{smallmatrix})$ for elements of $\Gamma(M)$, and if $A$ is not a subset of $B$, we mean $(\begin{smallmatrix} A \cup B \\ A \end{smallmatrix})$. Let

$$\Gamma_\mathfrak{P}(M) := \left\{ \begin{pmatrix} B \\ A \end{pmatrix} \in \Gamma(M) : \text{tp}(B/A) \perp \mathfrak{P} \perp \right\}.$$ 

Definition 4.16. For $(\begin{smallmatrix} B \\ A \end{smallmatrix}) \in \Gamma_\mathfrak{P}(M)$, a prime $(\mathfrak{X}, \mathfrak{P})$-decomposition over $(\begin{smallmatrix} B \\ A \end{smallmatrix})$ inside $M$, $\mathfrak{d} = \langle N_\eta, a_\eta : \eta \in I \rangle$, is a prime $(\mathfrak{X}, \mathfrak{P})$-decomposition inside $M$ in which $\langle 0 \rangle$ is the unique successor of $\langle \rangle$ in $I$, $A \subseteq N_{\langle \rangle}$, $B \subseteq N_{\langle 0 \rangle}$, and $B \subseteq \text{dcl}(a_{\langle 0 \rangle})$. By analogy with Definition 4.3

- such a $\mathfrak{d}$ is of $M$ if, for every $\eta \in I \neq \langle \rangle$, $\{a_\nu : \nu \in \text{Succ}_I(\eta)\}$ is a maximal $N_\eta$-independent subset of $C_\eta(M)$; and
- $\mathfrak{d}$ $\mathfrak{P}$-exhausts $M$ over $(\begin{smallmatrix} B \\ A \end{smallmatrix})$ if, for every $\eta \in I \neq \langle \rangle$, every regular $p \in S(N_\eta) \cap \mathfrak{P}$ orthogonal to $N_{\eta^-}$ (when $\eta \neq \langle \rangle$) and every $e \in p(\mathfrak{C})$, if $e \perp_{N_\eta} \{a_\nu : \nu \in \text{Succ}_I(\eta)\}$ then $e \perp_{N_\eta} M$.

The following lemma is straightforward. The verification of (5) is analogous to the proof of Lemma 4.4.

Lemma 4.17. Fix a $(\mathfrak{X}, \mathfrak{P})$-saturated model $M$ and $(\begin{smallmatrix} B \\ A \end{smallmatrix}) \in \Gamma_\mathfrak{P}(M)$.

1. If $N_{\langle \rangle} \preceq M$ is $(\mathfrak{X}, \mathfrak{P})$-prime over $\langle \rangle$, contains $A$, and $B \perp_{\mathfrak{A}} N_{\langle \rangle}$, and $N_{\langle 0 \rangle} \preceq M$ is $(\mathfrak{X}, \mathfrak{P})$-prime over $N_{\langle \rangle} \cup B$, then $\langle N_{\langle \rangle}, N_{\langle 0 \rangle} \rangle$ is a prime $(\mathfrak{X}, \mathfrak{P})$-decomposition over $(\begin{smallmatrix} B \\ A \end{smallmatrix})$ inside $M$.
2. A prime $(\mathfrak{X}, \mathfrak{P})$-decomposition $\mathfrak{d}$ over $(\begin{smallmatrix} B \\ A \end{smallmatrix})$ inside $M$ is a prime $(\mathfrak{X}, \mathfrak{P})$-decomposition over $(\begin{smallmatrix} B \\ A \end{smallmatrix})$ of $M$ if and only if $\mathfrak{d}$ has no proper $(\mathfrak{X}, \mathfrak{P})$-decomposition over $(\begin{smallmatrix} B \\ A \end{smallmatrix})$ end extending it.
(3) Every prime $\langle X, P \rangle$-decomposition over $\langle B \rangle$ inside $M$ has a (standardly regular) end extension to a prime $\langle X, P \rangle$-decomposition over $\langle B \rangle$ of $M$.

(4) Every prime $\langle X, P \rangle$-decomposition $\mathfrak{d}$ over $\langle B \rangle$ inside $M$ is a prime $\langle X, P \rangle$-decomposition inside $M$, hence has a (standardly regular) end extension to a prime $\langle X, P \rangle$-decomposition $\mathfrak{d}^*$ of $M$. Moreover, if $\mathfrak{d}$ is a decomposition over $\langle B \rangle$ of $M$ and is indexed by the tree $(I, \prec)$ and $\mathfrak{d}^*$ is indexed by $(J, \prec)$, then $\neg((0) \leq \eta)$ for all $\eta \in J \setminus I$.

(5) A $\langle X, P \rangle$-decomposition $\mathfrak{d}$ over $\langle B \rangle$ inside $M$ is of $M$ if and only if $\mathfrak{d}$ $P$-exhausts $M$ over $\langle B \rangle$.

5. Trees of subsets of an $\aleph_\varepsilon$-saturated model. Throughout this section, $T$ is superstable with $P$-NDOP, and $P$ is closed under automorphisms of $C$, non-orthogonality, and $P = P_{\text{active}}$. In addition, all models $M$ we consider will be $\aleph_\varepsilon$-saturated, and all decompositions we consider will be $\langle \aleph_\varepsilon, P \rangle$-decompositions inside/of $M$.

**Definition 5.1.** Fix an $\aleph_\varepsilon$-saturated model $M$ and $\langle B \rangle \in \Gamma_P(M)$. Let $\mathfrak{d} = \langle N_\eta, a_\eta : \eta \in I \rangle$ be a prime $\langle \aleph_\varepsilon, P \rangle$-decomposition over $\langle B \rangle$ of $M$. Then

$$P_P(\mathfrak{d}, M) = \{ p \in S(M) : p \in P, p \perp N_\emptyset, \text{ but } p \not\perp N_\eta \text{ for some } \eta \in I \setminus \{\langle \rangle\} \}.$$  

The goal for this section is Theorem 5.12, which asserts that $P_P(\mathfrak{d}_1, M) = P_P(\mathfrak{d}_2, M)$ for any two prime $\langle \aleph_\varepsilon, P \rangle$-decompositions $\mathfrak{d}_1, \mathfrak{d}_2$ of $M$ above $\langle B \rangle$. We begin by introducing another way of ‘increasing’ a decomposition.

**Definition 5.2.** A prime $\langle \aleph_\varepsilon, P \rangle$-decomposition $\mathfrak{d}_2 = \langle N_\eta^2, a_\eta^2 : \eta \in J \rangle$ inside $C$ is a blow up of the prime $\langle \aleph_\varepsilon, P \rangle$-decomposition $\mathfrak{d}_1 = \langle N_\eta^1, a_\eta^1 : \eta \in I \rangle$ inside $C$ if $J = I$, but for every $\eta \in I$, $N_\eta^1 \preceq N_\eta^2$ and, when $\eta \neq \langle \rangle$, $N_\eta^2$ is $(\aleph_\varepsilon, P)$-prime over $N_\eta^1 \cup N_\eta^2$.

**Lemma 5.3.** Suppose that $M$ is $\aleph_\varepsilon$-saturated, $\langle B \rangle \in \Gamma_P(M)$, $\mathfrak{d}_2 = \langle N_\eta^2, a_\eta : \eta \in I \rangle$ is a blow up of $\mathfrak{d}_1 = \langle N_\eta^1, a_\eta : \eta \in I \rangle$, $A \subseteq B \subseteq N_\emptyset$ and, for each $\eta$, $N_\eta^2 \preceq M$. Then:

1. If $\nu \in \text{Succ}_I(\eta)$, then $N_\eta^2 \nvdash N_\nu$.
2. If $Y = \{ \rho \in I : \neg(\eta \prec \rho) \}$, then $N_\eta^2 \nvdash \bigcup_{\rho \in Y} N_\rho$ and $\eta \prec \nu$ implies $N_\nu \nvdash \bigcup_{\rho \in Y} N_\rho$.
3. $\mathfrak{d}_2$ is an $\langle \aleph_\varepsilon, P \rangle$-decomposition inside $M$ above $\langle B \rangle$ if and only if $\mathfrak{d}_1$ is.
4. $\mathfrak{d}_2$ is an $\langle \aleph_\varepsilon, P \rangle$-decomposition of $M$ above $\langle B \rangle$ if and only if $\mathfrak{d}_1$ is.
Proof. This is exactly analogous to [5] Fact 1.20. In the proof of (4), we need to appeal to $P$-NDOP instead of NDOP.

**Lemma 5.4.** Suppose $M$ is $\aleph_\epsilon$-saturated and $\left(\begin{array}{c} B \\ A \end{array}\right) \in \Gamma_P(M)$. If $\mathfrak{d}_2$ is a blow up of $\mathfrak{d}_1$ and both $\mathfrak{d}_1, \mathfrak{d}_2$ are $(\aleph_\epsilon, P)$-decompositions of $M$ over $\left(\begin{array}{c} B \\ A \end{array}\right)$, then $P_\mathfrak{d}(\mathfrak{d}_1, M) = P_\mathfrak{d}(\mathfrak{d}_2, M)$.

Proof. This is very much like [5, Fact 1.22], but we give details. As notation, say $\mathfrak{d}_\ell = \langle N_{\eta}^\ell, a_{\eta}^\ell : \eta \in I_\ell \rangle$ for $\ell = 1, 2$. Fix $p \in S(M) \cap P$, so in particular, $p$ is regular. We must prove that

$$(p \perp N_{\eta}^1 \text{ and } p \Perp N_{\eta}^1) \iff (p \perp N_{\eta}^2 \text{ and } p \Perp N_{\eta}^2)$$

for every $\eta \neq \langle \rangle$.

First, assume $\eta \neq \langle \rangle$ and $p \perp N_{\eta}^1$ and $p \Perp N_{\eta}^1$. As $N_{\eta}^1 \leq N_{\eta}^2$, $p \Perp N_{\eta}^2$ trivially. Also, choose a regular $q \in S(N_{\eta}^1)$ with $p \Perp q$. Then $q \perp N_{\eta}^1$ since $p$ is, and it suffices to show that $q$ is orthogonal to $N_{\eta}^2$. But this follows immediately since $N_{\eta}^1 \perp N_{\eta}^2$.

Conversely, assume $\eta \neq \langle \rangle$ and $p \perp N_{\eta}^2$ and $p \Perp N_{\eta}^2$. Then, since $N_{\eta}^1 \leq N_{\eta}^2$, $p \perp N_{\eta}^1$. As well, $(N_{\eta}^1, N_{\eta}^2, N_{\eta}^1)$ form an independent triple of $\aleph_\epsilon$-saturated models (see Definition 3.1) and $N_{\eta}^2$ is $\aleph_\epsilon$-prime over their union. Thus, as $p \in P$, it follows from $P$-NDOP that $p \Perp N_{\eta}^1$.

**Lemma 5.5.** Suppose that $M$ is $\aleph_\epsilon$-saturated, $\left(\begin{array}{c} B \\ A \end{array}\right) \in \Gamma_P(M)$, and for $\ell = 1, 2$, $\mathfrak{d}_\ell = \langle N_{\eta}^\ell, a_{\eta}^\ell : \eta \in I_\ell \rangle$ are each prime $\aleph_\epsilon$-decompositions of $M$ above $\left(\begin{array}{c} B \\ A \end{array}\right)$. If $N_{\langle \rangle}^1 = N_{\langle \rangle}^2$ then $P_\mathfrak{d}(\mathfrak{d}_1, M) = P_\mathfrak{d}(\mathfrak{d}_2, M)$.

Proof. First, by Lemma 4.17(4), choose a prime, $\aleph_\epsilon$-prime decomposition $\mathfrak{d}_1^* = \langle N_{\eta}^1, a_{\eta}^1 : \eta \in J_1 \rangle$ of $M$ end extending $\mathfrak{d}_1$. As notation, let $H = J_1 \setminus I_1$ and for each $\eta \in H$, let $N_{\eta}^2 = N_{\eta}^1$ and $a_{\eta}^2 = a_{\eta}^1$. It is easily checked that $\mathfrak{d}_2^* = \langle N_{\eta}^2, a_{\eta}^2 : \eta \in I_2 \cup H \rangle$ is an $(\aleph_\epsilon, P)$-decomposition of $M$.

Now, for each $p \in S(M) \cap P$ with $p \perp N_{\langle \rangle}^1$ and for each $\ell = 1, 2$ there is a unique $\eta(p, \ell) \in I_\ell \cup H$ such that $p \Perp N_{\eta(p, \ell)}$, but $p \perp N_{\eta(p, 2)}$-. But, as $N_{\eta}^2 = N_{\eta}^1$ for each $\eta \in H$, $\eta(p, 1) \in H$ if and only if $\eta(p, 2) \in H$.

Thus, for each $p \in S(M) \cap P$ that is orthogonal to $N_{\langle \rangle}^1 = N_{\langle \rangle}^2$ we have $p \in P_\mathfrak{d}(\mathfrak{d}_1, M)$ if and only if $p \Perp N_{\langle \rangle}^2$. We come to the issue of the existence of blow-ups of decompositions. It is comparatively easy to blow up a decomposition inside an $\aleph_\epsilon$-saturated model $M$.

**Lemma 5.6.** Suppose that $M$ is $\aleph_\epsilon$-saturated and $\mathfrak{d} = \langle N_\eta, a_\eta : \eta \in I \rangle$ is a prime $(\aleph_\epsilon, P)$-decomposition inside $M$. For any $N^*$ satisfying $N_{\langle \rangle} \leq N^*$
Lemma 5.6. That is, \( \eta \) of \( (t_p, o_r) \) is a blow up of \( d \) and let \( N^* \) be any \( \aleph \)-saturated model over \( N^*_\eta \cup N^*_{\eta_i} \). Then it is easily checked that \( \mathfrak{d}^* = \langle N^*_\eta, a_\eta : \eta \in I \rangle \) is an \((\aleph, \mathbf{P})\)-decomposition inside \( M \) that is a blow up of \( \mathfrak{d} \). 

"Blowing down" a decomposition is more delicate and requires two technical lemmas, 3.12 and 3.13, that assert the existence of \( \aleph \)-submodels of a given \( \aleph \)-saturated structure with certain properties.

Lemma 5.7. Suppose that \( M \) is \( \aleph \)-saturated and \( \mathfrak{d} = \langle N_\eta, a_\eta : \eta \in I \rangle \) is a prime \((\aleph, \mathbf{P})\)-decomposition inside \( M \). Fix any \( \aleph \)-saturated \( N_0 \) such that for every \( \eta \in \text{Succ}_I(\langle \rangle) \), either \( \text{tp}(a_\eta/N_0) \) does not fork over \( N_0 \) or \( \text{tp}(a_\eta/N_0) \) is regular and non-orthogonal to \( N_0 \). Then there is a prime \((\aleph, \mathbf{P})\)-decomposition \( \mathfrak{d}_0 \) inside \( M \) with \( N^*_{\mathfrak{d}_0} = N_0 \) such that \( \mathfrak{d} \) is a blow up of \( \mathfrak{d}_0 \).

Proof. Choose an enumeration \( \langle \eta_i : i < i^* \rangle \) of \( I \) as in the proof of Lemma 5.6. That is, \( \eta_0 = \langle \rangle \), \( \eta_i \triangleleft \eta_j \) implies \( i < j \), and \( \eta_i \in \text{Succ}_I(\langle \rangle) \) if and only if \( 1 \leq i < \alpha^* \) for some \( \alpha^* \leq i^* \).

Put \( N^*_{\eta_i} = N_0 \). For \( 1 \leq i < i^* \) we inductively construct \( N^*_{\eta_i} \) to satisfy:

- \( N^*_{\eta_i} \leq N_{\eta_i} \) and \( N^*_{\eta_i}/N^*_{\eta_i} \)
- \( N_{\eta_i} \) is \( \aleph \)-prime over \( N^*_{\eta_i} \cup N_{\eta_i}^\dashv \) and
- \( a_{\eta_i} \in N^*_{\eta_i} \) and \( N^*_{\eta_i} \) is \( \aleph \)-prime over \( N^*_{\eta_i} \cup \{a_{\eta_i}\} \).

To accomplish this, for each \( 1 \leq i < \alpha^* \), use Lemma 3.13 to define \( N^*_{\eta_i} \) (where \( M_1 = N_0 \), \( M = N_{\eta_i} \)). We can take \( N^*_{\eta_i} \) to be the \( N \) there, and we can take \( a^* \) to be \( a_{\eta_i} \). Similarly, for \( \alpha^* \leq i < i^* \) we apply Lemma 3.12 where \( M \) is taken to be \( N_{\eta_i}^\dashv \), \( M_1 = N_{\eta_i}^\dashv \), \( M_2 = N_{\eta_i}^\dashv \), \( a \) is \( a_{\eta_i} \), and we take \( N^*_{\eta_i} \) to be the \( N \) produced there.

Definition 5.8. Suppose \( M \) is \( \aleph \)-saturated and \( (B_A) \in \mathcal{I}_P(M) \). We say that an \( \epsilon \)-finite subset \( W \subseteq M \) has a base \( W_0 \subseteq W \) respecting \( (B_A) \) if \( A \subseteq W_0 \), \( W_0 \downarrow A \), and \( W \) is dominated by \( B \) over \( W_0 \).
Lemma 5.9. If $\mathfrak{d} = \langle N_\eta, a_\eta : \eta \in I \rangle$ is an $(\aleph_\epsilon, \mathbf{P})$-decomposition inside $M$ over $(B_A^\eta)$ and $V \subseteq \bigcup_{\eta \in I} N_\eta$ is $\epsilon$-finite, then there is an $\epsilon$-finite $W$ with $V \subseteq W$ and $W \setminus V \subseteq N_0$ that has a base $W_0 \subseteq W \cap N_0$ respecting $(B_A^\eta)$.

Proof. Without loss of generality, we may assume $A \subseteq V$. It follows from the definition of an $(\aleph_\epsilon, \mathbf{P})$-decomposition inside $M$ over $(B_A^\eta)$ that $B_A \upharpoonright N_0$ and that $B$ dominates $\bigcup_{\eta \in I} N_\eta$ and hence $V$ over $N_0$. As both $B$ and $V$ are $\epsilon$-finite, it follows from superstability that there is an $\epsilon$-finite $C \subseteq N_0$ such that $BV \subseteq C\downarrow N_0$. So $B$ dominates $V$ over $C$. Again without loss of generality, $A \subseteq C$. Take $W = V \cup C$. Then $W_0 := W \cap N_0$ is a base respecting $(B_A^\eta)$. ■

Lemma 5.10. Suppose $W \subseteq M$ is $\epsilon$-finite and has a base $W_0 \subseteq W$ respecting $(B_A^\eta)$ and that $N \forcedby M$ is $\aleph_\epsilon$-prime over $\emptyset$, $W_0 \subseteq N_A$, with $N \joined \downharpoonright B$. Then there is $N[B] \forcedby M$ that is $\aleph_\epsilon$-prime over $N \cup B$ such that $W \subseteq N[B]$ and such that the two-element sequence $\langle N, N[B] \rangle$ is an $(\aleph_\epsilon, \mathbf{P})$-decomposition inside $M$ over $(B_A^\eta)$ (we take $a_{\langle 0 \rangle}$ to be $B$).

Proof. As $A, B, W$ are all $\epsilon$-finite, $N \downharpoonright B$, $W$ is dominated by $B$ over $W_0$, and $\text{tp}(W/\text{acl}(W_0 \cup B))$ is stationary, it follows that $\text{tp}(W/NB)$ is $\aleph_\epsilon$-isolated. Thus, $W \subseteq N[B]$ for some $\aleph_\epsilon$-prime model over $N \cup B$. Checking that $\langle N, N[B] \rangle$ is an $(\aleph_\epsilon, \mathbf{P})$-decomposition inside $M$ over $(B_A^\eta)$ is routine. ■

Proposition 5.11. If $M$ is $\aleph_\epsilon$-saturated, $(B_A^\eta) \in \Gamma_\mathbf{P}(M)$, and the $\epsilon$-finite set $W \subseteq M$ has a base $W_0 \subseteq W$ respecting $(B_A^\eta)$, then there is an $(\aleph_\epsilon, \mathbf{P})$-decomposition $\mathfrak{d}$ of $M$ over $(B_A^\eta)$ with $W_0 \subseteq N^a_{\langle 0 \rangle}$ and $W \subseteq N^a_{\langle 0 \rangle}$. Moreover, if $\mathfrak{d}_0$ is any $(\aleph_\epsilon, \mathbf{P})$-decomposition of $M$ over $(B_A^\eta)$, then $\mathfrak{d}$ can be chosen so that $\mathcal{P}_\mathbf{P}(\mathfrak{d}, M) = \mathcal{P}_\mathbf{P}(\mathfrak{d}_0, M)$.

Proof. Suppose $\mathfrak{d}_0 = \langle N_\eta, a_\eta : \eta \in I \rangle$ is given. As $\text{dcl}(a_{\langle 0 \rangle}) = \text{dcl}(B)$, we may assume that $a_{\langle 0 \rangle} = B$. Thus, $B \downharpoonright N_0$. Choose a finite $D \subseteq N_0$ such that $A \subseteq D$ and $W \downharpoonright D \subseteq N_0$. By e.g. [5, 1.18(9)] there is $N^1 \subseteq N_0$ that is $\aleph_\epsilon$-prime over $\emptyset$, $N^1 \downharpoonright D$, and $N_0$ is $\aleph_\epsilon$-prime over $N^1 \cup D$.

As for non-forking, we claim that the set $\{B, W_0, N^1\}$ is independent over $A$. To see this, first recall that $B \downharpoonright W_0$ since $W_0$ is a base respecting $(B_A^\eta)$. As well, $B \downharpoonright A \upharpoonright N_0$ since $\mathfrak{d}$ is over $(B_A^\eta)$. Thus, $B \downharpoonright N_0$. Hence, by our choice of $D$ and forking calculus, $W_0B \downharpoonright D \subseteq N_0$, so $W_0B \downharpoonright D$ since $N^1 \subseteq N_0$. But now, as $N^1 \downharpoonright D$, we have $N^1 \downharpoonright BW_0$, which gives the independence.

By Lemma 5.7, there is an $(\aleph_\epsilon, \mathbf{P})$-decomposition $\mathfrak{d}_1$ inside $M$ over $(B_A^\eta)$ with $N^a_{\langle 1 \rangle} = N^1$. By Lemma 5.3(4), $\mathfrak{d}_1$ is an $(\aleph_\epsilon, \mathbf{P})$-decomposition of $M$ over $(B_A^\eta)$, so by Lemma 5.4, $\mathcal{P}_\mathbf{P}(\mathfrak{d}_0, M) = \mathcal{P}_\mathbf{P}(\mathfrak{d}_1, M)$.
Next, let \( N^2 \leq M \) be \( \mathcal{R}_e \)-prime over \( N^1 \cup W_0 \). As \( B \perp_{N^1} W_0 \), by the \( \mathcal{R}_e \)-isolation of \( N^1 \) we have \( N^2 \perp_{N^1} B \). Thus, by Lemma 5.6 there is an \((\mathcal{R}_e, \mathcal{P})\)-decomposition \( \varnothing_2 \) inside \( M \) over \( (B^1_A) \) with \( N^0_\varnothing = N^2 \). Again by Lemma 5.3(4), \( \varnothing_2 \) is an \((\mathcal{R}_e, \mathcal{P})\)-decomposition of \( M \) over \( (B^0_A) \), and by Lemma 5.4 \( P_M(\varnothing_1, M) = P_M(\varnothing_2, M) \).

Put \( N := N^2 \). Clearly, \( W_0 \subseteq N \) and we showed \( B \perp_{N^1} N \). But, as \( B \) and \( N^1 \) are independent over \( A \), we see that \( B \perp_{N^1} N \). So, by Lemma 5.10 there is a \((\mathcal{R}_e, \mathcal{P})\)-decomposition \( \varnothing_3 \) of \( M \) over \( (B^0_A) \) extending \( (N, N^1) \) over \( (B^0_A) \).

Finally, by Lemma 4.17 there is an \((\mathcal{R}_e, \mathcal{P})\)-decomposition \( \varnothing_3 \) of \( M \) over \( (B^0_A) \) and extending \( (N, N^1) \) over \( (B^0_A) \). As \( N^0_\varnothing = N^0_\varnothing \), we conclude by Lemma 5.5 that \( P_M(\varnothing_3, M) = P_M(\varnothing_2, M) \). Thus, \( P_M(\varnothing_3, M) = P_M(\varnothing_0, M) \) and we finish.

We are finally ready to prove our main theorem.

**Theorem 5.12.** Suppose that \( M \) is \( \mathcal{R}_e \)-saturated and \( (B^0_A) \in \Gamma_P(M) \). Then \( P_M(\varnothing_1, M) = P_M(\varnothing_2, M) \) for any two prime \((\mathcal{R}_e, \mathcal{P})\)-decompositions \( \varnothing_1, \varnothing_2 \) of \( M \) over \( (B^0_A) \).

**Proof.** Suppose \( \varnothing_1 = (N_\eta, a_\eta : \eta \in I) \). By symmetry, it suffices to prove that every \( p \in P_M(\varnothing_1, M) \) is in \( P_M(\varnothing_2, M) \). Fix such a \( p \) and choose \( \eta \in I \backslash \{\} \) such that \( p \not\perp N_\eta \) but \( p \perp N_\eta \). Choose \( q \in S(N_\eta) \) regular such that \( p \not\perp q \) and choose a finite \( V \subseteq N_\eta \) on which \( q \) is based and stationary. By Lemma 5.9 there is a \( \epsilon \)-finite \( W \) such that \( V \subseteq W \subseteq M \) that has a subset \( W_0 = W \cap N_\emptyset \) respecting \( (B^0_A) \). Note that since \( p \perp N_\emptyset \), we have \( p \perp W_0 \), hence \( q \perp W_0 \). By applying Proposition 5.11 to \( W \) and \( \varnothing_2 \), we find that there is a prime \((\mathcal{R}_e, \mathcal{P})\)-decomposition \( \varnothing^* \) of \( M \) over \( (B^0_A) \) with \( P_M(\varnothing^*, M) = P_M(\varnothing_2, M) \). But, by construction, there is a type parallel to \( q \) (and hence non-orthogonal to \( p \)) in \( S(N^0_\emptyset) \). As \( B \) dominates \( W \) over \( W_0 \) and \( B \perp_{A} N_\emptyset \), we have \( W \perp_{W_0} N_\emptyset \). As \( q \) is based on \( W \) and \( q \perp W_0 \), we see that \( q \) (and hence \( p \)) is orthogonal to \( N_\emptyset \). Thus, \( p \in P_M(\varnothing^*, M) = P_M(\varnothing_2, M) \).

The previous theorem inspires the following definition.

**Definition 5.13.** For \( M \) \( \mathcal{R}_e \)-saturated and \( (B^0_A) \in \Gamma_P(M) \), \( P_M((B^0_A), M) = P_M(\varnothing, M) \) for some (equivalently for every) prime \((\mathcal{R}_e, \mathcal{P})\)-decomposition \( \varnothing \) of \( M \) over \( (B^0_A) \).

**Corollary 5.14.** Suppose that \( M \) is \( \mathcal{R}_e \)-saturated, \( (B^0_A) \in \Gamma_P(M) \), and that \( \varnothing = (N_\eta, a_\eta : \eta \in I) \) is a prime \((\mathcal{R}_e, \mathcal{P})\)-decomposition of \( M \) satisfying \( (1) \ N_\emptyset \) is \( \mathcal{R}_e \)-prime over \( A \); \( (2) \ B \perp_{A} N_\emptyset \); and \( (3) \ N_0 \) is \( \mathcal{R}_e \)-prime over \( A \).
Then, by Definition 4.16, there is an independent subset of $\eta$-saturated, this implies that there is a prime $(\kappa_e, P)$-decomposition of $M$ above $(B_1)$. But, by Theorem 5.12, for any $p \in S(M) \cap P$ we have
\begin{align*}
p \in \mathcal{P}_P((B_1), M) &\iff p \in \mathcal{P}_P(\emptyset, M) \iff \langle 0 \rangle \subseteq \eta(p).
\end{align*}

The following characterization is analogous to [5, Claim 1.24].

**Proposition 5.15.** Assume that $M_1 \preceq M_2$ are $\kappa_e$-saturated and $(B_1) \in \Gamma_P(M_1)$. Then the following are equivalent:

1. No $p \in \mathcal{P}_P((B_1), M_1)$ is realized in $M_2$.
2. There is a prime $(\kappa_e, P)$-decomposition of $M_1$ above $(B_1)$ that is also a prime $(\kappa_e, P)$-decomposition of $M_2$ above $(B_1)$.
3. Every prime $(\kappa_e, P)$-decomposition of $M_1$ above $(B_1)$ is also a prime $(\kappa_e, P)$-decomposition of $M_2$ above $(B_1)$.

**Proof.** (3) $\Rightarrow$ (2) is immediate since prime $(\kappa_e, P)$-decompositions of $M_1$ over $(B_1)$ exist.

(2) $\Rightarrow$ (1). Let $\emptyset = \langle N_\eta, a_\eta : \eta \in I \rangle$ be a prime $(\kappa_e, P)$-decomposition of $M_1$ above $(B_1)$ and assume that there is $e \in M_2 \setminus M_1$ such that $p = \text{tp}(e/M_1) \in \mathcal{P}_P(\emptyset, M_1)$. Choose $\eta \in I$ to be $\preceq$-minimal such that $p \not\preceq N_\eta$. Note that $\langle 0 \rangle \subseteq \eta$. By Fact 2.3(4) and because $N_\eta$, $M_1$, $M_2$ are $\kappa_e$-saturated, we can replace $e$ by the realization of a non-orthogonal regular type that satisfies $e \perp_{N_\eta} M_1$. As $e \in C_\eta(M_2)$, $\{a_\nu : \nu \in \text{Succ}_I(\eta)\}$ is not a maximal $N_\eta$-independent subset of $C_\eta(M_2)$, so $\emptyset$ is not a prime $(\kappa_e, P)$-decomposition of $M_2$ above $(B_1)$.

(1) $\Rightarrow$ (3). Let $\emptyset = \langle N_\eta, a_\eta : \eta \in I \rangle$ be a prime $(\kappa_e, P)$-decomposition of $M_1$ above $(B_1)$, and assume that it is not a prime $(\kappa_e, P)$-decomposition of $M_2$. Then, by Definition 4.16 there is $\eta \in I \setminus \{\emptyset\}$ such that $\{a_\nu : \nu \in \text{Succ}_I(\eta)\}$ is not a maximal $N_\eta$-independent subset of $C_\eta(M_2)$. As both $N_\eta$ and $M_2$ are $\kappa_e$-saturated, this implies that there is $e \in C_\eta(M_2)$ such that $\text{tp}(e/N_\eta) \in P$, but $e \perp_{N_\eta} \{a_\nu : \nu \in \text{Succ}_I(\eta)\}$. By Lemma 4.17(5), $\emptyset$ $P$-exhausts $M_1$ over $(B_1)$ so $e \perp_{N_\eta} M_1$. Thus, $p = \text{tp}(e/M_1)$ is an element of $\mathcal{P}_P((B_1), M_1)$ that is realized in $M_2$. ■

For pairs $(B_1)$ and $(B_2)$ from $\Gamma(M)$, we consider two ways in which $(B_2)$ can extend $(B_1)$, corresponding to the former ‘having more information’ or ‘appearing higher up in a $P$-decomposition’.
First, write \((B_1) \leq_a (B_2)\) if both are from \(\Gamma(M)\), \(A_1 \subseteq A_2\), \(B_1 \subseteq B_2\), \(B_1 \perp A_2\), and \(B_2\) is dominated by \(B_1\) over \(A_2\). Intuitively, think of \((B_2)_{A_2}\) as being a ‘better approximation’ of \((N, N', a)\).

The next approximation, which should be thought of as ‘stepping up in the tree’, is given by \((B_1) \leq_b (B_2)\) if and only if \(A_2 = B_1\), and \(\text{tp}(B_2/A_2)\) is regular and is orthogonal to \(A_1\).

Finally, let \(\leq^*\) be the transitive closure of \(\leq_a \cup \leq_b\).

**Proposition 5.16.** Fix an \(\aleph\)-saturated model \(M\) and \((B_1)_{A_1}, (B_2)_{A_2}\) in \(\Gamma_P(M)\).

1. If \((B_1) \leq_a (B_2)_{A_2}\), then \(\mathcal{P}_P((B_1)_{A_1}, M) = \mathcal{P}_P((B_2)_{A_2}, M)\).
2. If \((B_1) \leq_b (B_2)_{A_2}\), then \(\mathcal{P}_P((B_2)_{A_2}, M)\) is a proper subset of \(\mathcal{P}_P((B_1)_{A_1}, M)\).
3. If \((B_1) \leq^* (B_2)_{A_2}\), then \(\mathcal{P}_P((B_2)_{A_2}, M) \subseteq \mathcal{P}_P((B_1)_{A_1}, M)\).
4. If \(A_1 = A_2\) (whose common value denoted by \(A\)), \(\text{tp}(B_1/A)\), \(\text{tp}(B_2, A)\) are both regular, and \(B_1 \perp B_2\), then \(\mathcal{P}_P((B_1)_{A_1}, M) = \mathcal{P}_P((B_2)_{A_2}, M)\).
5. If \(A_1 = A_2 = A\) and \(B_1 \perp B_2\), then \(\mathcal{P}_P((B_1)_{A_1}, M)\) and \(\mathcal{P}_P((B_2)_{A_2}, M)\) are disjoint.

**Proof.** (1) Let \(N_\emptyset \preceq M\) be \(\aleph\)-prime over \(\emptyset\) with \(A_2 \subseteq N_\emptyset\) and \(B_2 \perp A_1\). Let \(N_\emptyset\) be \(\aleph\)-prime over \(N_\emptyset \cup B_2\), let \(a_\langle 0 \rangle = B_2\), and let \(\mathfrak{d} = \langle N_\eta, a_\eta : \eta \in I \rangle\) be a prime \((\aleph, P)\)-decomposition of \(M\) over \((B_2)_{A_2}\) and extending \(\langle N_\emptyset, N_\emptyset \rangle\).

By the forking calculus, \(\mathfrak{d}\) is also a prime \((\aleph, P)\)-decomposition of \(M\) over \((B_1)_{A_1}\). Thus, two applications of Theorem 5.12 yield

\[ \mathcal{P}_P\left((B_2)_{A_2}, M\right) = \mathcal{P}_P(\mathfrak{d}, M) = \mathcal{P}_P\left((B_1)_{A_1}, M\right). \]

(2) Given \(A_1 \subseteq B_1 = A_2 \subseteq B_2 \subseteq M\) with \(\text{tp}(B_2/A_2) \perp A_1\), first choose an \(\aleph\)-prime \(N_\emptyset \preceq M\) containing \(A_1\) with \(B_2 \perp A_1\). Note that \(\text{tp}(B_2/A_2) \perp N_\emptyset\). Let \(a_\langle \rangle\) be an arbitrary element of \(N_\emptyset\), let \(a_\langle 0 \rangle := A_2\), and choose \(N_0 \preceq M\) to be \(\aleph\)-prime over \(N_\emptyset \cup A_2\), with \(N_0 \perp A_2\). Also, choose \(N_{0,0} \preceq M\) to be \(\aleph\)-prime over \(N_{0,0} \cup B_2\) and let \(a_{0,0} := B_2\).

Let \(\mathfrak{d}_0\) be the three-element prime \((\aleph, P)\)-decomposition \(\langle N_\eta, a_\eta : \eta \in \{\emptyset, 0, (0, 0)\}\rangle\) inside \(M\) above \((B_1)_{A_1}\). Next, by ‘collapsing’, let \(\mathfrak{d}'_0 = \langle N'_\eta, a'_\eta : \eta \in \{\emptyset, 0\}\rangle\) be the two-element prime \((\aleph, P)\)-decomposition inside \(M\) above \((B_2)_{A_2}\), where \(N'_0 := N_0\), \(a'_0 := a_0\), \(N'_0 := N_{0,0}\), and \(a'_0 := a_{0,0}\).

Now, choose a prime \((\aleph, P)\)-decomposition \(\mathfrak{d}' = \langle N'_\eta, a'_\eta : \eta \in I' \rangle\) of \(M\) above \((B_2)_{A_2}\) and extending \(\mathfrak{d}'_0\). It follows immediately from Theorem 5.12 that \(\mathcal{P}_P((B_2)_{A_2}, M) = \mathcal{P}_P(\mathfrak{d}', M)\), so to obtain the inclusion \(\mathcal{P}_P((B_2)_{A_2}, M) \subseteq \mathcal{P}_P((B_1)_{A_1}, M)\) it suffices to construct a prime \((\aleph, P)\)-decomposition \(\mathfrak{d} = \langle\ldots\rangle\).
\langle N, a : \eta \in J \rangle \text{ inside } M \text{ over } (B_{A_1}) \text{ such that, for any } p \in S(M) \cap \mathbf{P}, \text{ if } p \not\perp N'_{\eta} \text{ but } p \perp N_{\eta'} \text{ for some } \eta \in I' \text{ with } \langle 0 \rangle \unlhd \eta', \text{ then there is } \eta \in J \text{ such that } \langle 0 \rangle \unlhd \eta, p \not\perp N, \text{ but } p \perp N_{\eta}.

We accomplish this as follows: Recall that \( N_{\eta}, a_{\eta} \) were defined for \( \eta \in \{ \langle \rangle, \langle 0 \rangle, \langle 0, 0 \rangle \} \) above. Let \( J' \subseteq I' \) be \( \{ \langle \rangle \} \cup \{ \eta \in I' : \langle 0 \rangle \unlhd \eta \} \), and define a function \( h \) with domain \( J' \) by \( h(\eta) := \langle 0 \rangle \smallsetminus \eta \) if \( \eta \neq \langle \rangle \). That is, \( h \) is ‘undoing’ the collapse given above. Let \( J = \{ \langle \rangle, \langle 0 \rangle \} \cup \{ h(\eta) : \eta \in J' \} \), and for each \( \eta \in J' \), set \( N_{h(\eta)} := N'_{\eta} \) and \( a_{h(\eta)} := a_{\eta}' \). Then \( \mathcal{D} := \langle N, a_{\eta} : \eta \in J \rangle \) is a prime \((\mathcal{N}, \mathbf{P})\)-decomposition inside \( M \) above \((B_{A_1})\), and for any \( p \in \mathcal{P}(\mathcal{D}', M) \), if \( p \not\perp N'_{\eta} \) for some \( \eta \in J' \), then \( p \not\perp N_{h(\eta)} \). Thus, \( \mathcal{D} \) is as required.

To show that the inclusion is strict, choose any regular type \( q \in S(N_{\langle 0 \rangle}) \) that is non-orthogonal to \( \text{tp}(B_2/N_{\langle 0 \rangle}) \). It is easy to check that the non-forking extension of \( q \) to \( S(M) \) is an element of \( \mathcal{P}(B_{A_1}, M) \setminus \mathcal{P}(B_{A_2}, M) \).

(3) follows immediately from (1) and (2).

(4) By symmetry, it suffices to show that \( \mathcal{P}(B_{A_1}, M) \subseteq \mathcal{P}(B_{A_2}, M) \), so fix a regular type \( p \in S(M) \cap \mathbf{P} \setminus \mathcal{P}(B_{A_1}, M) \). We will eventually produce a prime \((\mathcal{N}, \mathbf{P})\)-decomposition \( \mathcal{D}_2 \) inside \( M \) over \((B_{A_2})\) with the property that \( p \not\perp N_{\eta} \) for some \( \eta \) satisfying \( \neg (\langle 0 \rangle \unlhd \eta) \), which suffices by Lemma 4.17(4) and Theorem 5.12.

We begin by choosing an \( \mathcal{N} \)-prime (over \( \emptyset \)) \( N := M \) that contains \( A \), but \( B_1 B_2 \not\perp N \). Note that \( B_1 \) and \( B_2 \) are domination equivalent over \( N \).

Let \( A \in N \) be arbitrary, let \( N^1 \) be \( \mathcal{N} \)-prime over \( N \cup B_1 \), and let \( a_{\langle 0 \rangle} := B_1 \). Then \( \mathcal{D}_1 := \langle N, a_{\eta} : \eta \in \{ \langle \rangle, \langle 0 \rangle \} \rangle \) is a two-element prime \((\mathcal{N}, \mathbf{P})\)-decomposition inside \( M \) over \((B_{A_1})\). Let \( \mathcal{D}_1' = \langle N, a_{\eta} : \eta \in I \rangle \) be a prime \((\mathcal{N}, \mathbf{P})\)-decomposition of \( M \) over \((B_{A_1})\) end extending \( \mathcal{D}_1 \). Next, let \( \mathcal{D}_1^* := \langle N, a_{\eta} : \eta \in J \rangle \) be a prime \((\mathcal{N}, \mathbf{P})\)-decomposition of \( M \) end extending \( \mathcal{D}_1' \). Let \( H = \{ \eta \in J : \neg (\langle 0 \rangle \unlhd \eta) \} \). Then \( H \) is a subtree of \( J \), whose intersection with \( I \) is \( \{ \} \). Furthermore, as \( p \in \mathbf{P} \), it follows from Corollary 4.12 that \( p \not\perp N_{\eta} \) for some \( \eta \in J \). However, since \( p \not\in \mathcal{P}(B_{A_1}, M) \), Theorem 5.12 implies that \( p \not\in \mathcal{P}(\mathcal{D}_1, M) \), hence \( p \not\perp N_{\eta} \) for some \( \eta \in H \).

But now, choose \( N \subseteq M \) to be \( \mathcal{N} \)-prime over \( N \cup B_2 \). Let \( \mathcal{D}_2 := \langle N, a_{\eta} : \eta \in H \rangle \cap (N^2, B_2) \). As \( B_1 \) and \( B_2 \) are domination equivalent over \( N \), it is easily checked that \( \mathcal{D}_2 \) is a prime \((\mathcal{N}, \mathbf{P})\)-decomposition inside \( M \) over \((B_{A_2})\). Let \( \mathcal{D}_2' = \langle N, a_{\eta} : \eta \in I_2 \rangle \) be any prime \((\mathcal{N}, \mathbf{P})\)-decomposition inside \( M \) end extending \( \mathcal{D}_2 \). But, as \( p \not\perp N_{\eta} \) for some \( \eta \in H \), it follows from independence that \( p \perp N_{\eta} \) for any \( \nu \in I_2 \) satisfying \( \langle 0 \rangle \unlhd \nu \). Thus, \( p \not\in \mathcal{P}(B_{A_2}, M) \) by Theorem 5.12 again.

(5) Let \( N \) be \( \mathcal{N} \)-prime over \( A \) with \( N \not\perp B_1 B_2 \) and choose an \( \epsilon \)-finite \( B_0 \in N \) arbitrarily. For \( \ell = 1, 2 \), choose \( N(\ell) \) to be \( \mathcal{N} \)-prime over \( N \cup B_\ell \). Clearly,
\[\varnothing' := \{(N_0, B_0), (N_0, B_1), (N_1, B_1)\}\]

is a three-element, prime \((N_\epsilon, \mathbf{P})\)-decomposition inside \(M\). By Lemma 4.15(2) there is a prime \((N_\epsilon, \mathbf{P})\)-decomposition \(\varnothing = \langle N_\eta, a_\eta : \eta \in I \rangle\) of \(M\) extending \(\varnothing'\). It is easily checked that \(\varnothing\) satisfies the hypotheses of Corollary 5.14 as does the modification formed by exchanging the roles of \(\langle 0 \rangle\) and \(\langle 1 \rangle\). Thus, for any \(p \in S(M) \cap \mathbf{P}\), we see that \(p \in \mathcal{P}_{\mathbf{P}}((B_1^\alpha, M)\) if and only if \(\langle 0 \rangle \leq \eta(p)\), and that \(p \in \mathcal{P}_{\mathbf{P}}((B_2^\alpha, M)\) if and only if \(\langle 1 \rangle \leq \eta(p)\). As the elements \(\langle 0 \rangle\) and \(\langle 1 \rangle\) are incompatible, it follows that the sets \(\mathcal{P}_{\mathbf{P}}((B_1^\alpha, M)\) and \(\mathcal{P}_{\mathbf{P}}((B_2^\alpha, M)\) are disjoint. 

**Corollary 5.17.** Suppose that \(M\) is \(N_\epsilon\)-saturated and that \(\varnothing = \langle N_\eta, a_\eta : \eta \in I \rangle\) is any weak \(\mathbf{P}\)-decomposition inside \(M\). Choose any incomparable nodes \(\eta_1, \eta_2 \in I\). If, for each \(\ell = 1, 2\), \(A_\ell \subseteq N_{\eta_\ell}^\epsilon\) is \(\epsilon\)-finite on which \(\text{tp}(a_{\eta_\ell}/N_{\eta_\ell}^\epsilon)\) is based and stationary and \(B_\ell = \text{acl}(A_\ell \cup \{a_{\eta_\ell}\})\), then the sets \(\mathcal{P}_{\mathbf{P}}((B_1^\alpha, M)\) and \(\mathcal{P}_{\mathbf{P}}((B_2^\alpha, M)\) are disjoint.

**Proof.** As \(\eta_1\) and \(\eta_2\) are incomparable, neither one is \(\langle \rangle\), so let \(\mu\) denote the meet \(\eta_1^\epsilon \wedge \eta_2^\epsilon\). By incomparability again, there are distinct ordinals \(\alpha_1 \neq \alpha_2\) such that \(\mu^\epsilon(\alpha_1) \leq \eta_1\), while \(\mu^\epsilon(\alpha_2) \leq \eta_2\). Choose an \(\epsilon\)-finite \(E \subseteq M_\mu\) over which both \(\text{tp}(a_{\mu^\epsilon(\alpha_1)}/M_\mu)\) are based and stationary, and let \(C_\ell = \text{acl}(a_{\mu^\epsilon(\alpha_1)} \cup E)\) for each \(\ell\). As \(C_1 \nsubseteq C_2\) it follows from Proposition 5.16(5) that the sets \(\mathcal{P}_{\mathbf{P}}((C_1_E^\alpha, M)\) and \(\mathcal{P}_{\mathbf{P}}((C_2_E^\alpha, M)\) are disjoint. But, by Proposition 5.16(3), \(\mathcal{P}_{\mathbf{P}}((B_1^\alpha, M) \subseteq \mathcal{P}_{\mathbf{P}}((C_1_E^\alpha, M)\) for each \(\ell\), and the result follows. 

**Proposition 5.18.** Suppose that \(M\) is \(N_\epsilon\)-saturated, and \(p_1 \in S(A_1)\) and \(p_2 \in S(A_2)\) are non-orthogonal, trivial, regular types over \(\epsilon\)-finite subsets of \(M\). If, for \(\ell = 1, 2\), \(I_\ell\) is a maximal, \(A_\ell\)-independent subset of \(p_\ell(M)\), then there are cofinite subsets \(J_\ell \subseteq I_\ell\) and a bijection \(h : J_1 \rightarrow J_2\) such that

\[\mathcal{P}_{\mathbf{P}}\left(\left(\begin{array}{c} c \\ A_1 \end{array}\right), M\right) = \mathcal{P}_{\mathbf{P}}\left(\left(\begin{array}{c} h(c) \\ A_2 \end{array}\right), M\right)\]

for every \(c \in J_1\).

**Proof.** Let \(D = A_1 \cup A_2\). For \(\ell = 1, 2\), let \(J_\ell := \{c \in I_\ell : c \upharpoonright D\}\) and let \(q_\ell\) denote the non-forking extension of \(p_\ell\) to \(S(D)\). Then \(J_\ell\) is a cofinite subset of \(I_\ell\) and is a maximal, \(D\)-independent subset of \(q_\ell(M)\). As the regular types are trivial and non-orthogonal, \(p_1\) and \(p_2\) are not almost orthogonal, so as \(M\) is \(N_\epsilon\)-saturated, we see that for every \(c \in q_1(M)\), there is \(c' \in q_2(M)\) such that \(c_1 \upharpoonright_D c_2\). It follows that there is a unique bijection \(h : J_1 \rightarrow J_2\) satisfying \(c_1 \upharpoonright_D h(c)\) for each \(c \in J_1\). Thus, \(\mathcal{P}_{\mathbf{P}}\left(\left(\begin{array}{c} c \\ A_1 \end{array}\right), M\right) = \mathcal{P}_{\mathbf{P}}\left(\left(\begin{array}{c} h(c) \\ A_2 \end{array}\right), M\right)\) by clauses (1) and (4) of Proposition 5.16. 

\[\mathbf{P}-\text{NDOP and } \mathbf{P}\text{-decompositions} \quad 69\]
6. Decompositions and non-saturated models. Until this point, we have been looking at various flavors of decompositions of $\aleph_\epsilon$-saturated models. Now it would be desirable to see what effect these results have on understanding decompositions of arbitrary models. In the first subsection, given an arbitrary model $M$ and a sufficiently saturated elementary extension $M^*$, one can produce an $(\aleph_\epsilon, \mathcal{P})$-decomposition $d = \langle M_\eta, a_\eta : \eta \in I \rangle$ of $M^*$ that ‘enumerates $M$ as slowly as possible’. In particular, given any $\epsilon$-finite $A \subseteq M$, there is a finite subtree $J \subseteq I$, an elementary submodel $M^J \preceq M^*$ that is $\aleph_\epsilon$-prime over $\bigcup_{\eta \in J} M_\eta$, and an $\epsilon$-finite $B$, $A \subseteq B \subseteq M$, that satisfy $M \upharpoonright_B M_J$.

In the second subsection, we obtain a weak uniqueness result for $\mathcal{P}$-decompositions of unsaturated models $M$ subject to certain constraints. Whereas these conditions seem contrived, Theorem 6.19 plays a major role in [2].

6.1. Large extensions of weak decompositions. As usual, we assume that $\mathcal{P}$ is a set of stationary, regular types closed under isomorphism and non-orthogonality, and we assume that our theory $T$ is superstable with $\mathcal{P}$-NDOP.

**Definition 6.1.** Suppose $M \preceq M^*$ are given, with $M$ arbitrary, but $M^*$ sufficiently saturated. A prime $(\aleph_\epsilon, \mathcal{P})$-decomposition $d^* = \langle N_\eta, a_\eta : \eta \in I \rangle$ of $M^*$ respects $M$ if there is a continuous, elementary chain $\langle M_\alpha : \alpha \leq \alpha^* \rangle$ of $\aleph_\epsilon$-saturated elementary substructures of $M^*$ with $\bigcup_{\alpha \leq \alpha^*} M_\alpha = M^*$; a sequence $\langle d_\alpha : \alpha \leq \alpha^* \rangle$ of prime $(\aleph_\epsilon, \mathcal{P})$-decompositions of $M_\alpha$ with $d_{\alpha^*} = d^*$; and a sequence $\langle a_\alpha : \alpha \leq \alpha^* \rangle$ of elements from $M^*$ that satisfy the following constraints:

1. $M_0 = N_\emptyset$ and the sets $M$ and $M_0$ are independent.
2. If $\beta \leq \alpha$ then $d_\alpha$ end extends $d_\beta$ with $d_\gamma = \bigcup_\alpha d_\alpha$ for $\gamma$ a limit ordinal.
3. The trees $I_\alpha$ indexing the decompositions $d_\alpha$ satisfy $|I_{\alpha+1} \setminus I_\alpha| \leq 1$ for each $\alpha < \alpha^*$.
4. If $I_{\alpha+1} \setminus I_\alpha = \{\eta\}$, then $N_\eta$ is $\aleph_\epsilon$-prime over $N_\eta - a_\alpha$ and $N_\eta \upharpoonright_{N_\eta - a_\alpha} M M_\alpha$.

**Lemma 6.2.** Suppose that $M \preceq M^*$, where $M^*$ is saturated and $\|M^*\| > \|M\| + 2^{|T|}$. Then a prime $(\aleph_\epsilon, \mathcal{P})$-decomposition $d^*$ of $M^*$ respecting $M$ exists.

**Proof.** We recursively construct sequences $\langle M_\alpha \rangle$, $\langle d_\alpha \rangle$ and $\langle a_\alpha \rangle$ with the additional constraint of $\|M_\alpha\| < \|M^*\|$ for each $\alpha < \alpha^*$ as follows. First, choose $N_\emptyset \preceq M^*$ to be $\aleph_\epsilon$-saturated with $N_\emptyset \upharpoonright M$, let $M_0 = N_\emptyset$, $I_0 = \{\emptyset\}$, and $d_0 = \langle N_\emptyset \rangle$. For $\alpha \leq \alpha^*$ a limit ordinal, simply take unions.

Next, fix an enumeration $\langle c_i : i < \lambda \rangle$ of $M^*$ with $\lambda = \|M^*\|$ and the elements of $M$ forming an initial segment, and assume that $M_\beta$ and $d_\beta$ have
been defined. Let c* be the least element of M* that is not an element of Mβ. There are now two cases, depending on \( \text{tp}(c^*/M_\beta) \).

**Case 1:** \( \text{tp}(c^*/M_\beta) \perp \mathcal{P}^{\text{active}} \). In this case, choose a regular type \( q \in S(M_\beta) \) non-orthogonal to \( \text{tp}(c^*/M_\beta) \). As \( M^* \) is saturated, choose an element \( a_\beta \in M^* \) realizing \( q \) with \( a_\beta \not\in c^* \). Let \( \varnothing_{\beta+1} = \varnothing_\beta \) and let \( M_{\beta+1} \preceq M^* \) be \( \aleph_\epsilon \)-prime over \( M_\beta \cup \{a_\beta\} \) and satisfying \( M_{\beta+1} \perp M_\beta \).

**Case 2:** \( \text{tp}(c^*/M_\beta) \not\perp \mathcal{P}^{\text{active}} \). In this case, choose a regular type \( q \in S(M_\beta) \cap \mathcal{P}^{\text{active}} \) non-orthogonal to \( \text{tp}(c^*/M_\beta) \). By Corollary 4.12, there is a unique \( \eta \in I_\beta \) such that \( q \not\perp N_\eta \), but \( q \perp N_\eta^- \) (if \( \eta \neq \langle \rangle \)). Without loss of generality, we may assume that \( q \) does not fork over \( N_\eta \). As \( M^* \) is saturated, we can choose an element \( a_\beta \in M^* \) realizing \( q \) with \( a_\beta \not\in c^* \).

Let \( \gamma \) be the least ordinal such that \( \nu := \eta \wedge \gamma \notin I_\beta \). Choose \( N_\nu \preceq M^* \) to be \( \aleph_\epsilon \)-prime over \( N_\eta \cup \{a_\beta\} \) and satisfying \( N_\nu \perp_{N_\eta \cup \{a_\beta\}} M M_\beta \). As \( N_\eta \) is \( \aleph_\epsilon \)-saturated, it follows by Fact 2.3(2) that \( N_\nu \perp_{N_\eta} M_\beta \). Choose \( M_{\beta+1} \preceq M^* \) to be \( \aleph_\epsilon \)-prime over \( M_\beta \cup N_\nu \) and satisfying \( M_{\beta+1} \perp_{M_\beta \cup N_\nu} M \). Let \( I_{\beta+1} = I_\beta \cup \{\nu\} \), let \( \varnothing_{\beta+1} = \varnothing_\beta \wedge N_\nu \), and let \( M_{\beta+1} \preceq M^* \) be \( \aleph_\epsilon \)-prime over \( M_\beta \cup N_\nu \).

Note that in either case, \( R^\infty(c^*/M_{\beta+1}) < R^\infty(c^*/M_\beta) \), so by continuing in this fashion, \( c^* \) will be contained in \( M_\beta+k \) for some finite \( k \).

Suppose that \( M \preceq M^* \), and that \( \varnothing^* \) is a prime \( (\aleph_\epsilon, \mathcal{P}) \)-decomposition of \( M^* \) respecting \( M \), as witnessed by the sequences \( \langle M_\alpha \rangle \), \( \langle \varnothing_\alpha \rangle \), \( \langle a_\alpha \rangle \). Consider the following statement:

\( (\ast)_\alpha \) For all finite sets \( A \subseteq M, B \subseteq M_\alpha \), and a finite subtree \( t \subseteq I_\alpha \), there is a finite set \( A^* \subseteq M \) containing \( A \) and a finite subtree \( t^* \subseteq I_\alpha \) containing \( t \) such that \( \text{tp}(B/\bigcup\{N_\rho : \rho \in t^*\}) \) is \( \aleph_\epsilon \)-isolated and \( M \downarrow_{A^*} \{N_\rho : \rho \in t^*\} B \).

**Lemma 6.3.** \( (\ast)_\alpha \) holds for all \( \alpha \leq \alpha^* \).

**Proof.** We prove this by induction on \( \alpha \). For \( \alpha = 0 \), this is immediate since \( M_0 = N_0 \) and is independent of \( M \) over \( \emptyset \), hence over any finite subset of \( M \). For \( \alpha \) a non-zero limit ordinal, the statement follows easily from superstability.

For the successor case, fix \( \alpha = \beta + 1 \) and assume that \( (\ast)_\beta \) holds. The verification of \( (\ast)_\alpha \) splits into two cases, depending on whether or not \( I_\beta \) is extended. Here, we discuss the case where \( I_\alpha = I_\beta \cup \{\nu\} \) and leave the other (easier) case to the reader. So \( N_\nu \) is \( \aleph_\epsilon \)-prime over \( N_{\nu^-} \cup \{a_\beta\} \), \( N_\nu \perp_{N_{\nu^-}} M_\beta \), and \( M_\alpha \) is \( \aleph_\epsilon \)-prime over both sets \( \bigcup\{N_\rho : \rho \in I_\alpha\} \) and \( M_\beta \cup N_\nu \).
Towards verifying \((*)_\alpha\), fix finite sets \(A \subseteq M\), \(B \subseteq M_\alpha\), and a finite subtree \(t \subseteq I_\alpha\). Begin by choosing finite sets \(C_\nu \subseteq N_\nu\) and \(C_\beta \subseteq M_\beta\) such that
\[
\text{stp}(B/C_\beta C_\nu) \vdash \text{stp}(B/M_\beta N_\nu).
\]
Without loss, we may assume \(a_\beta \in C_\nu\) and \(C_\eta \cup C_\beta \subseteq B\).

Next, by superstability choose finite sets \(D \subseteq \bigcup \{N_\rho : \rho \in I_\beta\}\) and \(A' \subseteq M\) containing \(A\) such that
\[
C_\nu \downarrow_{DA'} \bigcup \{N_\rho : \rho \in I_\beta\} M.
\]
Similarly, choose finite sets \(E_\beta \subseteq M_\beta\) and \(A'' \subseteq M\) containing \(A'\) such that
\[
B \downarrow_{E_\beta A''} M_\beta M.
\]
Without loss of generality, we may assume \(D \subseteq E_\beta\), \(\nu \in t\), and \(D \subseteq \bigcup \{N_\rho : \rho \in s^*\}\), where \(s := t \setminus \{\nu\}\).

Now apply \((*)_\beta\) to the triple \((A'', E_\beta, s)\) and get a finite set \(A^* \subseteq M\) and a finite tree \(s^* \subseteq I_\beta\). Let \(t^* := s^* \cup \{\nu\}\). We claim that \((A^*, t^*)\) are as desired in the statement of \((*)_\alpha\).

Claim 1. \(B/\bigcup \{N_\rho : \rho \in t^*\}\) is \(\aleph_\epsilon\)-isolated.

To see this, first note that \(C_\beta \subseteq M_\beta\) is \(\aleph_\epsilon\)-isolated over \(\bigcup \{N_\rho : \rho \in s^*\}\). Since \(M_\beta \downarrow_{N_\nu} N_\nu\) and \(N_\nu\) is \(\aleph_\epsilon\)-saturated, it follows that \(C_\beta\) is \(\aleph_\epsilon\)-isolated over \(\bigcup \{N_\rho : \rho \in t^*\}\) as well. Also, \(C_\nu \subseteq N_\nu\), so it follows immediately that \(C_\beta C_\nu/\bigcup \{N_\rho : \rho \in t^*\}\) is \(\aleph_\epsilon\)-isolated as well. But, as \(\text{stp}(B/C_\beta C_\nu) \vdash \text{stp}(B/\bigcup \{N_\rho : \rho \in t^*\})\), the result follows.

Claim 2. \(M \downarrow_{A^*} N_0 N_\nu B\), where \(N_0 := \bigcup \{N_\rho : \rho \in s^*\}\).

First, it follows from our application of \((*)_\beta\) that \(M \downarrow_{A^*} N_0 E_\beta\). We next consider \(C_\nu\). By the definition of \(E_\beta\) and \(A''\) we have \(C_\nu \downarrow_{E_\beta A''} M_\beta M\). So, by monotonicity, we see that \(C_\nu \downarrow_{E_\beta A'^*} N_0 M\), hence \(C_\nu \downarrow_{E_\beta A'^* N_0} M\). Thus, the transitivity of non-forking yields
\[
M \downarrow_{A^*} N_0 E_\beta C_\nu.
\]
Finally, our choice of \(N_\nu\) gives \(N_\nu \downarrow_{N_\nu M_\beta M}\). But \(a_\beta \in C_\nu \subseteq N_\nu\), so \(N_\nu \downarrow_{N_\nu C_\nu} N_0 E_\beta A'^* M\). As \(N_\nu \subseteq N_0\), monotonicity yields
\[
M \downarrow_{N_0 E_\beta A'^*} N_\nu,
\]
and we finish by invoking the transitivity of non-forking. \(\blacksquare\)
PROPOSITION 6.4. Suppose that $M \preceq M^*$ with $M^*$ saturated and $\|M^*\| > \|M\| + 2^{|T|}$. If $\mathfrak{d}^*$ is a prime $(\mathcal{N}_e, \mathbf{P})$-decomposition of $M^*$ respecting $M$, then for every finite $A \subseteq M$ and every finite subtree $t \subseteq I_{\mathfrak{d}^*}$, there is a finite set $A^* \subseteq M$ containing $A$, a finite subtree $t^* \subseteq I_{\mathfrak{d}^*}$ extending $t$, and $M_{t^*} \preceq M^*$ that is $\aleph_e$-prime over $\bigcup\{N_\rho : \rho \in t^*\}$ such that $A \subseteq M_{t^*}$, but $M \not\preceq_{A^*} M_{t^*}$.

Proof. Fix finite $A \subseteq M$ and $t \subseteq I_{\mathfrak{d}^*}$. If $M^* = M_{\alpha^*}$, then applying $(\ast)_{\alpha^*}$ to the triple $(A, A, t)$ yields a finite set $A^* \subseteq M$ containing $A$ and $t^*$ such that $\text{tp}(A/\bigcup\{N_\rho : \rho \in t^*\})$ is $\aleph_e$-isolated and $M \not\preceq_{A^*} \bigcup\{N_\rho : \rho \in t^*\}$. Thus, as $M^*$ is saturated, we can find $M_{t^*} \preceq M^*$ containing $A$ that is both $\aleph_e$-prime over $\bigcup\{N_\rho : \rho \in t^*\}$ and independent of $M$ over $A^*$.

6.2. A weak uniqueness theorem for $\mathbf{P}$-decompositions. The goal of this subsection is Theorem 6.19, which is used in [2]. As we only seek a sufficient condition, the statements and assumptions in Theorem 6.19 are inelegant at best. Additionally, throughout this subsection we assume

$T$ is totally transcendental with $\mathbf{P}$-NDOP and $\mathbf{P} = \mathbf{P}^{\text{active}}$.

The assumption of the theory $T$ being totally transcendental is only used in Lemma 6.7, and one could easily imagine it being replaced by much weaker assumptions.

We begin with a standard fact about superstable theories.

LEMMA 6.5. Suppose that $p \in S(A)$ is stationary and that $J$ is an infinite, $A$-independent set of realizations of $p$. Let $B \supseteq A \cup J$, let $p' \in S(B)$ denote the non-forking extension of $p$, and let $C \supseteq B$ be constructible over $B$. Then $p'$ has a unique extension to $S(C)$.

DEFINITION 6.6. Given any model $M$, a $\mathbf{P}$-$r$-decomposition $\mathfrak{d} = \langle M_\eta, a_\eta : \eta \in I \rangle$ inside $M$ is a weak $\mathbf{P}$-decomposition inside $M$ with the additional property that $\text{tp}(a_\nu/M_{\nu \eta}) \in \mathbf{P}$ (hence is regular) for every $\nu \in I \setminus \{()\}$. Further, $\mathfrak{d}$ is a $\mathbf{P}$-$r$-decomposition of $M$ if, in addition, for every $\eta \in I$, $\{a_\nu : \nu \in \text{Succ}(\eta)\}$ is a maximal $M_\eta$-independent set of realizations of types in $\mathbf{P}$. A $\mathbf{P}$-$r$-decomposition of $M$ is $\mathbf{P}$-finitely saturated if, for every $\epsilon$-finite $A \subseteq M$ and every $b \in M$ such that $\text{tp}(b/A) \in \mathbf{P}$, there is some $\eta \in I$ such that $\text{tp}(b/A) \not\in M_\eta$.

As notation, given a $\mathbf{P}$-$r$-decomposition $\mathfrak{d} = \langle M_\eta, a_\eta : \eta \in I \rangle$ of $M$, let $I' = I \setminus \{()\}$. For each $\eta \in I'$, let $p_\eta = \text{tp}(a_\eta/M_{\eta ^{-}})$ and fix an $\epsilon$-finite $A_\eta \subseteq M_{\eta ^{-}}$ over which $p_\eta$ is based and stationary. We let $\mathcal{P}_\mathbf{P}(a_\eta)$ abbreviate $\mathcal{P}_\mathbf{P}(\langle \text{acl}(A_\eta \setminus a_\eta), \mathcal{C} \rangle)$. Note that by Proposition 5.16(1), $\mathcal{P}_\mathbf{P}(a_\eta) = \mathcal{P}_\mathbf{P}(a_\eta)$ for any $\epsilon$-finite $A_\eta' \subseteq M_{\eta ^{-}}$ on which $p_\eta$ is based and stationary.

Let $C_\eta := \{\rho \in I' : \rho ^{-} = \eta ^{-} \text{ and } p_\rho = p_\eta\}$ and let $J_\eta := \{a_\rho : \rho \in C_\eta\}$.
LEMMA 6.7. Fix any $\mathbf{P}^r$-decomposition $\mathfrak{d} = (\langle M_\eta, a_\eta : \eta \in I \rangle)$ of $M$ and choose any $\eta \in I'$ for which $C_\eta$ is infinite. Denote $p_\eta, A_\eta, C_\eta, J_\eta$ by $p, A, C, J$, respectively. For any $b \in \mathfrak{C}$ realizing $p|_A$, if $b \perp_M M_{\eta^{-}}$ then $b \perp_A M$.

Proof. Fix any element $b$ such that $b \perp_M M_{\eta^{-}}$. Let $D := \bigcup \{M_{\rho} : \rho \in C\}$ and let $E := \bigcup \{M_\nu : \nu \in I\}$. First, as $J$ is infinite,

$$\text{tp}(b/M_{\eta^{-}} - J) \vdash \text{tp}(b/D)$$

by Lemma 6.5. Next, $\text{tp}(b/D) \vdash \text{tp}(b/E)$ by the independence of the tree, orthogonality, and the non-forking calculus. Now, form a maximal, continuous elementary chain $\langle M_\alpha : \alpha < \beta \rangle$ of submodels of $M$ such that $M_0$ is constructible over $E$, and given $M_\alpha$, $M_{\alpha+1}$ is constructible over $M_\alpha \cup \{b_\alpha\}$ for some $b_\alpha$ such that $\text{tp}(b_\alpha/M_\alpha)$ is regular. (Here is where we use the assumption that $T$ is totally transcendental.) Clearly, the maximality of the sequence implies that the union is all of $M$. However, by Lemma 6.5 and the fact that $\text{tp}(b_\alpha/M_\alpha) \perp \mathbf{P}$ (which follows from $\mathbf{P} = \mathbf{P}^{\text{active}}$) we conclude that $\text{tp}(b/E) \vdash \text{tp}(b/M)$.

That $b \perp M$ follows by the transitivity of non-forking. $\blacksquare$

LEMMA 6.8. Suppose $\mathfrak{d} = (\langle M_\eta, a_\eta : \eta \in I \rangle)$ is a $\mathbf{P}^r$-decomposition of $M$ and there is $q \in \mathbf{P}$ and $\eta \in \text{max}(I')$ such that $q \not\perp M_\eta$, but $q \perp M_{\eta^{-}}$. Then, for any $\nu \in I$,

$$\nu \leq \eta \quad \text{if and only if} \quad q \in \mathbf{P}_\mathbf{P}(a^{\nu}_{\eta}).$$

Proof. First, assume that $\nu \leq \eta$. Let $\mathfrak{d}_0 := (\langle M_\delta, a_\delta : \nu^{-} \leq \delta \leq \eta \rangle)$. As in the proof of Lemma 5.6, we can blow up $\mathfrak{d}_0$ to a sequence $\mathfrak{d}_0^\ast := (\langle M_\delta^\ast, a_\delta : \nu^{-} \leq \delta \leq \eta \rangle)$, where $\mathfrak{d}_0^\ast$ is an $(\mathfrak{s}, \mathbf{P})$-decomposition inside $\mathfrak{C}$, with $q \not\perp M_\eta^\ast$, but $q \perp M_{\eta^{-}}$. Thus, $q \in \mathbf{P}_\mathbf{P}(a^{\nu}_{\eta})$ by its definition and Lemma 4.17(3).

Conversely, assume for a contradiction that $q \in \mathbf{P}_\mathbf{P}(a^{\nu}_{\eta})$ but $\neg(\nu \leq \eta)$. As $\nu \neq \eta$ and $\eta \in \text{max}(I')$, $\nu$ and $\eta$ are incomparable. However, since $q \in \mathbf{P}_\mathbf{P}(a^{\nu}_{\eta})$ by the above, it follows from Corollary 5.17 that $\nu$ and $\eta$ are comparable. Thus, $\eta^{-} \leq \nu$. But then, as $q \perp M_{\eta^{-}}$ and $M_{\eta^{-}} \perp a_\nu M_{\nu^{-}}$, it follows that $q$ is orthogonal to any chain starting with $M_{\nu^{-}}$ and $a_\nu$. $\blacksquare$

DEFINITION 6.9. Suppose $S \subseteq \mathbf{P}$. A $\mathbf{P}^r$-decomposition $\mathfrak{d} = (\langle M_\eta, a_\eta : \eta \in I \rangle)$ (inside $\mathfrak{C}$) supports $S$ if, for every $q \in S$, there is a (unique) $\eta(q) \in \text{max}(I')$ such that $q \not\perp M_{\eta(q)}$, but $q \perp M_{\eta^{-}q}$ if $\eta(q) < \eta$. If $\mathfrak{d}$ supports $S$, we let

- $\text{Field}(S) := \{\eta(q) \in \text{max}(I') : q \in S\}$; and
- $I^S := \{\nu \in I : \nu \leq \eta \text{ for some } \eta \in \text{Field}(S)\}$.
Lemma 6.10. Suppose $S \subseteq \mathbf{P}$ and fix a $\mathbf{P}^r$-decomposition $\mathfrak{d} = \langle M_\eta, a_\eta : \eta \in I \rangle$ (inside $\mathfrak{C}$) that supports $S$. Then:

1. If $\nu \in I^S$, then $\text{tp}(a_\nu/M_{\nu^-})$ is trivial.
2. For $\nu \in I', \nu \in I^S$ if and only if $\mathcal{P}_\mathbf{P}(a_\nu) \cap S \neq \emptyset$.
3. If, for all $\delta \in I^S$, there is a single $\epsilon$-finite $A^* \subseteq M_\delta$ such that $\text{tp}(a_\nu/A^*)$ is based and stationary on $A^*$ for every $\nu \in \text{Succ}_I(\delta)$, then for any $\nu \in \text{Succ}_I(\delta)$ and any $b \in \mathfrak{C}$ realizing $\text{tp}(a_\nu/A^*)$, if $\mathcal{P}_\mathbf{P}(b) \cap S \neq \emptyset$, then $b \downarrow_{A^*} M_\delta$.

Proof. (1) It follows immediately from the definitions of $\mathbf{P}^r$-decomposition and $I^S$ that $\text{tp}(a_\nu/M_{\nu^-})$ is in $\mathbf{P}$ and has positive $\mathbf{P}$-depth. Hence, the type is trivial by Lemma 3.11.

(2) This is immediate from unpacking the definitions and Lemma 6.8.

(3) Choose $A^*, \delta, \nu$, and $b$ as required. Choose $r \in \mathcal{P}_\mathbf{P}(b) \cap S$ and look at $\eta(r) \in \max(I')$. By Lemma 6.8, $\delta \triangleleft \eta(r)$. Choose $\mu \in \text{Succ}_I(\delta)$ satisfying $\mu \triangleleft \eta(r)$. By our choice of $A^*$ and Lemma 6.8 again, $r \in \mathcal{P}_\mathbf{P}(a_\mu)$, so by Proposition 5.16(5), $b \notin a_\mu$. But then, as $\text{tp}(b/A^*)$ is a trivial regular type, $b$ is domination equivalent to $a_\mu$ over $A^*$. Since $a_\mu \downarrow_{A^*} M_\delta$, we conclude that the same holds for $b$. ■

Definition 6.11. Fix $S \subseteq \mathbf{P}$ and a model $M$. A $\mathbf{P}^r$-decomposition $\mathfrak{d} = \langle M_\eta, a_\eta : \eta \in I \rangle$ of $M$ is $S$-reasonable if

1. $\mathfrak{d}$ is $\mathbf{P}$-finitely saturated and supports $S$;
2. for each $\eta \in I'$:
   a. $C_\eta \cap I_S$ is infinite;
   b. $p_\rho = p_\eta$ iff $p_\rho \not\perp p_\eta$ for every $\rho \in I'$ such that $\rho^- = \eta^-$; and
   c. if $b \in \mathfrak{C}$ and $\text{tp}(b/A_\eta) = p_\eta|_{A_\eta}$ and $\mathcal{P}_\mathbf{P}(b) \cap S \neq \emptyset$, then $b \downarrow_{A_\eta} M_{\eta^-}$.

Definition 6.12. A weak bijection between two infinite sets $I$ and $J$ is a bijection $h : I' \to J'$, where $I', J'$ are cofinite subsets of $I, J$, respectively.

For $\eta \in I^S \setminus \{()\}$, let $J^S_\eta = \{a_\rho : \rho \in C_\eta \cap I^S\}$.

Proposition 6.13. Fix a set $S \subseteq \mathbf{P}$ and a model $M$. For $\ell = 1, 2$, let $\mathfrak{d}_\ell = \langle M_{\eta_\ell}, a_{\eta_\ell} : \eta_\ell \in I_\ell \rangle$ be two $S$-reasonable $\mathbf{P}^r$-decompositions of $M$. For any $\eta_\ell \in I^S_\ell$, choose $\eta_{3-\ell} \in I_{3-\ell}$ such that $p_{\eta_1} \not\perp p_{\eta_2}$. There is a weak bijection $h : J^S_{\eta_1} \to J^S_{\eta_2}$ satisfying $\mathcal{P}_\mathbf{P}(a) = \mathcal{P}_\mathbf{P}(b(a))$ for each $a$ in $\text{dom}(J^S_{\eta_1})$. 
Proof. For definiteness, assume that $\eta_1 \in I^S_1$. Let $E = A_{\eta_1} \cup A_{\eta_2}$. For $\ell = 1, 2$, let $p_\ell \in S(E)$ be parallel to $p_{\eta_\ell}$, let $J_\ell' = \{a \in J_{\eta_\ell} : a \perp E\},$ and let

$$J_\ell^S = \left\{a \in J_\ell' : \mathcal{P}_\mathbf{P}\left(a_{A_{\eta_\ell}}\right) \cap S \neq \emptyset\right\},$$

which is a cofinite subset of $J^S_{\eta_\ell}$. In particular, $J^S_1 \neq \emptyset$ since $C_{\eta_1} \cap I^S$ is infinite. As well, choose a maximal $E$-independent set $J^*_\ell$ of realizations of $p_\ell$ in $\mathfrak{C}$ extending $J_\ell$. As $p_1$ and $p_2$ are non-orthogonal trivial regular types, it follows from Proposition 5.18 that there is a unique bijection $h : J^*_1 \to J^*_2$ satisfying $h(a) \perp E a$ for each $a \in J^*_1$.

As $a_\ell \perp E$ for $\ell = 1, 2$ and every $a_\ell$ is in $J^*_\ell$, by Proposition 5.16(1) we have

$$\mathcal{P}_\mathbf{P}\left(a_{A_{\eta_1}}\right) = \mathcal{P}_\mathbf{P}\left(a_{E}\right) = \mathcal{P}_\mathbf{P}\left(h(a)_{E}\right) = \mathcal{P}_\mathbf{P}\left(h(a)_{A_{\eta_2}}\right)$$

for each $a \in J^*_\ell$.

CLAIM. For every $a \in J^S_1$, $h(a) \in J^S_2$.

Proof. Choose any $a \in J^S_1$. We first find an element $b \in J_{\eta_2}$ such that $h(a) \perp E b$. Since $a = a_\rho$ for some $\rho \in I^S_1$ satisfying $\rho^- = \eta_1^-, \mathcal{P}_\mathbf{P}\left(a_{A_{\eta_1}}\right) \cap S \neq \emptyset$. As the two sets are equal, $\mathcal{P}_\mathbf{P}\left(h(a)_{A_{\eta_2}}\right) \cap S \neq \emptyset$ as well. As $\mathfrak{d}_2$ is $S$-reasonable, this implies $h(a) \perp M_{\eta_2}$. Next, we argue that $h(a)$ must fork with $J_{\eta_2}$ over $M_{\eta_2}^-$. Indeed, if this were not the case, then by Lemma 6.7 we would have $h(a) \perp M$. But, as $a \perp h(a)$, the fact that $p_{\eta_2}$ has weight one would imply that $a \perp M$ in $E$, which is absurd since $a \in M$.

Thus, $h(a)$ forks with $J_{\eta_2}$ over $M_{\eta_2}^-$. By triviality, there is a unique $b \in J_{\eta_2}$ such that $h(a) \perp M_{\eta_2}^- b$. However, as both $h(a)$ and $b$ are free from $M_{\eta_2}$ over $A_{\eta_2}$, it follows that $h(a)$ and $b$ fork over $A_{\eta_2}$, completing the first part of our argument.

Next, since $h(a)$ realizes $p_2$, it is free from $E$ over $A_{\eta_2}$. As $p_{\eta_2}$ has weight one, the last two statements imply that $b$ is free from $E$ over $A_{\eta_2}$ as well. Thus, $b \in J'_2$. Also, we have $\mathcal{P}_\mathbf{P}\left(a_{E}\right) = \mathcal{P}_\mathbf{P}\left(b_{E}\right)$, so the latter has non-empty intersection with $S$. Thus, $b \in J^S_2$.

Finally, note that both $h(a)$ and $b$ are elements of $J^*_2$ that fork with each other over $E$. Thus, $h(a) = b$ by the $E$-independence of $J^*_2$. So $h(a) \in J^S_2$, completing the proof of the Claim. ■

It follows from the Claim that $J^S_2$ is non-empty. Once we know this, the situation becomes symmetric, so by running the Claim backwards, $h^{-1}$ maps $J^S_2$ into $J^S_1$. That is, the restriction of $h$ to $J^S_1$ is a bijection with $J^S_2$, which completes the proof of Proposition 6.13. ■
We set some notation for partial maps between trees. Given a tree $I$, a large subtree of $I$ is a non-empty (downward closed) subtree $J$ such that for every $\eta \in J$, $\text{Succ}_I(\eta) \setminus J$ is finite. Given two trees $J$ and $K$, an almost embedding $h$ from $J$ to $K$ has $\text{dom}(h)$ a large subtree of $J$, $\text{range}(h) \subseteq K$, $h(\langle \rangle) = \langle \rangle_K$, and for all $\eta, \nu \in \text{dom}(h)$,

$$\eta < \nu \quad \text{if and only if} \quad h(\eta) < h(\nu).$$

The trees $J$ and $K$ are almost isomorphic if there is an almost embedding $h$ from $J$ to $K$ in which $\text{range}(h)$ is a large subtree of $K$.

For $J$ any tree and $\nu \in J$, let $J_{\geq \nu}$ be the tree with root $\nu$ and universe $\{\eta \in J : \eta \supseteq \nu\}$. Given two trees $J$ and $K$ and $\nu \in J$, $\mu \in K$, an almost embedding $h$ from $J$ to $K$ over $(\nu, \mu)$ is an almost embedding from $J_{\geq \nu}$ to $K_{\geq \mu}$.

Finally, if $J$ and $K$ are trees indexing decompositions, we call a pair $(\eta, \nu) \in J \times K$ $\mathcal{P}_\eta$-equivalent if either $\eta = \langle \rangle = \nu$, or both $\eta, \nu \neq \langle \rangle$ and $\mathcal{P}_\eta(A^\eta_\nu) = \mathcal{P}_\nu(A^\nu_\eta)$. An almost $\mathcal{P}_\eta$-embedding from $J$ to $K$ is an almost embedding $h$ from $J$ to $K$ with the pair $(\eta, h(\eta))$ $\mathcal{P}_\eta$-equivalent for each $\eta \in \text{dom}(h)$. Note that if $h$ is an almost $\mathcal{P}_\eta$-embedding and $h(\eta) = \nu$, then the restriction of $h$ to $J_{\geq \eta} := \{\delta \in \text{dom}(h) : \delta \supseteq \eta\}$ is an almost $\mathcal{P}_\eta$-embedding over $(\eta, \nu)$.

Given all of this notation, the proof of the following corollary simply involves successively iterating Proposition 6.13, using the fact that each decomposition is $\mathcal{P}$-finitely saturated.

**Corollary 6.14.** Fix a set $S \subseteq \mathcal{P}$ and a model $M$. For $\ell = 1, 2$, suppose that $\mathfrak{a}_\ell = \langle M_{\eta_\ell}, a_{\eta_\ell} : \eta_\ell \in I_\ell \rangle$ are $S$-reasonable $\mathcal{P}_\ell^S$-decompositions of $M$ with the additional property that for each $\ell$ and $\nu_\ell \in I_\ell$,

$$\left\{ p : \text{there is } \eta_\ell \in \text{Succ}(\nu_\ell) \text{ such that } p_{\eta_\ell} = p \wedge \mathcal{P}_\ell(A_{\eta_\ell}^\eta_\ell) \cap S \neq \emptyset \right\}$$

is finite. Then:

1. For $\ell = 1, 2$, there is an almost $\mathcal{P}_\ell$-embedding $h$ from $I_\ell^S$ to $I_{3-\ell}^S$.
2. For $\ell = 1, 2$ and any $\mathcal{P}_\ell$-equivalent pair $(\eta_\ell, \eta_{3-\ell}) \in I_\ell^S \times I_{3-\ell}^S$ there is an almost $\mathcal{P}$-embedding from $I_\ell^S$ to $I_{3-\ell}^S$ over $(\eta_\ell, \eta_{3-\ell})$.

If we wish to conclude more, namely that the trees $I_1^S$ and $I_3^S$ are almost isomorphic, then we need to show that the almost embeddings given above preserve lengths, i.e., $\log(h(\eta)) = \log(\eta)$ for every $\eta \in \text{dom}(h)$. To accomplish this, we need to put additional constraints on the shapes of the trees $I^S$. The conditions we require are severe, but will be easily satisfied in our construction in [2].

**Definition 6.15.** A two-coloring of a tree $I$ is a sequence $\langle E_\eta : \eta \in I \rangle$ where each $E_\eta$ is an equivalence relation on $\text{Succ}(\eta)$ with at most two classes,
each of which is infinite. (If \( \text{Succ}(\eta) = \emptyset \), then of course \( E_\eta \) is empty as well.)

A node \( \eta \in I \) has uniform depth \( n \) if every branch of the tree \( I_{>\eta} \) has length exactly \( n \). A node \( \eta \) often has unbounded depth if every large subtree \( J \subseteq I_{>\eta} \) has an infinite branch. A node \( \eta \) is an \((m,n)\)-cusp if there are infinite sets \( A_m, A_n, B \subseteq \text{Succ}(\eta) \) such that

1. the set \( A_m \cup A_n \) is pairwise \( E_\eta \)-equivalent;
2. each \( \delta \in A_m \) has uniform depth \( m \);
3. each \( \rho \in A_n \) has uniform depth \( n \); and
4. each \( \gamma \in B \) is often unbounded.

A cusp is an \((m,n)\)-cusp for some \( m \neq n \).

Fix any function \( \Phi : \omega \to \omega \). We say the two-colored tree \( I \) is \( \Phi \)-proper if, for every node \( \eta \in I \),

1. either \( \eta \) has uniform depth \( n \) for some \( n \), or else \( \eta \) often has unbounded depth;
2. if \( \eta \) is an \((m,n)\)-cusp, then \( \lg(\eta) = \Phi(m - n) \);
3. if \( E_\eta \) has two classes, then \( \eta \) is a cusp; and
4. if \( J \) is a large subtree of \( I \), and \( \eta \in J \) is often unbounded, then there is a cusp \( \nu \in J \) with \( \nu \triangleright \eta \).

Note that if \( I \) is a two-colored tree satisfying the conditions above, then for every \( \gamma \in I \) that is of any uniform depth \( k \), there are unique \( \eta, \delta \) such that \( \delta \triangleright \gamma \), \( \eta = \delta^- \), \( \eta \) is a cusp, and \( \delta \) has uniform depth \( n \) for some \( n \geq k \).

**Lemma 6.16.** Suppose that \( M, S, \mathfrak{d}_1, \mathfrak{d}_2 \) satisfy the assumptions of Corollary 6.14 and additionally assume that both \( I^S_1, I^S_2 \), when two-colored by the relations \( E_\eta \) defined by \( E_\eta(\delta, \rho) \) iff \( \delta^- = \eta = \rho^- \) and \( p_\delta = p_\rho \), are \( \Phi \)-proper for the same function \( \Phi \). Then for every \( \mathcal{P}_P \)-equivalent pair \((\eta, \nu) \in I^S_1 \times I^S_2 \):

1. \( \eta \) is often unbounded in \( I^S_1 \) if and only if \( \nu \) is often unbounded in \( I^S_2 \).
2. For any \( n \), \( \eta \) has uniform depth \( n \) if and only if \( \nu \) has uniform depth \( n \).
3. If \( \lg(\eta) = \lg(\nu) \) and \( \eta \) has uniform depth \( n \) for some \( n \), then any almost \( \mathcal{P}_P \)-embedding over \((\eta, \nu)\) preserves lengths.
4. If \( \lg(\eta) \leq \lg(\nu) \) and \( \eta \) is an \((m,n)\)-cusp, then \( \nu \) is also an \((m,n)\)-cusp, \( \lg(\eta) = \lg(\nu) \), and for any almost \( \mathcal{P}_P \)-embedding \( h \) over \((\eta, \nu)\), \( \lg(h(\delta)) = \lg(\delta) \) for all \( \delta \in \text{dom}(h) \cap \text{Succ}(\eta) \) of uniform depth \( m \) or \( n \).
5. If \( \lg(\eta) = \lg(\nu) \), then every almost \( \mathcal{P}_P \)-embedding over \((\eta, \nu)\) preserves lengths.
6. If \( \lg(\eta) = \lg(\nu) \), then the number of \( E_\eta \)-classes in \( I^S_1 \) equals the number of \( E_\nu \)-classes in \( I^S_2 \).
Proof. (1) First assume that \( \eta \) is often unbounded. By Corollary 6.14, choose an almost \( P \)-embedding \( h \) from \( I^1_1 \) to \( I^2_2 \) over \( (\eta, \nu) \). Choose a strictly \( ^< \)-increasing sequence \( \langle \eta_n : n \in \omega \rangle \) from \( \text{dom}(h) \) with \( \eta_0 = \eta \). Then \( \langle h(\eta_n) : n \in \omega \rangle \) is a strictly \( ^< \)-increasing sequence in \( I^2_2 \) with \( h(\eta_0) = \nu \). Thus, \( \nu \) cannot have any finite uniform depth, so it must be often unbounded by properness. The converse is symmetric.

(2) Suppose that \( \nu \) has uniform depth \( n \). Then by (1), \( \eta \) has uniform depth \( m \) for some \( m \). Arguing as in (1) shows that \( m \leq n \), since if we choose any almost \( P \)-embedding \( h \) from \( I^1_1 \) to \( I^2_2 \) over \( (\eta, \nu) \), then the image of any strictly \( ^< \)-increasing sequence \( \langle \eta_i : i < m \rangle \) with \( \eta_0 = \eta \) would be a strictly \( ^< \)-increasing sequence of length \( m \) over \( \nu \). But then, by symmetry, we would also have \( n \leq m \), so \( n = m \). The converse is symmetric.

(3) Suppose that \( h \) is any almost \( P \)-embedding over \( (\eta, \nu) \), where \( \text{lg}(\eta) = \text{lg}(\nu) \) and \( \eta \) has uniform depth \( n \). Then \( \nu \) also has uniform depth \( n \). So, every maximal \( ^< \)-increasing sequence extending \( \eta \) has length \( n \), the image of any such sequence under \( h \) is also a strictly \( ^< \)-increasing sequence of length \( n \), but there is no strictly \( ^< \)-increasing sequence of length more than \( n \) extending \( \nu \). Thus, \( h \) must map immediate successors to immediate successors, and consequently preserve lengths.

(4) Suppose that \( \eta \) is an \( (m, n) \)-cusp and \( \text{lg}(\eta) \leq \text{lg}(\nu) \). Choose an almost \( P \)-embedding \( h \) from \( I^1_1 \) to \( I^2_2 \) over \( (\eta, \nu) \). Choose \( E_\eta \)-equivalent \( \delta \in \text{Succ}(\eta) \cap \text{dom}(h) \) of uniform depth \( m \) and \( \rho \in \text{Succ}(\eta) \cap \text{dom}(h) \) of uniform depth \( n \). Choose \( \mu \in I^2_2 \) and \( q \in S(M^2_\mu) \) such that \( p_\delta \) (which equals \( p_\rho \)) is non-orthogonal to \( q \). By the definition of \( h \), both \( h(\delta), h(\rho) \in \text{Succ}(\mu) \). We argue that \( \mu = h(\eta) \). To see this, first note that since \( h \) is \( ^< \)-preserving, \( h(\eta) \triangleleft h(\delta) \) and \( h(\eta) \triangleleft h(\rho) \), so \( h(\eta) \triangleleft \mu \). But it follows from (2) that \( h(\delta) \) is uniformly of depth \( m \) and \( h(\rho) \) is uniformly of depth \( n \). Thus, \( \mu \) is an \( (m, n) \)-cusp and hence \( \text{lg}(\mu) = \Phi(m - n) = \text{lg}(\eta) \). As we assumed that \( \text{lg}(\eta) \leq \text{lg}(\nu) \) and \( h(\eta) = \nu \), we see that \( \text{lg}(\mu) = \text{lg}(h(\eta)) \), hence \( \mu = h(\eta) = \nu \). This yields \( \text{lg}(\nu) = \text{lg}(\eta) \). Finally, the argument above showed that \( h(\delta) \in \text{Succ}(\nu) \) whenever \( \delta \in \text{dom}(h) \cap \text{Succ}(\eta) \) has uniform depth \( m \) or \( n \).

(5) Assume that \( \text{lg}(\eta) = \text{lg}(\nu) \) and fix any almost \( P \)-embedding \( h \) from \( I^1_1 \) to \( I^2_2 \) over \( (\eta, \nu) \). Note that \( \text{lg}(h(\mu)) \geq \text{lg}(\mu) \) for any \( \mu \in \text{dom}(h) \) simply because \( h \) is \( ^< \)-preserving. We first consider the often unbounded nodes \( \mu \in \text{dom}(h) \). Specifically, we argue by induction on \( k \) that \( \text{lg}(h(\mu)) = \text{lg}(\mu) \) for every often unbounded node \( \mu \in \text{dom}(h) \) for which there is a cusp \( \zeta \supseteq \mu \) with \( \zeta \in \text{dom}(h) \) and \( \text{lg}(\zeta) = \text{lg}(\mu) + k \).

When \( k = 0 \), this means that any such \( \mu \) is itself a cusp, so \( \text{lg}(h(\mu)) = \text{lg}(\mu) \) by (4). Next, assume that the statement holds for \( k \), and choose \( \mu \in \text{dom}(h) \) with some cusp \( \zeta \in \text{dom}(h) \) with \( \mu \subseteq \zeta \) and \( \text{lg}(\zeta) = \text{lg}(\mu) + k + 1 \). Choose \( \rho \in \text{Succ}(\mu) \) with \( \mu \subseteq \rho \subseteq \zeta \). Then \( \text{lg}(h(\rho)) = \text{lg}(\rho) \) by our inductive assumption, so \( h(\rho) \in \text{Succ}(h(\mu)) \), hence \( \text{lg}(h(\mu)) = \text{lg}(\mu) \) as well. Thus,
we have shown that lengths are preserved for all often unbounded nodes \( \mu \in \text{dom}(h) \).

Next, assume that \( \gamma \in \text{dom}(h) \) has uniform depth. By the remark following Definition 6.15 choose \( \mu \) and \( \delta \) such that \( \mu \) is a cusp, \( \mu = \delta^{-}, \delta \sqsubseteq \gamma \), and \( \delta \) has uniform depth \( n \) for some \( n \geq k \). The last sentence of (4) implies that \( \lg(h(\delta)) = \lg(\delta) \). Thus, \( \lg(h(\gamma)) = \lg(\gamma) \) follows from (3). So \( h \) is length-preserving.

(6) As the hypotheses are symmetric, it suffices to prove that the number of \( E_\eta \)-classes is at most the number of \( E_\nu \)-classes. Using Corollary 6.14 choose an almost \( \mathcal{P} \)-isomorphism \( h \) over \((\eta, \nu)\). By (5), \( h \) maps immediate successors of \( \eta \) to immediate successors of \( \nu \). As well, for each \( \delta \in \text{dom}(h) \cap \text{Succ}(\eta) \), \( p_\delta \nsubseteq p_{h(\delta)} \). As non-orthogonality is an equivalence relation on regular types, this implies that \( h \) maps \( E_\eta \)-classes to \( E_\nu \)-classes, and maps distinct \( E_\eta \)-classes to distinct \( E_\nu \)-classes. As there are at most two \( E_\nu \)-classes, the inequality follows. ■

**Theorem 6.17.** Fix a set \( S \subseteq \mathcal{P} \) and a model \( M \). For \( \ell = 1, 2 \), suppose that \( \mathcal{V}_\ell = \langle M_\eta, a_\eta : \eta \in I_\ell \rangle \) satisfy the hypotheses of Lemma 6.16. Then there is an almost \( \mathcal{P} \)-isomorphism \( h \) from \( I^S_1 \) to \( I^S_2 \).

**Proof.** Using Corollary 6.14 choose any almost \( \mathcal{P} \)-embedding \( h \) of \( I^S_1 \) into \( I^S_2 \) such that, for any \( \delta \in \text{dom}(h) \), \( \text{dom}(h) \cap C_\delta \) is a cofinite subset of \( C_\delta \) and \( \text{range}(h) \cap C_{h(\delta)} \) is a cofinite subset of \( C_{h(\delta)} \). From Lemma 6.16 we know that \( h \) preserves levels and, for each node \( \eta \in \text{dom}(h) \), the number of \( E_{h(\eta)} \)-classes is equal to the number of \( E_\eta \)-classes. It follows that \( \text{range}(h) \) is a large subtree of \( I^S_2 \), so \( h \) is an almost \( \mathcal{P} \)-isomorphism between \( I^S_1 \) and \( I^S_2 \). ■

Finally, we exhibit an extreme case, whose hypotheses are satisfied in [2].

**Definition 6.18.** Fix \( S \subseteq \mathcal{P} \), a model \( M \), and a function \( \Phi : \omega \rightarrow \omega \). A \( \mathcal{P}^r \)-decomposition \( \mathcal{V} = \langle M_\eta, a_\eta : \eta \in I \rangle \) of \( M \) is \((S, \Phi)\)-simple if

(1) \( \mathcal{V} \) supports \( S \) and \( \mathcal{P} \)-finitely saturates \( M \);

(2) for every \( \eta \in I^S \):
   (a) \( \text{Succ}_{I^S}(\eta) \) is empty or infinite, but \( E_\eta \) is trivial, i.e., \( p_\nu = p_\mu \) for all \( \nu, \mu \in \text{Succ}_{I^S}(\eta) \);
   (b) \( \eta \) is either of some finite uniform depth or is a cusp; and
   (c) if \( \eta \) is an \((m, n)\)-cusp, then \( \Phi(m - n) = \lg(\eta) \).

**Theorem 6.19.** Fix a set \( S \subseteq \mathcal{P} \) and a model \( M \), and a function \( \Phi : \omega \rightarrow \omega \). If \( \mathcal{V}_1 \) and \( \mathcal{V}_2 \) are both \((S, \Phi)\)-simple \( \mathcal{P}^r \)-decompositions of \( M \), then the trees \( I^S_1 \) and \( I^S_2 \) are almost \( \mathcal{P} \)-isomorphic. ■

**Proof.** Because of Theorem 6.17 we only need to verify that the hypotheses of Lemma 6.16 are satisfied for each of the decompositions. But
this is routine, once one notes that Clause (2)(b) of Definition 6.18 is satisfied because of the triviality of $E_\eta$ and Lemma 6.10(3).

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