All $\mathrm{CAT}(0)$ boundaries of a group of the form $H \times K$ are CE equivalent

by

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Abstract. M. Bestvina has shown that for any given torsion-free $\mathrm{CAT}(0)$ group G, all of its boundaries are shape equivalent. He then posed the question of whether they satisfy the stronger condition of being cell-like equivalent. In this article we prove that the answer is "Yes" in the situation where the group in question splits as a direct product with infinite factors. We accomplish this by proving an interesting theorem in shape theory.

1. Introduction. The $\mathrm{CAT}(0)$ condition is a geometric notion of non-positive curvature, similar to the definition of Gromov δ-hyperbolicity. A geodesic space X is called $\mathrm{CAT}(0)$ if it has the property that geodesic triangles in X are "no fatter" than geodesic triangles in euclidean space (see [6, Section II.1] for a precise definition). The *visual* or *ideal boundary* of X, denoted ∂X, is the collection of geodesic rays emanating from a chosen basepoint. It is well-known that ∂X is well-defined and independent of the choice of basepoint. Furthermore, when given the cone topology, $X \cup \partial X$ is a Z-set compactification for X. A group G is called $\mathrm{CAT}(0)$ if it acts geometrically (i.e. properly discontinuously and cocompactly by isometries) on some $\mathrm{CAT}(0)$ space X. In this setup, we call X a $\mathrm{CAT}(0)$ *G-space* and ∂X a $\mathrm{CAT}(0)$ *boundary* of G. We say that a $\mathrm{CAT}(0)$ group G is *rigid* if it has only one topologically distinct boundary.

It is well-known that if G is negatively curved (acts geometrically on a Gromov δ-hyperbolic space) or if G is free abelian then G is rigid. Apart from this, little is known concerning rigidity of groups. P. L. Bowers and K. Ruane showed that if G splits as the product of a negatively curved group with a free abelian group, then G is rigid [5]. Ruane proved later in [25] that if G splits as a product of two negatively curved groups, then G is rigid. T. Hosaka has extended this work to show that in fact it suffices

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to know that G splits as a product of rigid groups [17]. Another condition which guarantees rigidity is knowing that G acts on a CAT(0) space with isolated flats, which was proven by C. Hruska in [18].

Not all CAT(0) groups are rigid, however: C. Croke and B. Kleiner constructed in [10] an example of a non-rigid CAT(0) group G. Specifically, they showed that G acts on two different CAT(0) spaces whose boundaries admit no homeomorphism. J. Wilson proved in [30] that this same group has uncountably many boundaries. Furthermore, it is shown in [22] that the knot group G of any connected sum of two non-trivial torus knots has uncountably many CAT(0) boundaries. For a collection of non-rigid CAT(0) groups with boundaries of higher dimension, see [21].

On the other end of the spectrum, it has been observed by M. Bestvina in [3] ([1]) and R. Geoghegan in [14] that for a given CAT(0) group, all of its boundaries are shape equivalent. A proof has also been written up by P. Ontaneda in [24]. Bestvina asks in [3] (also in [2]) if all boundaries of a given CAT(0) group also satisfy the stronger condition of being cell-like equivalent. Bestvina's question has been answered in part by R. Ancel, C. Guilbault, and J. Wilson, who showed in [1] that all the currently known boundaries of Croke and Kleiner's original group have this property; they are all cell-like equivalent to the Hawaiian earring.

In this article, we give further evidence in favor of Bestvina's conjecture by proving the following theorem.

THEOREM 1. *Let G be a* CAT(0) *group which splits as a product $H \times K$ where H and K are infinite. Then all* CAT(0) *boundaries of G are cell-like equivalent through finite-dimensional compacta.*

Contrasting this with Hosaka's result, no assumption needs to be made about the factor groups.

In order to prove Theorem 1, we first prove an interesting result in shape theory. In [16], Hastings proves that if two spaces are shape equivalent, then their suspensions are cell-like equivalent. The proof of the next theorem was inspired by a geometric proof of Hastings' theorem shown to the author by Craig Guilbault.

THEOREM 2. *Joins of shape equivalent compacta are cell-like equivalent. That is, if $X \overset{\text{SH}}{\simeq} X'$ and $Y \overset{\text{SH}}{\simeq} Y'$, then*

$$X * Y \overset{\text{CE}}{\simeq} X' * Y'$$

Furthermore, if these four compacta are finite-dimensional, then the cell-like equivalence can be realized through finite dimensions.

([1]) Bestvina's proof uses the hypothesis that the group in question is torsion-free.

Here $*$ denotes the join operation, $\overset{\text{SH}}{\simeq}$ denotes shape equivalence, and $\overset{\text{CE}}{\simeq}$ denotes cell-like equivalence. For us, the term "compactum" means a compact metric space.

2. Equivalence of compacta

2.1. *Shape equivalence.* Shape theory was invented by K. Borsuk in the 1960's as a way to study spaces with bad local properties. The formal definition of shape equivalence is rather technical. We refer the reader to Borsuk's book [4] for details (Mardešić and Segal also give a nice treatment of the subject in [20]). Roughly speaking, two compacta X and Y are shape equivalent if whenever they are embedded in the Hilbert cube Q (or some other ANR), their neighborhood systems are homotopy equivalent in a sense that can be made precise with inverse sequences. We write $X \overset{\text{SH}}{\simeq} Y$ to denote that X and Y are shape equivalent. It is a standard fact that spaces which are homotopy equivalent are also shape equivalent (see, for example, [20, Chapter I, Section 4.1]).

In [9, Section VI], Chapman gives a characterization of shape equivalence by proving the Complement Theorem:

THEOREM. *Two compacta X and Y are* shape equivalent *iff whenever X and Y are imbedded as Z-sets in the Hilbert cube Q, then $Q - X$ is homeomorphic to $Q - Y$.*

A subspace Z of a space X is called a *Z-set* in X if there is a homotopy $H_t : X \to X$ such that $H_0 = \text{id}_X$ but $H_t(X) \subset X - Z$ for all $t > 0$. Embedding a compactum X as a Z-set in Q is easy: one simply embeds X in

$$\{0\} \times \prod_{i=2}^{\infty}[0,1] \subset \prod_{i=1}^{\infty}[0,1] = Q.$$

Similarly, finite-dimensional compacta can be embedded in finite-dimensional cubes, by [23, Chap. 7, Theorem 9.6].

2.2. *Cell-like equivalence.* A *cell-like compactum* is a compact metric space which is shape equivalent to a point. In particular, contractible compacta are cell-like. A *cell-like map* is a proper surjective map $X \to Y$ such that every fiber is a cell-like compactum.

We say that two compacta X and Y are *cell-like equivalent* and write $X \overset{\text{CE}}{\simeq} Y$ if there is a zig-zag of compacta and cell-like maps
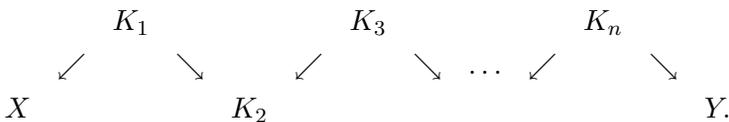
If all compacta in this zig-zag are finite-dimensional, then we say that X and Y are *cell-like equivalent through finite dimensions*, and write $X \overset{\mathrm{CE^f}}{\simeq} Y$.

2.3. *The finite-dimensional category.* If we restrict ourselves to the category of finite-dimensional compacta, then it is known that cell-like equivalence (that is, cell-like equivalence through finite dimensions) is strictly stronger than shape equivalence and strictly weaker than homotopy equivalence (denoted $\overset{\mathrm{HE}}{\simeq}$). Specifically, we have the following for finite-dimensional compacta X and Y.

FACTS 2.1.

(1) $X \overset{\mathrm{HE}}{\simeq} Y \Rightarrow X \overset{\mathrm{CE^f}}{\simeq} Y$ (proven by S. Ferry in [12, Theorem 2]).

(2) $X \overset{\mathrm{CE^f}}{\simeq} Y \nRightarrow X \overset{\mathrm{HE}}{\simeq} Y$.

(3) $X \overset{\mathrm{CE^f}}{\simeq} Y \Rightarrow X \overset{\mathrm{SH}}{\simeq} Y$ (proven by R. B. Sher in [26]).

(4) $X \overset{\mathrm{SH}}{\simeq} Y \nRightarrow X \overset{\mathrm{CE}}{\simeq} Y$ (S. Ferry gave a 1-dimensional counterexample in [13]).

A couple of notes about these facts: First of all, the theorem quoted in (1) does not explicitly mention the finite-dimensional case. However, a careful analysis of the intermediate space Z constructed in [12] reveals that it does indeed have finite dimension if X and Y are finite-dimensional ([2]). The second fact is a standard example; take X to be the topologist's sine curve and Y to be a point p. The map $X \to Y$ is cell-like, because X has the shape of a point, but X is certainly not contractible.

It is also important to observe that (3) does not hold if we leave the finite-dimensional category, as exhibited by J. Taylor in [28]. However, E. Swenson has shown in [27] that all boundaries of a CAT(0) space admitting a geometric group action are finite-dimensional. This is why Theorem 1 is stated in the finite-dimensional category.

There is one more proposition which we will need for the finite-dimensional version of Theorem 2. This proposition requires the finite-dimensional Complement Theorem of Chapman [8]:

THEOREM. *Let X and Y be compacta with dimension $\leq m$. Then for any integer $n \geq 2m + 2$ there exist copies $X', Y' \subset \mathbb{R}^n$ (of X and Y respectively) such that if $X \overset{\mathrm{SH}}{\simeq} Y$, then $\mathbb{R}^n - X' \approx \mathbb{R}^n - Y'$.*

PROPOSITION 2.2. *Let X, Y be finite-dimensional shape equivalent compacta. Then there is an n large enough and embeddings X' and Y' of X and Y respectively into the closed n-ball B^n such that $B^n - X'$ is homeomorphic to $B^n - Y'$.*

(^{2}) The author found formula (B) from [19, Section III.2] helpful in this analysis.

Proof. Let $n \geq 2 \max\{\dim X, \dim Y\} + 2$. By Chapman's finite-dimensional Complement Theorem, there exist embeddings X' and Y' of X and Y in $\mathbb{R}^n$ so that $\mathbb{R}^n - X'$ is homeomorphic to $\mathbb{R}^n - Y'$. By adding a point at infinity to $\mathbb{R}^n$, this homeomorphism extends to a homeomorphism $h : S^n - X' \to S^n - Y'$. Let $D, D_0 \subset S^n - X'$ be closed n-balls with $D_0 \subset \mathrm{int}\, D$. Choose D_0 to be tame so that the boundary of $h(D_0)$ is bicollared. We take $B = S^n - \mathrm{int}\, D_0$ and $B' = S^n - h(\mathrm{int}\, D_0)$. It is a consequence of the Generalized Schoenflies Theorem (proven by M. Brown in [7]) that B' is an n-ball. So we have n-balls B and B' and embeddings $X' \subset B$ and $Y' \subset B'$ such that the restriction of h to $B - X'$ is a homeomorphism onto $B' - Y'$. ∎

As a brief comment on the statement of Chapman's finite-dimensional Complement Thoerem, we observe that work has been done by Venema, Geoghegan, and Summerhill to improve the dimension n of the ambient space (see [29] and [15]).

3. Proofs of Theorems 1 and 2. Theorem 2 follows from this next proposition together with an easy transitivity argument.

PROPOSITION 3.1. *Let X, Y, and Z be compacta such that $X \overset{\mathrm{SH}}{\simeq} Y$. Then*
$$X * Z \overset{\mathrm{CE}}{\simeq} Y * Z.$$
Furthermore, if these compacta are finite-dimensional, then the cell-like equivalence may be obtained through finite dimensions.

Proof. We will begin by proving the proposition without the finite-dimensional hypothesis. The proof of the finite-dimensional case is obtained by a similar argument in which Q is replaced with a finite-dimensional cube.

Embed X in Q as a Z-set. For some fixed $z_0 \in Z$, we define the space
$$K_3 = (Q - X) \times (Z - z_0).$$
We will use the symbol "$\approx$" to mean that two spaces are homeomorphic. By Chapman's Complement Theorem, we have
$$K_3 \approx (Q - Y) \times (Z - z_0).$$
Therefore it suffices to prove that $X * Z \overset{\mathrm{CE}}{\simeq} K_3^\star$, where the $\star$ denotes one-point compactification.

Our cell-like equivalence zigzag between $X * Z$ and $K_3^\star$ will have two intermediate spaces and three cell-like maps:

$$
\begin{array}{ccccc}
X * Z & & K_2 & & \\
& \searrow^{\phi_1} & \nearrow^{\phi_2} & \searrow^{\phi_3} & \\
& K_1 & & & K_3^\star
\end{array}
$$

(see Figure 2) (^{3}). The first intermediate space is the quotient space

$$K_1 = X * Z / X * z_0.$$

The other is the union of $Q \times Z$ with a cone Γ on the complement of K_3 (see Figure 1):

$$K_2 = \Gamma \cup Q \times Z = p * (Q \times Z - K_3) \cup Q \times Z,$$

where p denotes the cone point of Γ. Note that $K_3^\star \approx K_2/\Gamma$. It is easy to see that K_2 is metrizable. The other spaces, K_1 and $K_3^\star$, are metrizable because they arise from finite decompositions of metrizable spaces into closed sets. In other words, each may be realized as the quotient space obtained when finitely many disjoint closed subsets of a metric space are identified to points (see [11]). The maps ϕ_1 and ϕ_3 are both quotient maps whose fibers are either points or cones. So these are cell-like maps.
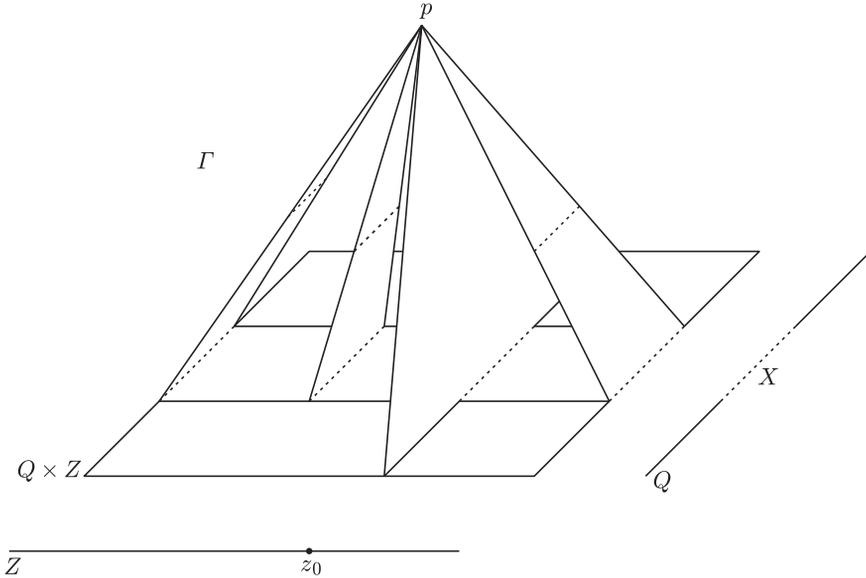

Fig. 1. K_2

We now realize ϕ_2 as a quotient map. Consider the following collection of subspaces of K_2:

$$\mathcal{S} = \big\{ Q \times z \mid z \neq z_0 \big\} \cup \big\{ p * (Q \times z_0) \big\}$$

and let ϕ_2 be the quotient map onto the decomposition space $K_2/\mathcal{S}$. Again, ϕ_2 is obviously cell-like, since fibers are contractible. It suffices to prove the following claim.

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(^{3}) Ric Ancel has suggested a variation on this proof which uses only one intermediate space and two cell-like maps.
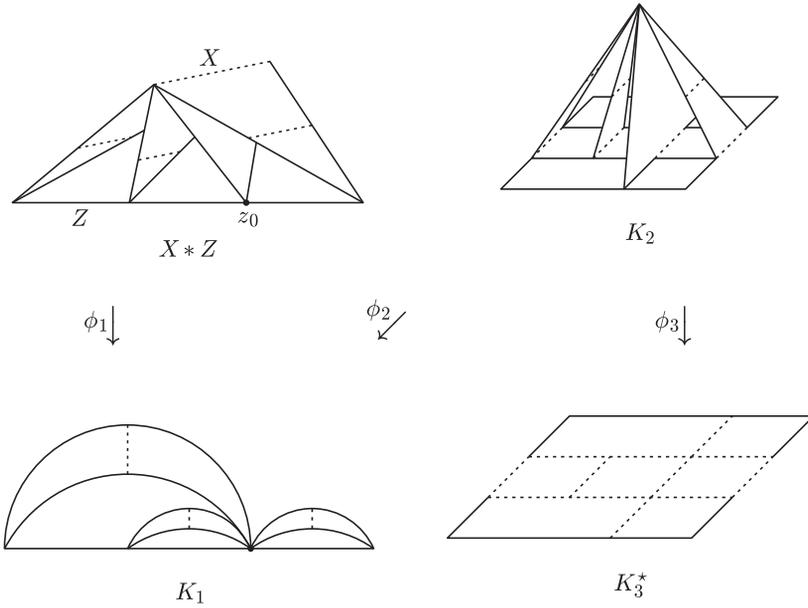
Fig. 2. $X * Z \overset{\mathrm{CE}}{\simeq} K_3^\star$

CLAIM. $K_2/\mathcal{S} \approx K_1$.

Consider the cone Γ' on $X \times Z$:

$$\Gamma' = p * (X \times Z).$$

Then $K_2/\mathcal{S} \approx \Gamma'/\mathcal{S}'$ where

$$\mathcal{S}' = \big\{ X \times z \mid z \neq z_0 \big\} \cup \big\{ p * (X \times z_0) \big\}.$$

We prove the Claim by showing that $\Gamma'/\mathcal{S}'$ and K_1 are the same when considered as quotients of the space of triples $X \times Z \times I$.

Begin by writing

$$\Gamma' = X \times Z \times I/\overset{2}{\sim}$$

where $\overset{2}{\sim}$ is generated by the rule

(a) $(x_1, z_1, 1) \sim (x_2, z_2, 1)$ for all $x_1, x_2 \in X$ and $z_1, z_2 \in Z$.

Here $X \times Z$ disappears at the top (at the point p). So

$$\Gamma'/\mathcal{S}' \approx X \times Z \times I/\overset{2'}{\sim}$$

where $\overset{2'}{\sim}$ is generated by (a) along with the following two additional rules:

(b) $(x_1, z, 0) \sim (x_2, z, 0)$ for all $x_1, x_2 \in X$ and $z \in Z$,
(c) $(x_1, z_0, t_1) \sim (x_2, z_0, t_2)$ for all $x_1, x_2 \in X$ and $t_1, t_2 \in I$,

Now we can also write

$$X * Z = X \times Z \times I/\overset{1}{\sim}$$

where $\overset{1}{\sim}$ is generated by (b) along with this rule:

(a') $(x, z_1, 1) \sim (x, z_2, 1)$ for all $x \in X$ and $z_1, z_2 \in Z$.

In other words, Z disappears at the top (at level 1) and X disappears at the bottom (at level 0). So

$$K_1 = X \times Z \times I / \overset{1'}{\sim}$$

where $\overset{1'}{\sim}$ is obtained from $\overset{1}{\sim}$ by adding rule (c). But the equivalence relations $\overset{1'}{\sim}$ and $\overset{2'}{\sim}$ are the same because (a) and (a') are equivalent in the context of (c). Therefore $K_2/\mathcal{S} \approx K_1$. This proves the infinite-dimensional version.

To get the finite-dimensional version of this proposition, we replace Q with a finite-dimensional cube and Chapman's Complement Theorem with Proposition 2.2 in the above proof. ∎

Along with Theorem 2, the proof of Theorem 1 requires two other results. The first is due to Hosaka.

THEOREM ([17, Theorem 3]). *Let $G = H \times K$ be a* CAT(0) *group with infinite factors and X be a* CAT(0) *G-space. Then there is a* CAT(0) *H-space Y and a* CAT(0) *K-space Z such that*

$$\partial X \approx \partial Y * \partial Z.$$

Note that this equation is exactly what one would expect in light of the equation

$$\partial(Y \times Z) \approx \partial Y * \partial Z$$

given in [6, Example II.8.11(6)]. In fact, Y and Z are constructed as subspaces of X. The action of H on Y and of K on Z is not immediate from the original action of $H \times K$ on X, however.

The second result is the observation of Geoghegan and Bestvina mentioned earlier. We refer the reader to the proof written by Ontaneda.

THEOREM ([24, Corollary B]). *Let G be any* CAT(0) *group and X and Y be* CAT(0) *G-spaces. Then $\partial X \overset{\mathrm{SH}}{\simeq} \partial Y$.*

The proof of Theorem 1 is now straightforward. Given any CAT(0) group $G = H \times K$ with infinite factors and any two CAT(0) G-spaces X and X', we use Hosaka's Theorem to write $\partial X \approx \partial Y * \partial Z$ and $\partial X' \approx \partial Y' * \partial Z'$ where Y and Y' are CAT(0) H-spaces and Z and Z' are CAT(0) K-spaces. By Ontaneda's Theorem, we have $\partial Y \overset{\mathrm{SH}}{\simeq} \partial Y'$ and $\partial Z \overset{\mathrm{SH}}{\simeq} \partial Z'$. By work of Swenson in [27], we know that boundaries of CAT(0) groups are finite-dimensional. So we apply Theorem 2 in the finite-dimensional category to get $\partial X \overset{\mathrm{CE^f}}{\simeq} \partial X'$.

In closing, we note the reason for requiring both factors to be infinite. If one of the factors, say H, is a finite group, then K acts geometrically on exactly the same family of CAT(0) spaces as G. In other words, any boundary of G is also a boundary of K and vice versa.

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