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MAD families with strong combinatorial properties

by

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Abstract. In his paper in Fund. Math. 178 (2003), Miller presented two conjectures regarding MAD families. The first is that CH implies the existence of a MAD family that is also a σ-set. The second is that under CH, there is a MAD family concentrated on a countable subset. These are proved in the present paper.

1. Introduction. Let $[\omega]^\omega$ denote the infinite subsets of the natural numbers ω. Two sets $a, b \in [\omega]^\omega$ are *almost disjoint* if $a \cap b$ is finite. A family $\mathcal{A} \subseteq [\omega]^\omega$ is *almost disjoint* if all its members are pairwise almost disjoint, and *maximal almost disjoint* (a *MAD family*, for short) if for all $x \in [\omega]^\omega$, $a \cap x$ is infinite for some $a \in \mathcal{A}$. In our work we construct MAD families with additional strong topological properties.

We think of MAD families as sets of reals and, accordingly, we identify elements of $[\omega]^\omega$ with their characteristic functions, i.e., with elements of 2^ω which are not eventually 0. Conversely, we usually equate $a \in 2^\omega$ with the corresponding set $\{i \in \omega : a(i) = 1\}$.

An uncountable set of reals $X \subseteq 2^\omega$ is a *Q-set* if every subset of X is a relative G_δ set, and a *σ-set* if every relative Borel subset is a relative G_δ set, i.e., for all Borel $B \subseteq 2^\omega$ there is a G_δ set $G \subseteq 2^\omega$ such that $B \cap X = G \cap X$. Every Q-set is a σ-set. Miller [8, Theorem 1] proved it is consistent with ZFC that there is a MAD Q-set. Such a set necessarily has size less than $\mathfrak{c} = |2^\omega|$. A modification of his argument showed it is consistent

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that there is a MAD σ-set of size $\mathfrak{c}$ where $\mathfrak{c}$ can be arbitrary [8, Theorem 3]. The existence of a MAD σ-set is a Σ^2_1 sentence. Thus, by Woodin's Σ^2_1 absoluteness (see [3, Theorem 3.2.1]) which asserts that if κ is a measurable Woodin cardinal, CH holds and $\mathbb{P}$ is a forcing notion of size less than κ then any Σ^2_1 sentence true in $V^{\mathbb{P}}$ is also true in V, Miller deduced that there is a MAD σ-set under CH + there is a measurable Woodin cardinal [8, Remark on p. 279]. Accordingly he conjectured such a set could be constructed under CH alone [8, Conjecture 4]. We prove this is indeed the case.

Theorem 1. *CH implies there is a MAD σ-set.*

Note that some assumption is necessary because there may be no σ-set [5, Theorem 22].

A set of reals $X \subseteq 2^\omega$ is *concentrated* on $Y \subseteq 2^\omega$ if for any open $U \supseteq Y$, $X \setminus U$ is at most countable. Miller [8, Theorem 5] also proved the generic MAD family adjoined by Hechler's standard forcing notion [1] is concentrated on a countable subset of itself. Using the same large cardinal considerations, he conjectured such a MAD family existed under CH [8, Conjecture 7]. We confirm this.

Theorem 2. *CH implies that there exists an infinite MAD family which is concentrated on a countable subset of itself.*

Again, this is not true in ZFC alone because all concentrated sets may be countable. (The latter holds, e.g., in Laver's model for the Borel conjecture [4] because every set concentrated on a countable set has strong measure zero [6, Theorem 3.1].)

We prove Theorem 2 in Section 2 by singling out one property of the generic MAD family (of [1]) used to prove Theorem 5 of [8] and then setting up a recursive construction which preserves this property along with creating a MAD family and turning it into a concentrated set. The proof of Theorem 1 is much harder. We use a topological argument, successively creating the members of the MAD family $\mathcal{A}$ as Cohen reals in an appropriate Polish space, roughly, the space of reals almost disjoint from previous members of $\mathcal{A}$, equipped with a natural topology finer than the standard topology. This approach has two advantages. First, we get maximality for free because for each real x, the set G_x of reals which have infinite intersection with x is a dense G_δ set, even in the finer topology. Second, we guarantee that the set of reals V_α on which a given Borel set B_α and a generically adjoined G_δ set U_α agree is also a dense G_δ set. Thus any later member of $\mathcal{A}$ will belong either to both B_α and U_α or to neither, and $B_\alpha \cap \mathcal{A} = U_\alpha \cap \mathcal{A}$ will follow. See Section 3 for details. In Section 4, we briefly discuss generalizations of our results under Martin's Axiom MA.

2. A MAD family concentrated on a countable subset. In this section, we prove Theorem 2. Assuming CH, we define a suitable MAD family inductively.

First choose an almost disjoint family $\langle a_n : n < \omega\rangle$ with the following property:

$$(\star)\qquad \forall F \subseteq \omega \text{ finite } \forall s \in 2^{<\omega}\ \exists n < \omega \quad \left(s \subseteq a_n \text{ and } \bigcup\{a_m : m \in F\} \cap a_n \subseteq |s|\right).$$

This technical property is a strengthening of denseness and is needed in the inductive definition of the MAD family. It is easy to construct such a family. In fact, the standard forcing for adding a countable almost disjoint family [1] generically adds a_n satisfying $(\star)$.

We now proceed by induction to produce a_α for $\omega \le \alpha < \omega_1$. Let $\langle U_\alpha : \alpha < \omega_1\rangle$ list all the open subsets of 2^ω that contain all of the a_n. That is, $\{a_n : n < \omega\} \subseteq U_\alpha$ for all $\alpha < \omega_1$. Let $\langle r_\alpha : \omega \le \alpha < \omega_1\rangle$ list all the infinite elements of 2^ω.

We construct a_α, $\alpha \ge \omega$, satisfying the following conditions.

(1) $\forall \beta < \alpha\ (|a_\beta \cap a_\alpha| < \aleph_0)$.
(2) $\forall \beta < \alpha\ (a_\alpha \in U_\beta)$.
(3) $\exists \beta \le \alpha\ (|r_\alpha \cap a_\beta| = \aleph_0)$.
(4) $\forall F \subseteq \alpha + 1$ finite $\forall s \in 2^{<\omega}\ \exists n < \omega$ ($s \subseteq a_n$ and $\bigcup\{a_\beta : \beta \in F\} \cap a_n \subseteq |s|$).

The last condition is the analogue of $(\star)$ above. Notice that (1) and (4) hold for $\alpha < \omega$ by construction. (Properties (2) and (3) are irrelevant for $\alpha < \omega$.)

We construct a_α by recursively producing countably many of its initial segments s_j, $j < \omega$, with $|s_j| \ge j$ as well as finite sets X_j, $j < \omega$, which will identify the a_β that must be avoided when we extend s_j. Let $\langle \beta_j : j < \omega\rangle$ enumerate α, and let $\langle (t_j, F_j) : j < \omega\rangle$ list all the pairs in $2^{<\omega} \times [\alpha]^{<\omega}$ in such a way that $|t_j| \le j$. As usual, $[s] = \{y \in 2^\omega : s \subseteq y\}$ denotes the clopen set defined by $s \in 2^{<\omega}$.

Stage 0. Let $s_0 \in 2^{<\omega}$ be such that $[s_0] \subseteq U_{\beta_0}$. Let $X_0 = \{a_{\beta_0}\}$.

Stage $k+1$. Assume that we have already defined s_j and X_j for all $j \le k$ such that $[s_j] \subseteq U_{\beta_j}$ and $a_{\beta_j} \in X_j$. We define $s_{k+1} \supseteq s'_{k+1} \supseteq s_k$ and $X_{k+1} \supseteq X_k$.

First we take care of property (2). Since X_k is finite, by the inductive hypothesis (using (4)) we can find $n_k < \omega$ such that $s_k \subseteq a_{n_k}$ and $\bigcup X_k \cap a_{n_k} \subseteq |s_k|$. Now let $j_k \ge \max\{k+1, |s_k|\}$ be such that $[a_{n_k} \restriction j_k] \subseteq U_{\beta_{k+1}}$. There must be such an j_k since $a_{n_k} \in U_{\beta_{k+1}}$. Let $s'_{k+1} = a_{n_k} \restriction j_k$.

Note that this will imply that once we have defined s_j for all $j < \omega$, $a_\alpha = \bigcup\{s_j : j < \omega\} \in U_{\beta_{k+1}}$.

Next we ensure that (4) holds for finite sets containing α. By the inductive hypothesis, we know that (4) holds for finite $F \subseteq \alpha$. Let $F'_k = F_k \cup \{n_k\}$. Given t_k and F'_k, there is i_k such that $t_k \subseteq a_{i_k}$ and $\bigcup\{a_\beta : \beta \in F'_k\} \cap a_{i_k} \subseteq |t_k|$. To obtain $\bigcup\{a_\beta : \beta \in F_k \cup \{\alpha\}\} \cap a_{i_k} \subseteq |t_k|$ it suffices to ensure that $a_\alpha \cap a_{i_k} = a_{n_k} \cap a_{i_k}$. We achieve this simply by adding a_{i_k} to X_k. Thus, let $X_{k+1} = X_k \cup \{a_{\beta_{k+1}}\} \cup \{a_{i_k}\}$.

If $r_\alpha \cap a_\beta$ is infinite for some $\beta < \alpha$, let $s_{k+1} = s'_{k+1}$. If not, we need to guarantee condition (3) as well: let $i \geq |s'_{k+1}|$ be minimal such that $r_\alpha(i) = 1$ but $a(i) = 0$ for all $a \in X_{k+1}$. There must be such an i since r_α is infinite and almost disjoint from all of the elements of X_{k+1}. Let s_{k+1} be a sequence of length $i+1$ extending s'_{k+1} and with $s_{k+1}(i) = 1$ and $s_{k+1}(j) = 0$ for $|s'_{k+1}| \leq j < i$. Note that this will imply that $a_\alpha = \bigcup\{s_j : j < \omega\}$ and r_α will have infinite intersection.

Once we have defined s_j for all $j < \omega$, we let $a_\alpha = \bigcup\{s_j : j < \omega\}$. We must now check that each of the conditions (1)–(4) hold.

(1) By construction, for all $k < \omega$, $a_{\beta_k} \cap a_\alpha \subseteq |s_k|$ because $a_{\beta_k} \in X_k$ and X_k is the set of reals which are avoided when extending s_k.

(2) and (3) are immediate as we observed during the induction.

(4) This follows from the fact that we put a_{i_k} into X_{k+1}. Indeed, the latter implies $a_\alpha \cap a_{i_k} \subseteq |s'_{k+1}|$. Also $a_\alpha \restriction j_k = s'_{k+1} = a_{n_k} \restriction j_k$ where $j_k \geq k+1$, and $a_{n_k} \cap a_{i_k} \subseteq |t_k| \leq k$. Thus $a_\alpha \cap a_{i_k} = a_{n_k} \cap a_{i_k}$ and we remarked earlier this was exactly what was needed to ensure (4).

This completes the recursive construction of the family $\langle a_\alpha : \alpha < \omega_1 \rangle$. By (1), it is an almost disjoint family, by (3) it is maximal, and by (2) it is concentrated on $\{a_n : n < \omega\}$ because for any $\alpha < \omega_1$, $\{\beta < \omega_1 : a_\beta \notin U_\alpha\} \subseteq \alpha + 1 \setminus \omega$, which is countable. This completes the proof of Theorem 2.

3. CH implies there exists a MAD σ-set

3.1. *The framework of the proof.* Before going into the actual combinatorial details, we describe the framework of the proof of Theorem 1.

Assume we have models $\langle M_\alpha : \alpha < \omega_1 \rangle$ and $\langle N_\alpha : \alpha < \omega_1 \rangle$ of ZFC such that:

- $M_\alpha \subseteq N_\alpha \subseteq M_{\alpha+1}$,
- M_α is countable in N_α,
- α is countable in M_α and all N_β, $\beta < \alpha$, are countable in M_α (so $\langle N_\beta : \beta < \alpha \rangle \in M_\alpha$ is countable in M_α),
- $2^\omega \subseteq \bigcup_{\alpha<\omega_1} M_\alpha = \bigcup_{\alpha<\omega_1} N_\alpha$.

We shall build perfect Polish spaces $(X_\alpha, \mathcal{S}_\alpha), (Y_\alpha, \mathcal{T}_\alpha)$ (where $\mathcal{S}_\alpha, \mathcal{T}_\alpha$ denote the respective topologies) such that:

- $X_{\alpha+1} \subseteq Y_\alpha \subseteq X_\alpha$,
- $X_\alpha = \bigcap_{\beta<\alpha} X_\beta = \bigcap_{\beta<\alpha} Y_\beta = Y_\alpha$ for limit α,
- the topology $\mathcal{T}_\alpha$ refines the topology $\mathcal{S}_\alpha$ (restricted to Y_α),
- the topology $\mathcal{S}_{\alpha+1}$ refines the topology $\mathcal{T}_\alpha$ (restricted to X_α),
- $\mathcal{T}_\alpha = \bigcup_{\beta<\alpha} \mathcal{T}_\beta = \bigcup_{\beta<\alpha} \mathcal{S}_\beta = \mathcal{S}_\alpha$ for limit α,
- $Y_\alpha \in M_\alpha$, $X_{\alpha+1} \in N_\alpha$,
- $X_0 = Y_0 = 2^\omega$ and $\mathcal{S}_0 = \mathcal{T}_0 =$ the standard topology.

Notice that for limit α, we indeed have $Y_\alpha = \bigcap_{\beta<\alpha} X_\beta \in M_\alpha$ because $\langle N_\beta : \beta < \alpha \rangle \in M_\alpha$. While we are mainly interested in $(X_\alpha, \mathcal{S}_\alpha)$ and $(Y_\alpha, \mathcal{T}_\alpha)$, we shall often think of $\mathcal{S}_\alpha$ and $\mathcal{T}_\alpha$ as refining the standard topology on 2^ω.

More explicitly, there will be sets $F^s_\alpha, F_\alpha, H^n_\alpha, H_\alpha \subseteq 2^\omega$ ($s \in 2^{<\omega}, n \in \omega$) such that:

- all F^s_α, H^n_α are closed in the standard topology,
- $F_\alpha = \bigcup_{s\in 2^{<\omega}} F^s_\alpha$ and $H_\alpha = \bigcup_{n\in\omega} H^n_\alpha$ are F_σ sets,
- $X_{\alpha+1} = Y_\alpha \cap F_\alpha$,
- the topology $\mathcal{S}_{\alpha+1}$ (on $X_{\alpha+1}$) is generated by (the restriction of) $\mathcal{T}_\alpha \cup \{F^s_\alpha : s \in 2^{<\omega}\}$ (so all the sets F^s_α are made clopen).

The description of the space $Y_{\alpha+1}$ and its topology $\mathcal{T}_{\alpha+1}$ is somewhat more difficult. There are closed $P_\alpha \subseteq X_{\alpha+1}$ and open $O_\alpha \subseteq X_{\alpha+1}$ (in the topology $\mathcal{S}_{\alpha+1}$) such that:

- $P_\alpha \cup O_\alpha = X_{\alpha+1}$,
- $P_\alpha \cap O_\alpha = \emptyset$,
- $Y_{\alpha+1} = P_\alpha \cup (X_{\alpha+1} \cap H_\alpha) = P_\alpha \cup (O_\alpha \cap H_\alpha)$,
- the topology $\mathcal{T}_{\alpha+1}$ (on $Y_{\alpha+1}$) is generated by (the restriction of) $\mathcal{S}_{\alpha+1}$ as well as sets of the form $F \cap H^n_\alpha$ where $F \cap X_{\alpha+1} \subseteq O_\alpha$ and F is open in $\mathcal{S}_{\alpha+1}$ and $n \in \omega$.

The latter stipulation means that the family $\mathcal{F}_{\alpha+1}$ of sets $F \in \mathcal{S}_{\alpha+1}$ with $F \cap X_{\alpha+1} \subseteq P_\alpha$ and of sets $F \cap \bigcap_{j<m} H^{n_j}_\alpha$ with $F \cap X_{\alpha+1} \subseteq O_\alpha$ and $F \in \mathcal{S}_{\alpha+1}$ is dense in the topology $\mathcal{T}_{\alpha+1}$. That is, $\mathcal{F}_{\alpha+1} \subseteq \mathcal{T}_{\alpha+1}$ and every $F \in \mathcal{T}_{\alpha+1}$ contains a member of $\mathcal{F}_{\alpha+1}$.

Also notice that, more generally, the topology $\mathcal{S}_\alpha$ is generated by the standard clopen sets together with F^s_β ($s \in 2^{<\omega}$, $\beta < \alpha$) and certain (*not all!*) intersections of the latter sets with sets of the form H^n_β ($n \in \omega$, $\beta < \alpha - 1$), where for limit α we set $\alpha - 1 = \alpha$. Similarly $\mathcal{T}_\alpha$ is generated by the standard clopen sets together with F^s_β ($s \in 2^{<\omega}$, $\beta < \alpha$) and some (*not all!*) intersections of the latter with H^n_β ($n \in \omega$, $\beta < \alpha$).

Observation 3.1. *The spaces* $(X_\alpha, \mathcal{S}_\alpha)$ *and* $(Y_\alpha, \mathcal{T}_\alpha)$ *are indeed Polish.*

Proof. By the characterization of $\mathcal{S}_\alpha$ directly preceding 3.1, all basic open sets of $\mathcal{S}_\alpha$ are closed in the standard topology $\mathcal{S}_0$. This means that by [2, Lemmata 13.2 and 13.3], $(X_0, \mathcal{S}_\alpha)$ is Polish. Similarly for the $\mathcal{T}_\alpha$. We prove by induction on $\beta \leq \alpha$ that all $(Z_\beta, \mathcal{U}_\alpha)$ are Polish as well where $Z \in \{X, Y\}$ and $\mathcal{U} \in \{\mathcal{S}, \mathcal{T}\}$ (and $(Y_\beta, \mathcal{S}_\alpha)$ is only considered for $\beta \leq \alpha - 1$).

If $\beta = \gamma + 1$ is successor, $(X_\beta, \mathcal{U}_\alpha)$ is Polish because $X_\beta = Y_\gamma \cap F_\gamma$ is open in $(Y_\gamma, \mathcal{U}_\alpha)$. Similarly, P_γ is closed and $O_\gamma \cap H_\gamma$ is open in $(X_\beta, \mathcal{U}_\alpha)$. So Y_β is G_δ in $(X_\beta, \mathcal{U}_\alpha)$ and thus Polish [2, Theorem 3.11] (here $\mathcal{U} = \mathcal{T}$ in case $\beta = \alpha$).

Let β be a limit ordinal. Since all $(X_\gamma, \mathcal{U}_\alpha)$, $\gamma < \beta$, are Polish, the X_γ form a decreasing sequence of G_δ subsets of $(X_0, \mathcal{U}_\alpha)$, their intersection $X_\beta = \bigcap_{\gamma<\beta} X_\gamma$ is still such a G_δ, and thus $(X_\beta, \mathcal{U}_\alpha)$ is Polish [2, Theorem 3.11]. ∎

We shall see below (Lemma 3.5) that all $(X_\alpha, \mathcal{S}_\alpha)$ and $(Y_\alpha, \mathcal{T}_\alpha)$ are also perfect.

Let $i_n = 2^n$ and put $I = \{i_n : n \in \omega\}$. (In fact, the exact nature of the i_n is irrelevant; what we need is that the sequence of i_n is increasing very fast.) Clearly $I \in M_0$. We will have sets $U^n_\alpha \subseteq 2^\omega$ $(n \in \omega)$ and U_α such that:

- $U^n_\alpha = 2^\omega \setminus H^n_\alpha$ is open,
- $U_\alpha = 2^\omega \setminus H_\alpha$, i.e. $U_\alpha = \bigcap_{n\in\omega} U^n_\alpha$ is G_δ,
- each U^n_α is a union of basic clopen sets $[s^{n,j}_\alpha]$, $j \in \omega$, such that:
  - $|s^{n,j}_\alpha| \in I$, $|s^{n,j}_\alpha| \geq i_{n+j}$,
  - for each $k \in \omega$, there is at most one $s^{n,j}_\alpha$ such that $|s^{n,j}_\alpha| = i_k$ (so $k \geq n + j$),
  - if $|s^{n,j}_\alpha| = i_k$ then there is $l \in (i_{k-1}, i_k)$ such that $s^{n,j}_\alpha(l) = 1$.

3.2. *The MAD family* $\mathcal{A} = \{a_\alpha : \alpha < \omega_1\}$ *(construction of the space* $X_{\alpha+1}$*).* We come now to the details of the construction. We begin with the construction of the space $X_{\alpha+1}$ and associated objects.

For each α let a_α be a Cohen-generic real belonging to the space Y_α over the model M_α in the model N_α (i.e. $M_\alpha[a_\alpha] \subseteq N_\alpha$). Such an a_α clearly exists because M_α is countable in N_α.

We let $F^s_\alpha = \{y : s \subseteq y \text{ and } (\forall l \geq |s|)\ (a_\alpha(l) = 1 \Rightarrow y(l) = 0)\}$, the set of reals y which contain s as an initial segment and which are disjoint from A_α beyond $|s|$. This is clearly closed, as required. Note also that $F^s_\alpha \subseteq [s]$.

Let $F_\alpha = \bigcup_{s\in 2^{<\omega}} F^s_\alpha$, the set of reals almost disjoint from a_α. Define $X_{\alpha+1}$ and $\mathcal{S}_{\alpha+1}$ as stipulated earlier.

Lemma 3.2. $\{a_\alpha : \alpha < \omega_1\}$ *is an almost disjoint family.*

Proof. For $\beta < \alpha$, F_β is the set of reals almost disjoint from a_β. Since $Y_\alpha \subseteq X_\alpha \subseteq F_\beta$ by construction, Y_α only contains reals almost disjoint from a_β. Thus a_α is almost disjoint from a_β. ∎

OBSERVATION 3.3. *A typical basic open set of* $(X_\alpha, \mathcal{S}_\alpha)$ *is of the form* $\bigcap_{j<m_0} F^s_{\beta_j} \cap \bigcap_{j<m_1} H^{n_j}_{\gamma_j} \neq \emptyset$ ($s \in 2^{<\omega}$, $n_j \in \omega$, $\beta_j < \alpha$, $\gamma_j < \alpha - 1$). *Similarly for* $(Y_\alpha, \mathcal{T}_\alpha)$.

Proof. Since $F^s_\beta \subseteq [s]$, there is no need to consider basic clopen sets of the standard topology, and a typical basic clopen set is of the form $F = \bigcap_{j<m_0} F^{s_j}_{\beta_j} \cap \bigcap_{j<m_1} H^{n_j}_{\gamma_j} \neq \emptyset$. For $j, j' < m_0$ we must have $s_j \subseteq s_{j'}$ or $s_{j'} \subseteq s_j$. Put $s = \bigcup_{j<m_0} s_j$. Clearly $F \subseteq [s]$. Since $F^{s_j}_{\beta_j} \cap [s] \neq \emptyset$, we must have $F^s_{\beta_j} \subseteq F^{s_j}_{\beta_j}$. In fact, $F^s_{\beta_j} \cap F = F^{s_j}_{\beta_j} \cap F$. Thus $F = \bigcap_{j<m_0} F^s_{\beta_j} \cap \bigcap_{j<m_1} H^{n_j}_{\gamma_j}$. ∎

The following is crucial for several subsequent results (see Lemmata 3.5 and 3.6).

LEMMA 3.4. *Given any* m, $\beta_j < \omega_1$ ($j < m$), n *and* k, *there is* $l \geq k$ *such that* $|(i_l, i_{l+1}) \setminus \bigcup_{j<m} a_{\beta_j}| \geq n$.

Proof. This is a standard Cohen-genericity argument, using the fact that $I \in M_0$. Fix n. We proceed by induction on m. Let β_j, $j < m$, be given such that $\beta_0 < \beta_1 < \cdots < \beta_{m-1}$. Assume the statement is true for $m - 1$ for all k. Put $\alpha = \beta_{m-1}$. Then $a_{\beta_j} \in M_\alpha$ for $j < m - 1$ and $a_\alpha = a_{\beta_{m-1}}$ is Cohen-generic over M_α in Y_α. By 3.3, a typical basic open set of the topology $\mathcal{T}_\alpha$ (equivalently, condition in the Cohen forcing) is of the form $p = \bigcap_{j<m_0} F^s_{\gamma_j} \cap \bigcap_{j<m_1} H^{n_j}_{\delta_j} \neq \emptyset$. Without loss of generality, we assume $|s| \in I$. Apply the induction hypothesis with k replaced by $\max\{|s|, k\}$ and find $l \geq \max\{|s|, k\}$ with $|(i_l, i_{l+1}) \setminus \bigcup_{j<m-1} a_{\beta_j}| \geq n$. Notice that $|s| \leq l < i_l$. Thus, we may strengthen the condition, replacing s by $t \supseteq s$ such that $|t| = i_{l+1}$ and $t(i) = 0$ for $i \in [|s|, i_{l+1})$, to get $q = \bigcap_{j<m_0} F^t_{\gamma_j} \cap \bigcap_{j<m_1} H^{n_j}_{\delta_j}$. To see that this works, notice that by the definition of the $F^s_{\gamma_j}$ and $H^{n_j}_{\delta_j}$, we must indeed have $[t] \cap \bigcap_{j<m_0} F^s_{\gamma_j} \cap \bigcap_{j<m_1} H^{n_j}_{\delta_j} \neq \emptyset$. The stronger condition q clearly forces $|(i_l, i_{l+1}) \setminus (\bigcup_{j<m-1} a_{\beta_j} \cup \dot{a}_\alpha)| \geq n$ so we are done. ∎

LEMMA 3.5. *All spaces* $(X_\alpha, \mathcal{S}_\alpha)$ *and* $(Y_\alpha, \mathcal{T}_\alpha)$ *are perfect Polish spaces.*

Proof. We already observed that $\mathcal{S}_\alpha$ and $\mathcal{T}_\alpha$ were Polish. So it suffices to show X_α and Y_α are perfect. Consider X_α, and let $\{\beta_j : j \in \omega\} = \alpha$, $\{\gamma_j : j \in \omega\} = \alpha - 1$. Recursively construct $\ell_j, n_j \in \omega$ and $s_j(\tau) \in 2^{<\omega}$ ($\tau \in 2^j$) such that

- $\ell_j < \ell_{j'}$ for $j < j'$,
- $|s_j(\tau)| = i_{\ell_j}$,

- if $j \le j'$ and $\tau \subseteq \tau'$, $\tau \in 2^j$, $\tau' \in 2^{j'}$, then $s_j(\tau) \subseteq s_{j'}(\tau')$, $s_{j'}(\tau') \in F_{\beta_j}^{s_j(\tau)}$ and $s_{j'}(\tau') \in H_{\gamma_j}^{n_j}$.

Here, $s \in F_{\beta_j}^{s_j(\tau)}$ ($s \in H_{\gamma_j}^{n_j}$, respectively) means that s belongs to the tree defining the closed set $F_{\beta_j}^{s_j(\tau)}$ ($H_{\gamma_j}^{n_j}$, resp.).

For $j = 0$, let $\ell_0 = 0$, choose $s_0(\langle\rangle)$ of length $i_0 = 2^0 = 1$ arbitrary and let n_0 be such that $s_0(\langle\rangle) \in H_{\gamma_0}^{n_0}$.

Suppose ℓ_j, n_j, and $s_j(\tau)$ have been defined. By Lemma 3.4, we can choose $\ell_{j+1} > \ell_j$ such that $|(i_{\ell_{j+1}-1}, i_{\ell_{j+1}}) \setminus \bigcup_{j' \le j} a_{\beta_{j'}}| \ge j+2$. Set $A = (i_{\ell_{j+1}-1}, i_{\ell_{j+1}}) \setminus \bigcup_{j' \le j} a_{\beta_{j'}}$. Fix $\tau \in 2^j$. Let $T_\tau = \{s : s_j(\tau) \subseteq s,\ |s| = i_{\ell_{j+1}}$ and $\forall i \in |s| \setminus (|s_j(\tau)| \cup A)\ (s(i) = 0)\}$. Clearly $|T_\tau| \ge 2^{j+2}$ and $s \in F_{\beta_{j'}}^{s_{j'}(\tau')}$ for all $j' \le j$, $\tau' \subseteq \tau$ and all $s \in T_\tau$. For each $j' \le j$, at most one $s \in T_\tau$ does not belong to $H_{\gamma_{j'}}^{n_{j'}}$. Since $2^{j+2} \ge j+3$, we can find $s_{j+1}(\tau \frown 0), s_{j+1}(\tau \frown 1) \in T_\tau \cap \bigcap_{j' \le j} H_{\gamma_{j'}}^{n_{j'}}$, as required. Finally, let n_{j+1} be such that $s_{j+1}(\tau) \in H_{\gamma_{j+1}}^{n_{j+1}}$ for all $\tau \in 2^{j+1}$. This completes the construction.

For $x \in 2^\omega$, define $y = y_x$ by $y \restriction i_{\ell_j} = s_j(x \restriction j)$ for all j. Then $y \in \bigcap_j F_{\beta_j}^{s_j(x \restriction j)} \cap \bigcap_j H_{\gamma_j}^{n_j} \subseteq \bigcap_j F_{\beta_j} \cap \bigcap_j H_{\gamma_j}$. Thus $\{y_x : x \in 2^\omega\} \subseteq \bigcap_j F_{\beta_j} \cap \bigcap_j H_{\gamma_j} \subseteq X_\alpha$ is a perfect set. Since $X_{\alpha+1} \subseteq Y_\alpha$, Y_α is perfect as well. ∎

In fact, a straightforward generalization shows that if $F \subseteq X_\alpha$ is a nonempty basic clopen set, then F contains a perfect subset. Similarly for Y_α.

For $x \in 2^\omega$ infinite (i.e. $x \in [\omega]^\omega$), let

$$G_x = \{y : \text{there are infinitely many } l \text{ such that } y(l) = x(l) = 1\}.$$

This is the set of all y which have infinite intersection with x. Clearly, G_x is a G_δ set. More explicitly, $G_x = \bigcap_{n \in \omega} G_x^n$, where

$$G_x^n = \{y : \exists l_0, \dots, l_{n-1} \text{ distinct such that } y(l_j) = x(l_j) = 1 \text{ for } j < n\}.$$

This is the set of all y whose intersection with x is of size at least n. Clearly, each G_x^n is dense open in the standard topology of 2^ω. So G_x is dense G_δ.

Lemma 3.6. *Assume x does not belong to the ideal generated by a_β, $\beta < \alpha$. (That is, x is not almost contained in a finite union of a_β, $\beta < \alpha$.) Then G_x^n is dense open in the space $(Y_\alpha, \mathcal{T}_\alpha)$.*

Proof. This is similar to the proof of Lemma 3.5. By 3.3, basic open sets of the topology $\mathcal{T}_\alpha$ are finite intersections of the form $\bigcap_{j<m_0} F_{\beta_j}^s \cap \bigcap_{j<m_1} H_{\gamma_j}^{n_j} \ne \emptyset$, where $s \in 2^{<\omega}$ and $\beta_j, \gamma_j < \alpha$. By extending s if necessary, we may assume $|s| \in I$ and, by Lemma 3.4, if we let $i_k = |s|$ then $|(i_k, i_{k+1}) \setminus \bigcup_{j<m_0} a_{\beta_j}| \ge (n+1)m_1$. Next choose $l_0, \dots, l_{n-1} \in x \setminus \bigcup_{j<m_0} a_{\beta_j}$ and $l > k+1$ with $i_{k+1} \le l_0 < l_1 < \dots < l_{n-1} < i_l$. Consider the set T of all

$t \supseteq s$ with $|t| = i_l$, $t(l_j) = 1$ for all $j < n$ and $t(i) = 0$ for all i such that $i \neq l_j$ $(j < n)$ and $i \notin (i_k, i_{k+1}) \setminus \bigcup_{j<m_0} a_{\beta_j}$.

Clearly, $|T| \geq 2^{(n+1)m_1}$. Also, $[t] \cap \bigcap_{j<m_0} F^s_{\beta_j} \neq \emptyset$ for all $t \in T$. By the definition of $H^{n_j}_{\gamma_j}$, it is easily seen that at most $n+1$ many $t \in T$ do not belong to the tree defining $H^{n_j}_{\gamma_j}$. Hence for at most $(n+1)m_1$ such $t \in T$, we may have $[t] \cap \bigcap_{j<m_0} F^s_{\beta_j} \cap \bigcap_{j<m_1} H^{n_j}_{\gamma_j} = \emptyset$. Since $2^{(n+1)m_1} > (n+1)m_1$, we can find $t \in T$ such that $[t] \cap \bigcap_{j<m_0} F^s_{\beta_j} \cap \bigcap_{j<m_1} H^{n_j}_{\gamma_j} \neq \emptyset$. Clearly, $[t] \subseteq G^n_x$. Thus, $\bigcap_{j<m_0} F^t_{\beta_j} \cap \bigcap_{j<m_1} H^{n_j}_{\gamma_j} \subseteq G^n_x$ and we are done. ∎

COROLLARY 3.7. *$\mathcal{A} = \{a_\alpha : \alpha < \omega_1\}$ is a MAD family.*

Proof. Let $x \in 2^\omega$. We need to show that there is an $\alpha < \omega_1$ such that $|x \cap a_\alpha| = \aleph_0$. Without loss of generality, we may assume that x does not belong to the ideal generated by the a_α. (Otherwise, the proof is trivial.) Find α such that $x \in M_\alpha$. By the previous lemma, G^n_x is dense open in $(Y_\alpha, \mathcal{T}_\alpha)$ for all $n \in \omega$. Since $a_\alpha \in Y_\alpha$ is Cohen-generic over M_α, it follows immediately that $a_\alpha \in G^n_x$ for all $n \in \omega$. Thus, $a_\alpha \in \bigcap_{n\in\omega} G^n_x = G_x$. Hence, $|a_\alpha \cap x| = \aleph_0$. ∎

3.3. *The G_δ sets U_α witnessing that $\mathcal{A}$ is a σ-set* (*construction of* $Y_{\alpha+1}$). We now consider the second part of the construction: the construction of the space $Y_{\alpha+1}$ and its associated objects.

Assume we have a list $\langle B_\alpha : \alpha < \omega_1 \rangle$ of all Borel sets such that $B_\alpha \in N_\alpha$. In N_α, $B_\alpha \cap X_{\alpha+1}$ has the property of Baire (because it is Borel) in the space $(X_{\alpha+1}, \mathcal{S}_{\alpha+1})$. Therefore there are disjoint sets P_α and O_α with P_α closed and O_α open, such that $P_\alpha \cup O_\alpha = X_{\alpha+1}$ and $B_\alpha \cap P_\alpha$ is comeager, while $B_\alpha \cap O_\alpha$ is meager. Let P^n_α, O^n_α be decreasing sequences of open sets in $(X_{\alpha+1}, \mathcal{S}_{\alpha+1})$ such that $P^0_\alpha = \mathrm{int}(P_\alpha)$, $O^0_\alpha = O_\alpha$, $P^n_\alpha \subseteq P^0_\alpha$ is dense, $O^n_\alpha \subseteq O_\alpha$ is dense, $\bigcap_{n\in\omega} P^n_\alpha \subseteq B_\alpha$, and $\bigcap_{n\in\omega} O^n_\alpha \cap B_\alpha = \emptyset$.

The forcing $\mathbb{P}$ consists of finite consistent sets p of conditions of the form:

- (n, a_β) where $\beta \leq \alpha$ and $a_\beta \notin B_\alpha$,
- (n, s) where $s \in 2^{<\omega}$ and $|s| \in I$,
- (n, F) where F is a typical basic clopen subset of $\mathcal{S}_{\alpha+1}$ (see Observation 3.3),

such that:

- if $(n, s) \in p$ and $|s| = i_l \in I$ then there is $i \in (i_{l-1}, i_l)$ such that $s(i) = 1$,
- for each $i \in I$ and $n \in \omega$ there is at most one s with $(n, s) \in p$ and $|s| = i$,
- if $(n, s) \in p$ and $|s| = i_l \in I$ then $l \geq n$,
- if $(n, a_\beta) \in p$ then $(n, a_\beta{\restriction}m) \notin p$ for all m,

- if $(n, F) \in p$ then $F \cap X_{\alpha+1} \subseteq P^n_\alpha$,
- if $(n, F) \in p$ then there is s such that $F \subseteq [s]$ and $(n, s) \in p$.

The ordering $\leq$ is by extension. That is, $q \leq p \Leftrightarrow q \supseteq p$. This is a modification of Silver's standard forcing notion for turning a given set into a relative G_δ (see [7, Section 5], see also [8]).

$\mathbb{P}$ is a countable forcing notion in N_α. (Recall that M_α is countable in N_α and so is $M_\alpha[a_\alpha]$, which contains $X_{\alpha+1}$ etc.)

Let us first check that we can always extend conditions appropriately.

LEMMA 3.8. *Assume $a_\beta \notin B_\alpha$ and $p \in \mathbb{P}$. Then there are $n \in \omega$ and $q \leq p$ such that $(n, a_\beta) \in q$.*

Proof. Choose n sufficiently large that no (n, s) appears in p and let $q = p \cup \{(n, a_\beta)\}$. ∎

LEMMA 3.9. *Assume $a_\beta \in B_\alpha$, $p \in \mathbb{P}$ and $n \in \omega$. Then there are $m \in \omega$ and $q \leq p$ such that $(n, a_\beta{\restriction}m) \in q$.*

Proof. First choose m_0 sufficiently large that:

- $a_\beta{\restriction}m_0 \neq a_\gamma{\restriction}m_0$ for all γ such that $(n, a_\gamma) \in p$,
- $m_0 \geq |s|$ for all s with $(n, s) \in p$,
- $m_0 \geq i_n$.

Then find $i_{l-1} < i < i_l$ with $m_0 \leq i_{l-1}$ such that $a_\beta(i) = 1$. (This is possible because $I \in M_0$ and such $i \notin I$ must exist by Cohen-genericity.) Let $m = i_l$ and $q = p \cup \{(n, a_\beta{\restriction}m)\}$. Clearly, all the requirements are satisfied. ∎

LEMMA 3.10. *Assume $F \cap X_{\alpha+1} \subseteq P^n_\alpha$ is non-empty open* (*in the sense of* $(X_{\alpha+1}, \mathcal{S}_{\alpha+1})$) *and $p \in \mathbb{P}$. Then there are $\emptyset \neq H \subseteq F$ and $q \leq p$ such that $(n, H) \in q$.*

Proof. Shrinking F if necessary, we may assume without loss that $a_\beta \notin F$ for all β with $(n, a_\beta) \in p$. Again choose m_0 such that

- $[a_\beta{\restriction}m_0] \cap F = \emptyset$ for all β such that $(n, a_\beta) \in p$,
- $m_0 \geq |s|$ for all s with $(n, s) \in p$,
- $m_0 \geq i_n$.

Then find $i_{l-1} < i < i_l$ with $m_0 \leq i_{l-1}$ and $t \in 2^{<\omega}$ with $t(i) = 1$, $|t| = i_l$ and $F \cap [t] \neq \emptyset$ (in $X_{\alpha+1}$). The argument showing there is such a t is similar to, but easier than, the proof of Lemma 3.6. Let $H = F \cap [t]$ and let $q = p \cup \{(n, t), (n, H)\}$. It is easy to see that q is indeed a condition and that $q \leq p$. ∎

Let G be $\mathbb{P}$-generic over N_α with $G \in M_{\alpha+1}$ (so $N_\alpha[G] \subseteq M_{\alpha+1}$). Such a G clearly exists because N_α is countable in $M_{\alpha+1}$.

Set

$$U^n_\alpha = \bigcup\{[s] : \exists p \in G\ ((n, s) \in p)\}, \qquad H^n_\alpha = 2^\omega \setminus U^n_\alpha.$$

Clearly, U_α^n is open in 2^ω and H_α^n is closed in 2^ω. Also, $U_\alpha = \bigcap_{n\in\omega} U_\alpha^n$ is a G_δ set and $H_\alpha = \bigcup_{n\in\omega} H_\alpha^n$ is an F_σ set (in the standard topology). It is immediate from the definition of the partial order $\mathbb{P}$ that the U_α^n and H_α^n satisfy all the stipulations required earlier.

Also set

$$V_\alpha^n = \Big(\bigcup\{F : \exists p \in G\ ((n,F) \in p)\} \cap X_{\alpha+1}\Big) \cup (O_\alpha^n \cap H_\alpha)$$

and let $V_\alpha = \bigcap_n V_\alpha^n$.

Finally, as stipulated earlier,

$$Y_{\alpha+1} = P_\alpha \cup (X_{\alpha+1} \cap H_\alpha) = P_\alpha \cup (O_\alpha \cap H_\alpha)$$

and $\mathcal{T}_{\alpha+1}$ is the topology generated by $\mathcal{S}_{\alpha+1}$ and by sets of the form $F \cap H_\alpha^n$ where $F \cap X_{\alpha+1} \subseteq O_\alpha$, $F \in \mathcal{S}_{\alpha+1}$.

COROLLARY 3.11. $\forall \beta \le \alpha\ (a_\beta \in U_\alpha \Leftrightarrow a_\beta \in B_\alpha)$.

Proof. ($\Rightarrow$) This follows by Lemma 3.8.

($\Leftarrow$) This follows by Lemma 3.9. ∎

LEMMA 3.12. *All V_α^n are dense open in $(Y_{\alpha+1}, \mathcal{T}_{\alpha+1})$. Consequently, V_α is dense G_δ in $(Y_{\alpha+1}, \mathcal{T}_{\alpha+1})$.*

Proof. For $(n,F) \in p$ with $p \in G$, $F \cap X_{\alpha+1}$ is open in $\mathcal{S}_{\alpha+1}$ and thus in $\mathcal{T}_{\alpha+1}$. Also all $O_\alpha^n \cap H_\alpha^m$, $m \in \omega$, are open in $\mathcal{T}_{\alpha+1}$. Hence V_α^n is indeed open in $Y_{\alpha+1}$.

Therefore it suffices to show that the V_α^n are dense. Let $F \in \mathcal{T}_{\alpha+1}$ be non-empty. We need to show $V_\alpha^n \cap F \neq \emptyset$. Without loss of generality, we may assume $F \cap X_{\alpha+1} \subseteq P_\alpha$ or $F \cap X_{\alpha+1} \subseteq O_\alpha$. In the first case, we must have $F \in \mathcal{S}_{\alpha+1}$, by definition of $\mathcal{T}_{\alpha+1}$. By further shrinking F if necessary, we may assume $F \cap X_{\alpha+1} \subseteq P_\alpha^n$. By Lemma 3.10 and genericity, there is a non-empty $H \subseteq F$, $H \in \mathcal{S}_{\alpha+1}$, such that $H \cap X_{\alpha+1} \subseteq V_\alpha^n$. Thus $V_\alpha^n \cap F \neq \emptyset$.

Therefore we may assume $F \cap X_{\alpha+1} \subseteq O_\alpha$. Then $F = F' \cap \bigcap_{j<m} H_\alpha^{n_j}$ where $F' \in \mathcal{S}_{\alpha+1}$ with $F' \cap X_{\alpha+1} \subseteq O_\alpha$.

Work in the model N_α, and assume $p \in \mathbb{P}$ forces $F' \cap \bigcap_{j<m} \dot{H}_\alpha^{n_j} \neq \emptyset$. By 3.3, $F' = \bigcap_{j<m_0} F_{\beta_j}^s \cap \bigcap_{j<m_1} H_{\gamma_j}^{k_j}$ with $\beta_j < \alpha+1$ and $\gamma_j < \alpha$. Without loss of generality $|s| \in I$. Since $F' \subseteq [s]$, we must have $(n_j, s') \notin q$ for any $j < m$, $s' \subseteq s$ and $q \le p$. (This means that for each such (n_j, s') with $|s'| = i_\ell \in I$, either $\ell < n_j$ or $(n_j, t) \in p$ for some $t \neq s'$ with $|t| = i_\ell$ or $s' \restriction (i_{\ell-1}, i_\ell) = 0$ or $(n_j, a_\beta) \in p$ for some β with $s' \subseteq a_\beta$. Otherwise $q = p \cup \{(n_j, s')\} \le p$, a contradiction.)

Let $s_0 \supseteq s$, $|s_0| \in I$, be such that $s_0(i) = 0$ for all i with $|s| \le i < |s_0|$ and $|s_0| \ge |s'|$ for all s' with $(n_j, s') \in p$ for some j. By the definition of $\mathbb{P}$, p still forces $[s_0] \cap F' \cap \bigcap_{j<m} \dot{H}_\alpha^{n_j} \neq \emptyset$. (The point here is that no (n_j, s') with $s' \subseteq s_0$ and $|s| < |s'| \le |s_0|$ can belong to any $q \le p$.)

Since $O^n_\alpha \subseteq O_\alpha$ is open dense (in the topology $\mathcal{S}_{\alpha+1}$), we may find $\emptyset \neq H' = \bigcap_{j<m_2} F^{s_1}_{\beta_j} \cap \bigcap_{j<m_3} H^{k_j}_{\gamma_j} \subseteq [s_0] \cap F'$ with $H' \cap X_{\alpha+1} \subseteq O^n_\alpha$ where $m_2 \geq m_0$, $m_3 \geq m_1$, and $s_0 \subseteq s_1$. Without loss of generality $|s_1| \in I$. Now strengthen p to q by adding appropriate conditions of the form (n_j, s') with $s' \not\subseteq s_1$, $|s'| \in I$, $|s_0| < |s'| \leq |s_1|$ so as to guarantee that $(n_j, s') \notin r$ for any $j < m$, $s' \subseteq s_1$ and $r \leq q$. This means that q forces $H' \cap \bigcap_{j<m} \dot{H}^{n_j}_\alpha \neq \emptyset$.

So, in the generic extension, we have $\emptyset \neq H' \cap \bigcap_{j<m} H^{n_j}_\alpha \cap Y_{\alpha+1} \subseteq F \cap O^n_\alpha \cap H_\alpha = F \cap V^n_\alpha$. This completes the proof of Lemma 3.12. ∎

Corollary 3.13. $V_\alpha \cap P_\alpha \subseteq U_\alpha \cap B_\alpha$.

Proof. Clearly $V^n_\alpha \cap P_\alpha \subseteq P^n_\alpha$ by definition of the forcing. Since $\bigcap_{n\in\omega} P^n_\alpha \subseteq B_\alpha$, it follows that $V_\alpha \cap P_\alpha = \bigcap_{n\in\omega}(V^n_\alpha \cap P_\alpha) \subseteq B_\alpha$. The definition of the forcing also gives $V^n_\alpha \cap P_\alpha \subseteq U^n_\alpha$. Thus, $V_\alpha \cap P_\alpha \subseteq U_\alpha$. ∎

Corollary 3.14. $(V_\alpha \cap O_\alpha) \cap (B_\alpha \cup U_\alpha) = \emptyset$.

Proof. It is immediate from the definition that $V_\alpha \cap O_\alpha = (\bigcap_{n\in\omega} O^n_\alpha) \cap H_\alpha \subseteq H_\alpha$. Since $U_\alpha = 2^\omega \setminus H_\alpha$, it follows that $(V_\alpha \cap O_\alpha) \cap U_\alpha = \emptyset$. Also, $\bigcap_{n\in\omega} O^n_\alpha \cap B_\alpha = \emptyset$ so $(V_\alpha \cap O_\alpha) \cap B_\alpha = \emptyset$. ∎

Corollary 3.15. *V_α is dense G_δ in $(Y_{\alpha+1}, \mathcal{T}_{\alpha+1})$ such that for all $x \in V_\alpha$, $x \in U_\alpha \Leftrightarrow x \in B_\alpha$.*

Proof. This is immediate from Lemma 3.12 and Corollaries 3.13 and 3.14. ∎

The point for having this result is that if we add x to $Y_{\alpha+1}$ by Cohen forcing (e.g. if we add $a_{\alpha+1}$) then x belongs to U_α if and only if it belongs to B_α. So we can hope that Corollary 3.11 also holds for $\beta > \alpha$. However, for this we need that the denseness of V_α is preserved along the construction.

Lemma 3.16. *For all $n \in \omega$, $\beta < \alpha$, $V^n_\beta \cap Y_\alpha$ is dense open in $(Y_\alpha, \mathcal{T}_\alpha)$ and $V^n_\beta \cap X_{\alpha+1}$ is dense open in $(X_{\alpha+1}, \mathcal{S}_{\alpha+1})$.*

Proof. Fix β and n. The proof is by induction on α.

Basic step: $\alpha = \beta + 1$. Then $V^n_\beta \cap Y_\alpha = V^n_\beta$ and the claim for Y_α follows from Lemma 3.12.

For $X_{\alpha+1}$, argue as follows. Let $s \in 2^{<\omega}$ and let $F = \bigcap_{j<m_0} F^s_{\gamma_j} \cap \bigcap_{j<m_1} H^{n_j}_{\delta_j} \neq \emptyset$ be a basic clopen set in $(Y_\alpha, \mathcal{T}_\alpha)$ where $\gamma_j, \delta_j < \alpha$ (see 3.3). Assume $|s| \in I$. We need to show that $F \cap F^s_\alpha \cap V^n_\beta \cap X_{\alpha+1} \neq \emptyset$.

Work in the model M_α. Let $p = \bigcap_{j<k_0} F^t_{\epsilon_j} \cap \bigcap_{j<k_1} H^{l_j}_{\zeta_j} \neq \emptyset$ be a condition in the Cohen forcing in the space $(Y_\alpha, \mathcal{T}_\alpha)$. Assume $|t| \in I$. We need to find a stronger condition $q \leq p$ forcing that $F \cap \dot{F}^s_\alpha \cap V^n_\beta \cap \dot{X}_{\alpha+1} \neq \emptyset$.

If $|s| \geq |t|$ then let $s_0 = s$. Otherwise, define s_0 as follows. Extend s to s_0 with $|s_0| = |t|$ and $s_0(i) = 0$ for all i with $|s| \leq i < |s_0|$. Notice that by definition of the $F^s_{\gamma_j}$ and $H^{n_j}_{\delta_j}$, we must have $[s_0] \cap \bigcap_{j<m_0} F^s_{\gamma_j} \cap \bigcap_{j<m_1} H^{n_j}_{\delta_j} \neq \emptyset$. That is, $\bigcap_{j<m_0} F^{s_0}_{\gamma_j} \cap \bigcap_{j<m_1} H^{n_j}_{\delta_j}$ is still basic open in $(Y_\alpha, \mathcal{T}_\alpha)$.

Since $V^n_\beta \cap Y_\alpha$ is dense open in Y_α (Lemma 3.12), we may find $\emptyset \neq H = \bigcap_{j<m_2} F^{s_1}_{\gamma_j} \cap \bigcap_{j<m_3} H^{n_j}_{\delta_j} \subseteq [s_0] \cap F$ with $H \cap Y_\alpha \subseteq V^n_\beta \cap Y_\alpha$ where $m_2 \geq m_0$, $m_3 \geq m_1$ and $s_0 \subseteq s_1$. Assume $|s_1| \in I$.

Extend t to t_1 with $|t_1|=|s_1|$ such that $t_1(i) = 0$ for all i with $|t| \leq i < |t_1|$. Again by the definition of the $F^t_{\epsilon_j}$ and $H^{l_j}_{\zeta_j}$, $q = [t_1] \cap p = \bigcap_{j<k_0} F^{t_1}_{\epsilon_j} \cap \bigcap_{j<k_1} H^{l_j}_{\zeta_j} \neq \emptyset$ is a condition strengthening p. Clearly, q forces $H \cap \dot{F}^{s_1}_\alpha \neq \emptyset$. Since $t_1(i) = 0$ for $|t| \leq i < |t_1|$, q also forces $\dot{F}^{s_1}_\alpha \subseteq \dot{F}^{s_0}_\alpha$. Furthermore, since $s_0(i) = 0$ for $|s| \leq i < |s_0|$, q forces $\dot{F}^{s_0}_\alpha \subseteq \dot{F}^s_\alpha$.

So, in the generic extension, we have $\emptyset \neq H \cap F^{s_1}_\alpha \cap X_{\alpha+1} \subseteq F \cap F^s_\alpha \cap V^n_\beta \cap X_{\alpha+1}$. This completes the basic step.

Induction step (successor): $\alpha = \alpha_0 + 1$. First deal with $Y_\alpha = Y_{\alpha_0+1}$. We assume $V^n_\beta \cap X_\alpha$ is dense open in $(X_\alpha, \mathcal{S}_\alpha)$. Let $F = F' \cap \bigcap_{j<m} H^{n_j}_{\alpha_0}$ where $F' \in \mathcal{S}_\alpha$. Work in the model N_{α_0} and repeat the second part of the argument of the proof of Lemma 3.12 with O^n_α replaced by V^n_β.

We leave the details to the reader.

The induction step for $X_{\alpha+1} = X_{\alpha_0+2}$ is like the basic step.

Induction step (limit): α is a limit ordinal. Then $X_\alpha = Y_\alpha = \bigcap_{\gamma<\alpha} X_\gamma = \bigcap_{\gamma<\alpha} Y_\gamma$. If F is a basic clopen in $(X_\alpha, \mathcal{S}_\alpha)$ then by construction there is $\gamma < \alpha$ such that F is basic clopen in $(X_\gamma, \mathcal{S}_\gamma)$. Thus, there is a basic clopen $H \subseteq F$ with $\emptyset \neq H \cap X_\gamma \subseteq F \cap V^n_\beta \cap X_\gamma$ by the induction hypothesis. Now simply notice that $H \cap X_\alpha \neq \emptyset$ (see Lemma 3.5 and the comment after its proof), so we are done.

This completes the proof of Lemma 3.16. ∎

Corollary 3.17. *$\forall \alpha, \beta < \omega_1$ $(a_\beta \in U_\alpha \Leftrightarrow a_\beta \in B_\alpha)$.*

Proof. For $\beta \leq \alpha$ this is simply Corollary 3.11. So assume $\beta > \alpha$. Fix n. By Lemma 3.16, $V^n_\alpha \cap Y_\beta$ is dense open in Y_β. Since a_β is Cohen-generic in Y_β over M_β, $a_\beta \in V^n_\alpha$ follows. Thus, $a_\beta \in \bigcap_{n\in\omega} V^n_\alpha = V_\alpha$. By Corollary 3.15, $a_\beta \in U_\alpha \Leftrightarrow a_\beta \in B_\alpha$. ∎

Corollary 3.18. *$\mathcal{A} = \{a_\alpha : \alpha < \omega_1\}$ is a σ-set.*

Proof. By Corollary 3.17, $\mathcal{A} \cap B_\alpha = \mathcal{A} \cap U_\alpha$ for all $\alpha < \omega_1$. Since for every $\alpha < \omega_1$, U_α is a G_δ set, we conclude that every Borel set is a relative G_δ and we are done. ∎

4. Generalizations. Theorem 2 can be generalized under the assumption that a large enough fragment of Martin's axiom MA holds. Say a set of reals $X \subseteq 2^\omega$ is $\mathfrak{c}$-*concentrated* on $Y \subseteq 2^\omega$ if for any open $U \supseteq Y$, we have $|X \setminus U| < \mathfrak{c}$ (see [6]).

THEOREM 4.1. *Assume* MA(σ*-centered*). *Then there is an infinite MAD family which is* $\mathfrak{c}$*-concentrated on a countable subset of itself.*

Sketch of proof. This is like the proof of Theorem 2 in Section 2, but we need to replace the recursive construction of the a_α by a forcing argument.

As before, we assume ($\star$) for $\langle a_n : n \in \omega\rangle$ and construct a_α, $\alpha \geq \omega$, satisfying conditions (1) through (4). At stage α, we consider the p.o. $\mathbb{R}$ which consists of pairs $\langle s, X\rangle$ where $s \in 2^{<\omega}$ and $X \subseteq \{a_\beta : \beta < \alpha\}$ is finite, ordered by $\langle t, Y\rangle \leq \langle s, X\rangle$ if $t \supseteq s$, $Y \supseteq X$, and $t(i) = 0$ for all $|s| \leq i < |t|$ with $i \in \bigcup X$. This is the standard σ-centered forcing notion for adding a set almost disjoint from all a_β, $\beta < \alpha$. The arguments in the proof of Theorem 2 now translate to density arguments which show that, if the generic a_α meets all relevant dense sets, then it will satisfy conditions (1) through (4). Thus, using MA(σ-centered), the construction can be carried out. ■

We do not know whether Theorem 1 can be generalized as well.

CONJECTURE 4.2. *Assume* MA(σ*-centered*). *Then there is a MAD σ-set.*

The approach taken in Section 3 does not seem to generalize easily: if $\alpha \geq \omega_1$, the spaces $(X_\alpha, \mathcal{S}_\alpha)$ and $(Y_\alpha, \mathcal{T}_\alpha)$ would not be second-countable (and thus not Polish) anymore, and while this does not affect much Subsections 3.1 and 3.2 (Cohen forcing would have to be replaced by the σ-centered partial order $\mathbb{Q}$ consisting of conditions of the form $p = \bigcap_{j<m_0} F^s_{\beta_j} \cap \bigcap_{j<m_1} H^{n_j}_{\gamma_j} \neq \emptyset$, of course), it does affect the argument at the beginning of Subsection 3.3: there we used the fact that $B_\alpha \cap X_{\alpha+1}$ has the property of Baire in the Polish space $(X_{\alpha+1}, \mathcal{S}_{\alpha+1})$.

Also we do not know to what extent the assumption MA(σ-centered) can be weakened in Theorem 4.1.

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