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Embedding products of graphs into Euclidean spaces

by

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Abstract. For any collection of graphs $G_1, \ldots, G_N$ we find the minimal dimension d such that the product $G_1 \times \ldots \times G_N$ is embeddable into $\mathbb{R}^d$ (see Theorem 1 below). In particular, we prove that $(K_5)^n$ and $(K_{3,3})^n$ are not embeddable into $\mathbb{R}^{2n}$, where K_5 and $K_{3,3}$ are the Kuratowski graphs. This is a solution of a problem of Menger from 1929. The idea of the proof is a reduction to a problem from so-called Ramsey link theory: we show that any embedding $\mathrm{Lk}\, O \to S^{2n-1}$, where O is a vertex of $(K_5)^n$, has a pair of linked $(n-1)$-spheres.

Introduction. By a *graph* we understand a finite compact one-dimensional polyhedron. We write $K \hookrightarrow \mathbb{R}^d$ if a polyhedron K is PL embeddable into $\mathbb{R}^d$. In this paper we solve the following problem: *for a given collection of graphs $G_1, \ldots, G_N$ find the minimal dimension d such that $G_1 \times \ldots \times G_N \hookrightarrow \mathbb{R}^d$*. A particular case of this problem was posed in [Men29].

The problem of embeddability of polyhedra into Euclidean spaces is of primary importance (e.g., see [Sch84, ReSk99, ARS01, Sko03]). Our special case is interesting because the complete answer can be obtained and is stated easily, but the proof is non-trivial and contains interesting ideas.

THEOREM 1. *Let $G_1, \ldots, G_n$ be connected graphs, distinct from a point, I and S^1. The minimal dimension d such that $G_1 \times \ldots \times G_n \times (S^1)^s \times I^i \hookrightarrow \mathbb{R}^d$ is*

$$d = \begin{cases} 2n + s + i & \textit{if either } i \neq 0 \textit{ or some } G_k \textit{ is planar}, & (1)\\ 2n + s + 1 & \textit{otherwise.} & (2) \end{cases}$$

Here the planarity of G_k can be checked easily by applying the Kuratowski graph planarity criterion.

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Remark. Theorem 1 remains true in the TOP category.

We prove this remark at the end of the paper. From now on till that moment we work in the PL category.

Theorem 1 was stated (without proof) in [Gal93] (see also [Gal92]). The proof of embeddability is trivial (see below). The non-embeddability has been proved earlier in some specific cases. For example, it was known that $Y^n \not\hookrightarrow \mathbb{R}^{2n-1}$, where Y is a *triod* (letter "Y"). (A nice proof of this folklore result is presented in [Sko03], cf. [ReSk01]). Also it was known that $K_5 \times S^1 \not\hookrightarrow \mathbb{R}^3$ (Tom Tucker, private communication). In [Um78] it is proved that $K_5 \times K_5 \not\hookrightarrow \mathbb{R}^4$; that proof contains about 10 pages of calculations involving spectral sequences. We obtain a shorter geometric proof of this result (see Example 2 and Lemma 2 below). The proof of the non-embeddability in case (2), namely, Lemma 2, is the main point of Theorem 1 (while case (1) is reduced easily to a result of van Kampen).

Our proof of Theorem 1 is quite elementary, in particular, we do not use any abstract algebraic topology. We use a reduction to a problem from so-called *Ramsey link theory* [S81, CG83, SeSp92, RST93, RST95, LS98, Neg98, SSS98, T00, ShTa]. Let us introduce some notation. Denote by K_n a *complete graph* on n vertices and by σ_n^m the m-skeleton of an n-simplex. For a polyhedron σ let σ^{*n} be the join of n copies of σ. Denote by $K_{n,n} = (\sigma_{n-1}^0)^{*2}$ a complete bipartite graph on $2n$ vertices. The classical Conway–Gordon–Sachs theorem of Ramsey link theory asserts that any embedding of K_6 into $\mathbb{R}^3$ has a pair of (homologically) linked cycles. In other words, K_6 *is not linklessly embeddable into* $\mathbb{R}^3$. The graph $K_{4,4}$ has the same property (the Sachs theorem, proved in [S81]). In our proof of Theorem 1 we use the following higher dimensional generalization of the Sachs theorem:

Lemma 1. *Any embedding $(\sigma_3^0)^{*n} \to \mathbb{R}^{2n-1}$ has a pair of linked $(n-1)$-spheres.*

Lemma 1 follows from Lemma 1′ below. For higher dimensional generalizations of the Conway–Gordon–Sachs theorem see [SeSp92, SSS98, T00].

The easy part of Theorem 1 and some heuristic considerations. Let us first prove all assertions of Theorem 1 except the non-embeddability in case (2).

Proof of the embeddability in Theorem 1. We need the following two simple results:

(∗) If a polyhedron $K \hookrightarrow \mathbb{R}^d$ and $d > 0$, then $K \times I$, $K \times S^1 \hookrightarrow \mathbb{R}^{d+1}$ (it is sufficient to prove this for $K = \mathbb{R}^d \cong \mathring{D}^d$, for which it is trivial).

(∗∗) For any d-polyhedron K^d the cylinder $K^d \times I \hookrightarrow \mathbb{R}^{2d+1}$ [RSS95].

Set $G = G_1 \times \ldots \times G_n$. By general position $G \hookrightarrow \mathbb{R}^{2n+1}$. If $i \neq 0$, then by $(**)$, $G \times I \hookrightarrow \mathbb{R}^{2n+1}$. And if, say, $G_1 \subset D^2$, then by $(**)$, $D^2 \times G_2 \times \ldots \times G_n \hookrightarrow \mathbb{R}^{2n}$, whence $G \hookrightarrow \mathbb{R}^{2n}$. Applying $(*)$ several times we get the embeddability assertion in all cases considered. ∎

Proof of the non-embeddability in case (1). Note that any connected graph, distinct from a point, I and S^1, contains a triod Y. So it suffices to prove that $Y^n \times I^{s+i} \not\hookrightarrow \mathbb{R}^{2n+s+i-1}$. Since $CK \times CL \cong C(K * L)$ and $K * \sigma_0^0 = CK$ for any polyhedra K and L, it follows that

$$Y^n \times I^{s+i} = (C\sigma_2^0)^n \times (C\sigma_0^0)^{s+i} \cong \underbrace{C \ldots C}_{s+i+1 \text{ times}} (\sigma_2^0)^{*n}.$$

If a polyhedron $K \not\hookrightarrow S^d$ then the cone $CK \not\hookrightarrow \mathbb{R}^{d+1}$ (because we work in the PL category). So the non-embeddability in case (1) follows from $(\sigma_2^0)^{*n} \not\hookrightarrow S^{2n-2}$ [Kam32] (and also from $Y^n \not\hookrightarrow S^{2n-1}$ [Sko03]). ∎

We are thus left with the proof of the non-embeddability in case (2). To make it clearer we precede it with a heuristic consideration of three simplest cases.

EXAMPLE 1. Let us first prove that the Kuratowski graph K_5 is not planar. Suppose to the contrary that $K_5 \subset \mathbb{R}^2$. Let O be a vertex of K_5 and D a small disc with center O. Then the intersection $K_5 \cap \partial D$ consists of 4 points. Denote them by A, B, C, D, in the order along the circle ∂D. Note that the pairs A, C and B, D are the ends of two disjoint arcs contained in $K_5 - \mathring{D}$, and, consequently, in $\mathbb{R}^2 - \mathring{D}$. Then the cycles $OAC, OBD \subset K_5$ intersect each other transversally at exactly one point O, which is impossible in the plane. So $K_5 \not\hookrightarrow \mathbb{R}^2$.

EXAMPLE 2. Now let us outline why $K_5 \times K_5 \not\hookrightarrow \mathbb{R}^4$. (Another proof is given in [Um78].) Recall that if K is a polyhedron and $O \in K$ is a vertex, then the *star* $\operatorname{St} O$ is the union of all closed cells of K containing O, and the *link* $\operatorname{Lk} O$ is the union of all cells of $\operatorname{St} O$ not containing O. In our previous example $\operatorname{Lk} O$ consists of 4 points and the proof is based on the fact that there are two pairs of points of $\operatorname{Lk} O$ linked in ∂D. Now take $K = K_5 \times K_5$. We get $\operatorname{Lk} O \cong K_{4,4}$. So by the Sachs theorem above any embedding $\operatorname{Lk} O \hookrightarrow \partial D^4$ has a pair of linked cycles $\alpha, \beta \in \operatorname{Lk} O$. Thus we can prove that $K \not\hookrightarrow \mathbb{R}^4$ analogously to Example 1, if we construct two disjoint 2-surfaces in $K - \operatorname{St} O$ with boundaries α and β respectively. This construction is easy; see the proof of Lemma 2 below for details. Analogously it can be shown that $\sigma_6^2 \not\hookrightarrow \mathbb{R}^4$. (Another proof is given in [Kam32].)

EXAMPLE 3. Let us show why $K_5 \times S^1 \not\hookrightarrow \mathbb{R}^3$. (Another proof was given by Tom Tucker.) Suppose that $K_5 \times S^1 \hookrightarrow \mathbb{R}^3$; then by $(*)$ we have

$K_5 \times S^1 \times S^1 \hookrightarrow \mathbb{R}^4$. But $S^1 \times S^1 \supset K_5$, so $K_5 \times K_5 \hookrightarrow \mathbb{R}^4$, which contradicts Example 2.

Proof of the non-embeddability in case (2) modulo some lemmas. Let us say that a PL map $f : K \to L$ between two polyhedra K and L with fixed triangulations is an *almost embedding* if for any two *disjoint* closed cells $a, b \subset K$ we have $fa \cap fb = \emptyset$ [FKT94].

LEMMA 2 (for $n = 2$ [Um78]). *The polyhedron* $(K_5)^n$ *is not almost embeddable into* $\mathbb{R}^{2n}$.

Proof of the non-embeddability in case (2) *of Theorem 1 modulo Lemma 2.* By the Kuratowski graph planarity criterion any non-planar graph contains a graph homeomorphic either to K_5 or to $K_{3,3}$. So we may assume that each G_k is either K_5 or $K_{3,3}$. Analogously to Example 3 we may assume that $s = 0$. Now we are going to replace all the graphs $K_{3,3}$ by K_5's.

Note that K_5 is almost embeddable in $K_{3,3}$ (Fig. 1). Indeed, map a vertex of K_5 into the middle point of an edge of $K_{3,3}$ and map the remaining four vertices to the four vertices of $K_{3,3}$ not belonging to this edge. Then map each edge e of K_5 onto the shortest (as regards the number of vertices) arc in $K_{3,3}$, joining the images of the ends of e, and the almost embedding is constructed.

Now note that a product of almost embeddings is an almost embedding, and also a composition of an almost embedding and an embedding is an almost embedding. Thus the non-embeddability in case (2) of Theorem 1 follows from Lemma 2. ∎

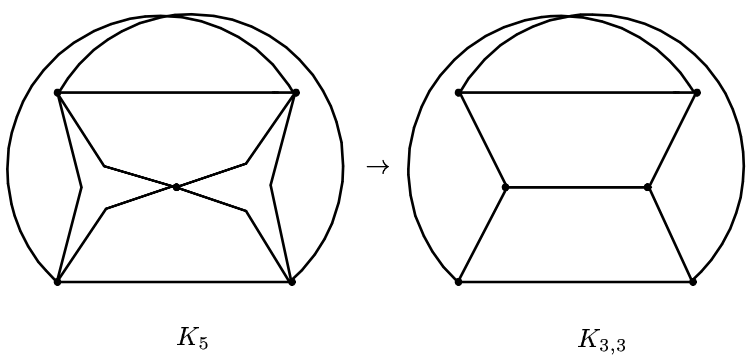

Fig. 1

For the proof of Lemma 2 we need the following notion. Let A, B be a pair of PL n-manifolds with boundary and let $f : A \to \mathbb{R}^{2n}$, $g : B \to \mathbb{R}^{2n}$ be a pair of PL maps such that $f\partial A \cap g\partial B = \emptyset$. Take a general position pair of PL maps $\bar{f} : A \to \mathbb{R}^{2n}$ and $\bar{g} : B \to \mathbb{R}^{2n}$ close to f and g respectively. The mod 2 *intersection index* $fA \cap gB$ is the number of points mod 2 in the set $\bar{f}A \cap \bar{g}B$. We are going to use the following simple result:

$(***)$ if both A and B are closed manifolds, then $fA \cap gB = 0$.

(This follows from the homology intersection form of $\mathbb{R}^{2n}$ being zero.) Lemma 2 will be deduced from the following generalization of Lemma 1:

LEMMA 1′. *Let $L = (\sigma_3^0)^{*n}$. Then for any almost embedding $CL \to \mathbb{R}^{2n}$ there exist two disjoint $(n-1)$-spheres $\alpha, \beta \subset L$ such that the intersection index $fC\alpha \cap fC\beta$ is* 1.

Proof of Lemma 2 modulo Lemma 1′. Assume that there exists an almost embedding $f : K = K_5 \times \ldots \times K_5 \to \mathbb{R}^{2n}$. Let $O = O_1 \times \ldots \times O_n$ be a vertex of K. By the well-known formula for links,

$$\operatorname{Lk} O \cong \operatorname{Lk} O_1 * \ldots * \operatorname{Lk} O_n, \qquad \operatorname{St} O = C \operatorname{Lk} O \cong C(\sigma_3^0)^{*n}.$$

Let $\alpha, \beta \subset \operatorname{Lk} O$ be a pair of $(n-1)$-spheres given by Lemma 1′. Identify $\operatorname{Lk} O$ and $\operatorname{Lk} O_1 * \ldots * \operatorname{Lk} O_n$. Since α and β are disjoint, it follows that for each $k = 1, \ldots, n$ the sets $\alpha \cap \operatorname{Lk} O_k$ and $\beta \cap \operatorname{Lk} O_k$ are disjoint and each of them consists of two points. By definition, put $\{A_k, C_k\} := \alpha \cap \operatorname{Lk} O_k$ and $\{B_k, D_k\} := \beta \cap \operatorname{Lk} O_k$. Consider two n-tori

$$T_\alpha = O_1A_1C_1 \times \ldots \times O_nA_nC_n, \qquad T_\beta = O_1B_1D_1 \times \ldots \times O_nB_nD_n$$

contained in K.

Clearly, $T_\alpha \supset C\alpha$, $T_\beta \supset C\beta$ and $T_\alpha \cap T_\beta = O$. Since f is an almost embedding, it follows that $fT_\alpha \cap fT_\beta = fC\alpha \cap fC\beta$. So $fT_\alpha \cap fT_\beta = 1$ by the choice of α and β. By $(***)$ we obtain a contradiction, so $K \not\to \mathbb{R}^{2n}$. ■

The proof of Lemma 1′. The proof is similar to that of the Conway–Gordon–Sachs theorem and applies the idea of [Kam32], only we use a more refined obstruction. The reader can restrict attention to the case when $n = 2$ and obtain an alternative proof of the Sachs theorem. (The proof for $n > 2$ is completely analogous to that for $n = 2$.)

We show that for any $(n-1)$-simplex c of L and any almost embedding $f : CL \to \mathbb{R}^{2n}$ there exists a pair of disjoint $(n-1)$-spheres $\alpha, \beta \subset L$ such that $\alpha \supset c$ and the intersection index $fC\alpha \cap fC\beta$ is 1.

For an almost embedding $f : CL \to \mathbb{R}^{2n}$ let $v(f) = \sum(fC\alpha \cap fC\beta) \bmod 2$ be the *van Kampen obstruction* to linkless embeddability. Here the sum is over all pairs of disjoint $(n-1)$-spheres $\alpha, \beta \subset L$ such that $c \subset \alpha$. It suffices to prove that $v(f) = 1$. Our proof is in two steps: first we show that $v(f)$ does not depend on f, and then we calculate $v(f)$ for certain “standard” embeddings $f : CL \to \mathbb{R}^{2n}$.

Let us prove that $v(f)$ does not depend on f (cf. [Kam32, CG83]). Take any two almost embeddings $F_0, F_1 : CL \to \mathbb{R}^{2n}$. By general position in the

PL category there exists a homotopy $F : I \times CL \to \mathbb{R}^{2n}$ between them such that

(1) there are only a finite number of *singular* times t, i.e. times $t \in I$ such that F_t is not an almost embedding;

(2) for each singular t there is exactly one pair of disjoint $(n-1)$-simplices $a, b \subset L$ such that $F_t Ca \cap F_t b \neq \emptyset$;

(3) the intersection $F_t Ca \cap F_t b$ is "transversal in time", i.e. $F(t \times Ca) \cap F([t-\varepsilon, t+\varepsilon] \times b)$ is transversal for some $\varepsilon > 0$.

Consider a singular time t. Property (3) implies that the intersection index $F_t C\alpha \cap F_t C\beta$ of a pair of disjoint $(n-1)$-spheres $\alpha, \beta \subset L$ changes with the increase of t if and only if either $\alpha \supset a$, $\beta \supset b$ or $\alpha \supset b$, $\beta \supset a$. Such pairs (α, β) satisfying the condition $\alpha \supset c$ are called *critical.* If $c \cap (a \cup b) = \emptyset$, then there are exactly two critical pairs. Indeed, we have either $\alpha \supset a \cup c$ or $\alpha \supset b \cup c$. Each of these determines a unique critical pair. If $c \cap (a \cup b) \neq \emptyset$, then there are two distinct vertices $v, w \in L - (a \cup b \cup c)$ belonging to the same copy of σ_3^0. Then there is an involution without fixed points on the set of critical pairs. Indeed, $\mathbb{Z}_2$ acts on the set of vertices of L by interchanging v and w, and it also acts on the set of critical pairs, because $v, w \notin a \cup b \cup c$. So the number of critical pairs is always even, therefore $v(F_0) = v(F_1)$.

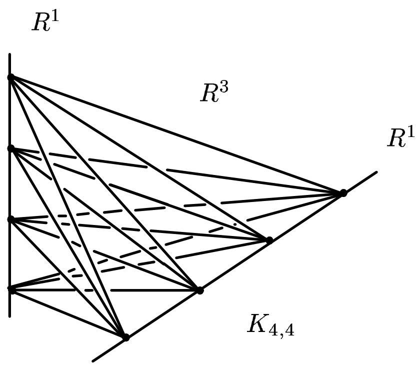

Fig. 2

Now let us prove that $v(f) = 1$ for a certain "standard" embedding $f : CL \hookrightarrow \mathbb{R}^{2n}$. To define it take a general position collection of n lines in $\mathbb{R}^{2n-1} \subset \mathbb{R}^{2n}$. For each $k = 1, \dots, n$ take a quadruple σ_k of distinct points on the kth line. Taking the join of all σ_k, we obtain an embedding $L \hookrightarrow \mathbb{R}^{2n-1}$ (Fig. 2 for $n = 2$). The standard embedding $f : CL \hookrightarrow \mathbb{R}^{2n}$ is defined to be the cone of this embedding. Below we omit f from the notation of f-images. Clearly, for a pair of disjoint $(n-1)$-spheres $\alpha, \beta \subset L$ we have $C\alpha \cap C\beta = \mathrm{lk}(\alpha, \beta) \bmod 2$. Let us show that $\mathrm{lk}(\alpha, \beta) = 1 \bmod 2$ if and only if for each $k = 1, \dots, n$ the 0-spheres $\alpha \cap \sigma_k$ and $\beta \cap \sigma_k$ are linked in the kth copy of $\mathbb{R}^1$. Indeed, let I be the segment between the pair of points of $\alpha\cap\sigma_1$. Set $D_\alpha = I*(\alpha\cap\sigma_2)*\ldots*(\alpha\cap\sigma_n)$. Then $\partial D_\alpha = \alpha$. The intersection $D_\alpha\cap\beta$ is not empty mod 2 if and only if the 0-spheres $\alpha\cap\sigma_1$ and $\beta\cap\sigma_1$ are linked in the first copy of $\mathbb{R}^1$. This intersection is transversal if and only if $\alpha\cap\sigma_k$ and $\beta\cap\sigma_k$ are linked in the remaining copies of $\mathbb{R}^1$. Now it is obvious that there exists exactly one pair α,β such that $\alpha\supset c$ and $C\alpha\cap C\beta = 1$ mod 2. So $v(f) = 1$, which proves the lemma. ■

We conclude our paper with the proof of Remark (due to the referee):

Proof of Theorem 1 in the TOP category. For codimension ≥ 3 the assertion of Theorem 1 in the TOP category follows from the one in the PL category by the result of Bryant [Bry72]. Analogously to Example 3, we reduce the codimension 1 and 2 cases to the codimension 3 case. ■

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