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Functions of Baire class one

by

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Abstract. Let K be a compact metric space. A real-valued function on K is said to be of Baire class one (Baire-1) if it is the pointwise limit of a sequence of continuous functions. We study two well known ordinal indices of Baire-1 functions, the oscillation index β and the convergence index γ. It is shown that these two indices are fully compatible in the following sense: a Baire-1 function f satisfies $\beta(f) \leq \omega^{\xi_1} \cdot \omega^{\xi_2}$ for some countable ordinals ξ_1 and ξ_2 if and only if there exists a sequence (f_n) of Baire-1 functions converging to f pointwise such that $\sup_n \beta(f_n) \leq \omega^{\xi_1}$ and $\gamma((f_n)) \leq \omega^{\xi_2}$. We also obtain an extension result for Baire-1 functions analogous to the Tietze Extension Theorem. Finally, it is shown that if $\beta(f) \leq \omega^{\xi_1}$ and $\beta(g) \leq \omega^{\xi_2}$, then $\beta(fg) \leq \omega^{\xi}$, where $\xi = \max\{\xi_1 + \xi_2, \xi_2 + \xi_1\}$. These results do not assume the boundedness of the functions involved.

1. Preliminaries. Let K be a compact metric space. A function $f : K \to \mathbb{R}$ is said to be of *Baire class one*, or simply *Baire-1*, if there exists a sequence (f_n) of real-valued continuous functions that converges pointwise to f. Let $\mathfrak{B}_1(K)$ (respectively, $\mathcal{B}_1(K)$) be the set of all real-valued (respectively, bounded real-valued) Baire-1 functions on K. Several authors have studied Baire-1 functions in terms of ordinal ranks associated to each function. (See, e.g., [2], [3] and [4].) In this paper, we study the relationship between two of these ordinal ranks, namely the oscillation rank β and the convergence rank γ.

We begin by recalling the definitions of the indices β and γ. Suppose that H is a compact metric space, and f is a real-valued function whose domain contains H. For any $\varepsilon > 0$, let $H^0(f,\varepsilon) = H$. If $H^\alpha(f,\varepsilon)$ is defined for some countable ordinal α, let $H^{\alpha+1}(f,\varepsilon)$ be the set of all those $x \in H^\alpha(f,\varepsilon)$ such that for every open set U containing x, there are two points x_1 and x_2 in $U \cap H^\alpha(f,\varepsilon)$ with $|f(x_1) - f(x_2)| \geq \varepsilon$. For a countable limit ordinal α, we let

$$H^\alpha(f,\varepsilon) = \bigcap_{\alpha'<\alpha} H^{\alpha'}(f,\varepsilon).$$

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The index $\beta_H(f,\varepsilon)$ is taken to be the least α with $H^\alpha(f,\varepsilon)=\emptyset$ if such an α exists, and ω_1 otherwise. The *oscillation index* of f is

$$\beta_H(f)=\sup\{\beta_H(f,\varepsilon):\varepsilon>0\}.$$

If the ambient space H is clear from the context, we write $\beta(f,\varepsilon)$ and $\beta(f)$ in place of $\beta_H(f,\varepsilon)$ and $\beta_H(f)$ respectively.

The γ index is defined analogously. If (f_n) is a sequence of real-valued functions such that $H\subseteq\bigcap_n \mathrm{dom}(f_n)$, let $H^0((f_n),\varepsilon)=H$ for any $\varepsilon>0$. If $H^\alpha((f_n),\varepsilon)$ has been defined for some countable ordinal α, let $H^{\alpha+1}((f_n),\varepsilon)$ be the set of all those $x\in H^\alpha((f_n),\varepsilon)$ such that for every open set U containing x and any $m\in\mathbb{N}$, there are two integers $n_1,\ n_2$ with $n_1>n_2>m$ and $x'\in U\cap H^\alpha((f_n),\varepsilon)$ such that $|f_{n_1}(x')-f_{n_2}(x')|\ge\varepsilon$. Define

$$H^\alpha((f_n),\varepsilon)=\bigcap_{\alpha'<\alpha}H^{\alpha'}((f_n),\varepsilon)$$

if α is a countable limit ordinal. Let $\gamma_H((f_n),\varepsilon)$ be the least α with $H^\alpha((f_n),\varepsilon)=\emptyset$ if such an α exists, and ω_1 otherwise. Finally, the *convergence index* of (f_n) is the ordinal

$$\gamma_H((f_n))=\sup\{\gamma_H((f_n),\varepsilon):\varepsilon>0\}.$$

Again, if there is no ambiguity about the space H, we write $\gamma((f_n),\varepsilon)$ and $\gamma((f_n))$ for $\gamma_H((f_n),\varepsilon)$ and $\gamma_H((f_n))$ respectively.

It is known that a function $f:K\to\mathbb{R}$ is Baire-1 if and only if $\beta(f)<\omega_1$. (See [3, Proposition 1.2].) Following [3], we define the set of functions of small Baire class ξ and the set of *bounded* functions of small Baire class ξ for each countable ordinal ξ as

$$\mathfrak{B}_1^\xi(K)=\{f\in\mathfrak{B}_1(K):\beta(f)\le\omega^\xi\}$$

and

$$\mathcal{B}_1^\xi(K)=\{f\in\mathcal{B}_1(K):\beta(f)\le\omega^\xi\}$$

respectively. In [4], the following results are shown.

THEOREM 1.1. *Let K be a compact metric space.*

(a) [4, Theorem 7] *If ξ is a finite ordinal, then a function f is in $\mathcal{B}_1^{\xi+1}(K)$ if and only if there exists a sequence (f_n) in $\mathcal{B}_1^1(K)$ converging pointwise to f such that $\gamma((f_n))\le\omega^\xi$.*

(b) [4, Corollary 9] *If ξ is an infinite countable ordinal, and $f\in\mathcal{B}_1(K)$ is the pointwise limit of a sequence (f_n) in $\mathcal{B}_1^1(K)$ such that $\gamma((f_n))\le\omega^\xi$, then $\beta(f)\le\omega^\xi$.*

One of our main results generalizes and unifies the two parts of Theorem 1.1.

THEOREM 1.2. *Let K be a compact metric space and let ξ_1, ξ_2 be countable ordinals. A function f is in $\mathfrak{B}_1^{\xi_1+\xi_2}(K)$, respectively, $\mathcal{B}_1^{\xi_1+\xi_2}(K)$, if and only if there exists a sequence (f_n) in $\mathfrak{B}_1^{\xi_1}(K)$, respectively, a bounded sequence (f_n) in $\mathcal{B}_1^{\xi_1}(K)$, converging pointwise to f such that $\gamma((f_n)) \leq \omega^{\xi_2}$.*

In the course of proving Theorem 1.2, we show that any Baire-1 function f on a closed subspace H of a compact metric space K can be extended to a Baire-1 function g on K such that $\beta_H(f) = \beta_K(g)$ (Theorem 3.6). When $\beta_H(f) = 1$, this is the familiar Tietze Extension Theorem. Proposition 2.1 and Theorem 2.3 in [3] imply that for a bounded Baire-1 function f, $\beta(f)$ is the smallest ordinal ξ such that there exists a sequence (f_n) of continuous functions converging pointwise to f and having $\gamma((f_n)) = \xi$. Theorem 5.5 below shows that the same result holds without the boundedness assumption on the function f. In the last section, we consider the product of Baire-1 functions. In contrast to the class $\mathcal{B}_1^{\xi}(K)$, the class $\mathfrak{B}_1^{\xi}(K)$ is not closed under multiplication. Theorem 6.5 shows that if $f \in \mathfrak{B}_1^{\xi_1}(K)$ and $g \in \mathfrak{B}_1^{\xi_2}(K)$, then $fg \in \mathfrak{B}_1^{\xi}(K)$, where $\xi = \max\{\xi_1 + \xi_2, \xi_2 + \xi_1\}$. It is also shown that this result is the best possible.

Our notation is standard. In what follows, K will always denote a compact metric space. If H is a closed subset of K, the derived set H' is the set of all limit points of H. A transfinite sequence of derived sets is defined in the usual manner. Let $H^{(0)} = H$ and $H^{(\alpha+1)} = (H^{(\alpha)})'$ for any ordinal α. If α is a limit ordinal, let

$$H^{(\alpha)} = \bigcap_{\alpha' < \alpha} H^{(\alpha')}.$$

It is easy to observe that $H^{\alpha}(f, \varepsilon) \subseteq H^{(\alpha)}$ and $H^{\alpha}((f_n), \varepsilon) \subseteq H^{(\alpha)}$, where $H^{\alpha}(f, \varepsilon)$ and $H^{\alpha}((f_n), \varepsilon)$ are the sets associated with the oscillation index and the convergence index respectively. Given real-valued functions f and g defined on a set S, we let

$$\|f - g\|_S = \sup\{|f(s) - g(s)| : s \in S\}.$$

When there is no cause for confusion, we write $\|f - g\|$ for $\|f - g\|_S$. Since we shall be dealing with unbounded functions in general, this functional can take the value ∞ and is not a "norm". However, it is compatible with the topology of uniform convergence on the set $\mathbb{R}^S$ of all real-valued functions on S in the sense that the sets

$$U(f, \varepsilon) = \{g : \|g - f\|_S < \varepsilon\}$$

form a basis for the said topology.

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2. Oscillation and convergence of Baire-1 functions. We begin by proving a result that yields an upper bound of the oscillation index of a Baire-1 function f as the product of the convergence index of a sequence of functions (f_n) converging pointwise to f, and the supremum of the oscillation indices of f_n's.

LEMMA 2.1. *Let U and L be sets such that $U \subseteq L \subseteq K$, where U is open in K and L is closed in K. Suppose f, f_n $(n \geq 1)$ are Baire-1 functions on K, $\alpha < \omega_1$, and $\varepsilon > 0$. Then*

(a) $L^\alpha(f,\varepsilon) \subseteq K^\alpha(f,\varepsilon) \cap L$,
(b) $L^\alpha((f_n),\varepsilon) \subseteq K^\alpha((f_n),\varepsilon) \cap L$,
(c) $K^\alpha(f,\varepsilon) \cap U \subseteq L^\alpha(f,\varepsilon)$,
(d) $K^\alpha((f_n),\varepsilon) \cap U \subseteq L^\alpha((f_n),\varepsilon)$.

Proof. We only prove (c). The proof is by induction on α. The statement is trivial if $\alpha = 0$ or a limit ordinal. Suppose the statement is true for all ordinals not greater than α. Let $x \in K^{\alpha+1}(f,\varepsilon) \cap U$. If N is a neighborhood of x in K, then $N \cap U$ is open in K. Thus there exist $x_1, x_2 \in (N \cap U) \cap K^\alpha(f,\varepsilon) = N \cap (U \cap K^\alpha(f,\varepsilon)) \subseteq N \cap L^\alpha(f,\varepsilon)$ such that $|f(x_1) - f(x_2)| \geq \varepsilon$. Hence $x \in L^{\alpha+1}(f,\varepsilon)$. ∎

PROPOSITION 2.2. *Let (f_n) be a sequence in $\mathfrak{B}_1(K)$ and let $\varepsilon > 0$. Suppose that $\beta(f_n,\varepsilon) \leq \beta_0$ for all $n \in \mathbb{N}$, and $\gamma((f_n),\varepsilon) \leq \gamma_0$. If (f_n) converges pointwise to a function f, then $\beta(f,3\varepsilon) \leq \beta_0 \cdot \gamma_0$.*

Proof. We first consider the case $\gamma_0 = 1$. Then $K^1((f_n),\varepsilon) = \emptyset$. For each $x \in K$, there exist an open neighborhood U_x of x and $p_x \in \mathbb{N}$ such that whenever $n > m > p_x$,

$$|f_n(x') - f_m(x')| < \varepsilon$$

for all $x' \in U_x$. By the compactness of K, there exist $x_1, \ldots, x_k$ such that

$$K \subseteq \bigcup_{i=1}^k U_{x_i}.$$

Let $p_0 = \max\{p_{x_1}, \ldots, p_{x_k}\}$. Then for all $n > m > p_0$ and $y \in K$, we have $y \in U_{x_i}$ for some i, $1 \leq i \leq k$. Since $n > m > p_{x_i}$,

$$|f_n(y) - f_m(y)| < \varepsilon.$$

Taking the limit as $n \to \infty$, we have

$$\|f - f_m\| \leq \varepsilon \quad \text{for all } m > p_0. \tag{2.1}$$

Using (2.1), it is easy to verify by induction that

$$K^\alpha(f,3\varepsilon) \subseteq K^\alpha(f_{p_0+1},\varepsilon)$$

for all $\alpha < \omega_1$. In particular,

$$K^{\beta_0}(f, 3\varepsilon) \subseteq K^{\beta_0}(f_{p_0+1}, \varepsilon) = \emptyset.$$

Hence $\beta(f, 3\varepsilon) \leq \beta_0 = \beta_0 \cdot \gamma_0$.

Suppose the assertion is true for some γ_0. Let (f_n) be a sequence in $\mathfrak{B}_1(K)$ that converges pointwise to a function f. Suppose there exists $\varepsilon > 0$ such that $\beta(f_n, \varepsilon) \leq \beta_0$ for all $n \in \mathbb{N}$ and $\gamma((f_n), \varepsilon) \leq \gamma_0 + 1$. We need to show $\beta(f, 3\varepsilon) \leq \beta_0 \cdot (\gamma_0 + 1)$. Since $\gamma((f_n), \varepsilon) \leq \gamma_0 + 1$, we have $K^{\gamma_0+1}((f_n), \varepsilon) = \emptyset$. For each $m \in \mathbb{N}$, let U_m denote the $1/m$-neighborhood of $K^{\gamma_0}((f_n), \varepsilon)$. Denote $K \setminus U_m$ by $\widetilde{K}_m$. From Lemma 2.1(a), (b), for each $n \in \mathbb{N}$, $\beta_{\widetilde{K}_m}(f_n, \varepsilon) \leq \beta_0$ and $\gamma_{\widetilde{K}_m}((f_n), \varepsilon) \leq \gamma_0$. By the inductive hypothesis, we see that

$$\beta_{\widetilde{K}_m}(f, 3\varepsilon) \leq \beta_0 \cdot \gamma_0.$$

From this and applying Lemma 2.1(c) with $U = K \setminus \overline{U}_m$, $L = \widetilde{K}_m$ for all $m \in \mathbb{N}$, we see that $K^{\beta_0 \cdot \gamma_0}(f, 3\varepsilon) \subseteq K^{\gamma_0}((f_n), \varepsilon)$. Let

$$\widetilde{K} = K^{\beta_0 \cdot \gamma_0}(f, 3\varepsilon) \subseteq K^{\gamma_0}((f_n), \varepsilon).$$

Then $\beta_{\widetilde{K}}(f_n, \varepsilon) \leq \beta_0$ and $\gamma_{\widetilde{K}}((f_n), \varepsilon) = 1$. Thus $\beta_{\widetilde{K}}(f, 3\varepsilon) \leq \beta_0$ by the case when $\gamma_0 = 1$. Therefore

$$K^{\beta_0 \cdot (\gamma_0+1)}(f, 3\varepsilon) = K^{\beta_0 \cdot \gamma_0 + \beta_0}(f, 3\varepsilon) = \widetilde{K}^{\beta_0}(f, 3\varepsilon) = \emptyset.$$

Hence

$$\beta(f, 3\varepsilon) \leq \beta_0 \cdot (\gamma_0 + 1).$$

Suppose $\gamma_0 < \omega_1$ is a limit ordinal and the statement holds for all ordinals $\gamma < \gamma_0$. Let $(f_n) \subseteq \mathfrak{B}_1(K)$ be a sequence that converges pointwise to a function f and let $\varepsilon > 0$ be given. Suppose that $\beta(f_n, \varepsilon) \leq \beta_0$ for all $n \in \mathbb{N}$, and $\gamma((f_n), \varepsilon) \leq \gamma_0$. Then $\gamma((f_n), \varepsilon) < \gamma_0$ since $\gamma((f_n), \varepsilon)$ must be a successor ordinal. Hence $\beta(f, 3\varepsilon) \leq \beta_0 \cdot \gamma((f_n), \varepsilon) < \beta_0 \cdot \gamma_0$. ■

THEOREM 2.3. *Let (f_n) be a sequence in $\mathfrak{B}_1(K)$ converging pointwise to a function f. Suppose* $\sup\{\beta(f_n) : n \in \mathbb{N}\} \leq \beta_0$ *and $\gamma((f_n)) \leq \gamma_0$. Then f is Baire-1 and $\beta(f) \leq \beta_0 \cdot \gamma_0$.*

For the next corollary, recall that DBSC(K) is the space of all differences of bounded semicontinuous functions on K. It is known that $\mathcal{B}_1^1(K)$ is the closure of DBSC(K) in the topology of uniform convergence ([3, Theorem 3.1]).

COROLLARY 2.4 ([4, Corollary 9]). *Let $f \in \mathcal{B}_1(K)$ be the pointwise limit of a sequence $(f_n) \subseteq$ DBSC(K). If $\gamma((f_n)) \leq \omega^\xi$ and $\omega \leq \xi < \omega_1$, then $\beta(f) \leq \omega^\xi$.*

3. Extension of Baire-1 functions. In this section, we establish several results regarding the extension of Baire-1 functions. They are analogs of the Tietze Extension Theorem for continuous functions. These results are applied in the next section in proving the converse of Theorem 2.3.

LEMMA 3.1. *Suppose that F is a closed subspace of K and that f is a Baire-1 function on F. For any $\varepsilon > 0$, there exists a continuous function $g : K \setminus F^1(f,\varepsilon) \to \mathbb{R}$ such that*

$$\|g - f\|_{F\setminus F^1(f,\varepsilon)} \le \varepsilon.$$

Proof. For any $x \in F \setminus F^1(f,\varepsilon)$, choose an open neighborhood U_x of x in K such that $U_x \cap F^1(f,\varepsilon) = \emptyset$ and $|f(x_1) - f(x_2)| < \varepsilon$ for all $x_1, x_2 \in U_x \cap F$. The collection $\mathcal{U} = \{U_x : x \in F \setminus F^1(f,\varepsilon)\} \cup \{K \setminus F\}$ is an open cover of $K \setminus F^1(f,\varepsilon)$. By [1, Theorems IX.5.3 and VIII.4.2], there exists a partition of unity $(\varphi_U)_{U\in\mathcal{U}}$ subordinate to $\mathcal{U}$. If $U = U_x \in \mathcal{U}$ for some $x \in F \setminus F^1(f,\varepsilon)$, let $a_U = f(x)$; if $U = K \setminus F$, let $a_U = 0$. Define $g : K \setminus F^1(f,\varepsilon) \to \mathbb{R}$ by $g = \sum_{U\in\mathcal{U}} a_U\varphi_U$. The sum is well defined and continuous since $\{\operatorname{supp}\varphi_U : U \in \mathcal{U}\}$ is locally finite. Let $x \in F \setminus F^1(f,\varepsilon)$. Then $\mathcal{V} = \{U \in \mathcal{U} : \varphi_U(x) \ne 0\}$ is a finite set, $\varphi_U(x) > 0$ for all $U \in \mathcal{V}$ and $\sum_{U\in\mathcal{V}} \varphi_U(x) = 1$. If $U \in \mathcal{V}$, then $x \in U \cap F$; hence $U \ne K \setminus F$. Therefore, $U = U_y$ for some $y \in F \setminus F^1(f,\varepsilon)$. But then $x, y \in U_y \cap F$ implies that $|a_U - f(x)| = |f(y) - f(x)| < \varepsilon$. It follows that

$$|g(x) - f(x)| = \Big|\sum_{U\in\mathcal{U}} a_U\varphi_U(x) - f(x)\Big| = \Big|\sum_{U\in\mathcal{V}} a_U\varphi_U(x) - \sum_{U\in\mathcal{V}} f(x)\varphi_U(x)\Big|$$
$$\le \sum_{U\in\mathcal{V}} |a_U - f(x)|\varphi_U(x) < \varepsilon.$$

This shows that $\|g - f\|_{F\setminus F^1(f,\varepsilon)} \le \varepsilon$. ∎

THEOREM 3.2. *Suppose that F is a closed subspace of K and that f is a Baire-1 function on F. For any $1 \le \beta_0 < \omega_1$ and any $\varepsilon > 0$, there exists $g : K \setminus F^{\beta_0}(f,\varepsilon) \to \mathbb{R}$ such that*

$$\|g - f\|_{F\setminus F^{\beta_0}(f,\varepsilon)} \le \varepsilon$$

and

$$\beta_H(g) \le \beta_0 \quad \text{for all compact subsets } H \text{ of } K \setminus F^{\beta_0}(f,\varepsilon).$$

Proof. Let $h : K\setminus F^1(f,\varepsilon) \to \mathbb{R}$ be the function obtained from Lemma 3.1. If $1 \le \alpha < \beta_0$, let $\widetilde{K} = \widetilde{F} = F^\alpha(f,\varepsilon)$. Applying Lemma 3.1 with $\widetilde{K}$, $\widetilde{F}$, and the function f yields a continuous function $g_\alpha : F^\alpha(f,\varepsilon) \setminus F^{\alpha+1}(f,\varepsilon) \to \mathbb{R}$ such that

$$\|g_\alpha - f\|_{F^\alpha(f,\varepsilon)\setminus F^{\alpha+1}(f,\varepsilon)} \le \varepsilon.$$

Let $g = h \cup \bigcup_{\alpha<\beta_0} g_\alpha : K \setminus F^{\beta_0}(f,\varepsilon) \to \mathbb{R}$. Then $\|g - f\|_{F\setminus F^{\beta_0}(f,\varepsilon)} \le \varepsilon$.

Suppose that $\delta > 0$ and H is a compact subset of $K \setminus F^{\beta_0}(f,\varepsilon)$. We claim that

$$H^\alpha(g,\delta) \subseteq H \cap F^\alpha(f,\varepsilon) \quad \text{if } 1 \le \alpha \le \beta_0.$$

This claim will be verified by induction on α, the only nontrivial case being that for successor ordinals. Thus suppose that α is a successor ordinal $\le \beta_0$ and either $\alpha = 1$ or the claim holds for its immediate predecessor $\alpha - 1$. Let $x \in H^\alpha(g,\delta)$. If $x \notin F^\alpha(f,\varepsilon)$, then there exists an open neighborhood U of x such that

$$\overline{U} \cap F^\alpha(f,\varepsilon) = \emptyset.$$

If $\alpha \ne 1$, then $H^{\alpha-1}(g,\delta) \subseteq F^{\alpha-1}(f,\varepsilon)$ by the inductive hypothesis. Hence

$$\overline{U} \cap H^{\alpha-1}(g,\delta) \subseteq F^{\alpha-1}(f,\varepsilon) \setminus F^\alpha(f,\varepsilon).$$

In particular, $g = g_{\alpha-1}$ on $\overline{U} \cap H^{\alpha-1}(g,\delta)$ and hence is continuous on the same set. Similarly, if $\alpha = 1$, then $g = h$ is continuous on $\overline{U} \cap H \subseteq K \setminus F^1(f,\varepsilon)$. By Lemma 2.1(c),

$$H^\alpha(g,\delta) \cap U = (H^{\alpha-1}(g,\delta))^1(g,\delta) \cap U \subseteq (H^{\alpha-1}(g,\delta) \cap \overline{U})^1(g,\delta) = \emptyset.$$

Thus, $x \notin H^\alpha(g,\delta)$. This shows that $H^\alpha(g,\delta) \subseteq H \cap F^\alpha(f,\varepsilon)$. It follows from the claim that $H^{\beta_0}(g,\delta) = \emptyset$. Hence $\beta_H(g) \le \beta_0$, as required. ∎

We obtain the following corollaries by taking $F = K$ and $\beta_0 = \beta_F(f)$ respectively.

Corollary 3.3. *Let f be a Baire-1 function on K such that $\beta(f,\varepsilon) \le \beta_0$ for some $1 \le \beta_0 < \omega_1$ and $\varepsilon > 0$. Then there exists $g : K \to \mathbb{R}$ such that*

$$\|g - f\| \le \varepsilon, \qquad \beta(g) \le \beta_0.$$

Corollary 3.4. *Let F be a closed subspace of K. If f is a Baire-1 function on F, then for every $\varepsilon > 0$ there exists a Baire-1 function g on K such that*

$$\|g - f\|_F \le \varepsilon, \qquad \beta_K(g) \le \beta_F(f).$$

Next we show that Corollary 3.4 can be improved to an exact extension theorem (i.e., the case $\varepsilon = 0$). In the statement of Lemma 3.5, the vacuous sum $\sum_{j=1}^0 g_j$ is taken to be the zero function.

Lemma 3.5. *Let F be a closed subspace of K and let f be a Baire-1 function on F. Then there exists a sequence (g_n) of Baire-1 functions on K such that*

(a) *g_n is continuous on $K \setminus F^1(f - \sum_{j=1}^{n-1} g_j, 1/2^{n-1})$ for all $n \in \mathbb{N}$,*
(b) $\|f - \sum_{j=1}^n g_j\|_{F \setminus F^1(f,1/4^{n-1})} \le 1/2^{n-1}$, $n \in \mathbb{N}$,
(c) *$\|g_n\|_K \le 1/2^{n-2}$ if $n \ge 2$, and*
(d) *$F^1(f - \sum_{j=1}^n g_j, \delta) \subseteq F^1(f, \delta/2^n)$ if $0 < \delta \le 1/2^{n-2}$, $n \in \mathbb{N}$.*

Proof. The functions (g_n) are constructed inductively. By Lemma 3.1, there exists a continuous function $g_1 : K \setminus F^1(f,1) \to \mathbb{R}$ such that $\|f - g_1\|_{F\setminus F^1(f,1)} \le 1$. Extend g_1 to a function on K by defining it to be 0 on $F^1(f,1)$. Then (a) and (b) hold. Condition (c) holds vacuously. Moreover, if $x \in F \setminus F^1(f,\delta/2)$, $0 < \delta \le 2$, then there exists a neighborhood U_1 of x in F such that $|f(x_1) - f(x_2)| < \delta/2$ for all $x_1, x_2 \in U_1$. Note that since $x \in F \setminus F^1(f,\delta/2)$, g_1 is continuous at x. Hence there exists a neighborhood U_2 of x in F such that $|g_1(x_1) - g_1(x_2)| < \delta/2$ for all $x_1, x_2 \in U_2$. Let $U = U_1 \cap U_2$. Then U is a neighborhood of x in F. For all $x_1, x_2 \in U$,

$$|(f - g_1)(x_1) - (f - g_1)(x_2)| < \delta.$$

Hence $x \notin F^1(f - g_1, \delta)$. This proves (d).

Suppose that $g_1, \ldots, g_n$ have been chosen. By Lemma 3.1, there exists a continuous function $h : K \setminus F^1(f - \sum_{j=1}^n g_j, 1/2^n) \to \mathbb{R}$ such that

$$\Big\| f - \sum_{j=1}^n g_j - h \Big\|_{F\setminus F^1(f - \sum_{j=1}^n g_j, 1/2^n)} \le \frac{1}{2^n}.$$

Define $\widetilde{h}$ on $K \setminus F^1(f - \sum_{j=1}^n g_j, 1/2^n)$ by $\widetilde{h} = (h \wedge 1/2^{n-1}) \vee (-1/2^{n-1})$. Then $\widetilde{h}$ is continuous. By (d), $F^1(f - \sum_{j=1}^n g_j, 1/2^n) \subseteq F^1(f, 1/4^n)$. Hence $\widetilde{h}$ is defined and continuous on $K \setminus F^1(f, 1/4^n)$. Moreover, it follows from (b) that

$$\Big\| f - \sum_{j=1}^n g_j \Big\|_{F\setminus F^1(f,1/4^n)} \le \frac{1}{2^{n-1}}. \tag{3.1}$$

From inequality (3.1) and the definition of $\widetilde{h}$, we have

$$\Big\| f - \sum_{j=1}^n g_j - \widetilde{h} \Big\|_{F\setminus F^1(f,1/4^n)} \le \Big\| f - \sum_{j=1}^n g_j - h \Big\|_{F\setminus F^1(f,1/4^n)}.$$

Therefore, $\|f - \sum_{j=1}^n g_j - \widetilde{h}\|_{F\setminus F^1(f,1/4^n)} \le 1/2^n$. Now define

$$g_{n+1} = \begin{cases} \widetilde{h} & \text{on } K \setminus F^1(f - \sum_{j=1}^n g_j, 1/2^n), \\ 0 & \text{otherwise.} \end{cases}$$

Then g_{n+1} is continuous on $K \setminus F^1(f - \sum_{j=1}^n g_j, 1/2^n)$. This proves (a). Furthermore,

$$\Big\| f - \sum_{j=1}^{n+1} g_j \Big\|_{F\setminus F^1(f,1/4^n)} = \Big\| f - \sum_{j=1}^n g_j - \widetilde{h} \Big\|_{F\setminus F^1(f,1/4^n)} \le \frac{1}{2^n}.$$

This proves (b). Also,

$$\|g_{n+1}\|_K \le \|\widetilde{h}\|_{K\setminus F^1(f - \sum_{j=1}^n g_j, 1/2^n)} \le \frac{1}{2^{n-1}}$$

by the definition of $\widetilde{h}$. This proves (c). Finally, suppose $0 < \delta \le 1/2^{n-1}$. Assume that $x \in F \setminus F^1(f, \delta/2^{n+1})$. Then $x \notin F^1(f - \sum_{j=1}^n g_j, \delta/2)$. Thus there exists a neighborhood U_1 of x in F such that

$$\Big|\Big(f - \sum_{j=1}^{n} g_j\Big)(x_1) - \Big(f - \sum_{j=1}^{n} g_j\Big)(x_2)\Big| < \frac{\delta}{2}$$

whenever $x_1, x_2 \in U_1$. Note that since $x \in F \setminus F^1(f - \sum_{j=1}^n g_j, \delta/2)$, g_{n+1} is continuous at x. Therefore, there exists a neighborhood U_2 of x in F such that $|g_{n+1}(x_1) - g_{n+1}(x_2)| < \delta/2$ for all $x_1, x_2 \in U_2$. Let $U = U_1 \cap U_2$. Then U is a neighborhood of x in F such that

$$\Big|\Big(f - \sum_{j=1}^{n+1} g_j\Big)(x_1) - \Big(f - \sum_{j=1}^{n+1} g_j\Big)(x_2)\Big| < \delta$$

whenever $x_1, x_2 \in U$. Hence $x \notin F^1(f - \sum_{j=1}^{n+1} g_j, \delta)$. This proves (d). ∎

THEOREM 3.6. *Let F be a closed subspace of K and let f be a Baire-1 function on F. Then there exists a Baire-1 function g on K such that*

$$g_{|F} = f, \qquad \beta(g) = \beta_F(f).$$

Proof. Let (g_n) be the sequence given by Lemma 3.5. Define g on K by

$$g = \begin{cases} \sum_{j=1}^\infty g_j & \text{on } K \setminus F, \\ f & \text{on } F. \end{cases}$$

Note that by Lemma 3.5(c), $\sum_{j=1}^\infty g_j$ converges uniformly on K. Hence g is well defined. Obviously, $g_{|F} = f$.

CLAIM. $K^1(g, 1/2^{n-3}) \subseteq F^1(f, 1/4^n)$ *for all* $n \in \mathbb{N}$.

Proof. Let $x \in K \setminus F^1(f, 1/4^n)$. We consider two cases. Suppose $x \notin F$. By Lemma 3.5(a), g_j is continuous on $K \setminus F$ for all j. Since $\sum_{j=1}^\infty g_j$ converges uniformly to g on $K \setminus F$, and $K \setminus F$ is an open subset of K, it follows that g is continuous at x. Hence $x \notin K^1(g, 1/2^{n-3})$. Now suppose $x \in F$. Then $x \in F \setminus F^1(f, 1/4^n)$. There is a neighborhood U_1 of x in K such that $|f(x) - f(x')| < 1/4^n$ for all $x' \in U_1 \cap F$. Also, for $1 \le k \le n$,

$$\begin{aligned} F^1\Big(f - \sum_{j=1}^{k} g_j, 1/2^k\Big) &\subseteq F^1(f, 1/4^k) \quad \text{by Lemma 3.5(d)} \\ &\subseteq F^1(f, 1/4^n). \end{aligned}$$

Since g_{k+1} is continuous on $K \setminus F^1(f - \sum_{j=1}^k g_j, 1/2^k)$, it follows that g_{k+1} is continuous on $K \setminus F^1(f, 1/4^n)$ for all k, $1 \le k \le n$. Similarly, $F^1(f, 1) \subseteq F^1(f, 1/4^n)$ and g_1 is continuous on $K \setminus F^1(f, 1)$ by Lemma 3.5(a); thus, g_1 is continuous on $K \setminus F^1(f, 1/4^n)$. Hence there exists a neighborhood U_2 of

x in K such that $U_2 \subseteq K \setminus F^1(f, 1/4^n)$ and

$$\Big|\sum_{j=1}^{n+1} g_j(x') - \sum_{j=1}^{n+1} g_j(x)\Big| < 1/2^n \quad \text{for all } x' \in U_2.$$

Let $U = U_1 \cap U_2$. Then U is a neighborhood of x in K. If $x' \in U \cap F$, then $x' \in U_1 \cap F$. Thus $|g(x') - g(x)| = |f(x') - f(x)| < 1/4^n < 1/2^{n-2}$. If $x' \in U \setminus F$, then

$$\begin{aligned}
|g(x') - g(x)| & \\
&= \Big|\sum_{j=1}^{\infty} g_j(x') - f(x)\Big| \\
&\le \Big|\sum_{j=1}^{n+1} g_j(x') - \sum_{j=1}^{n+1} g_j(x)\Big| + \Big|\sum_{j=1}^{n+1} g_j(x) - f(x)\Big| + \Big|\sum_{j=n+2}^{\infty} g_j(x')\Big| \\
&< \frac{1}{2^n} + \Big|\sum_{j=1}^{n+1} g_j(x) - f(x)\Big| + \sum_{j=n+2}^{\infty} \|g_j\| \quad \text{since } x' \in U_2 \\
&\le \frac{1}{2^n} + |\sum_{j=1}^{n+1} g_j(x) - f(x)| + \sum_{j=n+2}^{\infty} \frac{1}{2^{j-2}} \quad \text{by Lemma 3.5(c)} \\
&\le \frac{1}{2^n} + \frac{1}{2^n} + \frac{1}{2^{n-1}} \quad \text{by Lemma 3.5(b) since } x \in F \setminus F^1(f, 1/4^n) \\
&= \frac{1}{2^{n-2}}.
\end{aligned}$$

Thus $|g(x') - g(x)| < 1/2^{n-2}$ if $x' \in U$. Hence $|g(x_1) - g(x_2)| < 1/2^{n-3}$ whenever $x_1, x_2 \in U$. Therefore $x \notin K^1(g, 1/2^{n-3})$. This proves the Claim.

It follows by induction that

$$K^\alpha(g, 1/2^{n-3}) \subseteq F^\alpha(f, 1/4^n) \quad \text{for } 1 \le \alpha < \omega_1.$$

Indeed, the Claim yields the assertion for $\alpha = 1$. If the inclusion holds for some α, $1 \le \alpha < \omega_1$, let $\widetilde{F} = F^\alpha(f, 1/4^n)$. Then $K^{\alpha+1}(g, 1/2^{n-3}) \subseteq \widetilde{F}^1(g, 1/2^{n-3}) = \widetilde{F}^1(f, 1/2^{n-3}) \subseteq \widetilde{F}^1(f, 1/4^n) = F^{\alpha+1}(f, 1/4^n)$. Hence the inclusion holds for $\alpha + 1$. If the inclusion holds for all $1 \le \alpha' < \alpha$, where $\alpha < \omega_1$ is a limit ordinal, then

$$K^\alpha(g, 1/2^{n-3}) = \bigcap_{1 \le \alpha' < \alpha} K^{\alpha'}(g, 1/2^{n-3}) \subseteq \bigcap_{1 \le \alpha' < \alpha} F^{\alpha'}(f, 1/4^n) = F^\alpha(f, 1/4^n).$$

This proves the inclusion for $1 \le \alpha < \omega_1$.

In particular, if $\beta_F(f) = \beta_0$, then $K^{\beta_0}(g, 1/2^{n-3}) \subseteq F^{\beta_0}(f, 1/4^n) = \emptyset$. Thus $\beta_K(g, 1/2^{n-3}) \le \beta_0$ for all $n \in \mathbb{N}$. Hence $\beta_K(g) \le \beta_0$. Of course, since $g_{|F} = f$, we have $\beta_K(g) \ge \beta_F(f) \ge \beta_0$. Therefore $\beta_K(g) = \beta_0 = \beta_F(f)$. ∎

REMARK 3.7. If $\beta_F(f) = 1$, Theorem 3.6 is the familiar Tietze Extension Theorem. If $\beta_F(f)$ is transfinite, the conclusion of Theorem 3.6 can be obtained easily by defining the extension g to be 0 on $K \setminus F$. However, we do not see a simple proof for finite $\beta_F(f)$.

4. Decomposition of Baire-1 functions. In this section, we give a proof of Theorem 1.2. The extension results in §3 are employed in the course of the proof.

THEOREM 4.1. *Let f be a Baire-1 function on K, $1 \le \beta_0, \gamma_0 < \omega_1$ and $\varepsilon > 0$. Then there exist*

$$\widetilde{f} : K \setminus K^{\beta_0 \cdot \gamma_0}(f, \varepsilon) \to \mathbb{R} \quad \text{and} \quad f_n : K \setminus K^{\beta_0 \cdot \gamma_0}(f, \varepsilon) \to \mathbb{R}$$

such that (f_n) converges to $\widetilde{f}$ pointwise, $\|\widetilde{f} - f\|_{K \setminus K^{\beta_0 \cdot \gamma_0}(f,\varepsilon)} \le \varepsilon$, $\beta_H(f_n) \le \beta_0$ and $\gamma_H((f_n)) \le \gamma_0$ for all compact subsets H of $K \setminus K^{\beta_0 \cdot \gamma_0}(f, \varepsilon)$.

Proof. For $\alpha \le \gamma_0$, let $K_\alpha = K^{\beta_0 \cdot \alpha}(f, \varepsilon)$. If $\alpha < \gamma_0$, it follows from Theorem 3.2 that there exists $g_\alpha : K_\alpha \setminus K_{\alpha+1} \to \mathbb{R}$ such that $\|g_\alpha - f\|_{K_\alpha \setminus K_{\alpha+1}} \le \varepsilon$ and $\beta_H(g_\alpha) \le \beta_0$ for all compact subsets H of $K_\alpha \setminus K_{\alpha+1}$. List the ordinals in $[0, \gamma_0)$ in a (possibly finite) sequence $(\alpha_n)_{n=1}^p$. Here $p \in \mathbb{N}$ or $p = \infty$. For all n and α, let U_n^α be the $1/n$-neighborhood of K_α in K and set $F_n = \bigcup_{j=1}^{n \wedge p} (K_{\alpha_j} \setminus U_n^{\alpha_j+1})$. Then F_n is a closed subset of K. It is also easy to see that $K_\alpha \setminus U_n^{\alpha+1}$ and $K_{\alpha'} \setminus U_n^{\alpha'+1}$ are disjoint if $\alpha \ne \alpha'$. Thus $(K_{\alpha_j} \setminus U_n^{\alpha_j+1})_{j=1}^{n \wedge p}$ is a partition of F_n into clopen (in F_n) subsets.

Now define $\widetilde{g}_n : F_n \to K$ to be $\bigcup_{j=1}^{n \wedge p} g_{\alpha_j | K_{\alpha_j} \setminus U_n^{\alpha_j+1}}$. Since $H = K_{\alpha_j} \setminus U_n^{\alpha_j+1}$ is a compact subset of $K_{\alpha_j} \setminus K_{\alpha_j+1}$, we have $\beta_H(g_{\alpha_j}) \le \beta_0$. As the partition $(K_{\alpha_j} \setminus U_n^{\alpha_j+1})_{j=1}^{n \wedge p}$ is clopen, it follows readily that $\beta_{F_n}(\widetilde{g}_n) \le \beta_0$. By Theorem 3.6, there exists a function f_n' on K such that $f'_{n|F_n} = \widetilde{g}_n$ and $\beta_K(f_n') \le \beta_0$.

Finally, define f_n to be $f'_{n|K \setminus K_{\gamma_0}}$ and $\widetilde{f}$ to be $\bigcup_{\alpha < \gamma_0} g_{\alpha | K_\alpha \setminus K_{\alpha+1}}$. It follows from the choice of the g_α's that $\|f - \widetilde{f}\|_{K \setminus K_{\gamma_0}} \le \varepsilon$. Since $\bigcup_{n=1}^\infty F_n = K \setminus K_{\gamma_0}$ and the sets F_n are increasing, $\lim f_n = \widetilde{f}$ pointwise on $K \setminus K_{\gamma_0}$. Suppose H is a compact subset of $K \setminus K_{\gamma_0}$. Then $\beta_H(f_n) \le \beta_K(f_n') \le \beta_0$.

To complete the proof, we claim that

$$H^\gamma((f_n), \delta) \subseteq K_\gamma$$

for any $\delta > 0$ and any $\gamma \le \gamma_0$. This is proved by induction on γ. The case $\gamma = 0$ and the limit case are trivial. Now assume that the claim holds for some $\gamma < \gamma_0$. Let $x \in H^\gamma((f_n), \delta) \setminus K_{\gamma+1}$. Choose $j_1, j_2 \in \mathbb{N}$ such that $\alpha_{j_1} = \gamma$ and $d(x, K_{\gamma+1}) \ge 1/j_2$, where d is the metric on K. Denote $H^\gamma((f_n), \delta)$ by L and the $1/(2j_0)$-ball in K centered at x by U, where $j_0 = \max\{j_1, 2j_2\}$. Note that $L \subseteq K_\gamma$ by the inductive hypothesis. For all $n \geq j_0 = \max\{j_1, 2j_2\}$,

$$L \cap U \subseteq L \cap \overline{U} \subseteq K_{\alpha_{j_1}} \setminus U_n^{\alpha_{j_1}+1} \subseteq F_n.$$

This implies that $f_{n|L\cap\overline{U}} = \widetilde{g}_{n|L\cap\overline{U}} = g_{\alpha_{j_1}|L\cap\overline{U}} = g_{\gamma|L\cap\overline{U}}$ for all $n \geq j_0$. Thus $(L \cap \overline{U})^1((f_n), \delta) = \emptyset$. By Lemma 2.1(d),

$$L^1((f_n), \delta) \cap (L \cap U) = \emptyset.$$

In particular,

$$x \notin L^1((f_n), \delta) = H^{\gamma+1}((f_n), \delta).$$

Since $x \in H^\gamma((f_n), \delta) \setminus K_{\gamma+1}$ is arbitrary, this shows that $H^{\gamma+1}((f_n), \delta) \subseteq K_{\gamma+1}$. ∎

In particular, if $\beta_K(f) \leq \beta_0 \cdot \gamma_0$, we have the following.

THEOREM 4.2. *Let f be a Baire-1 function on K, $1 \leq \beta_0, \gamma_0 < \omega_1$, and $\beta(f) \leq \beta_0 \cdot \gamma_0$. For any $\varepsilon > 0$, there exist $\widetilde{f} : K \to \mathbb{R}$ and a sequence of functions $f_n : K \to \mathbb{R}$ such that (f_n) converges to $\widetilde{f}$ pointwise, $\|\widetilde{f} - f\| \leq \varepsilon$, $\beta(f_n) \leq \beta_0$ for all $n \in \mathbb{N}$, and $\gamma((f_n)) \leq \gamma_0$.*

A couple more preparatory steps will allow us to improve Theorem 4.2 to an exact result (i.e., with $\varepsilon = 0$) when γ_0 is of the right form.

THEOREM 4.3 ([3, Lemma 2.5]). *If (f_n) and (g_n) are two sequences of real-valued functions on K such that $\gamma((f_n)) \leq \omega^\xi$ and $\gamma((g_n)) \leq \omega^\xi$ for some $\xi < \omega_1$, then $\gamma((f_n + g_n)) \leq \omega^\xi$.*

PROPOSITION 4.4. *For $1 \leq \xi < \omega_1$, $\mathfrak{B}_1^\xi(K) = \{f \in \mathbb{R}^K : \beta(f) \leq \omega^\xi\}$ is a vector subspace of $\mathbb{R}^K$ that is closed under the topology of uniform convergence.*

We postpone the proof of Proposition 4.4 until the next section. We are now ready to prove the converse of Theorem 2.3 in certain cases.

THEOREM 4.5. *If $f \in \mathfrak{B}_1(K)$ and $\beta(f) \leq \beta_0 \cdot \omega^{\gamma_0}$ for some $1 \leq \beta_0 < \omega_1$ and $\gamma_0 < \omega_1$, then there exists $(f_n) \subseteq \mathfrak{B}_1(K)$ such that (f_n) converges pointwise to f, $\beta(f_n) \leq \beta_0$ for all $n \in \mathbb{N}$ and $\gamma((f_n)) \leq \omega^{\gamma_0}$.*

Proof. First we assume β_0 is of the form ω^{α_0}, where $\alpha_0 < \omega_1$. By Theorem 4.2 there exist a sequence $(f_n^1) \subseteq \mathfrak{B}_1(K)$ and a function $f^1 \in \mathfrak{B}_1(K)$ such that $\beta(f_n^1) \leq \omega^{\alpha_0}$ for all n, (f_n^1) converges pointwise to f^1, $\|f^1 - f\| \leq 1/2$, and $\gamma((f_n^1)) \leq \omega^{\gamma_0}$. Then $\beta(f^1) \leq \omega^{\alpha_0} \cdot \omega^{\gamma_0} = \omega^{\alpha_0+\gamma_0}$ by Theorem 2.3. This implies that $\beta(f - f^1) \leq \omega^{\alpha_0+\gamma_0}$ by Proposition 4.4. Hence there exist $(f_n^2) \subseteq \mathfrak{B}_1(K)$ and f^2 such that $\beta(f_n^2) \leq \omega^{\alpha_0}$ for all $n \in \mathbb{N}$, (f_n^2) converges pointwise to f^2, $\|f - f^1 - f^2\| \leq 1/2^2$, and $\gamma((f_n^2)) \leq \omega^{\gamma_0}$. We may assume that $\|f_n^2\| \leq 1/2$ for all $n \in \mathbb{N}$, for otherwise, simply replace

f_n^2 by $\widehat{f}_n^2 = (f_n^2 \vee (-1/2)) \wedge 1/2$. Continuing, we obtain f^m and $(f_n^m)_{n=1}^\infty$ for each m such that

- $\|f_n^m\| \le 1/2^{m-1}$, $m \ge 2$,
- $\beta(f_n^m) \le \omega^{\alpha_0}$ for all $m, n \in \mathbb{N}$,
- $\gamma((f_n^m)_n) \le \omega^{\gamma_0}$ for all $m \in \mathbb{N}$,
- $f^m = \lim_n f_n^m$ (pointwise) for all $m \in \mathbb{N}$,
- $\sum_{m=1}^\infty f^m$ converges uniformly to f on K.

Let $g_n^m = f_n^1 + f_n^2 + \ldots + f_n^m$ and $g_n = \sum_{m=1}^\infty f_n^m$. By Theorem 4.3, $\gamma((g_n^m)_n) \le \omega^{\gamma_0}$ for all $m \in \mathbb{N}$. Given $\varepsilon > 0$, there exists m_0 such that $\|g_n^{m_0} - g_n\| \le \varepsilon$ for all $n \in \mathbb{N}$. Then $K^{\omega^{\gamma_0}}((g_n), 3\varepsilon) \subseteq K^{\omega^{\gamma_0}}((g_n^{m_0}), \varepsilon) = \emptyset$. Therefore $\gamma((g_n)) \le \omega^{\gamma_0}$. By Proposition 4.4, $\beta(g_n^m) \le \omega^{\alpha_0}$ for all m, n. Therefore, $\beta(g_n) \le \omega^{\alpha_0}$ by Proposition 4.4. Moreover,

$$\lim_n g_n = \lim_n \lim_m g_n^m = \lim_m \lim_n g_n^m$$
$$= \lim_m \sum_{k=1}^m f^k = f \quad \text{pointwise.}$$

This proves the theorem in case $\beta_0 = \omega^{\alpha_0}$, with (g_n) in place of (f_n).

For a general nonzero countable ordinal β_0, write β_0 in Cantor normal form as

$$\beta_0 = \omega^{\beta_1} \cdot m_1 + \omega^{\beta_2} \cdot m_2 + \ldots + \omega^{\beta_k} \cdot m_k,$$

where $k, m_1, \ldots, m_k \in \mathbb{N}$, $\omega_1 > \beta_1 > \ldots > \beta_k$. If $\gamma_0 \ne 0$, then $\beta_0 \cdot \omega^{\gamma_0} = \omega^{\beta_1} \cdot \omega^{\gamma_0}$. By the previous case, there exists $(f_n) \subseteq \mathfrak{B}_1(K)$ such that $\beta(f_n) \le \omega^{\beta_1} \le \beta_0$, $\gamma((f_n)) \le \omega^{\gamma_0}$ and (f_n) converges pointwise to f. If $\gamma_0 = 0$, take $f_n = f$ for all n. Then $\beta(f_n) \le \beta_0$ for all n, $\gamma((f_n)) = 1 = \omega^{\gamma_0}$ and (f_n) converges pointwise to f. ∎

The combination of Theorem 2.3 and Corollary 4.6 yields Theorem 1.2. It is clear from the proof of Theorem 4.5 that the sequence (f_n) in the statement of Theorem 4.5 may be chosen to be bounded if f is bounded.

COROLLARY 4.6. *Let $f \in \mathfrak{B}_1^\xi(K)$, respectively, $\mathcal{B}_1^\xi(K)$, for some $\xi < \omega_1$. For all countable ordinals μ, ν such that $\mu + \nu \ge \xi$, there exists a sequence $(f_n) \subseteq \mathfrak{B}_1^\mu(K)$, respectively, a bounded sequence $(f_n) \subseteq \mathcal{B}_1^\mu(K)$, such that $f_n \to f$ pointwise and $\gamma((f_n)) \le \omega^\nu$.*

We do not know if Theorem 4.5 holds without the restriction on the form of the ordinal $\gamma((f_n))$.

PROBLEM 4.7. *Is it true that if $f \in \mathfrak{B}_1(K)$ with $\beta(f) \le \beta_0 \cdot \gamma_0$ for some countable ordinals β_0 and γ_0, then there exists a sequence (f_n) converging pointwise to f so that* $\sup_n \beta(f_n) \le \beta_0$ *and $\gamma((f_n)) \le \gamma_0$?*

As another application of our results, we give the proof of another characterization of the classes $\mathcal{B}_1^\xi(K)$ due to Kechris and Louveau.

Definition 4.8 ([3, Section 4]). A family $\{\Phi_\xi : 0 \le \xi < \omega_1\}$ of real-valued functions on K is defined as follows:

$$\Phi_0 = C(K),$$

$$\Phi_{\xi+1} = \{f : f \text{ is the pointwise limit of a bounded sequence } (f_n) \subseteq \Phi_\xi \text{ such that } \gamma((f_n)) \le \omega\},$$

and for limit ordinals λ,

$$\Phi_\lambda = \{f : f \text{ is the uniform limit of a bounded sequence } (f_n) \subseteq \textstyle\bigcup_{\xi<\lambda} \Phi_\xi\}.$$

Corollary 4.9 ([3, Theorem 4.2]). *For each $\xi < \omega_1$, $\mathcal{B}_1^\xi(K) = \Phi_\xi$.*

Proof. The case $\xi = 0$ is trivial. Suppose the assertion holds for some $\xi < \omega_1$. If $f \in \mathcal{B}_1^{\xi+1}(K)$, it follows from Corollary 4.6 that f is the pointwise limit of a bounded sequence (f_n) in $\mathcal{B}_1^\xi(K)$ such that $\gamma((f_n)) \le \omega$. Since $\mathcal{B}_1^\xi(K) = \Phi_\xi$ by the inductive hypothesis, $f \in \Phi_{\xi+1}$. Conversely, if $f \in \Phi_{\xi+1}$, then f is the pointwise limit of a sequence (f_n) in Φ_ξ with $\gamma((f_n)) \le \omega$. Since $\Phi_\xi = \mathcal{B}_1^\xi(K)$, it follows that $\beta(f) \le \omega^{\xi+1}$ by Theorem 2.3. Thus $f \in \mathcal{B}_1^{\xi+1}(K)$.

Now assume that the assertion holds for all $\xi' < \xi$, where ξ is a countable limit ordinal. Let $f \in \Phi_\xi$. By the inductive hypothesis, $\Phi_{\xi'} = \mathcal{B}_1^{\xi'}(K) \subseteq \mathcal{B}_1^\xi(K)$ for $\xi' < \xi$. Hence f is the uniform limit of a sequence in $\mathcal{B}_1^\xi(K)$, and thus belongs to $\mathcal{B}_1^\xi(K)$. Conversely, assume that $f \in \mathcal{B}_1^\xi(K)$. For every $n \in \mathbb{N}$, there exists $\xi_n < \xi$ such that $\beta(f, 1/n) \le \omega^{\xi_n}$. By Corollary 3.3, there exists $f_n \in \mathcal{B}_1^{\xi_n}(K) = \Phi_{\xi_n}$ such that $\|f - f_n\| \le 1/n$. Thus $f \in \Phi_\xi$, as required. ∎

Remark 4.10. If a family $\{\Psi_\xi : 0 \le \xi < \omega_1\}$ is defined in a similar way to $\{\Phi_\xi : 0 \le \xi < \omega_1\}$ except that the boundedness condition on the sequence (f_n) is removed, then $\Psi_\xi = \mathfrak{B}_1^\xi(K)$ for all $\xi < \omega_1$.

5. Optimal limit of continuous functions. In this section we prove the equivalence of the indices β and γ for functions in $\mathfrak{B}_1(K)$ in the same sense that was established for $\mathcal{B}_1(K)$ in Theorem 2.3 of [3]. Namely, it is shown that for all $f \in \mathfrak{B}_1(K)$, $\beta(f)$ is the smallest ordinal γ_0 for which there exists a sequence (f_n) in $C(K)$ converging pointwise to f and satisfying $\gamma((f_n)) \le \gamma_0$. Note that this result is also the converse of Theorem 2.3 when $\beta_0 = 1$.

Definition 5.1. Let $(f_n) \subseteq \mathbb{R}^K$ and $f \in \mathbb{R}^K$. We write:

(a) $(g_n) \prec (f_n)$ if (g_n) is a convex block combination of (f_n), i.e., there exist a sequence of nonnegative real numbers (a_k) and a strictly increasing sequence (p_n) in $\mathbb{N}$ such that $\sum_{k=p_{n-1}+1}^{p_n} a_k = 1$ and $g_n = \sum_{k=p_{n-1}+1}^{p_n} a_k f_k$ for all n ($p_0 = 0$),

(b) $(g_n) \overset{a}{\prec} (f_n)$ if there exists $m \in \mathbb{N}$ such that $(g_n)_{n=m}^\infty \prec (f_n)$,

(c) $[f]_{-M}^{M} = (f \vee -M) \wedge M$, where $0 \le M \in \mathbb{R}$.

The easy proof of the next lemma is left to the reader.

LEMMA 5.2. *If* $(g_n) \overset{a}{\prec} (f_n)$, *then* $\gamma((g_n), \varepsilon) \le \gamma((f_n), \varepsilon)$ *for all* $\varepsilon > 0$.

LEMMA 5.3. *Let f be a Baire-1 function on K. Suppose $\mathcal{H}$ is a countable collection of compact subsets of K such that $\|f\|_H < \infty$ for all $H \in \mathcal{H}$ and $\bigcup_{H\in\mathcal{H}} H = K$. Then there exists $(f_n) \subseteq C(K)$ such that*

(i) $f_n \to f$ *pointwise*,

(ii) $(f_{n|H})$ *is a bounded subset of* $C(H)$ *for all* $H \in \mathcal{H}$.

Proof. Write $\mathcal{H}$ as a sequence $(H_m)_{m=1}^\infty$. Without loss of generality, assume that $H_m \subseteq H_{m+1}$ for all $m \in \mathbb{N}$. Since f is Baire-1, there exists $(f_n^0) \subseteq C(K)$ such that (f_n^0) converges pointwise to f. Assume that $(f_n^{m-1})_n \subseteq C(K)$ has been chosen so that $\lim_n f_n^{m-1} = f$ pointwise. If $m, n \in \mathbb{N}$, let U_n^m be the $1/n$-neighborhood of H_m in K and let $M_m = \|f\|_{H_m}$. For all n, the function $[f_n^{m-1}]_{-M_m|H_m}^{M_m} \cup f_{n|K\setminus U_n^m}^{m-1}$ is continuous on $H_m \cup (K \setminus U_n^m)$. Let f_n^m be its continuous extension onto K. If $x \in H_m$, then $\lim_n f_n^m(x) = \lim_n [f_n^{m-1}(x)]_{-M_m}^{M_m} = [f(x)]_{-M_m}^{M_m} = f(x)$ since $\|f\|_{H_m} = M_m$. If $x \notin H_m$, then there exists n_0 such that $x \in K \setminus U_{n_0}^m$; thus $x \in K \setminus U_n^m$ for all $n \ge n_0$. Therefore $f_n^m(x) = f_n^{m-1}(x)$ for all $n \ge n_0$. Hence $\lim_n f_n^m(x) = f(x)$. Thus $\lim_n f_n^m = f$ pointwise. Now for each $n \in \mathbb{N}$, let $f_n = f_n^n$. Since $H_m \subseteq H_n$ for all $n \ge m$, on H_m we have

$$\begin{aligned} f_n = f_n^n &= [f_n^{n-1}]_{-M_n}^{M_n} \\ &= [[f_n^{n-2}]_{-M_{n-1}}^{M_{n-1}}]_{-M_n}^{M_n} = \ldots = [\ldots[[f_n^{m-1}]_{-M_m}^{M_m}]_{-M_{m+1}}^{M_{m+1}} \ldots]_{-M_n}^{M_n} \\ &= [f_n^{m-1}]_{-M_m}^{M_m} \quad \text{as } M_m \le M_{m+1} \le \ldots \le M_n. \end{aligned}$$

Thus $f_n = [f_n^{m-1}]_{-M_m}^{M_m}$ on H_m for all $n \ge m$. In particular, on the set H_m,

$$\lim_n f_n = [\lim_n f_n^{m-1}]_{-M_m}^{M_m} = [f]_{-M_m}^{M_m} = f$$

since $\|f\|_{H_m} = M_m$. As $K = \bigcup H_m$, we see that $f_n \to f$ pointwise. Also, for each m, $(f_{n|H_m})_{n=m}^\infty$ is bounded (by M_m) in $C(H_m)$; thus $(f_{n|H_m})_{n=1}^\infty$ is bounded in $C(H_m)$. ∎

For the next lemma, recall that for a real-valued function f defined on a set S, $\operatorname{osc}(f, S) = \sup\{|f(s_1) - f(s_2)| : s_1, s_2 \in S\}$.

LEMMA 5.4. *Let (f_n) be bounded in $C(H)$, where H is a compact metric space. Suppose (f_n) converges pointwise to f and $H^1(f, \varepsilon) = \emptyset$ for some $\varepsilon > 0$. Then there exists $(g_n) \prec (f_n)$ such that $H^1((g_n), 7\varepsilon) = \emptyset$.*

Proof. By Corollary 3.3, there exists $\widetilde{f} \in C(H)$ such that $\|f - \widetilde{f}\|_H \le \varepsilon$. Then $(f_n - \widetilde{f})$ is bounded in $C(H)$, $f_n - \widetilde{f} \to f - \widetilde{f}$ pointwise and $\operatorname{osc}(f - \widetilde{f}, H) \le 2\varepsilon$. By the first statement in the proof of Theorem 2.3 in [3], there exists $(h_n) \prec (f_n - \widetilde{f})$ such that $\|h_n - (f - \widetilde{f})\|_H \le 3\varepsilon$. Let $g_n = h_n + \widetilde{f}$ for all $n \in \mathbb{N}$. Then $(g_n) \prec (f_n)$ and $\|g_n - f\|_H \le 3\varepsilon$ for all $n \in \mathbb{N}$. It follows that $H^1((g_n), 7\varepsilon) = \emptyset$. ∎

THEOREM 5.5. *Let f be a Baire-1 function on K. There exists a sequence $(f_n) \subseteq C(K)$ such that (f_n) converges pointwise to f and $\gamma((f_n)) = \beta(f)$.*

Proof. Let $\beta_0 = \beta(f)$. For each $\alpha < \beta_0$, and all $m, j \in \mathbb{N}$, let $U^{\alpha}_{m,j}$ be the $1/j$-neighborhood of $K^{\alpha}(f, 1/m)$ in K. Define

$$\mathcal{H} = \{K^{\alpha}(f, 1/m) \setminus U^{\alpha+1}_{m,j} : \alpha < \beta_0,\ m, j \in \mathbb{N}\}.$$

Then $\mathcal{H}$ is a countable collection of compact subsets of K such that $\bigcup_{H \in \mathcal{H}} H = K$. If $\alpha < \beta_0$ and $m, j \in \mathbb{N}$, by Lemma 3.1, there is a continuous function g on $H = K^{\alpha}(f, 1/m) \setminus U^{\alpha+1}_{m,j}$ such that $\|g - f\|_H \le 1/m$. Hence $\|f\|_H < \infty$ for all $H \in \mathcal{H}$. By Lemma 5.3, there exists $(g_n) \subseteq C(K)$ such that (g_n) converges pointwise to f and $(g_{n|H})$ is bounded in $C(H)$ for all $H \in \mathcal{H}$.

List the elements of $\mathcal{H}$ in a sequence $(H_k)_{k=1}^{\infty}$. Take $\varepsilon_k = 1/m$ if H_k is of the form $K^{\alpha}(f, 1/m) \setminus U^{\alpha+1}_{m,j}$ for some α, m, j. Let $(g^0_n) = (g_n)$. Suppose $(g^{k-1}_n)_n \prec (g_n)_n$ has been chosen. Then $(g^{k-1}_n)_n$ converges to f pointwise, $(g^{k-1}_{n|H_k})$ is a bounded sequence in $C(H_k)$, and $(H_k)^1(f, \varepsilon_k) = \emptyset$. By Lemma 5.4, there exists $(g^k_n)_n \prec (g^{k-1}_n)_n$ such that $(H_k)^1((g^k_n)_n, 7\varepsilon_k) = \emptyset$. Let $f_n = g^n_n$ for all $n \in \mathbb{N}$. Then $(f_n) \prec (g_n)$. Therefore $(f_n) \subseteq C(K)$ and (f_n) converges pointwise to f.

We claim that for all $m \in \mathbb{N}$ and for all $\alpha \le \beta_0$,

$$K^{\alpha}((f_n), 7/m) \subseteq K^{\alpha}(f, 1/m).$$

We prove the claim by induction on α. The claim is trivial if $\alpha = 0$ or α is a limit ordinal. Assume that $\alpha \le \beta_0$ is a successor ordinal and that the claim holds for $\alpha - 1$. Let $x \in K^{\alpha}((f_n), 7/m)$. Then $x \in K^{\alpha-1}((f_n), 7/m) \subseteq K^{\alpha-1}(f, 1/m)$. If $x \notin K^{\alpha}(f, 1/m)$, then there exists $j \in \mathbb{N}$ such that $d(x, K^{\alpha}(f, 1/m)) > 1/j$. Choose k such that $H_k = K^{\alpha-1}(f, 1/m) \setminus U^{\alpha}_{m,j}$. Then $(f_n) \overset{a}{\prec} (g^k_n)_n$ and $\gamma_{H_k}((g^k_n)_n, 7\varepsilon_k) \le 1$ since $(H_k)^1((g^k_n)_n, 7\varepsilon_k) = \emptyset$. By Lemma 5.2, $(H_k)^1((f_n), 7\varepsilon_k) = \emptyset$. Thus $(H_k)^1((f_n), 7/m) = \emptyset$. But since $d(x, K^{\alpha}(f, 1/m)) > 1/j$, there exists an open set U in $\widetilde{K} = K^{\alpha-1}(f, 1/m)$ such that $x \in U \subseteq H_k \subseteq \widetilde{K}$. By Lemma 2.1(d), $(\widetilde{K})^1((f_n), 7/m) \cap U \subseteq (H_k)^1((f_n), 7/m) = \emptyset$. Therefore $x \notin (\widetilde{K})^1((f_n), 7/m) \supseteq K^{\alpha}((f_n), 7/m)$, a contradiction. This proves the claim.

From the claim, $K^{\beta_0}((f_n), 7/m) \subseteq K^{\beta_0}(f, 1/m) = \emptyset$ for all $m \in \mathbb{N}$. Therefore $\gamma((f_n)) \le \beta_0$. Since $\gamma((f_n)) \ge \beta_0$ by [3, Proposition 2.1] (or Theorem 2.3), $\gamma((f_n)) = \beta_0 = \beta(f)$. ∎

REMARK 5.6. Theorem 2.3 of [3] actually implies that if (f_n) is a bounded sequence in $C(K)$ converging pointwise to some $f \in \mathcal{B}_1(K)$, then there exists $(g_n) \prec (f_n)$ such that $\gamma((g_n)) = \beta(f)$. This does not hold in general for unbounded sequences of functions. Indeed, let $K = [0,1]$ and for each $n \in \mathbb{N}$ let f_n be a continuous function that vanishes outside $[1/(n+1), 1/n]$ such that $\int_K f_n = 1$. Then (f_n) converges pointwise to $f = 0$. If $(g_n) \prec (f_n)$, then $\int_K g_n = 1$ for all $n \in \mathbb{N}$. Thus (g_n) does not converge uniformly to f, i.e., $\gamma((g_n)) > 1 = \beta(f)$.

Proof of Proposition 4.4. It is easy to see that $af \in \mathfrak{B}_1^{\xi}(K)$ for all $f \in \mathfrak{B}_1^{\xi}(K)$ and $a \in \mathbb{R}$. If $f, g \in \mathfrak{B}_1^{\xi}(K)$, then by Theorem 5.5 there exist two sequences (f_n) and (g_n) of continuous functions converging pointwise to f and g respectively such that $\gamma((f_n)) \le \omega^{\xi}$ and $\gamma((g_n)) \le \omega^{\xi}$. According to Theorem 4.3, $\gamma((f_n + g_n)) \le \omega^{\xi}$. Hence by Theorem 2.3, $f + g \in \mathfrak{B}_1^{\xi}(K)$. Finally, given $f \in \overline{\mathfrak{B}_1^{\xi}(K)}$ and $\varepsilon > 0$, choose $g \in \mathfrak{B}_1^{\xi}(K)$ such that $\|f - g\| \le \varepsilon/3$. Then $K^{\omega^{\xi}}(f, \varepsilon) \subseteq K^{\omega^{\xi}}(g, \varepsilon/3) = \emptyset$. Thus $f \in \mathfrak{B}_1^{\xi}(K)$. ∎

6. Product of Baire-1 functions. In [3], it is observed that the classes $\mathcal{B}_1^{\xi}(K)$, $\xi < \omega_1$, are closed under multiplication. However, it is relatively easy to see that this fails for the classes $\mathfrak{B}_1^{\xi}(K)$. In this section, we show that if $f \in \mathfrak{B}_1^{\xi_1}(K)$ and $g \in \mathfrak{B}_1^{\xi_2}(K)$, then $fg \in \mathfrak{B}_1^{\xi}(K)$, where $\xi = \max\{\xi_1 + \xi_2, \xi_2 + \xi_1\}$. It is also shown that the result is sharp. The proof of the next lemma is left to the reader.

LEMMA 6.1. *If f is bounded and $\gamma((g_n)) \le \xi$, then $\gamma((fg_n)) \le \xi$.*

LEMMA 6.2. *If $f \in \mathcal{B}_1^{\xi_1}(K)$ and $g \in \mathfrak{B}_1^{\xi_2}(K)$, then $fg \in \mathfrak{B}_1^{\xi_1+\xi_2}(K)$.*

Proof. By Theorem 5.5, there exists a sequence $(g_n) \subseteq C(K)$ converging to g pointwise such that $\gamma((g_n)) \le \omega^{\xi_2}$. For each $n \in \mathbb{N}$, $g_n \in C(K) \subseteq \mathcal{B}_1^{\xi_1}(K)$ and $f \in \mathcal{B}_1^{\xi_1}(K)$. By [3] (see the remark in [3, p. 217]), we have $fg_n \in \mathcal{B}_1^{\xi_1}(K)$. Lemma 6.1 implies that $\gamma((fg_n)) \le \omega^{\xi_2}$. Since (fg_n) converges to fg pointwise, it follows from Theorem 2.3 that $\beta(fg) \le \omega^{\xi_1+\xi_2}$, i.e., $fg \in \mathfrak{B}_1^{\xi_1+\xi_2}(K)$. ∎

Now suppose $f \in \mathfrak{B}_1^{\xi_1}(K)$ and $g \in \mathfrak{B}_1^{\xi_2}(K)$. By Lemma 3.1, for all $\alpha < \omega^{\xi_2}$, there is a continuous function $g_\alpha : K^{\alpha}(g,1) \setminus K^{\alpha+1}(g,1) \to \mathbb{R}$ such that

$$\|g_\alpha - g\|_{K^{\alpha}(g,1) \setminus K^{\alpha+1}(g,1)} \le 1.$$

Let $h = \bigcup_{\alpha<\omega^{\xi_2}} g_\alpha$. It follows from the proof of Theorem 3.2 that $\beta(h) \le \omega^{\xi_2}$. Given a closed set $H \subseteq K$, we write

$$\mathrm{d}_f(H) = \{x \in H : \limsup_{\substack{y\to x\\ y\in H}} |f(y)| = \infty\}.$$

It is easy to see that $\mathrm{d}_f(H)$ is a closed subset of H such that $\mathrm{d}_f(H) \subseteq H^1(f,\varepsilon)$ for any $\varepsilon > 0$.

Lemma 6.3. *Suppose that $\alpha < \omega_1$, $\delta > 0$ and $s > 2$. If $x \in [K\setminus K^1(g,1)] \cap K^\alpha(fh,\delta)$, then*

$$x \in K^\alpha\biggl(f, \frac{\delta}{s(|h(x)|+1)} \wedge 1\biggr).$$

Proof. The proof is by induction on α. For brevity, set

$$K^\alpha_{s,x} = K^\alpha\biggl(f, \frac{\delta}{s(|h(x)|+1)} \wedge 1\biggr).$$

The result is clear if $\alpha = 0$ or a limit ordinal. Assume that the lemma holds for some $\alpha < \omega_1$. Suppose $\delta > 0$ and $s > 2$ are given. Let $x \in [K \setminus K^1(g,1)] \cap K^{\alpha+1}(fh,\delta)$. If $x \in \mathrm{d}_f(K^\alpha_{s,x})$, then $x \in K^{\alpha+1}_{s,x}$ and we are done. Otherwise, assume that $x \notin \mathrm{d}_f(K^\alpha_{s,x})$. Then there exist a neighborhood U_1 of x in K and $M < \infty$ such that $|f(y)| \le M$ for all $y \in U_1 \cap K^\alpha_{s,x}$. Since $h = g_0$ on $K \setminus K^1(g,1)$, and g_0 is continuous on $K \setminus K^1(g,1)$, there exists a neighborhood U_2 of x in K such that $|h(x_1) - h(x_2)| \le \delta/(2M)$ and $2(|h(x_1)|+1) < s(|h(x)|+1)$ for all $x_1, x_2 \in U_2$. Set $U = (U_1 \cap U_2) \setminus K^1(g,1)$. Then U is a neighborhood of x.

Claim. $K^\alpha(fh,\delta) \cap U \subseteq K^\alpha_{s,x}$.

To see this, note that if $y \in U$, then $y \in U_2$. Hence there exists $t > 2$ such that $t(|h(y)|+1) \le s(|h(x)|+1)$. Also, $y \in K^\alpha(fh,\delta) \cap U$ implies that $y \in [K \setminus K^1(g,1)] \cap K^\alpha(fh,\delta)$. Thus $y \in K^\alpha_{t,y}$ by the inductive hypothesis. Since

$$\frac{\delta}{t(|h(y)|+1)} \wedge 1 \ge \frac{\delta}{s(|h(x)|+1)} \wedge 1,$$

we have $y \in K^\alpha_{s,x}$, as required.

Now if V is a neighborhood of x in K, then there exist $x_1, x_2 \in U \cap V \cap K^\alpha(fh,\delta)$ such that

$$\begin{aligned}\delta &\le |f(x_1)h(x_1) - f(x_2)h(x_2)|\\ &\le |f(x_1) - f(x_2)|\,|h(x_1)| + |h(x_1) - h(x_2)|\,|f(x_2)|\\ &\le |f(x_1) - f(x_2)|\,|h(x_1)| + \frac{\delta}{2M}\cdot M,\end{aligned}$$

where, in the last inequality, $|f(x_2)| \le M$ since $x_2 \in U \cap K^\alpha_{s,x}$ by the claim. Therefore,

$$|f(x_1) - f(x_2)| \geq \frac{\delta}{s(|h(x)|+1)} \wedge 1.$$

By the claim, $x_1, x_2 \in V \cap K^{\alpha}_{s,x}$. Since V is an arbitrary neighborhood of x, this shows that $x \in K^{\alpha+1}_{s,x}$ and completes the induction. ∎

It follows from Lemma 6.3 that

$$K^{\omega^{\xi_1}}(fh, \delta) \subseteq K^1(g, 1).$$

Repeating the argument in Lemma 6.3 inductively yields

LEMMA 6.4. *$K^{\omega^{\xi_1}\cdot\alpha}(fh,\delta) \subseteq K^{\alpha}(g,1)$ for all $\alpha < \omega_1$ and $\delta > 0$.*

In particular, $K^{\omega^{\xi_1}\cdot\omega^{\xi_2}}(fh,\delta) = \emptyset$ for all $\delta > 0$, i.e., $fh \in \mathfrak{B}_1^{\xi_1+\xi_2}(K)$.

THEOREM 6.5. *If $f \in \mathfrak{B}_1^{\xi_1}(K)$ and $g \in \mathfrak{B}_1^{\xi_2}(K)$, then $fg \in \mathfrak{B}_1^{\xi}(K)$, where $\xi = \max\{\xi_1+\xi_2,\ \xi_2+\xi_1\}$.*

Proof. From the above, we obtain a function h on K such that $\|g-h\| \leq 1$, $\beta(h) \leq \omega^{\xi_2}$ and $fh \in \mathfrak{B}_1^{\xi_1+\xi_2}(K)$. Since $g, h \in \mathfrak{B}_1^{\xi_2}(K)$, it follows from Proposition 4.4 that $g - h \in \mathfrak{B}_1^{\xi_2}(K)$. As $g-h$ is bounded, we see that $g - h \in \mathcal{B}_1^{\xi_2}(K)$. By Lemma 6.2, $(g-h)f \in \mathfrak{B}_1^{\xi_2+\xi_1}(K) \subseteq \mathfrak{B}_1^{\xi}(K)$. Also, $fh \in \mathfrak{B}_1^{\xi_1+\xi_2}(K) \subseteq \mathfrak{B}_1^{\xi}(K)$. Applying Proposition 4.4 again gives $fg = f(g-h) + fh \in \mathfrak{B}_1^{\xi}(K)$. ∎

Theorem 6.9 below shows that the result is sharp. First we show a strong result in this direction on spaces of ordinals. For $\alpha < \omega_1$, denote the ordinal interval $[0, \omega^{\alpha}]$ by I_{α}.

PROPOSITION 6.6. *Let $0 < \alpha, \xi_1 < \omega_1$, and $h : I_{\omega^{\xi_1}\cdot\alpha} \to \mathbb{R}$ be a bounded function. Then there exist $f, g : I_{\omega^{\xi_1}\cdot\alpha} \to \mathbb{R}$ so that*

(a) $\beta(f) \leq \omega^{\xi_1}$,
(b) *g takes values in $\mathbb{N}$,*
(c) $g(\omega^{\omega^{\xi_1}\cdot\alpha}) = 1$,
(d) $(I_{\omega^{\xi_1}\cdot\alpha})^{\alpha}(g,\varepsilon) \subseteq \{\omega^{\omega^{\xi_1}\cdot\alpha}\}$ *for all* $\varepsilon > 0$,
(e) $fg = h$.

The proof of Proposition 6.6 is postponed to the end of the section. The next two lemmas allow us to transplant the result onto general compact metric spaces.

LEMMA 6.7. *Let K be a compact metric space. If $x_0 \in K^{(\alpha)}$ for some $\alpha < \omega_1$, then there is a countable compact subspace $H \subseteq K$ such that $\{x_0\} = H^{(\alpha)}$.*

Proof. The proof is by induction on α. The result is obvious if $\alpha = 0$.

Suppose that the lemma is true for some α. If $x_0 \in K^{(\alpha+1)}$, then there exists a sequence (x_n) in $K^{(\alpha)}$, $x_n \neq x_0$ for all n, that converges to x_0. We may also find a sequence (U_n) of open sets with disjoint closures so that $x_n \in U_n$ for all $n \in \mathbb{N}$ and $\operatorname{diam}(U_n) \to 0$. Note that $x_n \in K^{(\alpha)} \cap U_n \subseteq (\overline{U}_n)^{(\alpha)}$. Therefore by the inductive hypothesis, for all $n \in \mathbb{N}$, there is a countable compact set $H_n \subseteq \overline{U}_n$ such that $(H_n)^{(\alpha)} = \{x_n\}$. It is clear that $H = \{x_0\} \cup \bigcup_{n=1}^{\infty} H_n$ is a countable compact set such that $H^{(\alpha+1)} = \{x_0\}$.

Suppose the lemma is true for all $\alpha' < \alpha$, where $\alpha < \omega_1$ is a limit ordinal. If $x_0 \in K^{(\alpha)}$, then there exist a sequence $(\alpha_n)_{n=0}^{\infty}$ of ordinals that strictly increases to α, a sequence (x_n) converging to x_0 and a sequence (U_n) of open sets with disjoint closures so that $x_n \in (\overline{U}_n)^{(\alpha_n)}$ for all $n \in \mathbb{N}$ and $\operatorname{diam}(U_n) \to 0$. By the inductive hypothesis, for all $n \in \mathbb{N}$, there is a countable compact set $H_n \subseteq \overline{U}_n$ such that $(H_n)^{(\alpha_n)} = \{x_n\}$. It is clear that $H = \{x_0\} \cup \bigcup_{n=1}^{\infty} H_n$ is a countable compact set such that $H^{(\alpha)} = \{x_0\}$. ■

LEMMA 6.8. *Let K be a compact metric space. If $K^{(\alpha)} \neq \emptyset$ for some $0 < \alpha < \omega_1$, then there is a subspace $L \subseteq K$ such that L is homeomorphic to I_α.*

Proof. Suppose that $x_0 \in K^{(\alpha)}$. By Lemma 6.7, there is a countable compact set H such that $H^{(\alpha)} = \{x_0\}$. Since H is countable and compact, by a theorem of Mazurkiewicz and Sierpiński (see, e.g., [5, Theorem 8.6.10]), H is homeomorphic to an ordinal interval $[0, \beta]$. Since $H^{(\alpha)} = \{x_0\}$, it follows that $\omega^\alpha \le \beta < \omega^\alpha \cdot 2$. Therefore H is homeomorphic to $[0, \omega^\alpha] = I_\alpha$. ■

A consequence of Proposition 6.6 and Lemma 6.8 is the following.

THEOREM 6.9. *Suppose that ξ_1, ξ_2 are countable ordinals, and let*

$$\xi = \max\{\xi_1 + \xi_2, \xi_2 + \xi_1\}.$$

If K is a compact metric space such that $K^{(\xi)} \neq \emptyset$, then

$$\sup\{\beta(fg) : f \in \mathfrak{B}_1^{\xi_1}(K),\, g \in \mathfrak{B}_1^{\xi_2}(K)\} = \omega^\xi.$$

Proof. We may of course assume that neither ξ_1 nor ξ_2 is 0, and that $\xi = \xi_1 + \xi_2$. Let $0 < \alpha < \omega^{\xi_2}$. The assumption on K yields a subspace $H \subseteq K$ that is homeomorphic to $I_{\omega^{\xi_1} \cdot \alpha}$ (Lemma 6.8) and a $\{0,1\}$-valued function h in $\mathfrak{B}_1(H)$ such that $H^{\omega^{\xi_1} \cdot \alpha}(h, 1) \neq \emptyset$. Applying Proposition 6.6 to $h : H \to \mathbb{R}$, we obtain $f, g : H \to \mathbb{R}$ with properties as given in the proposition. Extend f, g, and h to K by defining them to be 0 on $K \setminus H$. Then $\beta_K(h) \ge \beta_H(h) \ge \omega^{\omega^{\xi_1} \cdot \alpha}$. Also, $K^1(f, \varepsilon) \subseteq H$ for all $\varepsilon > 0$. Hence $K^{1+\omega^{\xi_1}}(f, \varepsilon) \subseteq H^{\omega^{\xi_1}}(f, \varepsilon) = \emptyset$. Therefore $\beta(f) \le \omega^{\xi_1}$. Likewise, $\beta(g) \le 1 + \alpha + 1 \le \omega^{\xi_2}$. Summarizing, we have functions f and g such that $f \in \mathfrak{B}_1^{\xi_1}(K)$, $g \in \mathfrak{B}_1^{\xi_2}(K)$ and $\beta(fg) \ge \omega^{\xi_1} \cdot \alpha$. Since $\alpha < \omega^{\xi_2}$ is arbitrary, the theorem is proved. ■

Proof of Proposition 6.6. The proof is by induction on α. Let $h : I_{\omega^{\xi_1} \cdot \alpha} \to \mathbb{R}$ be a function so that $|h| < M$ for some $M > 0$. Choose a sequence

$(\tau_k)_{k=0}^{\infty}$ of ordinals with $\tau_0 = 0$ that strictly increases to ω^{ξ_1}. Define a function $G : I_{\omega^{\xi_1}} \to \mathbb{R}$ by

$$G(t) = \begin{cases} k & \text{if } \omega^{\tau_{k-1}} < t \le \omega^{\tau_k},\ k \in \mathbb{N}, \\ 1 & \text{if } t = 0, 1, \text{ or } \omega^{\omega^{\xi_1}}. \end{cases}$$

If $\alpha = 1$, let $g = G$ and $f = h/g$. Note that g is constant on each of the open sets $(\omega^{\tau_{k-1}}, \omega^{\tau_k}]$. Therefore, for all $\varepsilon > 0$, $(I_{\omega^{\xi_1}})^1(g,\varepsilon) \subseteq \{\omega^{\omega^{\xi_1}}\}$. If $t_1, t_2 > \omega^{\tau_{k-1}}$, then $|f(t_1) - f(t_2)| < 2M/k$. This implies that $(I_{\omega^{\xi_1}})^1(f, 2M/k) \subseteq [0, \omega^{\tau_{k-1}}] \cup \{\omega^{\omega^{\xi_1}}\}$. Hence $(I_{\omega^{\xi_1}})^{1+\tau_{k-1}+1}(f, 2M/k) \subseteq [0, \omega^{\tau_{k-1}}]^{(\tau_{k-1}+1)} = \emptyset$. Thus $(I_{\omega^{\xi_1}})^{\omega^{\xi_1}}(f, 2M/k) = \emptyset$ for all k, which means that $\beta(f) \le \omega^{\xi_1}$.

Suppose that the proposition is true for some α. For each $\lambda < \omega^{\omega^{\xi_1}}$, let J_λ denote the clopen ordinal interval $(\omega^{\omega^{\xi_1}\cdot\alpha}\cdot\lambda, \omega^{\omega^{\xi_1}\cdot\alpha}\cdot(\lambda+1)]$. Since J_λ is homeomorphic to $I_{\omega^{\xi_1}\cdot\alpha}$ for all $\lambda < \omega^{\omega^{\xi_1}}$, by the inductive hypothesis, there are functions $f_\lambda, g_\lambda : J_\lambda \to \mathbb{R}$ such that

(a) $\beta(f_\lambda) \le \omega^{\xi_1}$,
(b) g_λ takes values in $\mathbb{N}$,
(c) $g_\lambda(\omega^{\omega^{\xi_1}\cdot\alpha}\cdot(\lambda+1)) = 1$,
(d) $(J_\lambda)^\alpha(g_\lambda, \varepsilon) \subseteq \{\omega^{\omega^{\xi_1}\cdot\alpha}\cdot(\lambda+1)\}$ for all $\varepsilon > 0$,
(e) $f_\lambda g_\lambda = h_{|J_\lambda}$.

Define $g : I_{\omega^{\xi_1}\cdot(\alpha+1)} \to \mathbb{R}$ by

$$g(t) = \begin{cases} G(\lambda) g_\lambda(t) & \text{if } t \in J_\lambda,\ 0 \le \lambda < \omega^{\omega^{\xi_1}}, \\ 1 & \text{if } t = 0 \text{ or } \omega^{\omega^{\xi_1}\cdot(\alpha+1)}, \end{cases}$$

and let $f = h/g$. It suffices to verify properties (a) and (d). For each $\lambda < \omega^{\omega^{\xi_1}}$,

$$\begin{aligned}(I_{\omega^{\xi_1}\cdot(\alpha+1)})^\alpha(g,\varepsilon) \cap J_\lambda &\subseteq (J_\lambda)^\alpha(G(\lambda)g_\lambda, \varepsilon) \quad \text{by Lemma 2.1(c)} \\ &= (J_\lambda)^\alpha(g_\lambda, \varepsilon/G(\lambda)) \subseteq \{\omega^{\omega^{\xi_1}\cdot\alpha}\cdot(\lambda+1)\}.\end{aligned}$$

Therefore

$$(I_{\omega^{\xi_1}\cdot(\alpha+1)})^\alpha(g,\varepsilon) \subseteq \{\omega^{\omega^{\xi_1}\cdot\alpha}\cdot\lambda : \lambda \le \omega^{\omega^{\xi_1}}\}.$$

Using the fact that $g(\omega^{\omega^{\xi_1}\cdot\alpha}\cdot(\lambda+1)) = G(\lambda)$ for all $\lambda < \omega^{\omega^{\xi_1}}$ and the fact, proved above, that $(I_{\omega^{\xi_1}})^1(G,\varepsilon) \subseteq \{\omega^{\omega^{\xi_1}}\}$, we see that $(I_{\omega^{\xi_1}\cdot(\alpha+1)})^{\alpha+1}(g,\varepsilon) \subseteq \{\omega^{\omega^{\xi_1}\cdot(\alpha+1)}\}$.

Now consider f. Note that on J_λ, $f = h/g = h/(G(\lambda)g_\lambda) = f_\lambda/G(\lambda)$ and that $G(\lambda) \ge 1$. As a result,

$$(I_{\omega^{\xi_1}\cdot(\alpha+1)})^{\omega^{\xi_1}}(f,\varepsilon) \cap J_\lambda \subseteq (J_\lambda)^{\omega^{\xi_1}}(f_\lambda,\varepsilon) = \emptyset. \tag{6.1}$$

Since $|f| < M/G(\lambda)$ on J_λ, we have $|f| < M/k$ on $\bigcup\{J_\lambda : \omega^{\tau_{k-1}} < \lambda < \omega^{\omega^{\xi_1}}\}$. Hence

$$(I_{\omega^{\xi_1}\cdot(\alpha+1)})^1(f,2M/k) \subseteq \bigcup\{J_\lambda : \lambda \le \omega^{\tau_{k-1}}\} \cup \{\omega^{\omega^{\xi_1}\cdot(\alpha+1)}\}.$$

In particular, $\omega^{\omega^{\xi_1}\cdot(\alpha+1)} \notin (I_{\omega^{\xi_1}\cdot(\alpha+1)})^2(f,2M/k)$ for all $k \in \mathbb{N}$. This fact together with (6.1) implies that $(I_{\omega^{\xi_1}\cdot(\alpha+1)})^{\omega^{\xi_1}}(f,\varepsilon) = \emptyset$ for all $\varepsilon > 0$. This proves the proposition for $\alpha+1$.

Suppose $\alpha < \omega_1$ is a limit ordinal and the statement holds for all ordinals $\alpha' < \alpha$. Choose a sequence of ordinals (η_k) with $\eta_0 = 0$ that strictly increases to α. For each $k \in \mathbb{N}$, let L_k be the ordinal interval $(\omega^{\omega^{\xi_1}\cdot\eta_{k-1}}, \omega^{\omega^{\xi_1}\cdot\eta_k}]$, which is homeomorphic to $I_{\omega^{\xi_1}\cdot\eta_k}$. By the inductive hypothesis, for each $k \in \mathbb{N}$, there are f_k, $g_k : L_k \to \mathbb{R}$ such that

(a) $\beta(f_k) \le \omega^{\xi_1}$,
(b) g_k takes values in $\mathbb{N}$,
(c) $g_k(\omega^{\omega^{\xi_1}\cdot\eta_k}) = 1$,
(d) $(L_k)^{\eta_k}(g_k,\varepsilon) \subseteq \{\omega^{\omega^{\xi_1}\cdot\eta_k}\}$,
(e) $f_k g_k = h_{|L_k}$.

Define $g : I_{\omega^{\xi_1}\cdot\alpha} \to \mathbb{N}$ by

$$g(t) = \begin{cases} k g_k(t), & t \in L_k, \\ 1, & t = 0, 1, \text{ or } \omega^{\omega^{\xi_1}\cdot\alpha}, \end{cases}$$

and set $f = h/g$. Clearly,

$$\begin{aligned}(I_{\omega^{\xi_1}\cdot\alpha})^{\eta_k}(g,\varepsilon) \cap L_k &\subseteq (L_k)^{\eta_k}(g_k,\varepsilon) \quad \text{by Lemma 2.1(c)} \\ &\subseteq \{\omega^{\omega^{\xi_1}\cdot\eta_k}\}.\end{aligned}$$

Hence

$$(I_{\omega^{\xi_1}\cdot\alpha})^{\alpha}(g,\varepsilon) \subseteq \{\omega^{\omega^{\xi_1}\cdot\alpha}\}.$$

Since $|h| < M$, we have $|f(t)| < M/k$ whenever $t \in \bigcup_{j=k}^{\infty} L_j$. Therefore

$$(I_{\omega^{\xi_1}\cdot\alpha})^1(f,2M/k) \subseteq \bigcup_{j=1}^{k-1} L_j \cup \{\omega^{\omega^{\xi_1}\cdot\alpha}\}.$$

Since $\omega^{\omega^{\xi_1}\cdot\alpha}$ is an isolated point in $\bigcup_{j=1}^{k-1} L_j \cup \{\omega^{\omega^{\xi_1}\cdot\alpha}\}$, it follows that $\omega^{\omega^{\xi_1}\cdot\alpha} \notin (I_{\omega^{\xi_1}\cdot\alpha})^2(f,2M/k)$ for all $k \in \mathbb{N}$. Finally, $(I_{\omega^{\xi_1}\cdot\alpha})^{\omega^{\xi_1}}(f,2M/k) \cap L_k \subseteq (L_k)^{\omega^{\xi_1}}(f,2M/k) = \emptyset$. Hence $(I_{\omega^{\xi_1}\cdot\alpha})^{\omega^{\xi_1}}(f,\varepsilon) = \emptyset$ for all $\varepsilon > 0$. This completes the induction. ∎

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