Quasi-bounded trees and analytic inductions

by

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Abstract. A tree $T$ on $\omega$ is said to be cofinal if for every $\alpha \in \omega^\omega$ there is some branch $\beta$ of $T$ such that $\alpha \leq \beta$, and quasi-bounded otherwise. We prove that the set of quasi-bounded trees is a complete $\Sigma^1_1$-inductive set. In particular, it is neither analytic nor co-analytic.

In a recent joint work with G. Debs, we were led to study the complexity of the set of cofinal trees as a subset of the compact set of all trees on $\omega$, in fact to show that this set is not $\Pi^1_1$. The aim of this paper is to compute the exact complexity of this set, which appears to be beyond the $\sigma$-algebra generated by the analytic sets. We also prove similar results concerning the set of cofinal or quasi-bounded closed subsets of the Baire space with respect to the Effros Borel structure on the set $\mathcal{F}(\omega^\omega)$ of closed nonempty subsets of $\omega^\omega$.

Most of the definitions and results we recall here can be found in [4], which we refer to for all undefined notions and basic properties of classical descriptive classes.

Sequences and trees. For any set $E$ we denote by $\text{Seq}(E)$ the set of finite sequences of elements of $E$. If $s = \langle e_0, e_1, \ldots, e_{k-1} \rangle \in \text{Seq}(E)$ we denote by $|s|$ its length $k$. As usual, for any two $s = \langle e_0, e_1, \ldots, e_{k-1} \rangle$ and $t = \langle a_0, a_1, \ldots, a_{l-1} \rangle$ in $\text{Seq}(E)$ we say that $t$ extends $s$ or that $s$ is a beginning of $t$, and write $s \prec t$ if $|s| < |t|$ and $e_i = a_i$ for $i < |s|$. And we write $s \preceq t$ iff $s \prec t$ or $s = t$. When $s \in \text{Seq}(E)$ and $k \leq |s|$, we denote by $s|_k$ the sequence $s'$ of length $k$ such that $s' \preceq s$. Also we denote by $s \bowtie t$ the concatenation of $s$ and $t$, that is, the sequence $\langle e_0, e_1, \ldots, e_{k-1}, a_0, a_1, \ldots, a_{l-1} \rangle$ whose length is $|s| + |t|$.

For $s$ and $t$ in $\text{Seq}(\omega)$ we write $s \preceq t$ if $s$ and $t$ have the same length and moreover $s(i) \leq t(i)$ for every $i < |s|$.
We extend these notations to infinite sequences: for $\alpha = (a_n) \in E^{\omega}$ we denote by $\alpha_{|k}$ the sequence $t = \langle a_0, a_1, \ldots, a_{k-1} \rangle$, and write $t \prec \alpha$. For $s \in \text{Seq}(E)$ of length $k$ and $\alpha \in E^{\omega}$ the concatenation $s \concat \alpha$ is the infinite sequence $\beta$ such that $s \prec \beta$ and $\beta(k + i) = \alpha(i)$ for all $i \in \omega$. It will also be convenient for $s \in \text{Seq}(\omega)$, $\alpha \in \omega^{\omega}$ and $\beta \in \omega^{\omega}$ to write $\alpha \leq \beta$ iff $\alpha(i) \leq \beta(i)$ for all $i$, and $s \leq \alpha$ iff $s \leq \alpha_{|k}$ where $k = |s|$.

For any countable set $I$ we identify the set $\mathcal{P}(I)$ of subsets of $I$ with the compact space $2^I = \{0, 1\}^I$ by associating to each subset $J$ of $I$ its characteristic function $\chi_J : I \to \{0, 1\}$. In particular, if $a$ and $b$ are two members of $2^\omega$, we will write $a \leq b$ as well as $a \subset b$.

By a tree $T$ on $E$ we mean a nonempty subset of $\text{Seq}(E)$ which is left hereditary with respect to $\preceq$, that is, $(s \preceq t$ and $t \in T) \Rightarrow s \in T$. So the empty sequence $\emptyset$ belongs to any tree. An infinite branch (or a branch for short) of $T$ is an infinite sequence $\alpha \in E^{\omega}$ such that $\alpha_{|k} \in T$ for all $k$ (or equivalently for infinitely many $k$'s). We denote by $[T]$ the set of branches of $T$, which is a closed subset of $E^{\omega}$ equipped with the product topology when $E$ itself has the discrete topology. Conversely, for any closed subset $F$ of $E^{\omega}$ there are trees $T$ such that $[T] = F$.

A tree $T$ is said to be well-founded if it has no infinite branch, and ill-founded otherwise.

A tree $T$ on $\omega$ is said to be monotone if whenever $s \leq t$ and $s \in T$ then $t \in T$. It is clear that if $T$ is monotone and $\alpha$ is any branch of $T$ then $\beta \in [T]$ whenever $\beta \in \omega^{\omega}$ and $\alpha \leq \beta$.

We denote by $T$ the set of all trees on $\omega$ and by $T^+$ the set of all monotone trees on $\omega$, which are both closed subsets of $\mathcal{P}(\text{Seq}(\omega))$, hence compact metrizable spaces. It is a well known and fundamental fact that the set $\text{WF}$ of well-founded trees on $\omega$ is a complete $\Pi^1_1$-subset of $T$.

If $E$ and $F$ are two sets, a finite sequence $s$ of length $n$ of elements of $E \times F$ can be canonically identified with a pair $(t, u)$ with $t \in \text{Seq}(E)$, $u \in \text{Seq}(F)$ and $|t| = |u| = n$. Then a tree $T$ on $E \times F$ can be viewed as a set of pairs $(t, u) \in \text{Seq}(E) \times \text{Seq}(F)$ satisfying $|t| = |u|$. So we will say that $t \in \text{Seq}(E)$ and $u \in \text{Seq}(F)$ are $T$-compatible if $(t_{|k}, u_{|k}) \in T$, where $k = \min(|t|, |u|)$. In the same way, for $t \in \text{Seq}(E)$ and $\beta \in F^{\omega}$, we say that $t$ and $\beta$ are $T$-compatible if $(t, \beta_{|k}) \in T$, where $k = |t|$.

It is easy to check that, for $\beta \in F^{\omega}$, the set

$$T(\beta) := \{t \in \text{Seq}(E) : t \text{ is } T\text{-compatible with } \beta\}$$

is a tree on $E$ and that $\alpha \in [T(\beta)]$ if and only if $(\alpha, \beta) \in [T]$.

**Inductions.** Let $I$ and $P$ be sets, with $I$ countable. A mapping $\Phi : \mathcal{P}(I) \times P \to \mathcal{P}(I)$ is called an induction if it is monotone with respect to the first variable for every $x \in P$, i.e., $a \subset b \subset I \Rightarrow \Phi(a, x) \subset \Phi(b, x)$. 


For such a mapping, one can define inductively on $\xi \in \omega_1$ subsets $\Phi^\xi(x)$ of $I$, for fixed $x \in P$, by
\[
\Phi^0(x) = \emptyset, \quad \Phi^{\xi+1}(x) := \Phi(\Phi^\xi(x), x) \quad \Phi^\lambda(x) = \bigcup_{\xi<\lambda} \Phi^\xi(x) \text{ for limit } \lambda.
\]
It is easily shown that $\Phi^\xi(x) \subset \Phi^{\xi+1}(x)$ for all $\xi$, and $\Phi^\eta(x) \subset \Phi^\xi(x)$ for $\eta \leq \xi$. Since $I$ is countable, there is for each $x \in P$ a countable ordinal $\zeta$ such that $\Phi^{\zeta+1}(x) = \Phi^\zeta(x)$, thus $\Phi^\xi(x) = \Phi^\zeta(x)$ for all $\xi \geq \zeta$. We set $\Phi^\omega(x) := \Phi^\zeta(x) = \bigcup_{\xi \in \omega_1} \Phi^\xi(x)$. Thus $a := \Phi^\omega(x)$ is a fixed point for $\Phi(\cdot, x)$, i.e. $\Phi(a, x) = a$. Conversely, if $a$ is any fixed point for $\Phi(\cdot, x)$, it is immediate by induction on $\xi$ that $\Phi^\xi(x) \subset a$ for all $x$, hence $\Phi^\omega(x) \subset a$. This implies that $\Phi^\omega(x)$ is the least fixed point for $\Phi(\cdot, x)$.

If $i^*$ is a fixed element of $I$, the inductive set $\text{Ind}(\Phi, i^*)$ is defined as
\[
\text{Ind}(\Phi, i^*) := \{x \in P : i^* \in \Phi^\omega(x)\}
\]
and it follows easily from what precedes that $x \notin \text{Ind}(\Phi, i^*)$ is equivalent to
\[(*) \quad \exists a \in \mathcal{P}(I) \quad i^* \notin a \text{ and } (\forall i \in I \quad i \in a \text{ or } i \notin \Phi(a, x)).
\]

If $P$ is a Polish space and $\Gamma$ is a class, the induction $\Phi$ is said to be a $\Gamma$-induction if for every $i \in I$ the set $E_i := \{(a, x) : i \in \Phi(a, x)\}$ is a $\Gamma$-subset of $\mathcal{P}(I) \times P$, identified with the Polish space $2^I \times P$. In particular, if $\Phi$ is a $\Delta^1_1$-induction, or even a $\Pi^1_1$-induction, it follows immediately from $(*)$ that $\text{Ind}(\Phi, i^*)$ is $\Pi^1_1$.

A subset $X$ of the Polish space $P$ is said to be $\Sigma^1_1$-inductive if there is a countable set $I$, a $\Sigma^1_1$-induction $\Phi$ on $\mathcal{P}(I) \times P$ and an $i^* \in I$ such that $X = \text{Ind}(\Phi, i^*)$. We shall denote by $\Sigma^1_1$-IND the class of $\Sigma^1_1$-inductive sets.

**The game quantifier.** Let $P$ be a Polish space and $A$ a Borel subset of $\omega^\omega \times P$. For each fixed $x \in P$ the set $A_x := \{\alpha \in \omega^\omega : (\alpha, x) \in A\}$ can be viewed as the payoff of a Borel game on $\omega$. So by Martin’s Borel Determinacy Theorem this game $A_x$ is determined: if we denote by $\partial A$ the set
\[
\{x \in P : \text{Player I has a winning strategy in } A_x\},
\]
the complement of $\partial A$ in $P$ is the set
\[
\{x \in P : \text{Player II has a winning strategy in } A_x\},
\]
whence we deduce that both $\partial A$ and $P \setminus \partial A$ are $\Sigma^1_2$.

If $\Gamma$ is a class of Borel sets, we denote by $\partial \Gamma$ the class $\{\partial A : A \subset \omega^\omega \times \omega^\omega, A \in \Gamma\}$. It is well known that $\partial \Sigma^0_1 = \Pi^1_1$.

For $\Gamma = \Sigma^0_2$, it follows from Wolfe’s proof of $\Sigma^0_2$ determinacy (see for example [4, 6A.3]) that if $A \subset \omega^\omega \times P$ is $\Sigma^0_2$ one can define an analytic induction $\Phi : \mathcal{P}(I) \times P \rightarrow \mathcal{P}(I)$ (where $I$ is the countable set $\{s \in \text{Seq}(\omega) : |s| \text{ even}\}$) such that Player I has a winning strategy in the game $A_x$ if and
only if the empty sequence \( \emptyset \) belongs to \( \Phi^\infty(x) \). This shows that \( \mathcal{E} \Sigma^0_2 \subset \Sigma^1_1\text{-IND} \). Conversely, it was shown by R. Solovay (see [4, 7C.10]) that any \( \Sigma^1_1\)-inductive set is \( \mathcal{E} \Sigma^0_2 \), that is, \( \mathcal{E} \Sigma^0_2 = \Sigma^1_1\text{-IND} \).

**Cofinal and quasi-bounded trees.** As we said in the abstract, a tree \( T \) on \( \omega \) is said to be cofinal if for every \( \alpha \in \omega^\omega \) there is an infinite branch \( \beta \) of \( T \) such that \( \alpha \leq \beta \). We will say that such a branch \( \beta \) is above \( \alpha \).

If a tree \( T \) is not cofinal there is an \( \alpha \in \omega^\omega \) such that no branch of \( T \) (if any) is above \( \alpha \). Such an \( \alpha \) need not be a bound for the branches of \( T \), which would mean that “for all \( \beta \in [T], \beta \leq \alpha \)”, and we shall say that \( \alpha \) is a quasi-bound for \( T \), and that \( T \) is quasi-bounded.

It is well known that trees on \( \omega \) and closed subsets of \( \omega^\omega \) are closely related. As above a subset \( A \) of \( \omega^\omega \) is said to be cofinal (sometimes also dominating) if for every \( \alpha \in \omega^\omega \) there is some \( \beta \geq \alpha \) in \( A \). The subsets of \( \omega^\omega \) which are not cofinal will also be called quasi-bounded. The structure of cofinal subsets of \( \omega^\omega \) was already studied by several people (see [5], [1] or [2]).

The aim of this paper is to prove that the set \( \text{QB} \) of quasi-bounded trees on \( \omega \) is a \( \mathcal{E} \Sigma^0_2 \)-complete subset of \( T \). First we will prove that \( \text{QB} \) is \( \mathcal{E} \Sigma^0_2 \), hence \( \Sigma^1_1\text{-inductive} \). Then we will show that every \( \Sigma^1_1\)-inductive subset of \( \omega^\omega \) is continuously reducible to \( \text{QB} \). This will complete the proof that \( \text{QB} \) is \( \Sigma^1_1\text{-IND}-complete \). In fact this will also prove that any \( \Sigma^1_1\)-inductive set is \( \mathcal{E} \Sigma^0_2 \), hence will yield a new (but more complicated) proof of Solovay’s result.

We will also consider the set \( \text{QBC} \) of closed quasi-bounded subsets of the Baire space, equipped with the Effros Borel structure. This set was already studied by S. Solecki ([5]), in connection with Haar null sets of a non-locally compact Polish group. He showed this set is \( \Delta^1_2 \) but not \( \Sigma^1_1 \). We shall prove here that it is \( \Sigma^1_1\text{-IND-complete} \).

There are only very few examples in the literature of true \( \mathcal{E} \Sigma^0_2 \) sets. The most important one is given by Kechris in [3], where he shows that \( \Sigma^1_1\text{-IND} \) is the exact maximum complexity of \( \sigma \)-ideals of compact sets with \( \Sigma^1_1 \) bases.

The main interest of our result is to yield a “natural” and combinatorially simple example of a \( \mathcal{E} \Pi^0_2 \) set. It could be used to prove that a set \( X \) is not \( \mathcal{E} \Pi^0_2 \) by reducing continuously \( \text{QB} \) to it, in the same way as one can prove that a set is not \( \Sigma^1_1 \) by constructing a continuous reduction of WF to it.

**Definition 1.** For any tree \( T \) on \( \omega \), we denote by \( T^o \) the tree defined by

\[
s \in T^o \iff (s = \emptyset \text{ or } |s| \leq s(0) \text{ or } s = \langle k \rangle^{\sim} t \text{ with } t_{|k|} \notin T).
\]

It is clear from the definition that if \( \langle k \rangle^{\sim} t \) belongs to \( T^o \) and \( k \leq l \) then \( \langle l \rangle^{\sim} t \) also belongs to \( T^o \).

**Lemma 2.** Let \( T \) be a monotone tree on \( \omega \). Then the tree \( T^o \) is quasi-bounded if and only if \( T \) is ill-founded. Moreover, for any branch \( \alpha \) of \( T \), \( \langle 0 \rangle^{\sim} \alpha \) is a quasi-bound for \( T^o \).
Then we claim that $\langle 0 \rangle \prec \alpha$ is a quasi-bound for $T^\circ$.

Indeed, assume by contradiction that $\langle k \rangle \prec \beta$ is a branch of $T^\circ$ above $\langle 0 \rangle \prec \alpha$; then $t := \beta |_k \notin T$. But since $\alpha \leq \beta$ we have $s := \alpha |_k \leq \beta |_k = t$. So $s \in T$ since $\alpha \in [T]$, $t \notin T$ and $s \leq t$, in contradiction with $T \in T^+$.

Assume now $T$ is well-founded and $\langle m \rangle \prec \alpha \in \omega^\omega$. We claim that $T^\circ$ possesses a branch above $\langle m \rangle \prec \alpha$.

Indeed, $\alpha \notin [T] = \emptyset$. Hence there is some integer $k$ such that $\alpha |_k \notin T$. Replacing $k$ by $\max(k, m)$ if necessary, we can assume $m \leq k$. Then $\langle k \rangle \prec \alpha$ is a branch of $T^\circ$, and $\langle m \rangle \prec \alpha \leq \langle k \rangle \prec \alpha$. ■

**Theorem 3.** The set $\text{QB}$ is $\exists \Sigma^0_2$.

**Proof.** Define the mapping $\psi : \text{Seq}(2) \to \text{Seq}(\omega)$ by counting the blocks of contiguous 0’s inside $s$: if $s = \langle n_0, n_1, \ldots, n_{k-1} \rangle$ for some $s \in \text{Seq}(2)$, then the sequence $s$ contains $k$ terms equal to 1, with $n_0$ zeros before the first 1, $n_1$ zeros between the first and the second 1, $\ldots$, $n_{k-1}$ zeros between the last two 1’s.

So $\psi$ is defined inductively by letting

$$
\begin{align*}
\psi(\emptyset) &= \emptyset, \\
\psi(\langle 1 \rangle) &= \langle 0 \rangle, \\
\psi(\langle s \prec \langle 0 \rangle \rangle) &= \psi(s), \\
\psi(\langle s \prec \langle 1, 1 \rangle \rangle) &= \psi(\langle s \prec \langle 1 \rangle \rangle \prec \langle 0 \rangle), \\
\psi(\langle s \prec \langle 1 \rangle \rangle) &= u \prec \langle p \rangle \Rightarrow \psi(\langle s \prec \langle 0, 1 \rangle \rangle) = u \prec \langle p + 1 \rangle.
\end{align*}
$$

Then it is clear that $|\psi(s)| \leq |s|$ and that for any two sequences $s$ and $s'$ such that $s \prec s'$ we have $\psi(s) \preceq \psi(s')$.

Denote by $P_\infty$ the set of those $\gamma$’s in $2^\omega$ which have infinitely many coordinates equal to 1. For $\gamma \in P_\infty$ there is a unique $\beta \in \omega^\omega$ which we denote by $\hat{\psi}(\gamma)$ such that $s \prec \gamma \Rightarrow \psi(s) \prec \beta$. It is easily checked and well known that $2^\omega \setminus P_\infty$ is countable and that $\hat{\psi}$ is a homeomorphism from $P_\infty$ onto $\omega^\omega$.

For $T$ a given tree we define the game $G_{qb}(T)$ where Player I plays integers $n_0, n_1, \ldots$, and Player II plays $c_0, c_1, \ldots$ in $\{0, 1\}$ with the following two rules:

**R**$_{1}$: for every $k$, $\psi(\langle c_0, c_1, \ldots, c_{k-1} \rangle) \in T$.

**R**$_{2}$: for every $k$, $\langle n_0, n_1, \ldots, n_{p-1} \rangle \leq \psi(\langle c_0, c_1, \ldots, c_{k-1} \rangle)$, where $p$ is the length of $\psi(\langle c_0, c_1, \ldots, c_{k-1} \rangle)$.

The run where Player I plays $(n_k)$ and Player II plays $(c_k)$ is won by Player II iff $(c_k) \in P_\infty$. 

Clearly the set
\[ A := \{((n_k), (c_k), T) : \text{Player II respects the rules and } (c_k) \notin P_\omega \} \]
is \( \Sigma_2^0 \) in \( \omega^\omega \times 2^\omega \times T \). Hence the set \( \exists A \) is \( \exists \Sigma_2^0 \). Theorem 3 will then follow from the next two lemmas.

**Lemma 4.** If Player II has a winning strategy in the game \( G_{q\beta}(T) \), then the tree \( T \) is continuously cofinal, i.e. there is a continuous function \( f : \omega^\omega \to [T] \) such that \( f(\alpha) \geq \alpha \) for every \( \alpha \in \omega^\omega \). In particular, \( T \) is cofinal.

If \( \tau \) is a winning strategy for Player II, it defines a continuous function 
\[ g : \omega^\omega \to 2^\omega \]
such that for every \( \alpha \) in \( \omega^\omega \) and every \( s = (n_0, n_1, \ldots, n_{k-1}) \prec \alpha \) played by Player I the answer \( \langle c_0, c_1, \ldots, c_{k-1} \rangle \) of Player II under \( \tau \) satisfies 
\[ \langle c_0, c_1, \ldots, c_{k-1} \rangle \prec g(\alpha). \]
It then follows from the rule \( R_1 \) that we have 
\[ \psi((\langle c_0, c_1, \ldots, c_{k-1} \rangle)) \in T. \]
Moreover, since Player II wins, the run \( g(\alpha) \) is in \( P_\infty \). Hence \( \hat{\psi}(g(\alpha)) \in \omega^\omega \) and \( \hat{\psi}(g(\alpha)) \mid p \in T \) for arbitrarily large \( p \), whence we conclude that \( f(\alpha) := \hat{\psi}(g(\alpha)) \in [T] \). Since \( \psi \) is continuous on \( P_\infty \), \( f = \hat{\psi} \circ g \) itself is continuous. Finally, it follows from the rule \( R_2 \) that \( f(\alpha) \mid k \geq \alpha \mid k \) for arbitrarily large \( k \), hence \( f(\alpha) \geq \alpha \).

**Lemma 5.** If Player I has a winning strategy in \( G_{q\beta}(T) \), then \( T \) is quasi-bounded.

If \( \sigma \) is a winning strategy for Player I, it induces as above a continuous function 
\[ h : 2^\omega \to \omega^\omega \]
then the range \( K := h(2^\omega) \) is a compact subset of \( \omega^\omega \), and one can define for all \( n \) the integer \( \alpha(n) = \sup_{x \in K} x(n) \). We claim that this \( \alpha \) is a quasi-bound for \( T \).

Indeed, if \( \beta \) were a branch of \( T \) such that \( \alpha \leq \beta \), then Player II could play the following infinite run \( \gamma : \beta(0) \) times 0, then 1, then \( \beta(1) \) times 0, then 1, \ldots. This would respect the rule \( R_1 \) since \( \psi(\gamma \mid k) \prec \beta \) for all \( k \). And since \( \gamma \in P_\infty \), we would have \( \beta = \hat{\psi}(\gamma) \). Moreover, since \( h(\gamma) \in K \), we would have \( h(\gamma) \leq \alpha \leq \beta = \hat{\psi}(\gamma) \); this shows that the rule \( R_2 \) would also be respected. Finally, since \( \gamma \in P_\infty \), Player II would win the run against the strategy \( \sigma \). This contradiction completes the proof of the lemma.

Thus the proof of Theorem 3 is complete. One can notice that a similar game was used in [2] in order to prove that any cofinal \( \Sigma_1^1 \) subset of \( \omega^\omega \) is continuously cofinal.

**Remark.** It follows from the previous proof that a quasi-bound for \( T \) can be computed continuously from a winning strategy for Player I in \( G_{q\beta}(T) \). Conversely, a quasi-bound \( \alpha \) for \( T \) yields a simple strategy \( \sigma \) for Player I: he plays \( \alpha \) whatever Player II is answering. This strategy is clearly winning: in any run compatible with \( \sigma \) a position \( (\alpha \mid k, (c_0, c_1, \ldots, c_{k-1})) \) is
reached for which no extension of \( \psi(\langle c_0, c_1, \ldots, c_{k-1} \rangle) \) can be found in \( T \) above \( \alpha \); and beyond this position Player II must always play 0.

We now intend to show that QB has complexity at least \( \Sigma^1_1 \)-IND.

**Theorem 6.** If \( X \) is a \( \Sigma^1_1 \)-IND subset of \( \omega^\omega \), there exists a continuous mapping \( x \mapsto S(x) \) from \( \omega^\omega \) to \( T \) such that \( S(x) \in \text{QB} \) if and only if \( x \in X \).

**Proof.** Without loss of generality we assume that \( \Phi: 2^\omega \times \omega^\omega \to 2^\omega \) is a \( \Sigma^1_1 \)-induction on \( \omega \) and that

\[
x \in X \iff 0 \in \Phi^\infty(x).
\]

Then for each \( n \) the set \( E_n := \{(a, x) \in 2^\omega \times \omega^\omega : n \in \Phi(a, x)\} \) is \( \Sigma^1_1 \) and there is some tree \( T_n \) on \( 2 \times \omega \times \omega \) such that

\[
(a, x) \in E_n \iff \exists \beta \in \omega^\omega (a, \beta, x) \in [T_n]
\]

where we identify the subset \([T_n] \) of \((2\times\omega\times\omega)^\omega \) with a subset of \( 2^\omega \times \omega^\omega \times \omega^\omega \).

Identifying \( \text{Seq}(2 \times \omega \times \omega) \) with the set

\[
\{(s, t, u) \in \text{Seq}(2) \times \text{Seq}(\omega) \times \text{Seq}(\omega) : |s| = |t| = |u| \}
\]

we now define trees \( \hat{T}_n \) and \( U_n \) on \( 2 \times \omega \times \omega \) by

\[
(s, t, u) \in \hat{T}_n \iff \exists s' \exists t' s' \leq s, t' \leq t, \ (s', t', u) \in T_n,
\]

\[
(s, t, u) \in U_n \iff \quad \begin{cases} (s, t, u) = (\emptyset, \emptyset, \emptyset) \\ \text{or } |s| = |t| = |u| \leq t(0) \\ \text{or else } t = \langle k \rangle \sim t^* \text{ with } (s|_k, t^*_k, u|_k) \notin \hat{T}_n. \end{cases}
\]

Fix a bijection \((n, p) \mapsto n \ast p \) from \( \omega \times \omega \) onto \( \omega \) which is separately increasing with respect to each variable and satisfies \( n \ast 0 \leq 0 \ast n \). Then we necessarily have \( 0 \ast 0 = 0 \). For example we can put

\[
n \ast p = \frac{(n + p)(n + p + 1)}{2} + p.
\]

Then, for each \( s \in \text{Seq}(\omega) \) and each \( n \in \omega \), we define the sequence \( \theta_n(s) \in \text{Seq}(\omega) \) by

\[
\theta_n(s) = \langle s(n \ast 0), s(n \ast 1), \ldots s(n \ast (k - 1)) \rangle \quad \text{where } n \ast (k - 1) < |s| \leq n \ast k.
\]

In particular we get \( \theta_n(s) = \emptyset \) if \( |s| \leq n \ast 0 \). Define also \( \theta^*(s) \in \text{Seq}(2) \) by

\[
\theta^*(s) = \langle c_0, c_1, \ldots, c_{p-1} \rangle \quad \text{where } \begin{cases} (p - 1) \ast 0 < |s| \leq p \ast 0 \\ \text{and } c_i = 1 \iff s(i \ast 0) \text{ is odd} \end{cases}
\]

Observe that for any \( s \in \text{Seq}(\omega) \) and any \( n \), if \( k = |\theta_n(s)| \) and \( p = |\theta^*(s)| \), we have

\[
0 \ast (k - 1) \leq n \ast (k - 1) < |s| \leq p \ast 0 \leq 0 \ast p,
\]
hence $k - 1 < p$, thus $|\theta_n(s)| \leq |\theta^*(s)|$. Moreover it is clear that if $s \prec s'$ we have $\theta_n(s) \leq \theta_n(s')$ for all integers $n$, and $\theta^*(s) \leq \theta^*(s')$. We extend $\theta_n$ and $\theta^*$ to $\omega^\omega$ by letting

$$\hat{\theta}_n(\alpha)(k) = \alpha(n \cdot k),$$
$$\hat{\theta}^*(\alpha)(i) = \begin{cases} 1 & \text{if } s(i \cdot 0) \text{ is odd}, \\ 0 & \text{otherwise}. \end{cases}$$

We now define, for $x \in \omega^\omega$, a tree $S(x)$ by

$$s \in S(x) \iff \begin{cases} s = \emptyset \text{ or } (s(0) = 0 \text{ and } \forall n < |\theta^*(s)| \\ (\theta^*(s)(n) = 1 \text{ or } (\theta^*(s)|_k, \theta_n(s), x|_k) \in U_n) ), \end{cases}$$

where $k = |\theta_n(s)|$.

The theorem will follow from the next four lemmas.

**Lemma 7.** The mapping $x \mapsto S(x)$ is continuous from $\omega^\omega$ to $T$.

**Proof.** For any $s \in \text{Seq}(\omega)$ define $k := |\theta^*(s)|$. Then “$s \in S(x)$” depends only on $x|_k$. Hence $\{x \in \omega^\omega : s \in S(x)\}$ is open and closed. This shows that the mapping $x \mapsto S(x)$ is continuous from $\omega^\omega$ to $T$. ■

**Lemma 8.** For $a \in 2^\omega$ and $x \in \omega^\omega$ one has

$$n \in \Phi(a, x) \iff \exists \beta \in \omega^\omega \ (a, \beta, x) \in [\hat{T}_n] \iff U_n(a, x) \in \text{QB}.$$ 

Moreover, if $\beta$ is any branch of $\hat{T}_n(a, x)$, then $(0) \triangleleft \beta$ is a quasi-bound for $U_n(a, x)$.

**Proof.** Notice that $T_n \subset \hat{T}_n$. Thus if $n \in \Phi(a, x)$, then $(a, x) \in E_n$, hence there exists a $\beta$ such that $(a, \beta, x) \in [T_n] \subset [\hat{T}_n]$.

Conversely, if $(a, \beta, x) \in [\hat{T}_n]$ then for every integer $k$, $(a|_k, \beta|_k, x|_k)$ belongs to $\hat{T}_n$. Hence there are $s \in 2^k$ and $t \in \omega^k$ such that $(s, t, x|_k) \in T_n$, $s \leq a|_k$ and $t \leq \beta|_k$. It follows that the set

$$V := \{(s, t, u) \in T_n : |s| = |t| = |u|, s \leq a, t \leq \beta, u \prec x\}$$

is an infinite and finitely branching tree. By König’s Lemma the tree $V$ is ill-founded. If $(a', \beta', x')$ is a branch of $V$, one necessarily has $a' \leq a$ and $x' = x$. Thus $(a', x) \in E_n$, hence $n \in \Phi(a', x) \subset \Phi(a, x)$.

Notice that for any $a \in 2^\omega$ and any $x \in \omega^\omega$, $U_n(a, x) = \hat{T}_n(a, x)^\circ$ and $\hat{T}_n(a, x)$ is monotone. Then it follows from Lemma 2 that $U_n(a, x) = \hat{T}_n(a, x)^\circ$ is quasi-bounded if and only if $\hat{T}_n(a, x)$ is ill-founded, that is, if and only if $n \in \Phi(a, x)$. ■

**Lemma 9.** If $x \notin X$ then $S(x)$ is cofinal.
Proof. Assume \( x \notin X \) and let \( a = \Phi^\infty(x) \). Then \( 0 \notin a \) and for all \( n \notin a \) we have \( n \notin \Phi(a, x) \). Let \( \alpha \in \omega^\omega \) and define \( \alpha_n = \hat{\theta}_n(\alpha) \) for all \( n \). We will produce a branch \( \beta \) of \( S(x) \) such that \( \beta \geq \alpha \).

For \( n \in a \) we define \( \beta_n = \alpha_n \). For \( n \notin a \), since \( U_n(a, x) \) is cofinal, by Lemma 8 we can find \( \beta_n \in [U_n(a, x)] \) such that \( \alpha_n \leq \beta_n \). Replacing if necessary \( \beta_n(0) \) by \( \beta_n(0) + 1 \), we can assume that \( \beta_n(0) \) is odd for \( n \in a \) and even for \( n \notin a \). Then defining \( \beta \) by

\[
\forall n \forall p \quad \beta(n \ast p) = \beta_n(p)
\]

we get \( \hat{\theta}_n(\beta) = \beta_n \geq \alpha_n = \hat{\theta}_n(\alpha) \), \( \beta(0) = \beta_0(0) \) is even and \( \hat{\sigma}^*(\beta) = a \).

It follows easily that \( \beta \geq \alpha \) and that for each \( l, \beta_l \in S(x) \), hence \( \beta \in [S(x)] \).

**Lemma 10.** If \( x \in X \) then \( S(x) \) is quasi-bounded.

**Proof.** If \( x \in X \), then \( 0 \in \Phi^\infty(x) \), and we can define for every \( n \in \Phi^\infty(x) \) the rank \( \varrho_n := \min\{\xi : n \in \Phi^\xi(x)\} \in \omega_1 \) and then \( a_n := \{p \in \omega : \varrho_p \varrho_n\} \). Thus, for \( n \in \Phi^\infty(x) \), we have \( n \in \Phi(a_n, x) \). It follows that \( \hat{T}_n(a_n, x) \) is ill-founded. Then we can choose a branch \( \alpha_n \) of \( \hat{T}_n(a_n, x) \) and let \( \alpha_n := (0)^\ast \). For \( n \notin \Phi^\infty(x) \) we choose \( \alpha_n \) equal to the null sequence \( 0 \). Finally, defining \( \alpha \) by

\[
\forall n \forall p \quad \alpha(n \ast p) = \alpha_n(p)
\]

we get \( \hat{\theta}_n(\alpha) = \alpha_n \) for all \( n \).

We claim that \( \alpha \) is a quasi-bound for \( S(x) \). Indeed, assuming by contradiction that \( \beta \) is a branch of \( S(x) \) above \( \alpha \), we should have \( \beta_n := \hat{\theta}_n(\beta) \geq \hat{\theta}_n(\alpha) = \alpha_n \). Then put \( a := \hat{\theta}^*(\beta) \in 2^\omega \). Since \( \beta \in [S(x)] \), we should have \( \beta(0) \) even, hence \( 0 \in \Phi^\infty(x) \setminus a \). It follows that \( \varrho_n \in \Phi^\infty(x) \setminus a \) should be nonempty. Thus there would be an integer \( m \in \Phi^\infty(x) \setminus a \) such that \( \varrho_m = \min\{\varrho_n : n \in \Phi^\infty(x) \setminus a\} \). In particular \( m \notin a \), hence \( \beta_m \in [U_m(a, x)] \).

By minimality of \( \varrho_m \) we would have \( a_m = \{p : \varrho_p < \varrho_m\} \subset a \), hence \( m \in \Phi(a, x) \). Since \( \alpha_m^* \in [\hat{T}_m(a_n, x)] \), this would also imply that \( \alpha_m^* \in [U_m(a, x)] \), hence \( \alpha_m \) would be a quasi-bound for \( U_m(a, x) \) by Lemma 8, in contradiction with \( \alpha_m \leq \beta_m \) and \( \beta_m \in [U_m(a, x)] \).

This completes the proof of Theorem 6.

**Quasi-bounded closed subsets of the Baire space.** Now we are interested in closed subsets of \( \omega^\omega \) and will denote by \( \mathcal{F}(\omega^\omega) \) the set of nonempty closed subsets of \( \omega^\omega \) which we equip with the Effros Borel structure. As for trees on \( \omega \), we shall say that a closed subset \( F \) of \( \omega^\omega \) is cofinal if for every \( \alpha \in \omega^\omega \) there is some \( \beta \geq \alpha \) in \( F \), and that \( F \) is quasi-bounded otherwise. We shall say that \( \alpha \) is a quasi-bound for \( F \) if \( F \cap \{\beta : \beta \geq \alpha\} = \emptyset \).
We will denote by QBC the subset of $\mathcal{F}(\omega^\omega)$ consisting of the quasi-bounded closed subsets of $\omega^\omega$.

We shall show in the following theorem that QBC behaves with respect to Borel reducibility in the same way as QB does with respect to continuous reducibility.

**Theorem 11.** QBC is $\Sigma^1_1$-IND-complete.

This follows immediately from the next two lemmas.

**Lemma 12.** If $P$ is a Polish space and $F : P \to \mathcal{F}(\omega^\omega)$ a Borel mapping then $F^{-1}(\text{QBC})$ is $\Sigma^1_1$-inductive.

**Proof.** For each $s \in \text{Seq}(\omega)$ we denote by $N_s$ the basic open set $\{ \alpha \in \omega^\omega : s \prec \alpha \}$. For $x \in P$ define
$$T(x) := \{ s \in \text{Seq}(\omega) : N_s \cap F(x) \neq \emptyset \},$$
which is clearly a tree on $\omega$ such that $[T(x)] = F(x)$. By definition of the Effros Borel structure, $\{ H : N_s \cap H \neq \emptyset \}$ is Borel in $\mathcal{F}(\omega^\omega)$, thus $\{ x \in P : s \in T(x) \}$ is Borel for all $s$. Hence the mapping $f : x \mapsto T(x)$ is Borel from $P$ to $\mathcal{T}$. It is immediate from the definitions that $f(x) \in \text{QB} \iff F(x) \in \text{QBC}$. So $F^{-1}(\text{QBC}) = f^{-1}(\text{QB})$.

As QB is $\Sigma^1_1$-inductive in $\mathcal{T}$, there is an analytic induction $\Phi : \mathcal{P}(\omega) \times \mathcal{T} \to \mathcal{P}(\omega)$ such that $T \in \text{QB} \iff 0 \in \Phi^\infty(T)$. For $a \in \mathcal{P}(\omega)$ and $x \in P$ define
$$\Psi(a, x) := \Phi(a, f(x)).$$
Then $\Psi$ is an induction and clearly $\Psi^\xi(x) = \Phi^\xi(f(x))$ for each $\xi$, hence $\Psi^\infty(x) = \Phi^\infty(f(x))$ and
$$F(x) \in \text{QBC} \iff f(x) \in \text{QB} \iff 0 \in \Phi^\infty(f(x)) \iff 0 \in \Psi^\infty(x) \iff x \in \text{Ind}(\Psi, 0).$$
Then $n \in \Psi(a, x) \iff \exists T \in \mathcal{T} \ (T = f(x) \text{ and } n \in \Phi(a, T))$, whence we conclude that $\Psi$ is $\Sigma^1_1$ and finally that $F^{-1}(\text{QBC})$ is $\Sigma^1_1$-inductive.

**Lemma 13.** If $X$ is a $\Sigma^1_1$-IND subset of $\omega^\omega$, then there exists a Borel reduction of $X$ to QBC.

**Proof.** By Theorem 6 there is a continuous function $S$ from $\omega^\omega$ to $\mathcal{T}$ such that $S(x) \in \text{QB} \iff x \in X$. Denote for $n \in \omega$ by $z_n$ the null sequence of length $n$, and
$$\tilde{S}(x) := \{ s \upharpoonright z_n : s \in S(x), n \in \omega \}.$$
Clearly if $\alpha \in \omega^\omega$ is any sequence such that $\alpha(n) > 0$ for all $n$, then for all $\beta \in \omega^\omega$ we have
$$\beta \geq \alpha \text{ and } \beta \in [S(x)] \iff \beta \geq \alpha \text{ and } \beta \in [\tilde{S}(x)],$$
hence $S(x) \in \text{QB} \iff \tilde{S}(x) \in \text{QB}$. Since 
\[
s \in \tilde{S}(x) \iff (\exists k, l \leq |s|) \ s|_k \in S(x) \text{ and } s = s|_k \upharpoonright z_l,
\]
one sees that $\tilde{S}$ is continuous and that $X = \tilde{S}^{-1}(\text{QB})$. Then define $F(x) := [\tilde{S}(x)]$. It is immediate that $F(x)$ is a quasi-bounded closed subset of $\omega^\omega$ iff $\tilde{S}(x) \in \text{QB}$, i.e. iff $x \in X$. Finally, it is enough to notice that for each $s \in \text{Seq}(\omega)$ and each $x \in \omega^\omega$,
\[
F(x) \cap N_s \neq \emptyset \Rightarrow s \in \tilde{S}(x) \Rightarrow \forall n \ s \upharpoonright z_n \in \tilde{S}(x) \Rightarrow s \upharpoonright 0 \in [\tilde{S}(x)] \Rightarrow F(x) \cap N_s \neq \emptyset,
\]
so that $\{x : N_s \cap F(x) \neq \emptyset\}$ is clopen, and hence $F$ is Borel.

**References**


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