FUNDAMENTA
MATHEMATICAE
191 (2006)

Quasi-bounded trees and analytic inductions

by

Jean Saint Raymond (Paris)

Abstract. A tree T on ω is said to be *cofinal* if for every $\alpha \in \omega^\omega$ there is some branch β of T such that $\alpha \leq \beta$, and *quasi-bounded* otherwise. We prove that the set of quasi-bounded trees is a complete $\mathbf{\Sigma}^1_1$-inductive set. In particular, it is neither analytic nor co-analytic.

In a recent joint work with G. Debs, we were led to study the complexity of the set of cofinal trees as a subset of the compact set of all trees on ω, in fact to show that this set is not $\mathbf{\Pi}^1_1$. The aim of this paper is to compute the exact complexity of this set, which appears to be beyond the σ-algebra generated by the analytic sets. We also prove similar results concerning the set of cofinal or quasi-bounded closed subsets of the Baire space with respect to the Effros Borel structure on the set $\mathcal{F}(\omega^\omega)$ of closed nonempty subsets of ω^ω.

Most of the definitions and results we recall here can be found in [4], which we refer to for all undefined notions and basic properties of classical descriptive classes.

Sequences and trees. For any set E we denote by $\mathrm{Seq}(E)$ the set of finite sequences of elements of E. If $s = \langle e_0, e_1, \ldots, e_{k-1}\rangle \in \mathrm{Seq}(E)$ we denote by $|s|$ its *length* k. As usual, for any two $s = \langle e_0, e_1, \ldots, e_{k-1}\rangle$ and $t = \langle a_0, a_1, \ldots, a_{l-1}\rangle$ in $\mathrm{Seq}(E)$ we say that t *extends* s or that s is a *beginning* of t, and write $s \prec t$ if $|s| < |t|$ and $e_i = a_i$ for $i < |s|$. And we write $s \preceq t$ iff $s \prec t$ or $s = t$. When $s \in \mathrm{Seq}(E)$ and $k \leq |s|$, we denote by $s_{|k}$ the sequence s' of length k such that $s' \preceq s$. Also we denote by $s^\frown t$ the *concatenation* of s and t, that is, the sequence $\langle e_0, e_1, \ldots, e_{k-1}, a_0, a_1, \ldots, a_{l-1}\rangle$ whose length is $|s| + |t|$.

For s and t in $\mathrm{Seq}(\omega)$ we write $s \leq t$ if s and t have the same length and moreover $s(i) \leq t(i)$ for every $i < |s|$.

2000 *Mathematics Subject Classification*: 03E15, 54H05.
Key words and phrases: dominating trees, induction, Borel games.

We extend these notations to infinite sequences: for $\alpha = (a_n) \in E^\omega$ we denote by $\alpha_{|k}$ the sequence $t = \langle a_0, a_1, \ldots, a_{k-1}\rangle$, and write $t \prec \alpha$. For $s \in \mathrm{Seq}(E)$ of length k and $\alpha \in E^\omega$ the concatenation $s^\frown\alpha$ is the infinite sequence β such that $s \prec \beta$ and $\beta(k+i) = \alpha(i)$ for all $i \in \omega$. It will also be convenient for $s \in \mathrm{Seq}(\omega)$, $\alpha \in \omega^\omega$ and $\beta \in \omega^\omega$ to write $\alpha \le \beta$ iff $\alpha(i) \le \beta(i)$ for all i, and $s \le \alpha$ iff $s \le \alpha_{|k}$ where $k = |s|$.

For any countable set I we identify the set $\mathcal{P}(I)$ of subsets of I with the compact space $2^I = \{0,1\}^I$ by associating to each subset J of I its characteristic function $\chi_J : I \to \{0,1\}$. In particular, if a and b are two members of 2^ω, we will write $a \le b$ as well as $a \subset b$.

By a *tree* T on E we mean a nonempty subset of $\mathrm{Seq}(E)$ which is left hereditary with respect to $\preceq$, that is, $(s \preceq t$ and $t \in T) \Rightarrow s \in T$. So the empty sequence $\emptyset$ belongs to any tree. An *infinite branch* (or a *branch* for short) of T is an infinite sequence $\alpha \in E^\omega$ such that $\alpha_{|k} \in T$ for all k (or equivalently for infinitely many k's). We denote by $\lceil T \rceil$ the set of branches of T, which is a closed subset of E^ω equipped with the product topology when E itself has the discrete topology. Conversely, for any closed subset F of E^ω there are trees T such that $\lceil T \rceil = F$.

A tree T is said to be *well-founded* if it has no infinite branch, and *ill-founded* otherwise.

A tree T on ω is said to be *monotone* if whenever $s \le t$ and $s \in T$ then $t \in T$. It is clear that if T is monotone and α is any branch of T then $\beta \in \lceil T \rceil$ whenever $\beta \in \omega^\omega$ and $\alpha \le \beta$.

We denote by $\mathcal{T}$ the set of all trees on ω and by $\mathcal{T}^+$ the set of all monotone trees on ω, which are both closed subsets of $\mathcal{P}(\mathrm{Seq}(\omega))$, hence compact metrizable spaces. It is a well known and fundamental fact that the set WF of well-founded trees on ω is a complete $\mathbf{\Pi}^1_1$-subset of $\mathcal{T}$.

If E and F are two sets, a finite sequence s of length n of elements of $E \times F$ can be canonically identified with a pair (t, u) with $t \in \mathrm{Seq}(E)$, $u \in \mathrm{Seq}(F)$ and $|t| = |u| = n$. Then a tree T on $E \times F$ can be viewed as a set of pairs (t, u) in $\mathrm{Seq}(E) \times \mathrm{Seq}(F)$ satisfying $|t| = |u|$. So we will say that $t \in \mathrm{Seq}(E)$ and $u \in \mathrm{Seq}(F)$ are T*-compatible* if $(t_{|k}, u_{|k}) \in T$, where $k = \min(|t|, |u|)$. In the same way, for $t \in \mathrm{Seq}(E)$ and $\beta \in F^\omega$, we say that t and β are T-compatible if $(t, \beta_{|k}) \in T$, where $k = |t|$.

It is easy to check that, for $\beta \in F^\omega$, the set

$$T(\beta) := \{t \in \mathrm{Seq}(E) : t \text{ is } T\text{-compatible with } \beta\}$$

is a tree on E and that $\alpha \in \lceil T(\beta) \rceil$ if and only if $(\alpha, \beta) \in \lceil T \rceil$.

Inductions. Let I and P be sets, with I countable. A mapping $\Phi : \mathcal{P}(I) \times P \to \mathcal{P}(I)$ is called an *induction* if it is monotone with respect to the first variable for every $x \in P$, i.e., $a \subset b \subset I \Rightarrow \Phi(a, x) \subset \Phi(b, x)$.

For such a mapping, one can define inductively on $\xi \in \omega_1$ subsets $\Phi^\xi(x)$ of I, for fixed $x \in P$, by

$$\Phi^0(x) = \emptyset, \quad \Phi^{\xi+1}(x) := \Phi(\Phi^\xi(x), x) \quad \Phi^\lambda(x) = \bigcup_{\xi<\lambda} \Phi^\xi(x) \text{ for limit } \lambda.$$

It is easily shown that $\Phi^\xi(x) \subset \Phi^{\xi+1}(x)$ for all ξ, and $\Phi^\eta(x) \subset \Phi^\xi(x)$ for $\eta \leq \xi$. Since I is countable, there is for each $x \in P$ a countable ordinal ζ such that $\Phi^{\zeta+1}(x) = \Phi^\zeta(x)$, thus $\Phi^\xi(x) = \Phi^\zeta(x)$ for all $\xi \geq \zeta$. We set $\Phi^\infty(x) := \Phi^\zeta(x) = \bigcup_{\xi\in\omega_1} \Phi^\xi(x)$. Thus $a := \Phi^\infty(x)$ is a fixed point for $\Phi(\cdot, x)$, i.e. $\Phi(a, x) = a$. Conversely, if a is any fixed point for $\Phi(\cdot, x)$, it is immediate by induction on ξ that $\Phi^\xi(x) \subset a$ for all x, hence $\Phi^\infty(x) \subset a$. This implies that $\Phi^\infty(x)$ is the least fixed point for $\Phi(\cdot, x)$.

If i^* is a fixed element of I, the *inductive set* $\mathrm{Ind}(\Phi, i^*)$ is defined as

$$\mathrm{Ind}(\Phi, i^*) := \{x \in P : i^* \in \Phi^\infty(x)\}$$

and it follows easily from what precedes that $x \notin \mathrm{Ind}(\Phi, i^*)$ is equivalent to

$$(*) \qquad \exists a \in \mathcal{P}(I) \quad i^* \notin a \text{ and } (\forall i \in I \;\; i \in a \text{ or } i \notin \Phi(a, x)).$$

If P is a Polish space and Γ is a class, the induction Φ is said to be a *Γ-induction* if for every $i \in I$ the set $E_i := \{(a, x) : i \in \Phi(a, x)\}$ is a Γ-subset of $\mathcal{P}(I) \times P$, identified with the Polish space $2^I \times P$. In particular, if Φ is a $\mathbf{\Delta}^1_1$-induction, or even a $\mathbf{\Pi}^1_1$-induction, it follows immediately from $(*)$ that $\mathrm{Ind}(\Phi, i^*)$ is $\mathbf{\Pi}^1_1$.

A subset X of the Polish space P is said to be $\mathbf{\Sigma}^1_1$-*inductive* if there is a countable set I, a $\mathbf{\Sigma}^1_1$-induction Φ on $\mathcal{P}(I) \times P$ and an $i^* \in I$ such that $X = \mathrm{Ind}(\Phi, i^*)$. We shall denote by $\mathbf{\Sigma}^1_1$-IND the class of $\mathbf{\Sigma}^1_1$-inductive sets.

The game quantifier. Let P be a Polish space and A a Borel subset of $\omega^\omega \times P$. For each fixed $x \in P$ the set $A_x := \{\alpha \in \omega^\omega : (\alpha, x) \in A\}$ can be viewed as the payoff of a Borel game on ω. So by Martin's Borel Determinacy Theorem this game A_x is determined: if we denote by $\Game A$ the set

$$\{x \in P : \text{Player I has a winning strategy in } A_x\},$$

the complement of $\Game A$ in P is the set

$$\{x \in P : \text{Player II has a winning strategy in } A_x\},$$

whence we deduce that both $\Game A$ and $P \setminus \Game A$ are $\mathbf{\Sigma}^1_2$.

If Γ is a class of Borel sets, we denote by $\Game\Gamma$ the class $\{\Game A : A \subset \omega^\omega \times \omega^\omega,\ A \in \Gamma\}$. It is well known that $\Game\mathbf{\Sigma}^0_1 = \mathbf{\Pi}^1_1$.

For $\Gamma = \mathbf{\Sigma}^0_2$, it follows from Wolfe's proof of $\mathbf{\Sigma}^0_2$ determinacy (see for example [4, 6A.3]) that if $A \subset \omega^\omega \times P$ is $\mathbf{\Sigma}^0_2$ one can define an analytic induction $\Phi : \mathcal{P}(I) \times P \to \mathcal{P}(I)$ (where I is the countable set $\{s \in \mathrm{Seq}(\omega) : |s| \text{ even}\}$) such that Player I has a winning strategy in the game A_x if and only if the empty sequence $\emptyset$ belongs to $\Phi^\infty(x)$. This shows that $\Game\mathbf{\Sigma}^0_2 \subset \mathbf{\Sigma}^1_1$-IND. Conversely, it was shown by R. Solovay (see [4, 7C.10]) that any $\mathbf{\Sigma}^1_1$-inductive set is $\Game\mathbf{\Sigma}^0_2$, that is, $\Game\mathbf{\Sigma}^0_2 = \mathbf{\Sigma}^1_1$-IND.

Cofinal and quasi-bounded trees. As we said in the abstract, a tree T on ω is said to be *cofinal* if for every $\alpha \in \omega^\omega$ there is an infinite branch β of T such that $\alpha \leq \beta$. We will say that such a branch β is *above* α.

If a tree T is not cofinal there is an $\alpha \in \omega^\omega$ such that no branch of T (if any) is above α. Such an α need not be a bound for the branches of T, which would mean that "for all $\beta \in \lceil T \rceil$, $\beta \leq \alpha$", and we shall say that α is a *quasi-bound* for T, and that T *is quasi-bounded.*

It is well known that trees on ω and closed subsets of ω^ω are closely related. As above a subset A of ω^ω is said to be *cofinal* (sometimes also *dominating*) if for every $\alpha \in \omega^\omega$ there is some $\beta \geq \alpha$ in A. The subsets of ω^ω which are not cofinal will also be called quasi-bounded. The structure of cofinal subsets of ω^ω was already studied by several people (see [5], [1] or [2]).

The aim of this paper is to prove that the set QB of quasi-bounded trees on ω is a $\Game\mathbf{\Sigma}^0_2$-complete subset of $\mathcal{T}$. First we will prove that QB is $\Game\mathbf{\Sigma}^0_2$, hence $\mathbf{\Sigma}^1_1$-inductive. Then we will show that every $\mathbf{\Sigma}^1_1$-inductive subset of ω^ω is continuously reducible to QB. This will complete the proof that QB is $\mathbf{\Sigma}^1_1$-IND-complete. In fact this will also prove that any $\mathbf{\Sigma}^1_1$-inductive set is $\Game\mathbf{\Sigma}^0_2$, hence will yield a new (but more complicated) proof of Solovay's result.

We will also consider the set QBC of closed quasi-bounded subsets of the Baire space, equipped with the Effros Borel structure. This set was already studied by S. Solecki ([5]), in connection with Haar null sets of a non-locally compact Polish group. He showed this set is $\mathbf{\Delta}^1_2$ but not $\mathbf{\Sigma}^1_1$. We shall prove here that it is $\mathbf{\Sigma}^1_1$-IND-complete.

There are only very few examples in the literature of true $\Game\mathbf{\Sigma}^0_2$ sets. The most important one is given by Kechris in [3], where he shows that $\mathbf{\Sigma}^1_1$-IND is the exact maximum complexity of σ-ideals of compact sets with $\mathbf{\Sigma}^1_1$ bases.

The main interest of our result is to yield a "natural" and combinatorially simple example of a $\Game\mathbf{\Pi}^0_2$ set. It could be used to prove that a set X is not $\Game\mathbf{\Pi}^0_2$ by reducing continuously QB to it, in the same way as one can prove that a set is not $\mathbf{\Sigma}^1_1$ by constructing a continuous reduction of WF to it.

Definition 1. For any tree T on ω, we denote by T° the tree defined by

$$s \in T^\circ \;\Leftrightarrow\; (s = \emptyset \text{ or } |s| \leq s(0) \text{ or } s = \langle k \rangle ^\frown t \text{ with } t_{|k} \notin T).$$

It is clear from the definition that if $\langle k \rangle ^\frown t$ belongs to T° and $k \leq l$ then $\langle l \rangle ^\frown t$ also belongs to T°.

Lemma 2. *Let T be a monotone tree on ω. Then the tree T° is quasi-bounded if and only if T is ill-founded. Moreover, for any branch α of T, $\langle 0 \rangle ^\frown \alpha$ is a quasi-bound for T°.*

Proof. Assume first T is ill-founded and denote by α any branch of T. Then we claim that $\langle 0\rangle^\frown\alpha$ is a quasi-bound for T°.

Indeed, assume by contradiction that $\langle k\rangle^\frown\beta$ is a branch of T° above $\langle 0\rangle^\frown\alpha$; then $t := \beta_{|k} \notin T$. But since $\alpha \le \beta$ we have $s := \alpha_{|k} \le \beta_{|k} = t$. So $s \in T$ since $\alpha \in \lceil T\rceil$, $t \notin T$ and $s \le t$, in contradiction with $T \in \mathcal{T}^+$.

Assume now T is well-founded and $\langle m\rangle^\frown\alpha \in \omega^\omega$. We claim that T° possesses a branch above $\langle m\rangle^\frown\alpha$.

Indeed, $\alpha \notin \lceil T\rceil = \emptyset$. Hence there is some integer k such that $\alpha_{|k} \notin T$. Replacing k by $\max(k, m)$ if necessary, we can assume $m \le k$. Then $\langle k\rangle^\frown\alpha$ is a branch of T°, and $\langle m\rangle^\frown\alpha \le \langle k\rangle^\frown\alpha$. ■

THEOREM 3. *The set* QB *is* $\Game\mathbf{\Sigma}^0_2$.

Proof. Define the mapping $\psi : \mathrm{Seq}(2) \to \mathrm{Seq}(\omega)$ by counting the blocks of contiguous 0's inside s: if $\psi(s) = \langle n_0, n_1, \ldots, n_{k-1}\rangle$ for some $s \in \mathrm{Seq}(2)$, then the sequence s contains k terms equal to 1, with n_0 zeros before the first 1, n_1 zeros between the first and the second 1, …, n_{k-1} zeros between the last two 1's.

So ψ is defined inductively by letting

$$\begin{cases} \psi(\emptyset) = \emptyset, \\ \psi(\langle 1\rangle) = \langle 0\rangle, \\ \psi(s^\frown\langle 0\rangle) = \psi(s), \\ \psi(s^\frown\langle 1,1\rangle) = \psi(s^\frown\langle 1\rangle)^\frown\langle 0\rangle, \\ \psi(s^\frown\langle 1\rangle) = u^\frown\langle p\rangle \ \Rightarrow\ \psi(s^\frown\langle 0,1\rangle) = u^\frown\langle p+1\rangle. \end{cases}$$

Then it is clear that $|\psi(s)| \le |s|$ and that for any two sequences s and s' such that $s \prec s'$ we have $\psi(s) \preceq \psi(s')$.

Denote by P_∞ the set of those γ's in 2^ω which have infinitely many coordinates equal to 1. For $\gamma \in P_\infty$ there is a unique $\beta \in \omega^\omega$ which we denote by $\widehat{\psi}(\gamma)$ such that $s \prec \gamma \ \Rightarrow\ \psi(s) \prec \beta$. It is easily checked and well known that $2^\omega \setminus P_\infty$ is countable and that $\widehat{\psi}$ is a homeomorphism from P_∞ onto ω^ω.

For T a given tree we define the game $G_{\mathrm{qb}}(T)$ where Player I plays integers $n_0, n_1, \ldots$, and Player II plays $c_0, c_1, \ldots$ in $\{0,1\}$ with the following two rules:

- R_1: for every k, $\psi(\langle c_0, c_1, \ldots, c_{k-1}\rangle) \in T$.
- R_2: for every k, $\langle n_0, n_1, \ldots, n_{p-1}\rangle \le \psi(\langle c_0, c_1, \ldots, c_{k-1}\rangle)$, where p is the length of $\psi(\langle c_0, c_1, \ldots, c_{k-1}\rangle)$.

The run where Player I plays (n_k) and Player II plays (c_k) is won by Player II iff $(c_k) \in P_\infty$.

Clearly the set

$$A := \{((n_k), (c_k), T) : \text{Player II respects the rules and } (c_k) \notin P_\infty\}$$

is $\mathbf{\Sigma}^0_2$ in $\omega^\omega \times 2^\omega \times \mathcal{T}$. Hence the set $\Game A$ is $\Game\mathbf{\Sigma}^0_2$. Theorem 3 will then follow from the next two lemmas.

LEMMA 4. *If Player II has a winning strategy in the game $G_{\mathrm{qb}}(T)$, then the tree T is continuously cofinal, i.e. there is a continuous function $f : \omega^\omega \to \lceil T \rceil$ such that $f(\alpha) \geq \alpha$ for every $\alpha \in \omega^\omega$. In particular, T is cofinal.*

If τ is a winning strategy for Player II, it defines a continuous function $g : \omega^\omega \to 2^\omega$ such that for every α in ω^ω and every $s = \langle n_0, n_1, \ldots, n_{k-1}\rangle \prec \alpha$ played by Player I the answer $\langle c_0, c_1, \ldots, c_{k-1}\rangle$ of Player II under τ satisfies $\langle c_0, c_1, \ldots, c_{k-1}\rangle \prec g(\alpha)$. It then follows from the rule R_1 that we have $\psi(\langle c_0, c_1, \ldots, c_{k-1}\rangle) \in T$. Moreover, since Player II wins, the run $g(\alpha)$ is in P_∞. Hence $\widehat{\psi}(g(\alpha)) \in \omega^\omega$ and $\widehat{\psi}(g(\alpha))_{|p} \in T$ for arbitrarily large p, whence we conclude that $f(\alpha) := \widehat{\psi}(g(\alpha)) \in \lceil T \rceil$. Since ψ is continuous on P_∞, $f = \widehat{\psi} \circ g$ itself is continuous. Finally, it follows from the rule R_2 that $f(\alpha)_{|k} \geq \alpha_{|k}$ for arbitrarily large k, hence $f(\alpha) \geq \alpha$. ∎

LEMMA 5. *If Player I has a winning strategy in $G_{\mathrm{qb}}(T)$, then T is quasi-bounded.*

If σ is a winning strategy for Player I, it induces as above a continuous function $h : 2^\omega \to \omega^\omega$. Then the range $K := h(2^\omega)$ is a compact subset of ω^ω, and one can define for all n the integer $\alpha(n) = \sup_{x\in K} x(n)$. We claim that this α is a quasi-bound for T.

Indeed, if β were a branch of T such that $\alpha \leq \beta$, then Player II could play the following infinite run γ: $\beta(0)$ times 0, then 1, then $\beta(1)$ times 0, then $1, \ldots.$ This would respect the rule R_1 since $\psi(\gamma_{|k}) \prec \beta$ for all k. And since $\gamma \in P_\infty$, we would have $\beta = \widehat{\psi}(\gamma)$. Moreover, since $h(\gamma) \in K$, we would have $h(\gamma) \leq \alpha \leq \beta = \widehat{\psi}(\gamma)$; this shows that the rule R_2 would also be respected. Finally, since $\gamma \in P_\infty$, Player II would win the run against the strategy σ. This contradiction completes the proof of the lemma. ∎

Thus the proof of Theorem 3 is complete. One can notice that a similar game was used in [2] in order to prove that any cofinal $\mathbf{\Sigma}^1_1$ subset of ω^ω is continuously cofinal.

REMARK. It follows from the previous proof that a quasi-bound for T can be computed continuously from a winning strategy for Player I in $G_{\mathrm{qb}}(T)$. Conversely, a quasi-bound α for T yields a simple strategy σ for Player I: he plays α whatever Player II is answering. This strategy is clearly winning: in any run compatible with σ a position $(\alpha_{|k}, \langle c_0, c_1, \ldots, c_{k-1}\rangle)$ is reached for which no extension of $\psi(\langle c_0, c_1, \ldots, c_{k-1}\rangle)$ can be found in T above α; and beyond this position Player II must always play 0.

We now intend to show that QB has complexity at least $\mathbf{\Sigma}^1_1$-IND.

THEOREM 6. *If X is a $\mathbf{\Sigma}^1_1$-IND subset of ω^ω, there exists a continuous mapping $x \mapsto S(x)$ from ω^ω to $\mathcal{T}$ such that $S(x) \in$ QB if and only if $x \in X$.*

Proof. Without loss of generality we assume that $\Phi : 2^\omega \times \omega^\omega \to 2^\omega$ is a $\mathbf{\Sigma}^1_1$-induction on ω and that

$$x \in X \;\Leftrightarrow\; 0 \in \Phi^\infty(x).$$

Then for each n the set $E_n := \{(a, x) \in 2^\omega \times \omega^\omega : n \in \Phi(a, x)\}$ is $\mathbf{\Sigma}^1_1$ and there is some tree T_n on $2 \times \omega \times \omega$ such that

$$(a, x) \in E_n \;\Leftrightarrow\; \exists \beta \in \omega^\omega \;\; (a, \beta, x) \in \lceil T_n \rceil$$

where we identify the subset $\lceil T_n \rceil$ of $(2 \times \omega \times \omega)^\omega$ with a subset of $2^\omega \times \omega^\omega \times \omega^\omega$. Identifying $\mathrm{Seq}(2 \times \omega \times \omega)$ with the set

$$\{(s, t, u) \in \mathrm{Seq}(2) \times \mathrm{Seq}(\omega) \times \mathrm{Seq}(\omega) : |s| = |t| = |u|\}$$

we now define trees $\widehat{T}_n$ and U_n on $2 \times \omega \times \omega$ by

$$(s, t, u) \in \widehat{T}_n \;\Leftrightarrow\; \exists s' \; \exists t' \;\; s' \leq s,\, t' \leq t,\, (s', t', u) \in T_n,$$

$$(s, t, u) \in U_n \;\Leftrightarrow\; \begin{cases} (s, t, u) = (\emptyset, \emptyset, \emptyset) \\ \text{ or } \; |s| = |t| = |u| \leq t(0) \\ \text{or else } t = \langle k \rangle^\frown t^* \text{ with } (s_{|k}, t^*_{|k}, u_{|k}) \notin \widehat{T}_n. \end{cases}$$

Fix a bijection $(n, p) \mapsto n * p$ from $\omega \times \omega$ onto ω which is separately increasing with respect to each variable and satisfies $n * 0 \leq 0 * n$. Then we necessarily have $0 * 0 = 0$. For example we can put

$$n * p = \frac{(n + p)(n + p + 1)}{2} + p.$$

Then, for each $s \in \mathrm{Seq}(\omega)$ and each $n \in \omega$, we define the sequence $\theta_n(s) \in \mathrm{Seq}(\omega)$ by

$$\theta_n(s) = \langle s(n * 0), s(n * 1), \ldots s(n * (k - 1))\rangle \qquad \text{where } n * (k - 1) < |s| \leq n * k.$$

In particular we get $\theta_n(s) = \emptyset$ if $|s| \leq n * 0$. Define also $\theta^*(s) \in \mathrm{Seq}(2)$ by

$$\theta^*(s) = \langle c_0, c_1, \ldots, c_{p-1}\rangle \quad \text{where} \quad \begin{cases} (p - 1) * 0 < |s| \leq p * 0 \\ \text{ and } c_i = 1 \;\Leftrightarrow\; s(i * 0) \text{ is odd} \end{cases}$$

Observe that for any $s \in \mathrm{Seq}(\omega)$ and any n, if $k = |\theta_n(s)|$ and $p = |\theta^*(s)|$, we have

$$0 * (k - 1) \leq n * (k - 1) < |s| \leq p * 0 \leq 0 * p,$$

hence $k-1<p$, thus $|\theta_n(s)|\le|\theta^*(s)|$. Moreover it is clear that if $s\prec s'$ we have $\theta_n(s)\preceq\theta_n(s')$ for all integers n, and $\theta^*(s)\preceq\theta^*(s')$. We extend θ_n and θ^* to ω^ω by letting

$$\widehat{\theta}_n(\alpha)(k)=\alpha(n*k),$$
$$\widehat{\theta}^*(\alpha)(i)=\begin{cases}1 & \text{if } s(i*0) \text{ is odd},\\ 0 & \text{otherwise.}\end{cases}$$

We now define, for $x\in\omega^\omega$, a tree $S(x)$ by

$$s\in S(x)\ \Leftrightarrow\ \begin{cases}s=\emptyset \text{ or } (s(0)=0 \text{ and } \forall n<|\theta^*(s)|\\ (\theta^*(s)(n)=1 \text{ or } (\theta^*(s)_{|k},\theta_n(s),x_{|k})\in U_n)),\end{cases}$$

where $k=|\theta_n(s)|$.

The theorem will follow from the next four lemmas.

Lemma 7. *The mapping $x\mapsto S(x)$ is continuous from ω^ω to $\mathcal{T}$.*

Proof. For any $s\in\mathrm{Seq}(\omega)$ define $k:=|\theta^*(s)|$. Then "$s\in S(x)$" depends only on $x_{|k}$. Hence $\{x\in\omega^\omega : s\in S(x)\}$ is open and closed. This shows that the mapping $x\mapsto S(x)$ is continuous from ω^ω to $\mathcal{T}$. ∎

Lemma 8. *For $a\in 2^\omega$ and $x\in\omega^\omega$ one has*

$$n\in\Phi(a,x)\ \Leftrightarrow\ \exists\beta\in\omega^\omega\ \ (a,\beta,x)\in\lceil\widehat{T}_n\rceil\ \Leftrightarrow\ U_n(a,x)\in\mathrm{QB}.$$

Moreover, if β is any branch of $\widehat{T}_n(a,x)$, then $\langle 0\rangle^\frown\beta$ is a quasi-bound for $U_n(a,x)$.

Proof. Notice that $T_n\subset\widehat{T}_n$. Thus if $n\in\Phi(a,x)$, then $(a,x)\in E_n$, hence there exists a β such that $(a,\beta,x)\in\lceil T_n\rceil\subset\lceil\widehat{T}_n\rceil$.

Conversely, if $(a,\beta,x)\in\lceil\widehat{T}_n\rceil$ then for every integer k, $(a_{|k},\beta_{|k},x_{|k})$ belongs to $\widehat{T}_n$. Hence there are $s\in 2^k$ and $t\in\omega^k$ such that $(s,t,x_{|k})\in T_n$, $s\le a_{|k}$ and $t\le\beta_{|k}$. It follows that the set

$$V:=\{(s,t,u)\in T_n : |s|=|t|=|u|\,,\ s\le a,\ t\le\beta,\ u\prec x\}$$

is an infinite and finitely branching tree. By König's Lemma the tree V is ill-founded. If (a',β',x') is a branch of V, one necessarily has $a'\le a$ and $x'=x$. Thus $(a',x)\in E_n$, hence $n\in\Phi(a',x)\subset\Phi(a,x)$.

Notice that for any $a\in 2^\omega$ and any $x\in\omega^\omega$, $U_n(a,x)=\widehat{T}_n(a,x)^\circ$ and $\widehat{T}_n(a,x)$ is monotone. Then it follows from Lemma 2 that $U_n(a,x)=\widehat{T}_n(a,x)^\circ$ is quasi-bounded if and only if $\widehat{T}_n(a,x)$ is ill-founded, that is, if and only if $n\in\Phi(a,x)$. ∎

Lemma 9. *If $x\notin X$ then $S(x)$ is cofinal.*

Proof. Assume $x \notin X$ and let $a = \Phi^{\infty}(x)$. Then $0 \notin a$ and for all $n \notin a$ we have $n \notin \Phi(a, x)$. Let $\alpha \in \omega^{\omega}$ and define $\alpha_n = \widehat{\theta}_n(\alpha)$ for all n. We will produce a branch β of $S(x)$ such that $\beta \geq \alpha$.

For $n \in a$ we define $\beta_n = \alpha_n$. For $n \notin a$, since $U_n(a, x)$ is cofinal, by Lemma 8 we can find $\beta_n \in \lceil U_n(a, x)\rceil$ such that $\alpha_n \leq \beta_n$. Replacing if necessary $\beta_n(0)$ by $\beta_n(0) + 1$, we can assume that $\beta_n(0)$ is odd for $n \in a$ and even for $n \notin a$. Then defining β by

$$\forall n\ \forall p \quad \beta(n * p) = \beta_n(p)$$

we get $\widehat{\theta}_n(\beta) = \beta_n \geq \alpha_n = \widehat{\theta}_n(\alpha)$, $\beta(0) = \beta_0(0)$ is even and $\widehat{\theta}^*(\beta) = a$.

It follows easily that $\beta \geq \alpha$ and that for each l, $\beta_{|l} \in S(x)$, hence $\beta \in \lceil S(x)\rceil$. ∎

LEMMA 10. *If $x \in X$ then $S(x)$ is quasi-bounded.*

Proof. If $x \in X$, then $0 \in \Phi^{\infty}(x)$, and we can define for every $n \in \Phi^{\infty}(x)$ the rank $\varrho_n := \min\{\xi : n \in \Phi^{(\xi)}(x)\} \in \omega_1$ and then $a_n := \{p \in \omega : \varrho_p < \varrho_n\}$. Thus, for $n \in \Phi^{\infty}(x)$, we have $n \in \Phi(a_n, x)$. It follows that $\widehat{T}_n(a_n, x)$ is ill-founded. Then we can choose a branch α_n^* of $\widehat{T}_n(a_n, x)$ and let $\alpha_n := \langle 0\rangle^\frown \alpha_n^*$.

For $n \notin \Phi^{\infty}(x)$ we choose α_n equal to the null sequence $\mathbf{0}$. Finally, defining α by

$$\forall n\ \forall p \quad \alpha(n * p) = \alpha_n(p)$$

we get $\widehat{\theta}_n(\alpha) = \alpha_n$ for all n.

We claim that α is a quasi-bound for $S(x)$. Indeed, assuming by contradiction that β is a branch of $S(x)$ above α, we should have $\beta_n := \widehat{\theta}_n(\beta) \geq \widehat{\theta}_n(\alpha) = \alpha_n$. Then put $a := \widehat{\theta}^*(\beta) \in 2^{\omega}$. Since $\beta \in \lceil S(x)\rceil$, we should have $\beta(0)$ even, hence $0 \in \Phi^{\infty}(x) \setminus a$. It follows that $\{\varrho_n : n \in \Phi^{\infty}(x) \setminus a\}$ should be nonempty. Thus there would be an integer $m \in \Phi^{\infty}(x) \setminus a$ such that $\varrho_m = \min\{\varrho_n : n \in \Phi^{\infty}(x) \setminus a\}$. In particular $m \notin a$, hence $\beta_m \in \lceil U_m(a, x)\rceil$.

By minimality of ϱ_m we would have $a_m = \{p : \varrho_p < \varrho_m\} \subset a$, hence $m \in \Phi(a, x)$. Since $\alpha_m^* \in \lceil \widehat{T}_m(a_m, x)\rceil$, this would also imply that $\alpha_m^* \in \lceil \widehat{T}_m(a, x)\rceil$, hence α_m would be a quasi-bound for $U_m(a, x)$ by Lemma 8, in contradiction with $\alpha_m \leq \beta_m$ and $\beta_m \in \lceil U_m(a, x)\rceil$. ∎

This completes the proof of Theorem 6.

Quasi-bounded closed subsets of the Baire space. Now we are interested in closed subsets of ω^{ω} and will denote by $\mathcal{F}(\omega^{\omega})$ the set of nonempty closed subsets of ω^{ω} which we equip with the Effros Borel structure. As for trees on ω, we shall say that a closed subset F of ω^{ω} is cofinal if for every $\alpha \in \omega^{\omega}$ there is some $\beta \geq \alpha$ in F, and that F is quasi-bounded otherwise. We shall say that α is a quasi-bound for F if $F \cap \{\beta : \beta \geq \alpha\} = \emptyset$.

We will denote by QBC the subset of $\mathcal{F}(\omega^\omega)$ consisting of the quasi-bounded closed subsets of ω^ω.

We shall show in the following theorem that QBC behaves with respect to Borel reducibility in the same way as QB does with respect to continuous reducibility.

THEOREM 11. QBC *is* $\mathbf{\Sigma}^1_1$-IND-*complete.*

This follows immediately from the next two lemmas.

LEMMA 12. *If P is a Polish space and $F : P \to \mathcal{F}(\omega^\omega)$ a Borel mapping then $F^{-1}(\mathrm{QBC})$ is $\mathbf{\Sigma}^1_1$-inductive.*

Proof. For each $s \in \mathrm{Seq}(\omega)$ we denote by N_s the basic open set $\{\alpha \in \omega^\omega : s \prec \alpha\}$. For $x \in P$ define

$$T(x) := \{s \in \mathrm{Seq}(\omega) : N_s \cap F(x) \neq \emptyset\},$$

which is clearly a tree on ω such that $\lceil T(x) \rceil = F(x)$. By definition of the Effros Borel structure, $\{H : N_s \cap H \neq \emptyset\}$ is Borel in $\mathcal{F}(\omega^\omega)$, thus $\{x \in P : s \in T(x)\}$ is Borel for all s. Hence the mapping $f : x \mapsto T(x)$ is Borel from P to $\mathcal{T}$. It is immediate from the definitions that $f(x) \in \mathrm{QB} \Leftrightarrow F(x) \in \mathrm{QBC}$. So $F^{-1}(\mathrm{QBC}) = f^{-1}(\mathrm{QB})$.

As QB is $\mathbf{\Sigma}^1_1$-inductive in $\mathcal{T}$, there is an analytic induction $\Phi : \mathcal{P}(\omega) \times \mathcal{T} \to \mathcal{P}(\omega)$ such that $T \in \mathrm{QB} \Leftrightarrow 0 \in \Phi^\infty(T)$. For $a \in \mathcal{P}(\omega)$ and $x \in P$ define

$$\Psi(a, x) := \Phi(a, f(x)).$$

Then Ψ is an induction and clearly $\Psi^\xi(x) = \Phi^\xi(f(x))$ for each ξ, hence $\Psi^\infty(x) = \Phi^\infty(f(x))$ and

$$\begin{aligned} F(x) \in \mathrm{QBC} \ \Leftrightarrow\ f(x) \in \mathrm{QB} \ &\Leftrightarrow\ 0 \in \Phi^\infty(f(x)) \\ &\Leftrightarrow\ 0 \in \Psi^\infty(x) \ \Leftrightarrow\ x \in \mathrm{Ind}(\Psi, 0). \end{aligned}$$

Then $n \in \Psi(a, x) \Leftrightarrow \exists T \in \mathcal{T}\ (T = f(x) \text{ and } n \in \Phi(a, T))$, whence we conclude that Ψ is $\mathbf{\Sigma}^1_1$ and finally that $F^{-1}(\mathrm{QBC})$ is $\mathbf{\Sigma}^1_1$-inductive. ∎

LEMMA 13. *If X is a $\mathbf{\Sigma}^1_1$-*IND *subset of ω^ω, then there exists a Borel reduction of X to* QBC*.*

Proof. By Theorem 6 there is a continuous function S from ω^ω to $\mathcal{T}$ such that $S(x) \in \mathrm{QB} \Leftrightarrow x \in X$. Denote for $n \in \omega$ by z_n the null sequence of length n, and

$$\widetilde{S}(x) := \{s^\frown z_n : s \in S(x),\, n \in \omega\}.$$

Clearly if $\alpha \in \omega^\omega$ is any sequence such that $\alpha(n) > 0$ for all n, then for all $\beta \in \omega^\omega$ we have

$$\beta \geq \alpha \text{ and } \beta \in \lceil S(x) \rceil \ \Leftrightarrow\ \beta \geq \alpha \text{ and } \beta \in \lceil \widetilde{S}(x) \rceil,$$

hence $S(x) \in \mathrm{QB} \Leftrightarrow \widetilde{S}(x) \in \mathrm{QB}$. Since

$$s \in \widetilde{S}(x) \ \Leftrightarrow\ (\exists k, l \le |s| \ \ s_{|k} \in S(x) \text{ and } s = s_{|k}{}^\frown z_l),$$

one sees that $\widetilde{S}$ is continuous and that $X = \widetilde{S}^{-1}(\mathrm{QB})$. Then define $F(x) := \lceil \widetilde{S}(x) \rceil$. It is immediate that $F(x)$ is a quasi-bounded closed subset of ω^ω iff $\widetilde{S}(x) \in \mathrm{QB}$, i.e. iff $x \in X$. Finally, it is enough to notice that for each $s \in \mathrm{Seq}(\omega)$ and each $x \in \omega^\omega$,

$$\begin{aligned} F(x) \cap N_s \neq \emptyset \ \Rightarrow\ s \in \widetilde{S}(x) \ \Rightarrow\ \forall n \ \ s^\frown z_n \in \widetilde{S}(x) \\ \Rightarrow\ s^\frown \mathbf{0} \in \lceil \widetilde{S}(x) \rceil \ \Rightarrow\ F(x) \cap N_s \neq \emptyset, \end{aligned}$$

so that $\{x : N_s \cap F(x) \neq \emptyset\}$ is clopen, and hence F is Borel. ■

References

[1] J. Brendle, G. Hjorth and O. Spinas, *Regularity properties for dominating projective sets*, Ann. Pure Appl. Logic 72 (1995), 291–307.

[2] G. Debs and J. Saint Raymond, *Cofinal* $\mathbf{\Sigma}^1_1$ *and* $\mathbf{\Pi}^1_1$ *subsets of* ω^ω, Fund. Math. 159 (1999), 161–193.

[3] A. Kechris, *The descriptive set theory of σ-ideals of compact sets*, in: Logic Colloquium '88 (Padova, 1988), Stud. Logic Found. Math. 127, North-Holland, Amsterdam, 1989, 117–138.

[4] Y. N. Moschovakis, *Descriptive Set Theory*, North-Holland, Amsterdam, 1980.

[5] S. Solecki, *Haar null and non-dominating sets*, Fund. Math. 170 (2001), 197–217.

[6] O. Spinas, *Dominating projective sets in the Baire space*, Ann. Pure Appl. Logic 68 (1994), 327–342.

Analyse Fonctionnelle
Institut de Mathématique de Jussieu
Boîte 186
4, place Jussieu
75252 Paris Cedex 05, France
E-mail: jsr@ccr.jussieu.fr

Received 15 November 2005;
in revised form 3 February 2006