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Large superdecomposable $E(R)$-algebras

by

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In honour of Claus Michael Ringel on the occasion of his 60th birthday

Abstract. For many domains R (including all Dedekind domains of characteristic 0 that are not fields or complete discrete valuation domains) we construct arbitrarily large superdecomposable R-algebras A that are at the same time $E(R)$-algebras. Here "superdecomposable" means that A admits no (directly) indecomposable R-algebra summands $\neq 0$ and "$E(R)$-algebra" refers to the property that every R-endomorphism of the R-module A is multiplication by an element of A.

1. Introduction. Schultz [15] introduced the notion of an *E-ring* as a ring R such that the endomorphism ring of its additive group is isomorphic to R under the natural map $\eta \mapsto \eta(1)$, i.e. each endomorphism acts as multiplication by an element of R. E-rings have been investigated in several papers: see e.g. Dugas–Mader–Vinsonhaler [5], Dugas–Göbel [4], Göbel–Strüngmann [11], proving the existence of arbitrarily large E-rings, E-rings whose additive groups are $\aleph_1$-free abelian groups, etc.

Göbel–Strüngmann [11] discusses $E(R)$-*algebras*, i.e. algebras A over a domain R such that every endomorphism of A as an R-module is multiplication by an element of A. The existence of large $E(R)$-algebras over many domains R is established. Fuchs–Lee [7] constructs $E(R)$-algebras over certain domains R that are superdecomposable as R-algebras in the sense that they do not admit any algebra summand that is not a direct product of two non-zero subalgebras. In Theorem 5.3 we give a common generalization of these two results by proving the existence of arbitrarily large superdecomposable $E(R)$-algebras that are, in addition, $\aleph_1$-free in the sense that every countable subset is contained in a free R-submodule.

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Our proof is based on a version of Shelah's Black Box (see Theorem 3.1 below) which we borrow from Corner–Göbel [3]. (We emphasize that this principle is provable in ZFC.) Alternatively we could have used the "Strong Black Box" (see [13]) which has the advantage that some of the algebraic proofs are simpler, but has the drawback that the possible sizes of $E(R)$-algebras are more restricted. We work in an R-algebra $\widehat{F}$ that is a completion of a semigroup algebra $F = R[T]$ where the monoid T is appropriately chosen: T is a direct product of two monoids, one of which serves to guarantee that the R-algebra A to be constructed is superdecomposable, while the other will be responsible for the E-ring property of A. Our method follows closely the pattern of Corner–Göbel [3], which allows us to skip those details of the proofs that are obvious modifications of arguments in [3].

In Theorem 5.4 we prove the abundance of arbitrarily large superdecomposable $E(R)$-algebras. This, along with the similar result on indecomposable $E(R)$-algebras (cf. Dugas–Mader–Vinsonhaler [5]), shows that—as far as merely direct decompositions are concerned—$E(R)$-algebras do not display any particular behavior.

2. Superdecomposable algebras. Let R denote a commutative domain that contains a countable subsemigroup $\mathbb{S} = \{s_0 = 1, s_1, \ldots, s_n, \ldots\}$ (not containing 0) such that R is Hausdorff in the $\mathbb{S}$-topology (where the ideals Rq_n $(n \in \omega)$ form a base of neighborhoods of 0 in R), i.e. $\bigcap_{n\in\omega} Rq_n = 0$; here we have used the notation $q_n = s_0 s_1 \cdots s_n \in \mathbb{S}$. (Note that the Hausdorff property of the $\mathbb{S}$-topology is equivalent to the fact that the localization $R_{\mathbb{S}}$ of R at $\mathbb{S}$ is not a fractional ideal of R.) The symbol $\widehat{R}$ will denote the completion of R in its $\mathbb{S}$-topology. R is then a dense subalgebra of $\widehat{R}$.

Let μ denote an infinite cardinal; it is viewed as an initial ordinal, so we can talk about its subsets. We define a monoid T_1 whose elements are the finite subsets of μ and multiplication is defined via

$$\sigma \cdot \tau = \sigma \cup \tau$$

for all $\sigma, \tau \in T_1$. The empty set serves as the identity of T_1. (This monoid was inspired by Corner [1].)

Let F denote the semigroup algebra of T_1 over R, i.e.

$$F = R[T_1] = \bigoplus_{\tau \in T_1} R\tau;$$

this is an R-algebra with identity $\{\emptyset\}$. The $\mathbb{S}$-topology on F is Hausdorff. The $\mathbb{S}$-completion $\widehat{F}$ of F is an $\widehat{R}$-algebra containing F as a dense R-subalgebra whose elements $x \neq 0$ may be viewed as countable sums $x = \sum_{i\in\omega} r_i \tau_i$ with $r_i \in \widehat{R}$, $\tau_i \in T_1$, where for every $k \in \omega$ almost all (i.e. all but finitely many) coefficients r_i are divisible by q_k.

By the *support* $[x]$ of x is meant the set $\{\tau_i \mid r_i \neq 0\} \subseteq T_1$; this is always a countable subset, since $\mathbb{S}$ was assumed to be countable.

LEMMA 2.1. *Every R-algebra A that lies between the R-algebras $F = R[T_1]$ and $\widehat{F}$ constructed above for the infinite cardinal μ is superdecomposable as an R-algebra.*

Proof. Consider a non-zero algebra summand C of A; $A = C \oplus C'$. The C-coordinate of the identity of A is an idempotent element $0 \neq e \in A$.

CASE 1. If there is an ordinal $\alpha \in \mu$ not contained in any set in the support $[e]$, then $\{\alpha\} \in F$ is an idempotent which evidently satisfies $e\{\alpha\} \neq 0$. It also satisfies $e\{\alpha\} \neq e$, since for any $\tau \in [e]$ we have $\tau \cup \alpha \in [e\{\alpha\}] \setminus [e]$. The elements $e\{\alpha\}$ and $e - e\{\alpha\}$ are non-zero orthogonal idempotents in A with sum e, establishing the decomposability of C into the direct sum of two R-subalgebras.

CASE 2. If there is no ordinal α as in Case 1, then $\mu = \aleph_0$ and $\mu = \bigcup[e]$. Write $e = \sum_{\tau\in[e]} r_\tau \tau$ $(r_\tau \in \widehat{R})$ or $e = \sum_{\tau\in T_1} r_\tau \tau \in \widehat{F}$ with $r_\tau = 0$ for all $\tau \in T_1 \setminus [e]$. Pick any $\tau_0 \in [e]$ with $r_{\tau_0} \neq 0$. If $e\{\alpha\} = e$, then

$$\sum_{\tau\in T_1} r_\tau(\{\alpha\} \cup \tau) = \sum_{\tau\in T_1} r_\tau \tau.$$

If $\alpha \notin \tau_0$, then the comparison of the coefficients of $\{\alpha\} \cup \tau_0 \in T_1$ on both sides yields

$$r_{\tau_0} + r_{\{\alpha\}\cup\tau_0} = r_{\{\alpha\}\cup\tau_0}.$$

Hence $r_{\tau_0} = 0$, contradicting the choice of τ_0. Hence $e\{\alpha\} \neq e$ for all $\alpha \in \mu$.

Suppose, by way of contradiction, that $e\{\alpha\} = 0$ for all $\alpha \in \mu \setminus [\tau_0]$. Then $\sum_{\tau\in T_1} r_\tau(\{\alpha\} \cup \tau) = 0$, where the coefficient of $\{\alpha\} \cup \tau_0$ is $r_{\tau_0} + r_{\{\alpha\}\cup\tau_0} = 0$. Thus $r_{\{\alpha\}\cup\tau_0} = -r_{\tau_0}$ for all $\alpha \in \mu \setminus [\tau_0]$, which is obviously impossible. Consequently, there is always an $\alpha \in \mu$ such that $e\{\alpha\} \neq 0$ (in addition to $e\{\alpha\} \neq e$), completing the proof. ∎

We now construct another superdecomposable R-algebra as follows; we utilize an idea due to Corner [2].

Let μ be an infinite cardinal and T_2 the monoid with elements (α, p) where $\alpha \in \mu, 0 \le p \in \mathbb{Q}$, and multiplication is defined via

$$(\alpha, p)(\beta, q) = (\max\{\alpha, \beta\}, \max\{p, q\}) \quad ((\alpha, p), (\beta, q) \in T_2).$$

Let F denote the semigroup algebra $R[T_2]$ and $\widehat{F}$ its $\mathbb{S}$-completion. Now the element $(0, 0) \in \mu \times \mathbb{Q}$ is the identity of F. We have again:

LEMMA 2.2. *Every R-algebra A between the R-algebras $F = R[T_2]$ and $\widehat{F}$ just constructed for the infinite cardinal μ is a superdecomposable R-algebra.*

Proof. It suffices to verify that for every non-zero idempotent $e = \sum_{i\in I} r_i(\alpha_i, p_i) \in \widehat{F}$ $(0 \neq r_i \in \widehat{R}$, $(\alpha_i, p_i) \in T_2)$ (I is some index set) we can find an idempotent $e' = (\alpha, p) \in F$ such that $0 \neq e(\alpha, p) \neq e$. If not all the p_i are equal, then choose any $p \in \mathbb{Q}$ such that $p_i < p < p_j$ for some $i, j \in I$. In this case, $e' = (\alpha, p)$ is as desired for any choice of $\alpha \in \mu$. On the other hand, if all the p_i $(i \in I)$ are equal and if we can choose an ordinal α with $\alpha_i < \alpha < \alpha_j$ for some $i, j \in I$, then $e' = (\alpha, p_i) \in F$ is a good choice. In the remaining case, the idempotent e must be of the form $e = (\beta, q) \in T_2$ or $e = (\beta, q) - (\beta + 1, q)$. Then we can choose $e' = (\beta, p)$ for any $q < p \in \mathbb{Q}$. Consequently, we can always find an idempotent e' that establishes superdecomposability. ∎

It is straightforward to check:

REMARK 2.3. If we replace the monoid T_j ($j = 1$ or 2) by a monoid $T = T_j \times T'$, where T' is any monoid, then the preceding lemmas are still valid.

3. The Black Box. We turn our attention to the construction of a superdecomposable $E(R)$-algebra between F and $\widehat{F}$. For the construction we shall need a version of Shelah's Black Box principle. (For a general discussion of this principle, we refer to Göbel–Trlifaj [12]; for the strong black box see Eklof–Mekler [6, Chapter XIII].)

Let $R, \mathbb{S}$ have the same meaning as in the preceding section. Furthermore, let κ be a cardinal such that $|R| \leq \kappa$, and assume in addition that λ is a cardinal satisfying

$$\lambda^\kappa = \lambda.$$

Then we have $\mathrm{cf}\,\lambda > \kappa \geq \aleph_0$; see e.g. Jech [14, p. 28].

The set $L = {}^{\omega>}\lambda$ of all finite sequences $\varrho = (\alpha_0, \ldots, \alpha_{n-1})$ (of length n) with $\alpha_i \in \lambda$ (the empty sequence is included) is a tree of length ω under the natural ordering: $\varrho_1 \leq \varrho_2$ in L if and only if ϱ_1 is an initial segment of ϱ_2. Maximal linearly ordered subsets $\mathbf{b} = \{\varrho_0 < \varrho_1 < \cdots < \varrho_n < \cdots\}$ of L are called *branches*; here the length of ϱ_n is n. The set of branches of L will be denoted by $\mathrm{Br}(L)$. Clearly, $|\mathrm{Br}(L)| = \lambda^{\aleph_0} = \lambda$.

Let T_0 be the free commutative monoid generated by the symbols u_ϱ for all $\varrho \in L$. Define the monoid T as

$$T = M \times T_0,$$

where $M = T_1$ or $M = T_2$ as constructed above in Section 2 with the choice $\mu = \aleph_0$. Thus the elements of T are of the form $\theta = (\tau, u)$, where $\tau \in M$ and $u \in T_0$. The semigroup algebra $F = R[T] = \bigoplus_{\theta\in T} R\theta$, its $\mathbb{S}$-completion $\widehat{F}$ and any R-algebra A in between are superdecomposable by Remark 2.3.

We will distinguish three natural kinds of supports depending on T_0, L and λ respectively.

Each element $0 \neq x \in \widehat{F}$ can be expressed uniquely as a sum $x = \sum_{i\in I} r_i(\tau_i, u_i)$ (where I is an indexing set with $1 \leq |I| \leq \aleph_0$) such that $0 \neq r_i \in \widehat{R}$ and $(\tau_i, u_i) \in T$ for all $i \in I$. Then $[x] = \{u_i \mid i \in I\} \subseteq T_0$ denotes the *support* of x. (If we want to emphasize that this is a subset of T_0, we will say that $[x]$ is the T_0-*support* of x.) Every element $u_i \in [x]$ is the unique product of certain generators $u_{\varrho_{ij}}$ $(j \leq n_i)$. The collection of all these ϱ_{ij} $(i \in I,\ j \leq n_i)$ constitutes the L-*support* $[x]_L \subseteq L$ of x. Finally, by the λ-*support* is meant the set $[x]_\lambda \subseteq \lambda$ of all ordinals used in $[x]_L$. The *norm* of x is defined as $\|x\| = \sup\,[x]_\lambda$.

These notions extend naturally to subsets. If $X \subseteq \widehat{F}$ is a set of cardinality $\leq \kappa$, then $[X] = \bigcup_{x\in X}[x]$ is the support of X and $[X]_L, [X]_\lambda$ are defined similarly. Observe that the norm of X is a well defined ordinal $\|X\| = \sup\,[X]_\lambda \in \lambda$, because $\operatorname{cf}\lambda > \kappa$.

For a subset I of λ of size $\leq \kappa$, we define

$$P_I = \bigoplus_{\theta\in M\times I'} R\theta$$

as a *canonical R-subalgebra*, where I' denotes the submonoid of T_0 generated by the u_ϱ with finite sequences $\varrho = (\alpha_0, \ldots, \alpha_n) \in {}^{\omega>}I$. Evidently, P_I is a subalgebra of F with support I' (and L-support ${}^{\omega>}I$) that is an R-free summand of size $\leq \kappa$ of F with free complement. (We often write simply P rather than P_I if there is no need for specifying the index set.) There are λ canonical R-subalgebras of F.

We also consider order-preserving embeddings

$$f : {}^{\omega>}\kappa \to L.$$

By a *trap* is meant a triple (f, P, ϕ), where f is such an embedding, P is a canonical R-subalgebra, and ϕ is an R-homomorphism $P \to \widehat{P}$ subject to the following conditions:

(a) $[P]_L$ is a subtree of L; thus $\varrho \in [P]_L$ implies $\sigma \in [P]_L$ for all $\sigma \leq \varrho$;
(b) $\operatorname{cf}\|P\| = \omega$;
(c) $\operatorname{Im} f \subseteq [P]_L$;
(d) $\|\mathbf{b}\| = \|P\|$ for all $\mathbf{b} \in \operatorname{Br}(\operatorname{Im} f)$.

In the following theorem we assume that R is a domain such that

(i) R admits a countable semigroup $\mathbb{S}$ such that R is Hausdorff in the $\mathbb{S}$-topology;
(ii) R is torsion-free as an abelian group;
(iii) R is $\mathbb{S}$-*cotorsion-free*, where by the $\mathbb{S}$-cotorsion-freeness of an R-module N is meant the property that $\operatorname{Hom}_R(\widehat{R}, N) = 0$ (as above $\widehat{R}$ stands for the $\mathbb{S}$-completion of R).

Observe that from property (ii) it follows that all the R-subalgebras of the R-algebra $\widehat{F}$ are torsion-free as abelian groups.

We can now state:

THEOREM 3.1 (Black Box). *Let R be as stated. Given κ and λ as above, there exist a limit ordinal λ^* of cardinality λ and a sequence of traps $t_\alpha = (f_\alpha, P_\alpha, \phi_\alpha)$ $(\alpha \in \lambda^*)$ such that for all $\alpha, \beta \in \lambda^*$ we have*:

(a) *$\beta < \alpha$ implies $\|P_\beta\| \le \|P_\alpha\|$;*
(b) $\mathrm{Br}(\mathrm{Im}\, f_\alpha) \cap \mathrm{Br}(\mathrm{Im}\, f_\beta) = \emptyset$ *whenever $\alpha \ne \beta$;*
(c) *if $\beta + \kappa^{\aleph_0} \le \alpha$, then* $\mathrm{Br}\,(\mathrm{Im}\, f_\alpha) \cap \mathrm{Br}\,([P_\beta]_L) = \emptyset$;
(d) *if X is a subset of $\widehat{F}$ of cardinality $\le \kappa$ and $\phi \in \mathrm{End}(\widehat{F})$, then there is an ordinal $\alpha \in \lambda^*$ such that*

$$X \subseteq \widehat{P}_\alpha, \qquad \|X\| < \|P_\alpha\|, \qquad \phi\restriction P_\alpha = \phi_\alpha.$$

Proof. See appendix in Corner–Göbel [3] or Göbel–Trlifaj [12]. ∎

4. The construction. The method of constructing an $E(R)$-algebra A such that $F \subseteq A \subseteq_* \widehat{F}$ as the union of a continuous ascending chain of subalgebras A_α is described in the next theorem.

Let $\mathbf{b} \in \mathrm{Br}(L)$ be a branch in L and $F = R[T]$ the R-algebra as in Section 3. We associate with the branch $\mathbf{b} = (\varrho_0 < \cdots < \varrho_n < \cdots)$ the *branch element*

$$\widetilde{b} = \sum_{n \in \omega} q_n(1, u_{\varrho_n}) \in \widehat{F},$$

where the coefficients q_n are elements of $\mathbb{S}$ chosen in Section 2.

For an R-subalgebra $M \subseteq \widehat{F}$ and an element $x \in \widehat{F}$, the symbol $M[x]$ will denote the R-subalgebra of $\widehat{F}$ generated by M and x, while stars in subscripts designate the relatively divisible hull in $\widehat{F}$, i.e. $M[x]_*/M[x]$ is the torsion part of $\widehat{F}/M[x]$. For simplicity we write $A[g]_*$ for $(A[g])_*$.

THEOREM 4.1. *For a sequence of traps $t_\alpha = (f_\alpha, P_\alpha, \phi_\alpha)$ $(\alpha \in \lambda^*)$ as in Theorem 3.1, there exist R-subalgebras A_α of $\widehat{F}$, branches $\mathbf{a}_\alpha \in \mathrm{Br}(\mathrm{Im}\, f_\alpha)$, and elements $g_\alpha \in \widehat{F}(\alpha \in \lambda^*)$ such that*

(i) *for all $\beta \in \lambda^*$, $g_\beta = b_\beta \pi_\beta + \widetilde{a}_\beta$ for some $b_\beta \in \widehat{P}_\beta$ and $\pi_\beta \in \widehat{R}$;*
(ii) *$g_\beta \in \widehat{P}_\beta$ for each $\beta \in \lambda^*$;*
(iii) *for all $\beta < \alpha < \lambda^*$, $g_\beta \phi_\beta \notin A_\beta$ implies $g_\beta \phi_\beta \notin A_\alpha$;*
(iv) *$\{A_\alpha \mid \alpha \in \lambda^*\}$ is a continuous properly ascending chain of relatively divisible R-subalgebras of $\widehat{F}$, with $A_0 = F$;*
(v) *$A_{\beta+1} = A_\beta[g_\beta]_*$ for all $\beta \in \lambda^*$.*

Proof. In the proof we will make use of the following result proved in Corner–Göbel [3, p. 457, Lemma 3.6] and Dugas–Mader–Vinsonhaler [5, pp. 95–96].

PROPOSITION 4.2. *Assume that, for some ordinal α, A_α is an R-subalgebra of $\widehat{F}$ satisfying conditions* (i)–(v) *in Theorem* 4.1 *for all $\beta < \alpha$. Then there is a branch $\mathbf{a} \in \mathrm{Br}(\mathrm{Im}\, f_\alpha)$ such that for any $g = c + \widetilde{a}$ with $c \in \widehat{P}_\alpha$ satisfying $\|c\| < \|\mathbf{a}\|$ and for any $\beta < \alpha$, $g_\beta\phi_\beta \notin A_\beta$ implies $g_\beta\phi_\beta \notin A_\alpha[g]_*$.*

In order to verify the theorem, in view of the continuity of the chain of the A_α, it suffices to describe the step from α to $\alpha+1$. Suppose that the subalgebras A_β for all $\beta \le \alpha$ and the elements g_β for all $\beta < \alpha$ have already been constructed as required. To choose g_α and $A_{\alpha+1}$, we argue as follows.

Proposition 4.2 ensures that we can always find a branch $\mathbf{a}_\alpha \in \mathrm{Br}(\mathrm{Im}\, f_\alpha)$ and elements $b_\alpha \in P_\alpha$, $\pi_\alpha \in \widehat{R}$ such that $g = b_\alpha\pi_\alpha + \widetilde{a}_\alpha \in \widehat{P}_\alpha$ satisfies the condition that (iii) holds for this α. Then we set $g_\alpha = g$ with the proviso that—if possible—g should definitely be selected so as to satisfy $g\phi_\alpha \notin A_\alpha[g]_*$ as well. Once g_α has been chosen, it only remains to set $A_{\alpha+1} = A_\alpha[g_\alpha]_*$ to complete the proof. ∎

We also observe the following important fact about the R-algebras A_α just constructed.

LEMMA 4.3. *The R-algebras A_α constructed in the preceding theorem with the aid of the Black Box are $\aleph_1$-free, and thus also $\mathbb{S}$-cotorsion-free. The same holds for their union $A = \bigcup_{\alpha<\lambda^*} A_\alpha$.*

Proof. See Dugas–Mader–Vinsonhaler [5] or Göbel–Wallutis [13], where it is shown that the R-algebras A_α are $\mathbb{S}$-cotorsion-free. The same argument verifies their $\aleph_1$-freeness. Cf. also Göbel–Trlifaj [12]. (The $\aleph_1$-freeness is due to the freeness of F and the linear independence of different branch elements.) ∎

Let us point out that Göbel–Shelah–Strüngmann [10] proves the existence of $\aleph_1$-free E-rings of cardinality $\aleph_1$.

5. Proof of the main theorem. The R-algebras A constructed above need not be $E(R)$-algebras. In order to obtain an $E(R)$-algebra A, we have to ensure that there are no unwanted endomorphisms. To this end we have to show that we can always find an element $g_\alpha = g$ with the required properties that also satisfies $g\phi_\alpha \notin A_\alpha[g]_*$ provided that ϕ_α is not multiplication by an algebra element. This can be accomplished by the Step Lemma below.

Before stating the crucial Step Lemma, we prove a technical result.

LEMMA 5.1. *Assume the hypotheses of Proposition* 4.2, *and write the αth branch* (*defined in Proposition* 4.2) *as $\mathbf{a}_\alpha = (\varrho_0 < \cdots < \varrho_n < \cdots)$. Let* *k be a natural number and* $0 \neq x \in A_\alpha$. *Then there exists an element* $\theta \in T$ *such that for almost all* $n \in \omega$ *we have*

$$\theta(1, u^k_{\varrho n}) \in [x\widetilde{a}^k_\alpha].$$

Proof. Let $x = \sum_{\theta \in [x]} r_\theta \theta$ with $r_\theta \in \widehat{R}$. If $x \notin F$, then there exist an element $y \in F$ and an ordinal $\beta < \alpha$ such that $x - y \in A_\beta[g_\beta] \setminus A_\beta$ and $\|x - y\| \leq \|P_\beta\|$. Let the βth branch be $\mathbf{a}_\beta = (\sigma_0 < \cdots < \sigma_n < \cdots)$. We conclude that we can choose a $u^j_{\sigma_n}$ for some integer $j \geq 1$ and for large enough $n \in \omega$ such that $\theta = (\tau, u^j_{\sigma_n}) \in [x]$ for some $\tau \in M$. It follows that $(\tau, u^j_{\sigma_n})(1, u^k_{\varrho l}) = (\tau, u^j_{\sigma_n} u^k_{\varrho l}) \in [x\widetilde{a}^k_\alpha]$ for all large enough integers l.

If $0 \neq x \in F$, then $[x]$ is a non-empty finite subset of T. As above, we can choose $(\tau, u) \in [x]$ ($\tau \in M$, $u \in T_0$) such that $(\tau, u)(1, u^k_{\varrho l}) = (\tau, uu^k_{\varrho l}) \in [x\widetilde{a}^k_\alpha]$. Thus either $\theta = (\tau, u^j_{\sigma_n})$ or $\theta = (\tau, u)$ satisfies the requirements, and the lemma follows. ∎

Lemma 5.2 (Step Lemma). *For an* $\alpha \in \lambda^*$, *let the trap* $t_\alpha = (f_\alpha, P_\alpha, \phi_\alpha)$ *be given by the Black Box* 3.1, *and let* $A_\alpha \subseteq \widehat{F}$ *and* $\mathbf{a}_\alpha \in \mathrm{Br}(\mathrm{Im}\, f_\alpha)$ *be as in Theorem* 4.1. *If* $\phi_\alpha : P_\alpha \to A_\alpha$ *is not multiplication by an element of* A_α, *then there exist elements* $b \in P_\alpha$ *and* $\pi \in \widehat{R}$ *such that the following holds either for* $y = \widetilde{a}_\alpha$ *or for* $y = \pi b + \widetilde{a}_\alpha$.

(i) $A'_{\alpha+1} = A_\alpha[y]_*$ *is an* $\mathbb{S}$*-relatively divisible* R*-subalgebra of* $\widehat{F}$ *that is* $\aleph_1$*-free as an* R*-module*;

(ii) $y\phi_\alpha \notin A'_{\alpha+1}$.

Proof. Before entering into the proof, we observe that $A'_{\alpha+1}$ will be $\mathbb{S}$-cotorsion-free in view of (i) and the $\mathbb{S}$-cotorsion-freeness of R.

(i) is an immediate consequence of Lemma 4.3.

The branch element $\widetilde{a}_\alpha$ related to $\mathbf{a}_\alpha$ belongs to $\widehat{P}_\alpha$. Suppose that $y = \widetilde{a}_\alpha$ is not a good choice, that is, $\widetilde{a}_\alpha \phi_\alpha \in A_\alpha[\widetilde{a}_\alpha]_*$. This means that there are $k, n \in \omega$ and $r_i \in A_\alpha$ ($i \leq n$) such that

$$q_k \widetilde{a}_\alpha \phi_\alpha = \sum_{i \leq n} r_i \widetilde{a}^i_\alpha. \tag{1}$$

First let $n \leq 1$. Since ϕ_α was assumed not to be multiplication by any element of A_α, neither is $q_k \phi_\alpha$, thus $q_k \phi_\alpha \notin A_\alpha$. Consequently, we have $P_\alpha(q_k \phi_\alpha - r_1) \neq 0$, and so there exists an element b of P such that

$$0 \neq b(q_k \phi_\alpha - r_1) = q_k b \phi_\alpha - b r_1 \in A_\alpha.$$

From Lemma 4.3 it follows that A_α is $\mathbb{S}$-cotorsion-free, therefore for some $\pi \in \widehat{R}$ we have

$$\pi(q_k b \phi_\alpha - b r_1) \notin A_\alpha. \tag{2}$$

Suppose that $y = \widetilde{a}_\alpha + \pi b$ also satisfies $y\phi \in A_\alpha[y]_*$. Then $q_k y\phi_\alpha = q_k\widetilde{a}_\alpha\phi_\alpha + q_k\pi b\phi_\alpha = r_0 + r_1\widetilde{a}_\alpha + q_k\pi b\phi_\alpha = r_0 + r_1 y + (q_k\pi b\phi_\alpha - r_1\pi b)$, whence

$$\pi(q_k b\phi_\alpha - r_1 b) \in A_\alpha[y]_*.$$

There are $n' \in \omega$, $k \le l < \omega$, and $t_i \in A_\alpha$ $(i \le n')$ such that

$$q_l y\phi_\alpha = \sum_{i\le n'} t_i y^i.$$

Using (1) we obtain

$$q_l\pi b\phi_\alpha = q_l y\phi_\alpha - q_l\widetilde{a}_\alpha\phi_\alpha = \sum_{i\le n'} t_i(\widetilde{a}_\alpha + \pi b)^i - \frac{q_l}{q_k}(r_0 + r_1\widetilde{a}_\alpha).$$

Since $[\pi b] \subseteq [b]$, $[q_l\pi b\phi_\alpha] \subseteq [b\phi_\alpha]$ and $\{(1, u^i_{\varrho_n}) \mid n \in \omega\} \subseteq [\widetilde{a}^i_\alpha]$, from Lemma 5.1 we deduce that $n' = 1$ and $t_1 = (q_l/q_k)r_1$. Therefore,

$$q_l\pi b\phi_\alpha = t_0 - \frac{q_l}{q_k}r_0 + \frac{q_l}{q_k}r_1\pi b,$$

and so

$$\frac{q_l}{q_k}\pi(q_k b\phi_\alpha - r_1 b) = t_0 - \frac{q_l}{q_k}r_0 \in A_\alpha,$$

where $q_l/q_k \in \mathbb{S}$. Hence $\pi(q_k b\phi_\alpha - r_1 b) \in A_\alpha$, contradicting (2). This means that $y = \pi b + \widetilde{a}_\alpha$ satisfies (i) and (ii).

Now suppose $n > 1$ in (1). We may assume that $r_n \ne 0$, and therefore $0 \ne nr_n \in A_\alpha$ by the torsion-freeness of A_α. There is $\pi \in \widehat{R}$ satisfying

(3) $$\pi \cdot nr_n \notin A_\alpha.$$

Set $y = \widetilde{a}_\alpha + \pi$ (i.e. $b = 1 \in R \subseteq P \subseteq A_\alpha$), and suppose that $y\phi_\alpha \in A_\alpha[y]_*$. Thus $q_l y\phi_\alpha = \sum_{i\le n'} t_i y^i$ for some $n' \in \omega$, $k \le l < \omega$, and $t_i \in A_\alpha$ $(i \le n')$. Using (1) we obtain

$$q_l\pi\phi_\alpha = q_l y\phi_\alpha - q_l\widetilde{a}_\alpha\phi_\alpha = \sum_{i\le n'} t_i y^i - \frac{q_l}{q_k}\sum_{i\le n} r_i\widetilde{a}^i_\alpha.$$

Comparing the supports again, we deduce $n' = n$, $t_n = (q_l/q_k)r_n$, $t_{n-1} + t_n\pi n = (q_l/q_k)r_{n-1}$, and so

$$\frac{q_l}{q_k}r_n\pi n = \frac{q_l}{q_k}r_{n-1} - t_{n-1} \in A_\alpha.$$

We conclude that $r_n\pi n \in A_\alpha$, in contradiction to (3). Consequently, either $y = \widetilde{a}_\alpha$ or $y = \widetilde{a}_\alpha + \pi$ satisfies $y\phi_\alpha \notin A_\alpha[y]_*$. ∎

We are now ready to prove our main result:

THEOREM 5.3. *Assume R is a domain satisfying conditions* (i)–(iii) *of Section* 3, *and κ, λ are cardinals such that $|R| \le \kappa$ and $\lambda^\kappa = \lambda$. Then there exists a superdecomposable $\aleph_1$-free $E(R)$-algebra A of cardinality λ.*

Proof. Define A as the union of the well-ordered ascending chain of algebras A_α as stated in Theorem 4.1. Then A is evidently of cardinality λ, is superdecomposable by Lemma 2.2 and Remark 2.3, and is $\aleph_1$-free by Lemma 4.3. It only remains to show that A is an $E(R)$-algebra.

Multiplications by elements of A are evidently R-endomorphisms, so A may be viewed as a subring of its endomorphism ring. Suppose that ϕ is an R-endomorphism of A that is not multiplication by an element of A. It is clear that there must exist a canonical submodule $P \subset F$ such that $\phi\restriction P : P \to \widehat{P}$ also is not multiplication by an element in A.

We appeal to the Black Box to argue that there is a trap $t_\alpha = (f_\alpha, P_\alpha, \phi_\alpha)$ such that $P \subseteq P_\alpha$. Manifestly, $\phi\restriction P_\alpha = \phi_\alpha$ cannot be multiplication by any element of A. By virtue of the Step Lemma, there exists an element $g'_\alpha = b'\pi' + \widetilde{a}_\alpha$ ($b' \in P_\alpha,\ \pi' \in \widehat{R}$) that satisfies $g'_\alpha\phi_\alpha \notin A_\alpha[g'_\alpha]$. Because of the existence of such a g', the proof of Theorem 4.1 indicates that g_α had to be chosen so as to satisfy $g_\alpha\phi_\alpha \notin A_\alpha[g_\alpha] = A_{\alpha+1}$. But then from condition (iii) in the same theorem we conclude that $g_\alpha\phi = g_\alpha\phi_\alpha \notin A$ as well. Thus ϕ cannot be an endomorphism of A, and as a consequence, A is indeed an $E(R)$-algebra. ∎

Moreover, we can establish the existence of a fully rigid family of 2^λ superdecomposable $\aleph_1$-free $E(R)$-algebras of size λ.

THEOREM 5.4. *The algebra A constructed in Theorem* 5.3 *contains superdecomposable $\aleph_1$-free $E(R)$-subalgebras A_X for every $X \subseteq \lambda$ such that for all $X, Y \subseteq \lambda$ we have*

(i) $X \subseteq Y$ *implies* $A_X \subseteq A_Y$;
(ii) $\mathrm{Hom}_R(A_X, A_Y) = A_Y$ *if* $X \subseteq Y$ *and* 0 *otherwise.*

Proof. In order to find a family of $E(R)$-algebras satisfying conditions (i) and (ii), we change the definition of a trap and replace t_α in Theorem 3.1 by $t_\alpha = (f_\alpha, P_\alpha, \phi_\alpha, \xi_\alpha)$, where $\xi_\alpha \in \lambda$. Condition (d) of Theorem 3.1 now reads:

(d*) If X is a subset of $\widehat{F}$ of cardinality $\le \kappa$, $\xi \in \lambda$ and $\phi \in \mathrm{End}(\widehat{F})$, then there is an ordinal $\alpha \in \lambda^*$ such that

$$X \subseteq \widehat{P}_\alpha, \quad \|X\| < \|P_\alpha\|, \quad \phi\restriction P_\alpha = \phi_\alpha, \quad \xi = \xi_\alpha.$$

Recall from Theorem 5.3 that $A = F[g_\alpha : \alpha \in \lambda^*]_*$. If $X \subseteq \lambda$, then set $X^* = \{\alpha \in \lambda^* \mid \xi_\alpha \in X\} \subseteq \lambda^*$, and define

$$A_X = F[g_\alpha : \alpha \in X^*]_* \subseteq A.$$

The same proof as above shows that A_X is a superdecomposable $\aleph_1$-free $E(R)$-algebra. It is evident that $A_X \subseteq A_Y$ whenever $X \subseteq Y$. If $X, Y \subseteq \lambda$ are arbitrary subsets, then the argument in Corner–Göbel [3, p. 462, (4)] shows that $\mathrm{Hom}_R(A_X, A_Y) \neq 0$ implies $X \subseteq Y$, and in this case, (ii) holds true. ∎

6. Remarks. It is easy to characterize all Dedekind domains R that satisfy conditions (i)–(iii) of Section 3.

Evidently, R has to be of characteristic 0 and not a field. One can choose the monoid $\mathbb{S}$ generated by the (finite number of) generators of a maximal ideal of R. In order to exclude the case when R is not $\mathbb{S}$-cotorsion-free, it suffices to assume that R is not a complete discrete valuation domain. Thus,

Corollary 6.1. *There exist arbitrarily large $\aleph_1$-free superdecomposable $E(R)$-algebras over a Dedekind domain R that is not a field or a complete discrete valuation domain, and has characteristic* 0. ∎

The choice of $R = \mathbb{Z}$ leads us to the existence of large superdecomposable $\aleph_1$-free E-rings.

Next assume that R is a *Matlis domain* (i.e. its field of quotients, Q, as an R-module, is of projective dimension 1). If $R \neq Q$, then R contains a countable multiplicative monoid $\mathbb{S}$ such that R is Hausdorff in the $\mathbb{S}$-topology (cf. Fuchs–Salce [8, Lemma 4.3, p. 139]). Consequently,

Corollary 6.2. *There exist arbitrarily large superdecomposable $E(R)$-algebras over a Matlis domain R of characteristic* 0 *that is not a field and is not complete in any metrizable linear topology.* ∎

Observe that every domain S of characteristic 0 embeds in a ring R satisfying conditions (i)–(iii) mentioned above. In fact, we can choose the polynomial ring $R = S[x]$ with an indeterminate x and $\mathbb{S} = \{1, x, \dots, x^n, \dots\}$.

It is worth pointing out that if the ring R is of cardinality $< 2^{\aleph_0}$, then for its cotorsion-freeness it suffices to check that it is reduced (see Göbel–May [9]).

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