Large superdecomposable $E(R)$-algebras

by

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In honour of Claus Michael Ringel on the occasion of his 60th birthday

Abstract. For many domains $R$ (including all Dedekind domains of characteristic 0 that are not fields or complete discrete valuation domains) we construct arbitrarily large superdecomposable $R$-algebras $A$ that are at the same time $E(R)$-algebras. Here “superdecomposable” means that $A$ admits no (directly) indecomposable $R$-algebra summands $\neq 0$ and “$E(R)$-algebra” refers to the property that every $R$-endomorphism of the $R$-module $A$ is multiplication by an element of $A$.

1. Introduction. Schultz [15] introduced the notion of an $E$-ring as a ring $R$ such that the endomorphism ring of its additive group is isomorphic to $R$ under the natural map $\eta \mapsto \eta(1)$, i.e. each endomorphism acts as multiplication by an element of $R$. $E$-rings have been investigated in several papers: see e.g. Dugas–Mader–Vinsonhaler [5], Dugas–Gobel [4], Göbel–Strüngmann [11], proving the existence of arbitrarily large $E$-rings, $E$-rings whose additive groups are $\aleph_1$-free abelian groups, etc.

Göbel–Strüngmann [11] discusses $E(R)$-algebras, i.e. algebras $A$ over a domain $R$ such that every endomorphism of $A$ as an $R$-module is multiplication by an element of $A$. The existence of large $E(R)$-algebras over many domains $R$ is established. Fuchs–Lee [7] constructs $E(R)$-algebras over certain domains $R$ that are superdecomposable as $R$-algebras in the sense that they do not admit any algebra summand that is not a direct product of two non-zero subalgebras. In Theorem 5.3 we give a common generalization of these two results by proving the existence of arbitrarily large superdecomposable $E(R)$-algebras that are, in addition, $\aleph_1$-free in the sense that every countable subset is contained in a free $R$-submodule.
Our proof is based on a version of Shelah’s Black Box (see Theorem 3.1 below) which we borrow from Corner–Göbel [3]. (We emphasize that this principle is provable in ZFC.) Alternatively we could have used the “Strong Black Box” (see [13]) which has the advantage that some of the algebraic proofs are simpler, but has the drawback that the possible sizes of $E(R)$-algebras are more restricted. We work in an $R$-algebra $\hat{F}$ that is a completion of a semigroup algebra $F = R[T]$ where the monoid $T$ is appropriately chosen: $T$ is a direct product of two monoids, one of which serves to guarantee that the $R$-algebra $A$ to be constructed is superdecomposable, while the other will be responsible for the $E$-ring property of $A$. Our method follows closely the pattern of Corner–Göbel [3], which allows us to skip those details of the proofs that are obvious modifications of arguments in [3].

In Theorem 5.4 we prove the abundance of arbitrarily large superdecomposable $E(R)$-algebras. This, along with the similar result on indecomposable $E(R)$-algebras (cf. Dugas–Mader–Vinsonhaler [5]), shows that—as far as merely direct decompositions are concerned—$E(R)$-algebras do not display any particular behavior.

2. Superdecomposable algebras. Let $R$ denote a commutative domain that contains a countable subsemigroup $S = \{s_0 = 1, s_1, \ldots, s_n, \ldots\}$ (not containing 0) such that $R$ is Hausdorff in the $S$-topology (where the ideals $Rq_n$ $(n \in \omega)$ form a base of neighborhoods of 0 in $R$), i.e. $\bigcap_{n \in \omega} Rq_n = 0$; here we have used the notation $q_n = s_0s_1 \cdots s_n \in S$. (Note that the Hausdorff property of the $S$-topology is equivalent to the fact that the localization $R_S$ of $R$ at $S$ is not a fractional ideal of $R$.) The symbol $\hat{R}$ will denote the completion of $R$ in its $S$-topology. $R$ is then a dense subalgebra of $\hat{R}$.

Let $\mu$ denote an infinite cardinal; it is viewed as an initial ordinal, so we can talk about its subsets. We define a monoid $T_1$ whose elements are the finite subsets of $\mu$ and multiplication is defined via

$$\sigma \cdot \tau = \sigma \cup \tau$$

for all $\sigma, \tau \in T_1$. The empty set serves as the identity of $T_1$. (This monoid was inspired by Corner [1].)

Let $F$ denote the semigroup algebra of $T_1$ over $R$, i.e.

$$F = R[T_1] = \bigoplus_{\tau \in T_1} R\tau;$$

this is an $R$-algebra with identity $\{0\}$. The $S$-topology on $F$ is Hausdorff. The $S$-completion $\hat{F}$ of $F$ is an $\hat{R}$-algebra containing $F$ as a dense $R$-subalgebra whose elements $x \neq 0$ may be viewed as countable sums $x = \sum_{i \in \omega} r_i \tau_i$ with $r_i \in \hat{R}$, $\tau_i \in T_1$, where for every $k \in \omega$ almost all (i.e. all but finitely many) coefficients $r_i$ are divisible by $q_k$. 
By the support $[x]$ of $x$ is meant the set $\{\tau_i \mid r_i \neq 0\} \subseteq T_1$; this is always a countable subset, since $S$ was assumed to be countable.

**Lemma 2.1.** Every $R$-algebra $A$ that lies between the $R$-algebras $F = R[T_1]$ and $\hat{F}$ constructed above for the infinite cardinal $\mu$ is superdecomposable as an $R$-algebra.

**Proof.** Consider a non-zero algebra summand $C$ of $A$; $A = C \oplus C'$. The $C$-coordinate of the identity of $A$ is an idempotent element $0 \neq e \in A$.

**Case 1.** If there is an ordinal $\alpha \in \mu$ not contained in any set in the support $[e]$, then $\{\alpha\} \subseteq F$ is an idempotent which evidently satisfies $e\{\alpha\} \neq 0$. It also satisfies $e\{\alpha\} \neq e$, since for any $\tau \in [e]$ we have $\tau \cup \alpha \in [e\{\alpha\}] \setminus [e]$. The elements $e\{\alpha\}$ and $e - e\{\alpha\}$ are non-zero orthogonal idempotents in $A$ with sum $e$, establishing the decomposability of $C$ into the direct sum of two $R$-subalgebras.

**Case 2.** If there is no ordinal $\alpha$ as in Case 1, then $\mu = \aleph_0$ and $\mu = \bigcup[e]$. Write $e = \sum_{\tau \in [e]} r_{\tau \tau} \ (r_{\tau \tau} \in \hat{R})$ or $e = \sum_{\tau \in T_1} r_{\tau \tau} \in \hat{F}$ with $r_{\tau \tau} = 0$ for all $\tau \in T_1 \setminus [e]$. Pick any $\tau_0 \in [e]$ with $r_{\tau_0} \neq 0$. If $e\{\alpha\} = e$, then

$$\sum_{\tau \in T_1} r_{\tau \tau}\{\alpha\} = \sum_{\tau \in T_1} r_{\tau \tau}.$$

If $\alpha \notin \tau_0$, then the comparison of the coefficients of $\{\alpha\} \cup \tau_0 \in T_1$ on both sides yields

$$r_{\tau_0} + r_{\{\alpha\} \cup \tau_0} = r_{\{\alpha\} \cup \tau_0}.$$

Hence $r_{\tau_0} = 0$, contradicting the choice of $\tau_0$. Hence $e\{\alpha\} \neq e$ for all $\alpha \in \mu$.

Suppose, by way of contradiction, that $e\{\alpha\} = 0$ for all $\alpha \in \mu \setminus [\tau_0]$. Then

$$\sum_{\tau \in T_1} r_{\tau} \{\alpha\} \cup \tau = 0,$$

where the coefficient of $\{\alpha\} \cup \tau_0$ is $r_{\tau_0} + r_{\{\alpha\} \cup \tau_0} = 0$.

Thus $r_{\{\alpha\} \cup \tau_0} = -r_{\tau_0}$ for all $\alpha \in \mu \setminus [\tau_0]$, which is obviously impossible. Consequently, there is always an $\alpha \in \mu$ such that $e\{\alpha\} \neq 0$ (in addition to $e\{\alpha\} \neq e$), completing the proof. ■

We now construct another superdecomposable $R$-algebra as follows; we utilize an idea due to Corner [2].

Let $\mu$ be an infinite cardinal and $T_2$ the monoid with elements $(\alpha, p)$ where $\alpha \in \mu, 0 \leq p \in \mathbb{Q}$, and multiplication is defined via

$$(\alpha, p)(\beta, q) = (\max\{\alpha, \beta\}, \max\{p, q\}) \quad ((\alpha, p), (\beta, q) \in T_2).$$

Let $F$ denote the semigroup algebra $R[T_2]$ and $\hat{F}$ its $S$-completion. Now the element $(0, 0) \in \mu \times \mathbb{Q}$ is the identity of $F$. We have again:

**Lemma 2.2.** Every $R$-algebra $A$ between the $R$-algebras $F = R[T_2]$ and $\hat{F}$ just constructed for the infinite cardinal $\mu$ is a superdecomposable $R$-algebra.
Proof. It suffices to verify that for every non-zero idempotent $e = \sum_{i \in I} r_i(\alpha_i, p_i) \in \widehat{F}$ ($0 \neq r_i \in \widehat{R}$, $(\alpha_i, p_i) \in T_2$) ($I$ is some index set) we can find an idempotent $e' = (\alpha, p) \in F$ such that $0 \neq e(\alpha, p) \neq e$. If not all the $p_i$ are equal, then choose any $p \in \mathbb{Q}$ such that $p_i < p < p_j$ for some $i, j \in I$. In this case, $e' = (\alpha, p)$ is as desired for any choice of $\alpha \in \mu$. On the other hand, if all the $p_i$ ($i \in I$) are equal and if we can choose an ordinal $\alpha$ with $\alpha_i < \alpha < \alpha_j$ for some $i, j \in I$, then $e' = (\alpha, p_i) \in F$ is a good choice. In the remaining case, the idempotent $e$ must be of the form $e = (\beta, q) \in T_2$ or $e = (\beta, q) - (\beta + 1, q)$. Then we can choose $e' = (\beta, p)$ for any $q < p \in \mathbb{Q}$. Consequently, we can always find an idempotent $e'$ that establishes superdecomposability. \hfill \qed

It is straightforward to check:

Remark 2.3. If we replace the monoid $T_j$ ($j = 1$ or 2) by a monoid $T = T_j \times T'$, where $T'$ is any monoid, then the preceding lemmas are still valid.

3. The Black Box. We turn our attention to the construction of a superdecomposable $E(R)$-algebra between $F$ and $\widehat{F}$. For the construction we shall need a version of Shelah’s Black Box principle. (For a general discussion of this principle, we refer to Göbel–Trlifaj [12]; for the strong black box see Eklof–Mekler [6, Chapter XIII].)

Let $R, S$ have the same meaning as in the preceding section. Furthermore, let $\kappa$ be a cardinal such that $|R| \leq \kappa$, and assume in addition that $\lambda$ is a cardinal satisfying

$$\lambda^\kappa = \lambda.$$ 

Then we have cf $\lambda > \kappa \geq \aleph_0$; see e.g. Jech [14, p. 28].

The set $L = \omega^\kappa \geq \lambda$ of all finite sequences $\varrho = (\alpha_0, \ldots, \alpha_{n-1})$ (of length $n$) with $\alpha_i \in \lambda$ (the empty sequence is included) is a tree of length $\omega$ under the natural ordering: $\varrho_1 \leq \varrho_2$ in $L$ if and only if $\varrho_1$ is an initial segment of $\varrho_2$. Maximal linearly ordered subsets $b = \{\varrho_0 < \varrho_1 < \cdots < \varrho_n < \cdots\}$ of $L$ are called branches; here the length of $\varrho_n$ is $n$. The set of branches of $L$ will be denoted by $\text{Br}(L)$. Clearly, $|\text{Br}(L)| = \lambda^{\aleph_0} = \lambda$.

Let $T_0$ be the free commutative monoid generated by the symbols $u_\varrho$ for all $\varrho \in L$. Define the monoid $T$ as

$$T = M \times T_0,$$

where $M = T_1$ or $M = T_2$ as constructed above in Section 2 with the choice $\mu = \aleph_0$. Thus the elements of $T$ are of the form $\theta = (\tau, u)$, where $\tau \in M$ and $u \in T_0$. The semigroup algebra $F = R[T] = \bigoplus_{\theta \in T} R\theta$, its $S$-completion $\widehat{F}$ and any $R$-algebra $A$ in between are superdecomposable by Remark 2.3.

We will distinguish three natural kinds of supports depending on $T_0$, $L$ and $\lambda$ respectively.
Each element $0 \neq x \in \hat{F}$ can be expressed uniquely as a sum $x = \sum_{i \in I} r_i (\tau_i, u_i)$ (where $I$ is an indexing set with $1 \leq |I| \leq \aleph_0$) such that $0 \neq r_i \in \hat{R}$ and $(\tau_i, u_i) \in T$ for all $i \in I$. Then $[x] = \{ u_i \mid i \in I \} \subseteq T_0$ denotes the support of $x$. (If we want to emphasize this is a subset of $T_0$, we will say that $[x]$ is the $T_0$-support of $x$.) Every element $u_i \in [x]$ is the unique product of certain generators $u_{\phi ij}$ ($i \in I$, $j \leq n_i$). The collection of all these $\phi ij$ constitutes the $L$-support $[x]_L \subseteq L$ of $x$. Finally, the $\lambda$-support is meant the set $[x]_\lambda \subseteq \lambda$ of all ordinals used in $[x]_L$. The norm of $x$ is defined as $\|x\| = \sup [x]_\lambda$.

These notions extend naturally to subsets. If $X \subseteq \hat{F}$ is a set of cardinality $\leq \kappa$, then $[X] = \bigcup_{x \in X} [x]$ is the support of $X$ and $[X]_L, [X]_\lambda$ are defined similarly. Observe that the norm of $X$ is a well defined ordinal $\|X\| = \sup [X]_\lambda \in \lambda$, because $\text{cf} \lambda > \kappa$.

For a subset $I$ of $\lambda$ of size $\leq \kappa$, we define

$$P_I = \bigoplus_{\theta \in M \times I'} R \theta$$

as a canonical $R$-subalgebra, where $I'$ denotes the submonoid of $T_0$ generated by the $u_{\theta}$ with finite sequences $\theta = (\alpha_0, \ldots, \alpha_n) \in \omega > I$. Evidently, $P_I$ is a subalgebra of $F$ with support $I'$ (and $L$-support $\omega > I$) that is an $R$-free summand of size $\leq \kappa$ of $F$ with free complement. (We often write simply $P$ rather than $P_I$ if there is no need for specifying the index set.) There are $\lambda$ canonical $R$-subalgebras of $F$.

We also consider order-preserving embeddings

$$f : \omega > \kappa \rightarrow L.$$ 

By a trap is meant a triple $(f, P, \phi)$, where $f$ is such an embedding, $P$ is a canonical $R$-subalgebra, and $\phi$ is an $R$-homomorphism $P \rightarrow \hat{P}$ subject to the following conditions:

(a) $[P]_L$ is a subtree of $L$; thus $\sigma \in [P]_L$ implies $\sigma \subseteq [P]_L$ for all $\sigma \leq \varrho$;
(b) $\text{cf} \|P\| = \omega$;
(c) $\text{Im} f \subseteq [P]_L$;
(d) $\|b\| = \|P\|$ for all $b \in \text{Br(Im} f)$.

In the following theorem we assume that $R$ is a domain such that

(i) $R$ admits a countable semigroup $S$ such that $R$ is Hausdorff in the $S$-topology;
(ii) $R$ is torsion-free as an abelian group;
(iii) $R$ is $S$-cotorsion-free, where by the $S$-cotorsion-freeness of an $R$-module $N$ is meant the property that $\text{Hom}_R(\hat{R}, N) = 0$ (as above $\hat{R}$ stands for the $S$-completion of $R$).
Observe that from property (ii) it follows that all the \( R \)-subalgebras of the \( R \)-algebra \( \hat{F} \) are torsion-free as abelian groups.

We can now state:

**Theorem 3.1 (Black Box).** Let \( R \) be as stated. Given \( \kappa \) and \( \lambda \) as above, there exist a limit ordinal \( \lambda^* \) of cardinality \( \lambda \) and a sequence of traps \( t_\alpha = (f_\alpha, P_\alpha, \phi_\alpha) \ (\alpha \in \lambda^*) \) such that for all \( \alpha, \beta \in \lambda^* \) we have:

(a) \( \beta < \alpha \) implies \( \|P_\beta\| \leq \|P_\alpha\| \);

(b) \( \text{Br}(\text{Im} f_\alpha) \cap \text{Br}(\text{Im} f_\beta) = \emptyset \) whenever \( \alpha \neq \beta \);

(c) if \( \beta + \kappa^{\aleph_0} \leq \alpha \), then \( \text{Br}(\text{Im} f_\alpha) \cap \text{Br}([P_\beta]_L) = \emptyset \);

(d) if \( X \) is a subset of \( \hat{F} \) of cardinality \( \leq \kappa \) and \( \phi \in \text{End}(\hat{F}) \), then there is an ordinal \( \alpha \in \lambda^* \) such that

\[
X \subseteq \hat{P}_\alpha, \quad \|X\| < \|P_\alpha\|, \quad \phi|P_\alpha = \phi_\alpha.
\]


4. The construction. The method of constructing an \( E(R) \)-algebra \( A \) such that \( F \subseteq A \subseteq \hat{F} \) as the union of a continuous ascending chain of subalgebras \( A_\alpha \) is described in the next theorem.

Let \( b \in \text{Br}(L) \) be a branch in \( L \) and \( F = R[T] \) the \( R \)-algebra as in Section 3. We associate with the branch \( b = (\varrho_0 < \cdots < \varrho_n < \cdots) \) the branch element

\[
\hat{b} = \sum_{n \in \omega} q_n(1, u_{\varrho_n}) \in \hat{F},
\]

where the coefficients \( q_n \) are elements of \( S \) chosen in Section 2.

For an \( R \)-subalgebra \( M \subseteq \hat{F} \) and an element \( x \in \hat{F} \), the symbol \( M[x] \) will denote the \( R \)-subalgebra of \( \hat{F} \) generated by \( M \) and \( x \), while stars in subscripts designate the relatively divisible hull in \( \hat{F} \), i.e. \( M[x]_*/M[x] \) is the torsion part of \( \hat{F}/M[x] \). For simplicity we write \( A[g]_* \) for \( (A[g])_* \).

**Theorem 4.1.** For a sequence of traps \( t_\alpha = (f_\alpha, P_\alpha, \phi_\alpha) \ (\alpha \in \lambda^*) \) as in Theorem 3.1, there exist \( R \)-subalgebras \( A_\alpha \) of \( \hat{F} \), branches \( a_\alpha \in \text{Br}(\text{Im} f_\alpha) \), and elements \( g_\alpha \in \hat{F}(\alpha \in \lambda^*) \) such that

(i) for all \( \beta \in \lambda^* \), \( g_\beta = b_\beta \pi_\beta + \tilde{a}_\beta \) for some \( b_\beta \in \hat{P}_\beta \) and \( \pi_\beta \in \hat{R} \);

(ii) \( g_\beta \in \hat{P}_\beta \) for each \( \beta \in \lambda^* \);

(iii) for all \( \beta < \alpha < \lambda^* \), \( g_\beta \phi_\beta \notin A_\beta \) implies \( g_\beta \phi_\beta \notin A_\alpha \);

(iv) \( \{A_\alpha \mid \alpha \in \lambda^*\} \) is a continuous properly ascending chain of relatively divisible \( R \)-subalgebras of \( \hat{F} \), with \( A_0 = F \);

(v) \( A_{\beta+1} = A_\beta[g_\beta]_* \) for all \( \beta \in \lambda^* \).
Proof. In the proof we will make use of the following result proved in Corner–Göbel [3, p. 457, Lemma 3.6] and Dugas–Mader–Vinsonhaler [5, pp. 95–96].

**Proposition 4.2.** Assume that, for some ordinal \( \alpha \), \( A_\alpha \) is an \( R \)-subalgebra of \( \widehat{F} \) satisfying conditions (i)–(v) in Theorem 4.1 for all \( \beta < \alpha \). Then there is a branch \( \mathbf{a} \in \text{Br}(\text{Im} f_\alpha) \) such that for any \( g = c + \tilde{a} \) with \( c \in \widehat{P}_\alpha \) satisfying \( \|c\| < \|\mathbf{a}\| \) and for any \( \beta < \alpha \), \( g\beta \phi_\beta \notin A_\beta \) implies \( g\beta \phi_\beta \notin A_\alpha[g]* \).

In order to verify the theorem, in view of the continuity of the chain of the \( A_\alpha \), it suffices to describe the step from \( \alpha \) to \( \alpha + 1 \). Suppose that the subalgebras \( A_\beta \) for all \( \beta \leq \alpha \) and the elements \( g_\beta \) for all \( \beta < \alpha \) have already been constructed as required. To choose \( g_\alpha \) and \( A_{\alpha + 1} \), we argue as follows.

Proposition 4.2 ensures that we can always find a branch \( \mathbf{a}_\alpha \in \text{Br}(\text{Im} f_\alpha) \) and elements \( b_\alpha \in P_\alpha \), \( \pi_\alpha \in \widehat{R} \) such that \( g = b_\alpha \pi_\alpha + \tilde{a}_\alpha \in \widehat{P}_\alpha \) satisfies the condition that (iii) holds for this \( \alpha \). Then we set \( g_\alpha = g \) with the proviso that—if possible—\( g \) should definitely be selected so as to satisfy \( g\phi_\alpha \notin A_\alpha[g]* \) as well. Once \( g_\alpha \) has been chosen, it only remains to set \( A_{\alpha + 1} = A_\alpha[g_*] \) to complete the proof. \( \blacksquare \)

We also observe the following important fact about the \( R \)-algebras \( A_\alpha \) just constructed.

**Lemma 4.3.** The \( R \)-algebras \( A_\alpha \) constructed in the preceding theorem with the aid of the Black Box are \( \aleph_1 \)-free, and thus also \( S \)-cotorsion-free. The same holds for their union \( A = \bigcup_{\alpha < \lambda^*} A_\alpha \).

**Proof.** See Dugas–Mader–Vinsonhaler [5] or Göbel–Wallutis [13], where it is shown that the \( R \)-algebras \( A_\alpha \) are \( S \)-cotorsion-free. The same argument verifies their \( \aleph_1 \)-freeness. Cf. also Göbel–Trlifaj [12]. (The \( \aleph_1 \)-freeness is due to the freeness of \( F \) and the linear independence of different branch elements.) \( \blacksquare \)

Let us point out that Göbel–Shelah–Strüngmann [10] proves the existence of \( \aleph_1 \)-free \( E(R) \)-rings of cardinality \( \aleph_1 \).

5. **Proof of the main theorem.** The \( R \)-algebras \( A \) constructed above need not be \( E(R) \)-algebras. In order to obtain an \( E(R) \)-algebra \( A \), we have to ensure that there are no unwanted endomorphisms. To this end we have to show that we can always find an element \( g_\alpha = g \) with the required properties that also satisfies \( g\phi_\alpha \notin A_\alpha[g]* \) provided that \( \phi_\alpha \) is not multiplication by an algebra element. This can be accomplished by the Step Lemma below.

Before stating the crucial Step Lemma, we prove a technical result.

**Lemma 5.1.** Assume the hypotheses of Proposition 4.2, and write the \( \alpha \)th branch (defined in Proposition 4.2) as \( \mathbf{a}_\alpha = (\varrho_0 < \cdots < \varrho_n < \cdots) \). Let
$k$ be a natural number and $0 \neq x \in A_\alpha$. Then there exists an element $\theta \in T$ such that for almost all $n \in \omega$ we have

$$\theta(1, u^k_{\tilde{a}_n}) \in [x\tilde{a}^k_\alpha].$$

**Proof.** Let $x = \sum_{\theta \in [x]} r_\theta \theta$ with $r_\theta \in \hat{R}$. If $x \not\in F$, then there exist an element $y \in F$ and an ordinal $\beta < \alpha$ such that $x - y \in A_\beta[g_\beta] \setminus A_\beta$ and $\|x - y\| \leq \|P_\beta\|$. Let the $\beta$th branch be $a_\beta = (\sigma_0 < \cdots < \sigma_n < \cdots)$. We conclude that we can choose a $u^j_{\sigma_n}$ for some integer $j \geq 1$ and for large enough $n \in \omega$ such that $\theta = (\tau, u^j_{\sigma_n}) \in [x]$ for some $\tau \in M$. It follows that $(\tau, u^j_{\sigma_n})(1, u^k) = (\tau, u^j_{\sigma_n}u^k) \in [x\tilde{a}^k_\alpha]$ for all large enough integers $l$.

If $0 \neq x \in F$, then $[x]$ is a non-empty finite subset of $T$. As above, we can choose $(\tau, u) \in [x]$ $(\tau \in M, u \in T_0)$ such that $(\tau, u)(1, u^k) = (\tau, uu^k) \in [x\tilde{a}^k_\alpha]$. Thus either $\theta = (\tau, u^j_{\sigma_n})$ or $\theta = (\tau, u)$ satisfies the requirements, and the lemma follows. ■

**Lemma 5.2 (Step Lemma).** For an $\alpha \in \lambda^*$, let the trap $t_\alpha = (f_\alpha, P_\alpha, \phi_\alpha)$ be given by the Black Box 3.1, and let $A_\alpha \subseteq \hat{F}$ and $a_\alpha \in \text{Br}(\text{Im} f_\alpha)$ be as in Theorem 4.1. If $\phi_\alpha : P_\alpha \to A_\alpha$ is not multiplication by an element of $A_\alpha$, then there exist elements $b \in P_\alpha$ and $\pi \in \hat{R}$ such that the following holds either for $y = \tilde{a}_\alpha$ or for $y = b + \tilde{a}_\alpha$.

(i) $A'_{\alpha+1} = A_\alpha[y]_*$ is an $S$-relatively divisible $R$-subalgebra of $\hat{F}$ that is $\aleph_1$-free as an $R$-module;

(ii) $y\phi_\alpha \not\in A'_{\alpha+1}$. 

**Proof.** Before entering into the proof, we observe that $A'_{\alpha+1}$ will be $S$-cotorsion-free in view of (i) and the $S$-cotorsion-freeness of $R$.

(i) is an immediate consequence of Lemma 4.3.

The branch element $\tilde{a}_\alpha$ related to $a_\alpha$ belongs to $\hat{P}_\alpha$. Suppose that $y = \tilde{a}_\alpha$ is not a good choice, that is, $\tilde{a}_\alpha \phi_\alpha \not\in A_\alpha[\tilde{a}_\alpha]_*$. This means that there are $k, n \in \omega$ and $r_i \in A_\alpha$ ($i \leq n$) such that

$$q_k\tilde{a}_\alpha \phi_\alpha = \sum_{i \leq n} r_i \tilde{a}^i_\alpha.$$  

First let $n \leq 1$. Since $\phi_\alpha$ was assumed not to be multiplication by any element of $A_\alpha$, neither is $q_k \phi_\alpha$, thus $q_k \phi_\alpha \not\in A_\alpha$. Consequently, we have $P_\alpha(q_k \phi_\alpha - r_1) \neq 0$, and so there exists an element $b$ of $P$ such that

$$0 \neq b(q_k \phi_\alpha - r_1) = q_k b \phi_\alpha - br_1 \in A_\alpha.$$  

From Lemma 4.3 it follows that $A_\alpha$ is $S$-cotorsion-free, therefore for some $\pi \in \hat{R}$ we have

$$\pi(q_k b \phi_\alpha - br_1) \not\in A_\alpha.$$
Suppose that $y = \tilde{a}_\alpha + \pi b$ also satisfies $y \phi \in A_\alpha[y]_*$. Then
\[ q_k y \phi_\alpha = q_k \tilde{a}_\alpha \phi_\alpha + q_k \pi b \phi_\alpha = r_0 + r_1 \tilde{a}_\alpha + q_k \pi b \phi_\alpha = r_0 + r_1 y + (q_k \pi b \phi_\alpha - r_1 \pi b), \]
whence
\[ \pi(q_k b \phi_\alpha - r_1 b) \in A_\alpha[y]_* \]
There are $n' \in \omega$, $k \leq l < \omega$, and $t_i \in A_\alpha$ ($i \leq n'$) such that
\[ q_i y \phi_\alpha = \sum_{i \leq n'} t_i y^i. \]
Using (1) we obtain
\[ q_i \pi b \phi_\alpha = q_i y \phi_\alpha - q_i \tilde{a}_\alpha \phi_\alpha = \sum_{i \leq n'} t_i (\tilde{a}_\alpha + \pi b)^i - \frac{q_i}{q_k} (r_0 + r_1 \tilde{a}_\alpha). \]
Since $[\pi b] \subseteq [b]$, $[q_i \pi b \phi_\alpha] \subseteq [b \phi_\alpha]$ and $\{(1, u_n^i) | n \in \omega\} \subseteq [\tilde{a}^i]$, from Lemma 5.1 we deduce that $n' = 1$ and $t_1 = (q_i/q_k)r_1$. Therefore,
\[ q_i \pi b \phi_\alpha = t_0 - \frac{q_i}{q_k} r_0 + \frac{q_i}{q_k} r_1 \pi b, \]
and so
\[ \frac{q_i}{q_k} \pi(q_k b \phi_\alpha - r_1 b) = t_0 - \frac{q_i}{q_k} r_0 \in A_\alpha, \]
where $q_i/q_k \in S$. Hence $\pi(q_k b \phi_\alpha - r_1 b) \in A_\alpha$, contradicting (2). This means that $y = \pi b + \tilde{a}_\alpha$ satisfies (i) and (ii).

Now suppose $n > 1$ in (1). We may assume that $r_n \neq 0$, and therefore $0 \neq nr_n \in A_\alpha$ by the torsion-freeness of $A_\alpha$. There is $\pi \in \widehat{R}$ satisfying
\[ \pi \cdot nr_n \not\in A_\alpha. \]
Set $y = \tilde{a}_\alpha + \pi$ (i.e. $b = 1 \in R \subseteq P \subseteq A_\alpha$), and suppose that $y \phi_\alpha \in A_\alpha[y]_*$. Thus $q_i y \phi_\alpha = \sum_{i \leq n'} t_i y^i$ for some $n' \in \omega$, $k \leq l < \omega$, and $t_i \in A_\alpha$ ($i \leq n'$). Using (1) we obtain
\[ q_i \pi \phi_\alpha = q_i y \phi_\alpha - q_i \tilde{a}_\alpha \phi_\alpha = \sum_{i \leq n'} t_i y^i - \frac{q_i}{q_k} \sum_{i \leq n} r_i \tilde{a}_\alpha^i. \]
Comparing the supports again, we deduce $n' = n$, $t_n = (q_i/q_k)r_n$, $t_{n-1} + t_n \pi n = (q_i/q_k)r_{n-1}$, and so
\[ \frac{q_i}{q_k} r_n \pi n = \frac{q_i}{q_k} r_{n-1} - t_{n-1} \in A_\alpha. \]
We conclude that $r_n \pi n \in A_\alpha$, in contradiction to (3). Consequently, either $y = \tilde{a}_\alpha$ or $y = \tilde{a}_\alpha + \pi$ satisfies $y \phi_\alpha \not\in A_\alpha[y]_*$. ■

We are now ready to prove our main result:

**Theorem 5.3.** Assume $R$ is a domain satisfying conditions (i)–(iii) of Section 3, and $\kappa, \lambda$ are cardinals such that $|R| \leq \kappa$ and $\lambda^\kappa = \lambda$. Then there exists a superdecomposable $\aleph_1$-free $E(R)$-algebra $A$ of cardinality $\lambda$. 
Proof. Define $A$ as the union of the well-ordered ascending chain of algebras $A_\alpha$ as stated in Theorem 4.1. Then $A$ is evidently of cardinality $\lambda$, is superdecomposable by Lemma 2.2 and Remark 2.3, and is $\aleph_1$-free by Lemma 4.3. It only remains to show that $A$ is an $E(R)$-algebra.

Multiplications by elements of $A$ are evidently $R$-endomorphisms, so $A$ may be viewed as a subring of its endomorphism ring. Suppose that $\phi$ is an $R$-endomorphism of $A$ that is not multiplication by an element of $A$. It is clear that there must exist a canonical submodule $P = bP$ such that $P$ is not multiplication by an element in $A$. We appeal to the Black Box to argue that there is a trap $t = (f, P, \phi, \xi)$ such that $P$ is not multiplication by any element of $A$. By virtue of the Step Lemma, there exists an element $\tilde{g}_\alpha = b'\pi + \tilde{\alpha} (b' \in P_\alpha, \pi' \in \hat{R})$ that satisfies $\tilde{g}_\alpha \phi \notin A[g_\alpha]$. Because of the existence of such a $\tilde{g}_\alpha$, the proof of Theorem 4.1 indicates that $g_\alpha$ had to be chosen so as to satisfy $g_\alpha \phi \notin A[g_\alpha] = A_{\alpha+1}$. But then from condition (iii) in the same theorem we conclude that $g_\alpha \phi = g_\alpha \phi \notin A$ as well. Thus $\phi$ cannot be an endomorphism of $A$, and as a consequence, $A$ is indeed an $E(R)$-algebra.

Moreover, we can establish the existence of a fully rigid family of $2^\lambda$ superdecomposable $\aleph_1$-free $E(R)$-algebras of size $\lambda$.

Theorem 5.4. The algebra $A$ constructed in Theorem 5.3 contains superdecomposable $\aleph_1$-free $E(R)$-subalgebras $A_X$ for every $X \subseteq \lambda$ such that for all $X, Y \subseteq \lambda$ we have

(i) $X \subseteq Y$ implies $A_X \subseteq A_Y$;
(ii) $\text{Hom}_R(A_X, A_Y) = A_Y$ if $X \subseteq Y$ and 0 otherwise.

Proof. In order to find a family of $E(R)$-algebras satisfying conditions (i) and (ii), we change the definition of a trap and replace $t_\alpha$ in Theorem 3.1 by $t_\alpha = (f_\alpha, P_\alpha, \phi_\alpha, \xi_\alpha)$, where $\xi_\alpha \in \lambda$. Condition (d) of Theorem 3.1 now reads:

(d*) If $X$ is a subset of $\hat{F}$ of cardinality $\leq \kappa$, $\xi \in \lambda$ and $\phi \in \text{End}(\hat{F})$, then there is an ordinal $\alpha \in \lambda^*$ such that $X \subseteq \hat{P}_\alpha$, $\|X\| < \|P_\alpha\|$, $\phi|P_\alpha = \phi_\alpha$, $\xi = \xi_\alpha$.

Recall from Theorem 5.3 that $A = F[g_\alpha : \alpha \in \lambda]^*$. If $X \subseteq \lambda$, then set $X^* = \{\alpha \in \lambda^* \mid \xi_\alpha \in X\} \subseteq \lambda^*$, and define $A_X = F[g_\alpha : \alpha \in X^*]^* \subseteq A$.

The same proof as above shows that $A_X$ is a superdecomposable $\aleph_1$-free $E(R)$-algebra. It is evident that $A_X \subseteq A_Y$ whenever $X \subseteq Y$. If $X, Y \subseteq \lambda$ are arbitrary subsets, then the argument in Corner–Göbel [3, p. 462, (4)]
shows that $\text{Hom}_R(A_X, A_Y) \neq 0$ implies $X \subseteq Y$, and in this case, (ii) holds true.

6. Remarks. It is easy to characterize all Dedekind domains $R$ that satisfy conditions (i)–(iii) of Section 3.

Evidently, $R$ has to be of characteristic 0 and not a field. One can choose the monoid $S$ generated by the (finite number of) generators of a maximal ideal of $R$. In order to exclude the case when $R$ is not $S$-cotorsion-free, it suffices to assume that $R$ is not a complete discrete valuation domain. Thus,

**Corollary 6.1.** There exist arbitrarily large $\aleph_1$-free superdecomposable $E(R)$-algebras over a Dedekind domain $R$ that is not a field or a complete discrete valuation domain, and has characteristic 0.

The choice of $R = \mathbb{Z}$ leads us to the existence of large superdecomposable $\aleph_1$-free $E$-rings.

Next assume that $R$ is a Matlis domain (i.e. its field of quotients, $Q$, as an $R$-module, is of projective dimension 1). If $R \neq Q$, then $R$ contains a countable multiplicative monoid $S$ such that $R$ is Hausdorff in the $S$-topology (cf. Fuchs–Salce [8, Lemma 4.3, p. 139]). Consequently,

**Corollary 6.2.** There exist arbitrarily large superdecomposable $E(R)$-algebras over a Matlis domain $R$ of characteristic 0 that is not a field and is not complete in any metrizable linear topology.

Observe that every domain $S$ of characteristic 0 embeds in a ring $R$ satisfying conditions (i)–(iii) mentioned above. In fact, we can choose the polynomial ring $R = S[x]$ with an indeterminate $x$ and $S = \{1, x, \ldots, x^n, \ldots\}$.

It is worth pointing out that if the ring $R$ is of cardinality $< 2^{\aleph_0}$, then for its cotorsion-freeness it suffices to check that it is reduced (see Göbel–May [9]).

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