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G-ANR's with homotopy trivial fixed point sets

by

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*Dedicated to the memory of Professor Karol Borsuk
on the occasion of his centennial*

Abstract. Let G be a compact group and X a G-ANR. Then X is a G-AR iff the H-fixed point set X^H is homotopy trivial for each closed subgroup $H \subset G$.

1. Introduction. The purpose of this paper is to prove the following

Main Theorem. *Let G be a compact group and X a G-ANR. Then X is a G-AR iff the H-fixed point set X^H is homotopy trivial for each closed subgroup $H \subset G$.*

This is the equivariant version of a well known result in the theory of retracts asserting that an ANR is an AR iff it is homotopy trivial [14, Theorem 4.2.20].

Only the "if" part of the Main Theorem is nontrivial. Since a G-ANR is a G-AR iff it is G-contractible (see [1, Theorem 6]), one just needs to establish the G-contractibility of X. For G a compact Lie group this follows from the following James–Segal Theorem [10]: a G-map $f : X \to Y$ of G-ANR's is a G-homotopy equivalence if its restriction $f^H : X^H \to Y^H$ to the H-fixed point sets is an ordinary homotopy equivalence for every closed subgroup $H \subset G$. Validity of the Main Theorem for any compact (not necessarily Lie) group actions leads us to believe that the James–Segal Theorem should be true in this general case as well.

Our proof of the Main Theorem, even in the case of a compact Lie group G, relies neither on the James–Segal Theorem nor on its proof. We give a short proof based on the notion of an *approximate slice* (see Theorem 2.2 below) which is applicable at once to arbitrary compact group ac-

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tions. Another important ingredient in our proof is the Jaworowski–Lashof equivariant extension theorem [11], [13] (see Section 2).

The following example describes a situation where the Main Theorem applies in an essential way. Let X be a Peano continuum (i.e., a connected, locally connected, compact metric space containing more than one point) on which a compact group G acts nontransitively. Let $\exp X$ be the hyperspace of all nonempty compact subsets of X endowed with the Hausdorff metric topology and the induced action of G. Set $Y = (\exp X) \setminus \{X\}$. We claim that Y is a G-AR. Indeed, it is proved in [5, Proposition 3.1] that $\exp X$ is a G-AR, so Y is a G-ANR. Hence, to prove that Y is a G-AR, it suffices to show that Y is G-contractible. However, there is no canonical way to contract Y equivariantly to a G-fixed point of Y (although $\exp X$ is canonically G-contractible to its G-fixed point $\{X\}$).

But the Main Theorem gives an answer here. Namely, if $H \subset G$ is a closed subgroup, then $(\exp X)^H \cong \exp(X/H)$. Since the H-orbit space X/H is a Peano continuum (it is not a singleton because G acts nontransitively on X) we infer that, due to the Curtis–Schori–West Hyperspace Theorem (see [14, Theorem 4.2.27]), $\exp(X/H)$ is a Hilbert cube. Consequently, $Y^H = (\exp X)^H \setminus \{X\}$ is a Hilbert cube with a removed point and hence is contractible. Now the Main Theorem works.

In connection with this example it is worth recalling that every compact metrizable (not necessarily Lie) group G can act effectively (i.e., each $g \in G$ acts as a nontrivial homeomorphism) and nontransitively on a Peano continuum. What is more, such a group can even act effectively and nontransitively on the Hilbert cube. Indeed, according to a well-known result of Pontryagin [19, Ch. 8, Theorem 68], every compact metrizable group G can be represented as the limit of an inverse system $\{G_n, \pi_{n,n+1} : n = 1, 2, \dots\}$ of compact Lie groups G_n and their continuous epimorphisms $\pi_{n,n+1} : G_{n+1} \to G_n$ such that each limit homomorphism $G \to G_n$ is also an epimorphism. Then for each $n \geq 1$ we consider $\mathrm{Cone}(G_n)$, the cone with base G_n. Since G_n is a compact ANR, $\mathrm{Cone}(G_n)$ is a compact AR containing more than one point. Consequently, by a well-known result of West [20], $\prod_{n=1}^{\infty} \mathrm{Cone}(G_n)$ is homeomorphic to the Hilbert cube. On the other hand, since G_n is the quotient group G/H_n for a closed normal subgroup $H_n \subset G$, we see that G acts naturally (on the left) on G_n, and this action extends to an action of G on $\mathrm{Cone}(G_n)$. Next, since $\bigcap_{n=1}^{\infty} H_n$ only contains the unity of G, the induced diagonal action of G on the Hilbert cube $\prod_{n=1}^{\infty} \mathrm{Cone}(G_n)$ is effective. To complete our construction it remains to observe that this action has a (unique) G-fixed point, and hence, is nontransitive. By the way, no compact group can act transitively on the Hilbert cube; we leave the details of the proof to the reader as a stimulating exercise.

It is also interesting to mention that $\prod_{n=1}^{\infty} \mathrm{Cone}(G_n)$ is, in fact, a G-AR. This is because each coset $G_n = G/H_n$, being a Lie group, is a G-ANR (see Proposition 2.3 below). In turn, this implies that $\mathrm{Cone}(G_n)$, $n \geq 1$, is a G-AR (see [4, Proposition 2.2]), and hence $\prod_{n=1}^{\infty} \mathrm{Cone}(G_n)$ is also a G-AR. Thus, to each representation of a compact metrizable group G as the limit of an inverse system of compact Lie groups corresponds an effective (and nontransitive) G-action on the Hilbert cube, making the latter a G-AR.

Perhaps it is in order to recall here yet another (and more universal) source of G-ANR's: these are the mapping spaces. Namely, it is proved in [2, Theorem 8] that for any compact group G and any A(N)R space L, the space $C(G, L)$ of all continuous mappings $f : G \to L$, endowed with the compact-open topology, is a G-A(N)R. Here G acts on $C(G, L)$ according to the rule $(g, f) \mapsto gf$, where $(gf)(t) = f(tg)$, $t \in G$. In particular, if L is a normed linear space then $C(G, L)$ is a G-AR. Moreover, each G-A(N)R is a (neighborhood) G-retract of $C(G, L)$ for a suitably chosen normed linear space L [2, Corollary 5].

We conclude this introduction by considering yet another example where the Main Theorem substantially helps to establish that a given G-ANR space is, in fact, a G-AR. Let G be any compact group and L an infinite-dimensional normed linear space. Further, let X denote the above mentioned normed linear G-space $C(G, L)$ with the origin removed. Since $C(G, L)$ is a G-AR we see that X is a G-ANR. But in fact X is a G-AR, though, after removing the origin, the G-contractibility of X is not evident. First we observe that for any closed subgroup $H \subset G$, $X^H = C(G, L)^H \setminus \{0\}$. Further, it is easy to see that $C(G, L)^H$ is homeomorphic to $C(G/H, L)$, where G/H denotes the right coset space. Since $C(G/H, L)$ is an infinite-dimensional normed linear space, according to a result of Klee [12], the complement $X^H = C(G/H, L) \setminus \{0\}$ is homeomorphic to $C(G/H, L)$. It then follows that X^H is contractible, and hence homotopy trivial. It remains to apply the Main Theorem.

Finally, we refer the reader to [4] where the "Lie case" of the Main Theorem is applied in the study of proper actions (in the sense of R. Palais [18]) of noncompact Lie groups. We hope that availability of the Main Theorem for non-Lie groups will now foster further development of the theory of proper actions of arbitrary locally compact groups.

2. Preliminaries. Throughout the paper the letter G will denote a compact Hausdorff topological group unless otherwise stated; by e we denote the unit element of G.

The basic ideas and facts of the theory of G-spaces or topological transformation groups can be found in Bredon [7] and in Palais [17]. For the equivariant theory of retracts the reader can see, for instance, the papers

[1], [2] and [6]. Below, for the convenience of the reader, we recall some more special definitions and facts.

All spaces are assumed to be completely regular and Hausdorff. All G-maps are assumed to be continuous.

A space X is called *homotopy trivial* if for every $n \geq 0$, every continuous map from the sphere S^n to X can be continuously extended over the closed ball B^{n+1}.

A metrizable G-space Y is called a *G-equivariant absolute neighborhood retract* (for the class of all metrizable G-spaces) (notation: $Y \in G$-ANR) if, for any closed invariant subset A of a metrizable G-space X and any G-map $f : A \to Y$, there exist an invariant neighborhood U of A in X and a G-map $\psi\colon U \to Y$ that extends f. If, in addition, one can always take $U = X$, then we say that Y is a *G-equivariant absolute retract* (notation: $Y \in G$-AR). The map ψ is called a *G-extension* of f.

For a point x of a G-space X, the subgroup $G_x = \{g \in G \mid gx = x\}$ is called the *stabilizer* or *isotropy subgroup* of x. For a subgroup $H \subset G$, the set $X^H = \{x \in X|\ H \subset G_x\}$ is called the *H-fixed point set* of X. We denote by G/H the G-space of cosets $\{gH \mid g \in G\}$ under the action induced by left translations.

The family $(H) = \{gHg^{-1} \mid g \in G\}$ of all subgroups of G which are conjugate to a given subgroup $H \subset G$ is called a *G-orbit type*. One says that a G-space X has *finitely many G-orbit types* if there exist a finite number of closed subgroups $H_1, \dots, H_n$ of G such that $(G_x) \in \{(H_1), \dots, (H_n)\}$ for all $x \in X$.

If X is a G-space and $S \subset X$ then the set $G(S) = \{gs \mid g \in G,\ s \in S\}$ is called the *saturation* of S.

The notion of a slice is the key tool in our proofs, so let us recall it:

Definition 2.1 ([17]). Let X be a G-space and $H \subset G$ a closed subgroup. A subset $S \subset X$ is called an *H-slice* in X if:

(1) S is H-invariant,
(2) the saturation $G(S)$ is open in X,
(3) if $g \in G \setminus H$, then $gS \cap S = \emptyset$,
(4) S is closed in $G(S)$.

The saturation $G(S)$ is called an *H-tube*, and H is called a *slicing subgroup.*

Each H-slice S uniquely determines a G-map $f : G(S) \to G/H$ such that $f^{-1}(eH) = S$ [17, Theorem 1.7.7]. We will call such a G-map f the *slicing map.*

The classical Slice Theorem, which in its final form was proved in Mostow [16], asserts that if G is a compact Lie group and x any point of a given G-space X, then there exists a G_x-slice $S \subset X$ that contains x. This theorem is no longer true if the acting group is not Lie (see [3]). However, the following approximate version of the Slice Theorem remains true for arbitrary compact group actions:

THEOREM 2.2 (Approximate Slice Theorem [3]). *Let X be a G-space and $x \in X$. Then for any neighborhood U of x in X, there exist a large subgroup $K \subset G$ with $G_x \subset K$ and a K-slice S such that $x \in S \subset U$.*

Here a closed subgroup K of a compact group is called *large* [3] if there exists a closed normal subgroup N of G such that $N \subset K$ and G/N is a Lie group. Large subgroups are characterized by the following proposition:

PROPOSITION 2.3 ([3]). *For a closed subgroup $H \subset G$, the following conditions are equivalent*:

(1) *H is a large subgroup*,
(2) *G/H is a G-ANR*,
(3) *G/H is locally contractible*,
(4) *G/H is a smooth manifold.*

Let X be a G-space, $H \subset G$ a large subgroup, S an H-slice in X, and $O \subset G$ a neighborhood of the identity. The set $gOS = \{gps \mid p \in O, s \in S\}$, where $g \in G$, is called a *tubular segment of type H.*

A family

$$\mathcal{U} = \{gO_\mu S_\mu \mid g \in G,\ \mu \in \mathcal{M}\}$$

consisting of tubular segments with large slicing subgroups $\{H_\mu \mid \mu \in \mathcal{M}\}$ is called a *G-normal cover* of X if the family $\widetilde{\mathcal{U}} = \{G(S_\mu) \mid \mu \in \mathcal{M}\}$ of open tubes covers X and there exists an invariant locally finite partition of unity $\{\varphi_\mu : X \to [0,1] \mid \mu \in \mathcal{M}\}$ subordinated to $\widetilde{\mathcal{U}}$, i.e., every φ_μ is an invariant function with $\varphi_\mu^{-1}((0,1]) \subset G(S_\mu)$.

If X is paracompact then the orbit space X/G, being a closed image of X, is paracompact as well (see [9, Ch. VIII, Theorem 2.4]). In this case each cover of the form $\mathcal{U} = \{gO_\mu S_\mu \mid g \in G,\ \mu \in \mathcal{M}\}$, where S_μ is an H_μ-slice with a large slicing subgroup H_μ, is a G-normal cover.

The following result is based on the Approximate Slice Theorem 2.2 and plays an important role in this paper:

LEMMA 2.4 ([6]). *Let X be a paracompact G-space. Then for each open cover $\mathcal{V}$ of X there exists a G-normal cover $\mathcal{U} = \{gO_\mu S_\mu \mid g \in G,\ \mu \in \mathcal{M}\}$ of X with large slicing subgroups $\{H_\mu \mid \mu \in \mathcal{M}\}$ such that $\mathcal{U}$ is a refinement of $\mathcal{V}$.*

DEFINITION 2.5. Let X be a G-space, U an open invariant subset of X and $\mathcal{U} = \{gO_\mu S_\mu \mid g \in G,\ \mu \in \mathcal{M}\}$ a G-normal cover of U. Then $\mathcal{U}$ is called a *Dugundji G-cover* of U (with respect to X) if for any point $a \in X \setminus U$ and any neighborhood V_a of a in X, there exists a neighborhood $W_a \subset V_a$ of a in X such that every $gO_\mu S_\mu$ which meets W_a is contained in V_a.

LEMMA 2.6. *If X is a metrizable G-space, then every invariant open subset $U \subset X$ admits a Dugundji G-cover with respect to X.*

Proof. If $U = X$, then the one-element cover $\{X\}$ (with G as the slicing subgroup) is a Dugundji G-cover of U.

Let $U \neq X$. We consider an invariant metric ϱ on X (see, e.g., [17, Proposition 1.1.12]). In what follows we shall denote by $N(x, r)$ the open ball in X of radius r centered at $x \in X$. Let $r_x = (1/4)\varrho(x, X\backslash U)$. Consider the index set $\mathcal{M} = U/G$. In each orbit $\mu \in \mathcal{M}$ we choose a point x_μ; so $\mu = G(x_\mu)$. By continuity of the G-action on X, choose a neighborhood O_μ of the unity in G and a number $0 < \delta_\mu < r_{x_\mu}$ such that $O_\mu N(x_\mu, \delta_\mu) \subset N(x_\mu, r_{x_\mu})$ (recall that for $F \subset G$ and $A \subset X$ we denote by FA the set $\{ga \mid g \in F,\ a \in A\}$).

By Theorem 2.2, there exist a large subgroup H_μ of G with $G_{x_\mu} \subset H_\mu$ and an H_μ-slice S_μ such that $x_\mu \in S_\mu \subset N(x_\mu, \delta_\mu)$. Clearly, $O_\mu S_\mu \subset N(x_\mu, r_{x_\mu})$.

We claim that the G-normal cover $\mathcal{U} = \{gO_\mu S_\mu\} \mid g \in G,\ \mu \in \mathcal{M}\}$ is a Dugundji G-cover of U with large slicing subgroups $\{H_\mu \mid \mu \in \mathcal{M}\}$. Indeed, since the orbit space U/G is metrizable (see [17, Proposition 1.1.12]), and hence paracompact, there exists an invariant partition of unity subordinated to the cover $\{G(S_\mu) \mid \mu \in \mathcal{M}\}$.

Next, let $a \in X\backslash U$ and V_a a neighborhood of a. Choose $\varepsilon > 0$ such that $N(a, 2\varepsilon) \subset V_a$. We will show that $W_a = N(a, \varepsilon/2)$ is the desired neighborhood of a. In fact, let $gO_\mu S_\mu \cap W_a \neq \emptyset$, and let $y \in gO_\mu S_\mu \cap W_a$. Since $y \in W_a$, one has $\varrho(a, y) < \varepsilon/2$. On the other hand, $y \in gO_\mu S_\mu$, implying

$$\varrho(y, gx_\mu) \leq \operatorname{diam} gO_\mu S_{x_\mu}. \tag{2.1}$$

Since $O_\mu S_\mu \subset N(x_\mu, r_{x_\mu})$, we see that $\operatorname{diam} O_\mu S_\mu \leq 2r_{x_\mu}$. Now, by the invariance of the metric ϱ, we have

$$\begin{aligned}\operatorname{diam} gO_\mu S_\mu = \operatorname{diam} O_\mu S_\mu \leq 2r_{x_\mu} &= (1/2)\varrho(x_\mu, X\backslash U)\\ &= (1/2)\varrho(gx_\mu, X\backslash U).\end{aligned} \tag{2.2}$$

Since $a \in X\backslash U$, one has $\varrho(gx_\mu, X\backslash U) \leq \varrho(gx_\mu, a)$. Consequently, (2.1) and (2.2) yield

$$\varrho(y, gx_\mu) \leq (1/2)\varrho(gx_\mu, a).$$

Using this inequality and the triangle inequality, we get

$$\varrho(a, gx_\mu) \leq \varrho(a, y) + \varrho(y, gx_\mu) < \varepsilon/2 + (1/2)\varrho(a, gx_\mu).$$

This yields $\varrho(a, gx_\mu) < \varepsilon$, which together with (2.2) implies that

$$\operatorname{diam} gO_\mu S_\mu \leq (1/2)\varrho(gx_\mu, X\backslash U) \leq (1/2)\varrho(gx_\mu, a) < \varepsilon/2.$$

Consequently, since $gO_\mu S_\mu$ meets $W_a = N(a, \varepsilon/2)$, it follows from the inequality $\operatorname{diam} gO_\mu S_\mu < \varepsilon/2$ that $gO_\mu S_\mu \subset N(a, 2\varepsilon)$. Since $N(a, 2\varepsilon) \subset V_a$, the proof is finished. ∎

We shall need the following theorem which is a particular case of a result proved in Jaworowski [11] and Lashof [13]:

THEOREM 2.7 (Jaworowski–Lashof). *Let G be a compact Lie group, X a finite-dimensional, separable, metrizable G-space with a finite number of G-orbit types, and A a closed invariant subset of X. Suppose $f : A \to Y$ is a G-map into a metrizable G-space Y. If the H-fixed point set Y^H is an AR for each closed subgroup $H \subset G$, then f extends to a G-map $F : X \to Y$.*

3. Replacement by a G-nerve. First we recall the definition of the G-nerve of a G-normal cover [6].

Let X be a G-space and $\mathcal{U} = \{gO_\mu S_\mu \mid g \in G, \mu \in \mathcal{M}\}$ a G-normal cover of X with large slicing subgroups $\{H_\mu \mid \mu \in \mathcal{M}\}$. Let $\widetilde{\mathcal{N}}(\mathcal{U})$ be the ordinary nerve of the invariant cover $\{G(S_\mu) \mid \mu \in \mathcal{M}\}$. We shall denote by $\langle \mu_0, \ldots, \mu_n \rangle$ the closed n-simplex of $\widetilde{\mathcal{N}}(\mathcal{U})$ corresponding to the sets $G(S_{\mu_0}), \ldots, G(S_{\mu_n})$ with $G(S_{\mu_0}) \cap \cdots \cap G(S_{\mu_n}) \neq \emptyset$. Let $f_\mu : G(S_\mu) \to G/H_\mu$ be the corresponding slicing map (see Section 2). For any simplex $\sigma = \langle \mu_0, \ldots, \mu_n \rangle \subset \widetilde{\mathcal{N}}(\mathcal{U})$, we define the following subset of the product $\prod_{i=0}^n G/H_{\mu_i}$ endowed with the diagonal G-action:

$$F_\sigma = \Big\{ (f_{\mu_0}(x), \ldots, f_{\mu_n}(x)) \;\Big|\; x \in \bigcap_{i=0}^n G(S_{\mu_i}) \Big\}.$$

It follows from the equivariance of f_{μ_i} that F_σ is an invariant subset of the G-space $\prod_{i=0}^n G/H_{\mu_i}$. Observe that if τ is a subsimplex of σ, then $q_{\sigma\tau}(F_\sigma) \subset F_\tau$, where $q_{\sigma\tau} : \prod_{\mu\in\sigma} G/H_\mu \to \prod_{\mu\in\tau} G/H_\mu$ is the Cartesian projection.

For a simplex $\sigma = \langle \mu_0, \ldots, \mu_n \rangle \subset \widetilde{\mathcal{N}}(\mathcal{U})$, we denote by $\mathcal{J}(\sigma)$ the finite join

$$G/H_{\mu_0} * \cdots * G/H_{\mu_n}$$

in the sense of Milnor [15], equipped with the natural action of G. We shall use the notation $\sum_{i=0}^n t_{\mu_i} g_{\mu_i} H_{\mu_i}$ for the point in $\mathcal{J}(\sigma)$ determined by the elements $g_{\mu_i} H_{\mu_i} \in G_{\mu_i}/H_{\mu_i}$ and the numbers $t_{\mu_i} \geq 0$ with $\sum_{i=0}^n t_{\mu_i} = 1$. The t_{μ_i} are called the *barycentric coordinates* of $\sum_{i=0}^n t_{\mu_i} g_{\mu_i} H_{\mu_i}$.

Since each H_μ is a large subgroup, G/H_μ is a compact metrizable G-space (see Proposition 2.3). This implies that $\mathcal{J}(\sigma)$ is a compact metrizable G-space as well.

Further, we denote by $\Delta(\sigma, F_\sigma)$ the invariant subset of $\mathcal{J}(\sigma)$ consisting of all those $\sum_{i=0}^n t_{\mu_i} g_{\mu_i} p_{\mu_i} H_{\mu_i} \in \mathcal{J}(\sigma)$ for which

$$(g_{\mu_0} H_{\mu_0}, \ldots, g_{\mu_n} H_{\mu_n}) \in F_\sigma \quad \text{and} \quad p_{\mu_i} \in O_{\mu_i}, \quad 1 \leq i \leq n.$$

Set

$$\Delta(\sigma) = \bigcup\{\Delta(\tau, F_\tau) \mid \tau \text{ is a subsimplex of } \sigma\}.$$

Then $\Delta(\sigma)$ is a G-invariant subset of $\mathcal{J}(\sigma)$. We will consider the induced topology and G-action on $\Delta(\sigma)$.

We call $\Delta(\sigma)$ a *closed G-n-simplex* over the n-simplex σ associated with the families

$$\{H_{\mu_0}, \dots, H_{\mu_n}\}, \quad \{O_{\mu_0}, \dots, O_{\mu_n}\} \quad \text{and} \quad \{F_\tau \mid \tau \text{ is a subsimplex of } \sigma\}.$$

The set

$$\partial\Delta(\sigma) = \Delta(\sigma) \setminus \Big\{ \sum_{i=0}^{n} t_{\mu_i} g_{\mu_i} H_{\mu_i} \in \Delta(\sigma) \,\Big|\, t_{\mu_i} > 0,\, 0 \le i \le n \Big\}$$

is called the *G-boundary* of $\Delta(\sigma)$. The homogeneous spaces $G/H_{\mu_0}, \dots, G/H_{\mu_n}$ are called the *G-vertices* of the G-simplex $\Delta(\sigma)$.

Next, if $g_{\mu_0}H_{\mu_0} \in G/H_{\mu_0}, \dots, g_{\mu_n}H_{\mu_n} \in G/H_{\mu_n}$ are fixed elements, then the *closed* (resp., *open*) *n-cell* $\langle g_{\mu_0}H_{\mu_0}, \dots, g_{\mu_n}H_{\mu_n}\rangle$ is defined to be the subset of $\mathcal{J}(\sigma)$ consisting of all $\sum_{i=0}^{n} t_{\mu_i} g_{\mu_i} H_{\mu_i}$, where $t_i \ge 0$ (res., $t_i > 0$) and $\sum_{i=0}^{n} t_{\mu_i} = 1$. The corresponding *open n-cell* is denoted by $(g_{\mu_0}H_{\mu_0}, \dots, g_{\mu_n}H_{\mu_n})$.

Consider the union

$$\mathcal{N}(\mathcal{U}) = \bigcup\{\Delta(\sigma) \mid \sigma \text{ is a simplex of } \widetilde{\mathcal{N}}(\mathcal{U})\},$$

endowed with the weak topology determined by the family $\{\Delta(\sigma) \mid \sigma \subset \widetilde{\mathcal{N}}(\mathcal{U})\}$, i.e., a set $U \subset \mathcal{N}(\mathcal{U})$ is open in $\mathcal{N}(\mathcal{U})$ iff $U \cap \Delta(\sigma)$ is open in $\Delta(\sigma)$ for every simplex $\sigma \subset \widetilde{\mathcal{N}}(\mathcal{U})$. If σ and τ are two simplices of $\widetilde{\mathcal{N}}(\mathcal{U})$, then it is easy to see that $\Delta(\sigma) \cap \Delta(\tau) = \Delta(\sigma \cap \tau)$. This shows that $\Delta(\sigma) \cap \Delta(\tau)$ is closed in both $\Delta(\sigma)$ and $\Delta(\tau)$, implying that each G-simplex $\Delta(\sigma)$ retains its original topology and is a closed subset of $\mathcal{N}(\mathcal{U})$ (see, e.g., [9, Ch. VI, § 8]).

It is an easy exercise to show that the induced G-action on $\mathcal{N}(\mathcal{U})$ is continuous; thus $\mathcal{N}(\mathcal{U})$ is a G-space.

Definition 3.1 ([6]). The G-space $\mathcal{N}(\mathcal{U})$, endowed with the weak topology and the G-action determined by the family of its G-simplices

$$\{\Delta(\sigma) \mid \sigma \text{ is a simplex of } \widetilde{\mathcal{N}}(\mathcal{U})\},$$

is called the *equivariant nerve* or *G-nerve* of the G-normal cover $\mathcal{U}$.

In what follows we shall use the easily checked fact that a map $f : \mathcal{N}(\mathcal{U}) \to Z$ is continuous iff its restriction $f|_{\Delta(\sigma)}$ to every closed G-simplex $\Delta(\sigma)$ is continuous.

Lemma 3.2 ([6]). *Let Y be a G-space and $\mathcal{U} = \{gO_\mu S_\mu \mid g \in G,\, \mu \in \mathcal{M}\}$ a G-normal cover of Y. Then for each invariant partition of unity subordinated to the invariant cover $\{G(S_\mu) \mid \mu \in \mathcal{M}\}$, there exists a G-map*

$p : Y \to \mathcal{N}(\mathcal{U})$ such that the μth barycentric coordinate of $p(y)$ vanishes whenever $y \in Y \setminus G(S_\mu)$.

Generalizing an idea of Dugundji [8], we define the replacement by a G-nerve.

Let Y be a G-space, A an invariant closed subset of Y, and let

$$\mathcal{U} = \{gO_\mu S_\mu \mid g \in G,\, \mu \in \mathcal{M}\}$$

be a G-normal cover of $Y \setminus A$ with large slicing subgroups $\{H_\mu \mid \mu \in \mathcal{M}\}$. Let $\mathcal{N}(\mathcal{U})$ be the G-nerve of $\mathcal{U}$ and let $D(\mathcal{U})$ denote the disjoint union $A \cup \mathcal{N}(\mathcal{U})$. Topologize $D(\mathcal{U})$ as follows. Let $z \in D(\mathcal{U})$ be an arbitrary point. If $z \in \mathcal{N}(\mathcal{U})$, a basis of neighborhoods of z in $D(\mathcal{U})$ is taken to be all neighborhoods of z in $\mathcal{N}(\mathcal{U})$. If $z \in A$, we define a basis of neighborhoods of z in $D(\mathcal{U})$ to be all sets $\widetilde{W}$ defined as follows. For any neighborhood W of z in Y, let $\widetilde{W} \subset A \cup \mathcal{N}(\mathcal{U})$ be the set $A \cap W$ together with each open cell $(g_1 p_{\mu_1} H_{\mu_1}, \ldots, g_n p_{\mu_n} H_{\mu_n})$ in $\mathcal{N}(\mathcal{U})$, $g_i \in G$, $p_{\mu_i} \in O_{\mu_i}$, $1 \le i \le n$, such that $g_j O_{\mu_j} S_{\mu_j} \subset W$ for some $1 \le j \le n$.

It is easy to verify that $D(\mathcal{U})$ equipped with this topology is a Hausdorff space, and that both A and $\mathcal{N}(\mathcal{U})$, as subspaces of $D(\mathcal{U})$, preserve their original topologies. Moreover, $\mathcal{N}(\mathcal{U})$ is open in $D(\mathcal{U})$, and hence A is closed in $D(\mathcal{U})$. We equip $D(\mathcal{U})$ with the natural action of G in such a way that A and $N(\mathcal{U})$ are invariant subsets of $D(\mathcal{U})$, and we call the resulting G-space $D(\mathcal{U})$ the *Dugundji G-replacement* (associated with the G-normal cover $\mathcal{U}$).

LEMMA 3.3. *The natural action of G on $D(\mathcal{U})$ is continuous.*

Proof. One only needs to check continuity at each $a \in A$. Let $g \in G$ and $\widetilde{V}$ be a basic neighborhood of ga in $D(\mathcal{U})$. By continuity of the G-action on Y, there are a neighborhood W of a in Y and a neighborhood O of g in G such that $O \cdot W \subset V$. One easily sees that $O \cdot \widetilde{W} \subset \widetilde{V}$, completing the proof. ∎

LEMMA 3.4. *Let Y be a G-space, A a closed invariant subset of Y, and $\mathcal{U} = \{gO_\mu S_\mu \mid g \in G,\, \mu \in \mathcal{M}\}$ a Dugundji G-cover of $Y \setminus A$ with large slicing subgroups $\{H_\mu \mid \mu \in \mathcal{M}\}$. Then there exists a G-map $q : Y \to D(\mathcal{U})$ such that*

(1) *$q|_A$ is the identical homeomorphism,*
(2) *$q(Y \setminus A) \subset D(\mathcal{U}) \setminus q(A)$.*

Proof. Choosing an invariant partition of unity subordinated to the invariant cover $\{G(S_\mu) \mid \mu \in \mathcal{M}\}$, one can define a G-map $p : Y \setminus A \to \mathcal{N}(\mathcal{U})$ as in Lemma 3.2. Now we define $q : Y \to D(\mathcal{U})$ by setting

$$q(y) = \begin{cases} y & \text{if } y \in A, \\ p(y) & \text{if } y \in Y \setminus A. \end{cases}$$

Clearly, (1) and (2) hold. Let us check that q is continuous. For the continuity on $Y \setminus A$ we refer to Lemma 3.2. To prove the continuity on A, fix $a \in A$ and let $\widetilde{V}$ be a basic neighborhood of $q(a)$ in $D(\mathcal{U})$. By Definition 2.5, there exists a neighborhood $W \subset V$ of a in Y such that every $gO_\mu S_\mu$ which meets W is contained in V.

We claim that $q(W) \subset \widetilde{V}$. Indeed, if $y \in A \cap W$ then $q(y) = y \in A \cap W \subset A \cap V \subset \widetilde{V}$. Now let $y \in (Y \setminus A) \cap W$, and let $\mu_1, \ldots, \mu_n$ be all the indices such that $y \in g_{\mu_1} S_{\mu_1} \cap \cdots \cap g_{\mu_n} S_{\mu_n}$ for some $g_{\mu_1}, \ldots, g_{\mu_n} \in G$. Then each $g_{\mu_i} S_{\mu_i}$, $1 \leq i \leq n$, meets W at y, and hence

$$g_{\mu_1} O_{\mu_1} S_{\mu_1} \cup \cdots \cup g_{\mu_n} O_{\mu_n} S_{\mu_n} \subset V,$$

so that $\langle g_{\mu_1} p_{\mu_1} H_{\mu_1}, \ldots, g_{\mu_n} p_{\mu_n} H_{\mu_n} \rangle \subset \widetilde{V}$ for all $p_{\mu_i} \in O_{\mu_i}$. Since $q(y) = p(y) \in \langle g_{\mu_1} p_{\mu_1} H_{\mu_1}, \ldots, g_{\mu_n} p_{\mu_n} H_{\mu_n} \rangle$, we see that $q(y) \in \widetilde{V}$. The continuity of q is proved. Its equivariance follows from the invariance of φ_μ and the equivariance of f_μ, $\mu \in \mathcal{M}$. ■

Lemma 3.5. *Under the hypotheses of Lemma* 3.3, *for every neighborhood U of A in $D(\mathcal{U})$, there exists an invariant neighborhood V of A in $D(\mathcal{U})$ such that $\overline{V} \subset U$.*

Proof. One can assume that $U = \bigcup_{a \in A} \widetilde{U}_a$, where U_a is a neighborhood of a in Y, and $U_{ga} = gU_a$ for all $g \in G$ and $a \in A$.

By Definition 2.5, choose a neighborhood $V_a \subset U_a$ of a in Y such that every $gO_\mu S_\mu \in \mathcal{U}$ which meets V_a is contained in U_a. Clearly, one can assume that $V_{ga} = gV_a$ for all $g \in G$ and $a \in A$.

We claim that $V = \bigcup_{a \in A} \widetilde{V}_a$ is the desired neighborhood of A. Indeed, for every G-vertex G/H_μ, we define the *open star* by

$$\mathrm{St}(G/H_\mu, \mathcal{N}(\mathcal{U})) = \Big\{ \sum_{\lambda \in \mathcal{M}} t_\lambda g_\lambda H_\lambda \in \mathcal{N}(\mathcal{U}) \;\Big|\; t_\mu > 0 \Big\}.$$

It is not difficult to find that V is the union of A and all the open stars $\mathrm{St}(G/H_\mu, \mathcal{N}(\mathcal{U}))$ for which $O_\mu S_\mu \subset V_a$ for some $a \in A$. Then the closure $\overline{V}$ is just the union of A and all the corresponding *closed stars*

$$\overline{\mathrm{St}}\big(G/H_\mu, \mathcal{N}(\mathcal{U})\big) = \Big\{ \sum_{\lambda \in \mathcal{M}} t_\lambda g_\lambda H_\lambda \in \mathcal{N}(\mathcal{U}) \;\Big|\; t_\mu \geq 0 \Big\}.$$

Now let $\langle \underline{g_{\mu_0}} p_{\mu_0} H_{\mu_0}, \ldots, g_{\mu_n} g_{\mu_n} H_{\mu_n} \rangle$ be any closed cell contained in the closed star $\overline{\mathrm{St}}(G/H_{\mu_0}, \mathcal{N}(\mathcal{U}))$ with $O_{\mu_0} S_{\mu_0} \subset V_a$ for some $a \in A$. It suffices to show that $\langle g_{\mu_0} p_{\mu_0} H_{\mu_0}, \ldots, g_{\mu_n} g_{\mu_n} H_{\mu_n} \rangle$ is contained in U.

By the definition of the cell $\langle g_{\mu_0} p_{\mu_0} H_{\mu_0}, \ldots, g_{\mu_n} g_{\mu_n} H_{\mu_n} \rangle$, there exists a point $x \in \bigcap_{i=0}^n G(S_{\mu_i})$ such that $f_{\mu_i}(x) = g_{\mu_i} H_{\mu_i}$ for all $0 \leq i \leq n$, where $f_{\mu_i} : G(S_{\mu_i}) \to G/H_{\mu_i}$ is the slicing map (see Section 2). This shows, in turn, that $x \in \bigcap_{i=0}^n g_{\mu_i} S_{\mu_i}$, and in particular, $\bigcap_{i=0}^n g_{\mu_i} O_{\mu_i} S_{\mu_i} \neq \emptyset$.

But remember that $O_{\mu_0}S_{\mu_0} \subset V_a$ for some $a \in A$, so $g_{\mu_0}O_{\mu_0}S_{\mu_0} \subset g_{\mu_0}V_a = V_{g_{\mu_0}a}$. Now, due to the choice of the neighborhood $V_{g_{\mu_0}a}$, each $g_{\mu_i}O_{\mu_i}S_{\mu_i}$, $0 \le i \le n$, is contained in U_a because $g_{\mu_i}O_{\mu_i}S_{\mu_i} \cap V_{g_{\mu_0}a} \supset g_{\mu_i}O_{\mu_i}S_{\mu_i} \cap g_{\mu_0}O_{\mu_0}S_{\mu_0} \neq \emptyset$. This shows that the closed cell $\langle g_{\mu_0}p_{\mu_0}H_{\mu_0}, \ldots$ $\ldots, g_{\mu_n}g_{\mu_n}H_{\mu_n}\rangle$ is contained in $\widetilde{U}_a$, and hence in U, as required. ∎

4. Proof of Main Theorem

LEMMA 4.1. *Let (Y, A) be a G-pair, $\mathcal{U} = \{gO_\mu S_\mu \mid g \in G,\ \mu \in \mathcal{M}\}$ a Dugundji G-cover of $Y \setminus A$ with large slicing subgroups $\{H_\mu \mid \mu \in \mathcal{M}\}$, and let X be a G-ANR. Then every G-map $\varphi : A \to X$ admits a G-extension $\varphi' : U \to X$ over a neighborhood U of A in the Dugundji G-replacement $D(\mathcal{U})$.*

Proof. One can assume that X is a closed invariant subset of a G-space of the form $C(G, L)$, where L is a normed linear space and $C(G, L)$ is the normed linear G-space of all continuous maps $G \to L$, endowed with the sup-norm and the linear G-action defined by the formula $(gf)(x) = f(xg)$ for $g, x \in G$, $f \in C(G, L)$ (see [2, Theorem 3]).

Let $p : C(G, L) \to L$ be the evaluation map defined by $p(f) = f(e)$, where $f \in C(G, L)$ and e is the unity of G. Clearly p is continuous. Set $f = p\varphi$.

First we shall extend f to a continuous map $F : D(\mathcal{U}) \to L$. Then the map $f' : D(\mathcal{U}) \to C(G, L)$ defined by $f'(z)(g) = F(gz)$, $z \in Z$, $g \in G$, is a G-extension of f. Further, since X is a G-ANR, there exists a G-retraction $r : T \to X$ for some invariant neighborhood T of X in $C(G, L)$. Put $U = (f')^{-1}(T)$ and $\varphi' = rf'$. Then the map $\varphi' : U \to X$ is the desired G-extension of φ.

Now, let us proceed with the construction of $F : D(\mathcal{U}) \to L$.

Let $\mathcal{N}_k(\mathcal{U})$ denote the *G-k-skeleton* of $\mathcal{N}(\mathcal{U})$, i.e., $\mathcal{N}_k(\mathcal{U})$ is the union of all G-m-simplices of $\mathcal{N}(\mathcal{U})$ with $m \le k$.

First we extend f over each G-vertex $G/H_\mu \subset \mathcal{N}_0(\mathcal{U})$. To this end, for every G-vertex G/H_μ choose a finite open cover of the form

$$\{g_1O_\mu H_\mu, \ldots, g_nO_\mu H_\mu\},$$

where n and $g_1, \ldots, g_n$ depend upon $\mu \in \mathcal{M}$. Of course, g_k is not uniquely determined by $g_kO_\mu H_\mu$, but for our further constructions it suffices to fix just one such element. Let $\{\varphi_1, \ldots, \varphi_n\}$ be a partition of unity of G/H_μ subordinated to this cover, i.e., $\varphi_k^{-1}((0, 1]) \subset g_kO_\mu H_\mu$, $1 \le k \le n$. For each $\mu \in \mathcal{M}$ choose an $x_\mu \in S_\mu$ and associate to it a point $a_\mu \in A$ such that $\varrho(x_\mu, a_\mu) < 2\varrho(x_\mu, A)$. Since A is an invariant subset and ϱ is an invariant metric,

$$\varrho(gx_\mu, ga_\mu) < 2\varrho(gx_\mu, A) \quad \text{for all } g \in G \text{ and } \mu \in \mathcal{M}. \tag{4.1}$$

Define $\Phi : A \cup \mathcal{N}_0(\mathcal{U}) \to L$ by setting

$$\Phi(x) = \begin{cases} f(x), & \text{if } x \in A, \\ \varphi_1(x) f(g_1 a_\mu) + \cdots + \varphi_n(x) f(g_n a_\mu) & \text{if } x \in G/H_\mu. \end{cases}$$

As $f(g_1 a_\mu), \ldots, f(g_n a_\mu)$ do not depend on $x \in G/H_\mu$, and $\varphi_1, \ldots, \varphi_n$ are continuous functions on G/H_μ, we conclude that Φ is continuous at each point of G/H_μ. Consequently, Φ is continuous on $\mathcal{N}_0(\mathcal{U})$.

Let us verify continuity of Φ at each $a \in A$. Let M be a neighborhood of $\Phi(a) = f(a)$ in L. Of course, one can assume that M is an open ball centered at $f(a)$. By continuity of f, there exists a $\delta > 0$ such that $f(N(a, \delta) \cap A) \subset M$. Recall that $N(y, r)$ denotes the open ball in Y of radius r centered at $y \in Y$.

Let $V = N(a, \delta/3)$. By Definition 2.5, there exists a neighborhood $W \subset V$ of a in Y such that every $gO_\mu S_\mu$ that meets W is contained in V. We claim that $\Phi(\widetilde{W} \cap (A \cup \mathcal{N}_0(\mathcal{U}))) \subset M$, where $\widetilde{W}$ is the basic neighborhood of a in $D(\mathcal{U})$ defined by W (see Section 3, the paragraph after Lemma 3.2).

Indeed, $\widetilde{W} \cap A = W \cap A \subset N(a, \delta) \cap A$, so $f(\widetilde{W} \cap A) \subset M$.

If $x \in \widetilde{W} \cap \mathcal{N}_0(\mathcal{U})$ then $x = g_1 H_\mu \in G/H_\mu$ for some G-vertex G/H_μ and $g_1 O_\mu S_\mu \subset W$, where $g_1 \in G$. Let $g_1 O_\mu H_\mu, \ldots, g_k O_\mu H_\mu$ be all the elements of the cover $\{g_1 O_\mu H_\mu, \ldots, g_n O_\mu H_\mu\}$ which contain x. Then $\bigcap_{i=1}^k g_i O_\mu S_\mu \neq \emptyset$.

Since $g_1 O_\mu S_\mu \subset W$, we infer that $g_i O_\mu S_\mu \cap W \neq \emptyset$, $1 \le i \le k$. Hence, due to the choice of W, each $g_i O_\mu S_\mu$ must be contained in V.

Therefore, $\varrho(g_i x_\mu, a) < \delta/3$ for all $i = 1, \ldots, k$. Using (4.1), we get

$$\begin{aligned} \varrho(g_i a_\mu, a) &\le \varrho(g_i a_\mu, g_i x_\mu) + \varrho(g_i x_\mu, a) < 2\varrho(g_i x_\mu, A) + \varrho(g_i x_\mu, a) \\ &\le 2\varrho(g_i x_\mu, a) + \varrho(g_i x_\mu, a) < 2\delta/3 + \delta/3 = \delta. \end{aligned}$$

Hence all the points $g_i a_\mu$, $i = 1, \ldots, k$, belong to the ball $N(a, \delta)$, which yields $f(g_1 a_\mu), \ldots, f(g_n a_\mu) \in M$. Since M is convex, it then follows from the definition of $\Phi(x)$ that $\Phi(x) \in M$. Thus, Φ is continuous on $A \cup \mathcal{N}_0(\mathcal{U})$.

Since L is a linear space, we can extend Φ linearly over each G-simplex of $\mathcal{N}(\mathcal{U})$ to obtain a map $F : D(\mathcal{U}) \to X$. More precisely, if $\sigma = \langle \mu_0, \ldots, \mu_n \rangle$ is a simplex and $\sum_{i=0}^n t_{\mu_i} g_{\mu_i} p_{\mu_i} H_{\mu_i}$ is a point of the G-simplex $\Delta(\sigma) \subset \mathcal{N}(\mathcal{U})$, then we set

$$F\Big(\sum_{i=0}^n t_{\mu_i} g_{\mu_i} p_{\mu_i} H_{\mu_i}\Big) = \sum_{i=0}^n t_{\mu_i} \Phi(g_{\mu_i} p_{\mu_i} H_{\mu_i}).$$

Clearly, F is continuous on each G-simplex of $\mathcal{N}(\mathcal{U})$. Since $\mathcal{N}(\mathcal{U})$ is endowed with the weak topology defined by its closed G-simplices, F is continuous on $\mathcal{N}(\mathcal{U})$. It remains to check the continuity on A.

Fix $a \in A$ and let M be a neighborhood of $F(a) = \Phi(a)$ in L. Clearly, one can assume that M is an open ball centered at $f(a)$.

By continuity of Φ, there exists a basic neighborhood $\widetilde{V}$ of a in $\mathcal{N}(\mathcal{U})$ such that $\Phi(\widetilde{V} \cap (A \cup \mathcal{N}_0(\mathcal{U}))) \subset M$, where V is a neighborhood of a in Y. By Definition 2.5, there exists a neighborhood $W \subset V$ of a in Y such that every $gO_\mu S_\mu$ which meets W is contained in V. We claim that $F(\widetilde{W}) \subset M$.

Indeed, $\widetilde{W} \cap A = W \cap A \subset V \cap A$, so $F(\widetilde{W} \cap A) = \Phi(W \cap A) \subset \Phi(V \cap A) \subset M$.

If $x \in \widetilde{W} \cap \mathcal{N}(\mathcal{U})$ then x belongs to an open cell

$$(g_{\mu_0} p_{\mu_1} H_{\mu_1}, \ldots, g_{\mu_n} p_n H_{\mu_n}) \subset \widetilde{W} \cap \mathcal{N}(\mathcal{U})$$

with $g_{\mu_i} \in G$, $p_{\mu_i} \in O_{\mu_i}$, $0 \le i \le n$, such that $g_{\mu_j} O_{\mu_j} S_{\mu_j} \subset W$ for some $0 \le j \le n$.

It follows from the definition of $(g_{\mu_0} p_{\mu_0} H_{\mu_0}, \ldots, g_{\mu_n} p_n H_{\mu_n})$ that $\bigcap_{i=0}^{n} g_{\mu_i} O_{\mu_i} S_{\mu_i} \neq \emptyset$ (cf. the proof of Lemma 3.5). Since $g_{\mu_j} O_{\mu_j} S_{\mu_j} \subset W$, it follows that every $g_{\mu_i} O_{\mu_i} S_{\mu_i}$ meets W, and is contained in V. Consequently, each cell vertex $g_{\mu_i} p_{\mu_i} H_{\mu_i}$ belongs to $\widetilde{V} \cap \mathcal{N}_0(\omega)$, and so $\Phi(g_{\mu_i} p_{\mu_i} H_{\mu_i}) \in M$. But $F(x)$ is a convex combination of the points $\Phi(g_{\mu_i} p_{\mu_i} H_{\mu_i})$, $0 \le i \le n$, and since M is convex, we conclude that $F(x) \in M$. ∎

Proof of Main Theorem. The "only if" part is evident. To prove the "if" part, we first observe that the H-fixed point sets of a G-ANR are ANR's [1, Theorem 7]. On the other hand, it is well known that a homotopy trivial ANR is an AR [14, Theorem 4.2.20]. So, in our case each X^H is an AR. Using this fact, let us prove that X is a G-AR.

Indeed, let Y be a metrizable G-space, A a closed invariant subset of Y, and $\varphi : A \to X$ a G-map. We shall show that φ extends to a G-map $\Phi : Y \to X$.

Choose a Dugundji G-cover $\mathcal{U} = \{gO_\mu S_\mu \mid g \in G,\ \mu \in \mathcal{M}\}$ of $Y \setminus A$ with large slicing subgroups $\{H_\mu \mid \mu \in \mathcal{M}\}$ (see Lemma 2.6). Let $\mathcal{N}(\mathcal{U})$ denote the G-nerve of $\mathcal{U}$ and let $q : Y \to D(\mathcal{U})$ be a G-map as in Lemma 3.4. It is sufficient to prove that $\varphi : A \to X$ extends to a G-map $F : D(\mathcal{U}) \to X$. Then $\Phi = Fq$ is as desired.

Let $\mathcal{N}_k(\mathcal{U})$ denote the G-k-skeleton of $\mathcal{N}(\mathcal{U})$. Since X is a G-ANR, by Lemma 4.1, φ extends to a G-map $\varphi' : U \to X$, where U is a G-invariant neighborhood of A in $D(\mathcal{U})$. By Lemma 3.5, choose a G-invariant neighborhood V of A in $D(\mathcal{U})$ such that $\overline{V} \subset U$.

We will extend φ, by induction on the dimension k, to a G-map $f_k : \overline{V} \cup \mathcal{N}_k(\mathcal{U}) \to X$, $k \ge 0$, and thus obtain the desired G-extension $F : D(\mathcal{U}) \to X$.

First we define $f_0 : \overline{V} \cup \mathcal{N}_0(\mathcal{U}) \to X$. Since $X^G \neq \emptyset$, we choose $v \in X^G$. If G/H_μ is a G-vertex in $\mathcal{N}_0(\mathcal{U}) \setminus \overline{V}$, we set $f_0(gH_\mu) = v$, $gH_\mu \in G/H_\mu$. We also put $f_0|_{\overline{V}} = \varphi'|_{\overline{V}}$. Since f_0 is continuous at each G-vertex G/H_μ, it is continuous on $\mathcal{N}_0(\mathcal{U})$. Since it coincides with φ' on V, it is continuous on A as well. Moreover, it preserves the action of G, and hence it is a G-extension of φ.

Now suppose a G-extension $f_k : \overline{V} \cup \mathcal{N}_k(\mathcal{U}) \to X$ of f_{k-1}, $k \geq 1$, is already constructed. We shall construct a G-extension $f_{k+1} : \overline{V} \cup \mathcal{N}_{k+1}(\mathcal{U}) \to X$ of f_k.

Let $\sigma = \langle \mu_0, \ldots, \mu_{k+1} \rangle$ be a $k+1$-simplex in $\widetilde{\mathcal{N}}(\mathcal{U})$ such that the G-$(k+1)$-simplex $\Delta(\sigma) \subset \mathcal{N}_{k+1}(\mathcal{U})$ is not contained in $\overline{V}$. Consider the following closed G-invariant subset of $\Delta(\sigma)$:

$$B = (\overline{V} \cap \Delta(\sigma)) \cup \partial\Delta(\sigma),$$

where $\partial\Delta(\sigma)$ is the G-boundary of $\Delta(\sigma)$. Observe that f_k is defined on B because $\overline{V} \cap \Delta(\sigma) \subset \overline{V}$ and $\partial\Delta(\sigma) \subset \mathcal{N}_k(\mathcal{U})$.

Let N_{μ_i}, $0 \leq i \leq k+1$, be a closed normal subgroup of G such that $N_{\mu_i} \subset H_{\mu_i}$ and G/N_{μ_i} is a Lie group. Put $N = \bigcap_{i=0}^{k+1} N_{\mu_i}$. This is clearly a closed normal subgroup of G and the quotient group G/N is a Lie group (see e.g. [19, Ch. 8, Section 46 (A)]). It is clear that $N \subset G_b \subset G_{f_k(b)}$ for every $b \in B$. In other words, f_k maps B into the N-fixed point set X^N.

Since N is a normal subgroup of G, X^N is a G-invariant subset of X, and hence, a G-space. As N acts trivially on the join $\mathcal{J}(\sigma) = G/H_{\mu_0} * \cdots * G/H_{\mu_{k+1}}$ and on X^N, we can consider these G-spaces as G/N-spaces endowed with the induced G/N-action.

Now we aim at extending the G/N-map $f_k : B \to X^N$ to a G/N-map $f_\sigma : \Delta(\sigma) \to X^N$. Observe that for any closed subgroup $K' \subset G/N$, the set $(X^N)^{K'}$ of K'-fixed points in X^N is an AR. Indeed, let K be the preimage of K' under the natural homomorphism $G \to G/N$. Then it is clear that $(X^N)^{K'} = X^K$. But X^K is an AR by the hypothesis.

Claim. *The join $\mathcal{J}(\sigma)$ has finitely many G/N-orbit types.*

Proof. First we observe that G/H_{μ_i} is naturally G/N-homeomorphic to $(G/N)/(H_{\mu_i}/N)$. Since G/N is a Lie group, by [17, Proposition 1.4.1], G/H_{μ_i} can be G/N-equivariantly embedded into a Euclidean G/N-space E_i equipped with an orthogonal action of G/N. This gives rise to a natural G/N-equivariant embedding of $\mathcal{J}(\sigma)$ into $E = E_0 \oplus \cdots \oplus E_{k+1}$ endowed with the diagonal action of G/N. Since E is a Euclidean G/N-space, it has finitely many G/N-orbit types (see [17, Corollary 1.7.26]). This implies that $\mathcal{J}(\sigma)$ also has finitely many G/N-orbit types, proving the claim.

Further, since $\Delta(\sigma)$ is an invariant subset of $\mathcal{J}(\sigma)$, which in turn is a subset of a Euclidean G/N-space (see the proof of the Claim), we infer that $\Delta(\sigma)$ is a finite-dimensional, separable, metrizable G/N-space with finitely many G/N-orbit types. Thus, the hypotheses of Jaworowski–Lashof Theorem 2.7 are fulfilled, and so $f_k|_B$ admits a G/N-extension $f_\sigma : \Delta(\sigma) \to X^N$.

In what follows we will consider f_σ as a G-map from $\Delta(\sigma)$ to X.

Next, we define the map $f_{k+1} : \overline{V} \cup \mathcal{N}_{k+1}(\mathcal{U}) \to X$ by setting

$$f_{k+1}(z) = \begin{cases} f_\sigma(z) & \text{if } z \in \Delta(\sigma) \text{ with } \dim \sigma = k+1, \\ f_k(a) & \text{if } a \in \overline{V} \cup \mathcal{N}_k(\mathcal{U}). \end{cases}$$

Then f_{k+1} is well defined and extends f_k. Since $f_{k+1}|_{\Delta(\tau)}$ is continuous for every G-simplex $\Delta(\tau) \subset \mathcal{N}_{k+1}(\mathcal{U})$, we see that f_{k+1} is continuous on $\mathcal{N}_{k+1}(\mathcal{U})$. As f_{k+1} coincides with φ' on V, it is continuous at the points of A as well. It is also clear that f_{k+1} preserves the action of G. This completes the inductive step.

Now, we define the desired G-map $F : D(\mathcal{U}) \to X$ by setting

$$F(z) = f_k(z) \quad \text{whenever} \quad z \in A \cup \mathcal{N}_k(\mathcal{U}),\ k \geq 0.$$

Clearly, F is well defined and preserves the G-action. Its continuity on $\mathcal{N}(\mathcal{U})$ follows from the continuity of $F|_{\Delta(\tau)}$ for each G-simplex $\Delta(\tau) \subset \mathcal{N}(\mathcal{U})$. Since F coincides with φ' on the neighborhood V of A, it is continuous on A as well. This completes the proof of the Main Theorem.

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