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On closed sets with convex projections in Hilbert space

by

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Abstract. Let k be a fixed natural number. We show that if C is a closed and nonconvex set in Hilbert space such that the closures of the projections onto all k-hyperplanes (planes with codimension k) are convex and proper, then C must contain a closed copy of Hilbert space. In order to prove this result we introduce for convex closed sets B the set $\mathcal{E}^k(B)$ consisting of all points of B that are extremal with respect to projections onto k-hyperplanes. We prove that $\mathcal{E}^k(B)$ is precisely the intersection of all k-imitations C of B, i.e., closed sets C that have the same projections as B onto all k-hyperplanes. For every closed convex set B in ℓ^2 with nonempty interior we construct "minimal" k-imitations C, in the sense that $\dim(C \setminus \mathcal{E}^k(B)) \leq 0$. Finally, we show that whenever a compact set has convex projections onto all finite-dimensional planes, then it must be convex.

1. Introduction. Consider the vector space $\mathbb{R}^n$ for $n \geq 3$. Let us call the image of a subset X of $\mathbb{R}^n$ or Hilbert space under an orthogonal projection onto a hyperplane a *shadow* of X. Borsuk [3] has shown that there exist Cantor sets in $\mathbb{R}^n$ such that all their shadows contain $(n-1)$-dimensional convex bodies. In contrast, Cobb [5] showed that every compactum C in $\mathbb{R}^n$ with the property that all its shadows are convex bodies contains an arc. Dijkstra, Goodsell, and Wright [6] improved on this result by showing that such a C must contain an $(n-2)$-sphere, so in this case projections cannot raise dimension by more than one.

The starting point of the present paper are the results in Barov, Cobb, and Dijkstra [1]. In that paper closed sets C in $\mathbb{R}^n$ that have convex projections onto all k-dimensional planes are considered. If the projections of C are proper in a sufficient number of directions, then it is proved that C contains a closed subset that is a $(k-1)$-manifold without boundary. Also, for every closed and convex set $B \subset \mathbb{R}^n$ with nonempty interior "minimal imitations"

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are constructed, which are closed sets that have the same projections onto k-planes and are minimal with respect to dimension. A natural question is whether one can get similar results when the underlying space is the real Hilbert space ℓ^2 instead of $\mathbb{R}^n$. The answer to this question is positive and the main purpose of this paper is to formulate and prove these results.

In order to formulate the main theorems we need some definitions. If $A \subset \ell^2$ then we define

$$A^{\perp} = \{v \in \ell^2 : v \cdot x = v \cdot y \text{ for all } x, y \in A\},$$

where $\cdot$ denotes the inner product. Also we define

$$\operatorname{codim} A = \dim A^{\perp} \in \{0, 1, \ldots, \infty\}.$$

A *plane* in ℓ^2 is a closed affine subspace of ℓ^2 and a plane L is called a *k-plane* if $\dim L = k$. A *k-hyperplane* H is a plane with $\operatorname{codim} H = k$. If L is a plane then $\mathfrak{p}_L : \ell^2 \to L$ denotes the orthogonal projection onto L, defined by $\{\mathfrak{p}_L(x)\} = L \cap (x + L^{\perp})$ for $x \in \ell^2$. A *basis* for ℓ^2 is a set of linearly independent vectors whose linear hull is dense in ℓ^2. Finally, $\overline{A}$ denotes the closure of A in ℓ^2.

THEOREM 1. *Let $k \in \mathbb{N}$, let B be a closed convex subset of ℓ^2, and let C be a closed set in ℓ^2 such that $B \neq C$. Assume that $\overline{p(C)} = \overline{p(B)}$ for every projection p of ℓ^2 onto a k-hyperplane. If there exists a basis $\mathcal{B}$ for ℓ^2 such that $\overline{\mathfrak{p}_{L^{\perp}}(C)} \neq L^{\perp}$ for every linear space L generated by k elements of $\mathcal{B}$, then C contains a closed set homeomorphic to ℓ^2.*

If $k \in \mathbb{N}$ then two subsets A and B of ℓ^2 are called *k-imitations* of each other if they have identical projections onto all k-hyperplanes or, equivalently, a k-plane meets A if and only if it meets B. In order to prove Theorem 1 we introduce for closed convex sets B the sets $\mathcal{E}^k(B)$ consisting of points of B that are "extremal with respect to projections onto k-hyperplanes". We prove that $\mathcal{E}^k(B)$ is precisely the intersection of all closed k-imitations of B (Corollary 23) and we find the required copy of ℓ^2 in this set.

In the final section we construct minimal imitations of B:

THEOREM 2. *If $k \in \mathbb{N}$ and if B is a closed convex subset of ℓ^2 such that $\operatorname{codim} B \neq k$, then there exists a closed k-imitation C of B such that $\dim(C \setminus \mathcal{E}^k(B)) \leq 0$.*

In the process of proving our results we follow the general approach of [1], which in turn was based on the method of Dijkstra, Goodsell, and Wright [6]. However, some of the arguments in [1] rely on properties of finite-dimensional spaces that are not valid in Hilbert space such as the fact that in $\mathbb{R}^n$ the interior of a convex set in its affine hull is nonempty and that every closed set in $\mathbb{R}^n$ is σ-compact. This calls for a different approach or a more complicated argument in some places. In particular, the role of compacta is very different in ℓ^2 and is discussed in §4, a section with no analogue in [1].

Our paper is organized as follows. In §2 we establish the terminology and we present basic lemmas. Theorem 1 is proved in §3. We deal with projections onto finite-dimensional planes and the role of compacta in §4. §5 is about hiding sets behind zero-dimensional sets, and the results from that section are then used to prove Theorem 2 in §6.

2. Definitions and preliminaries. In this section we set up our terminology and we give the basic lemmas in preparation for the proof of the main theorems. Throughout this paper the underlying space will be the real Hilbert space ℓ^2, defined as follows:

$$\ell^2 = \Big\{x = (x_n)_{n=1}^{\infty} : x_n \in \mathbb{R} \text{ and } \sum_{i=1}^{\infty} x_i^2 < \infty\Big\}.$$

The origin of ℓ^2 will be denoted by $\mathbf{0}$. Let $u=(u_1, u_2, \dots)$ and $v=(v_1, v_2, \dots)$ be elements of ℓ^2. We shall use the standard dot product: $u \cdot v = \sum_{i=1}^{\infty} u_i v_i$. The norm on ℓ^2 is given by $\|u\| = \sqrt{u \cdot u}$ and the metric d by $d(u, v) = \|v - u\|$. Throughout this paper $B_\varepsilon(x)$ stands for the open ε-neighbourhood of the point x. Let $\{e^1, e^2, \dots\}$ denote the standard orthonormal basis for ℓ^2, that is, e^i is the unit vector in the positive direction of the x_i-axis.

A *plane* in ℓ^2 is a closed affine subspace of ℓ^2, thus planes have the form $v + L$ where $v \in \ell^2$ and L is a closed linear subspace of ℓ^2. Note that the set $A^{\perp}$ as defined in the introduction is a closed linear subspace of ℓ^2. If L is a plane in ℓ^2, then $L^{\perp}$ is called the *orthocomplement* of L. Note that we have extended the usual definition of orthocomplement from linear spaces to affine spaces in such a way that $L^{\perp} = (v + L)^{\perp}$. A *$k$-plane* in ℓ^2 is a k-dimensional affine subspace of ℓ^2 and a *k-subspace* is a k-dimensional linear subspace of ℓ^2. We will identify the space $\mathbb{R}^k$ with the k-subspace $\{x \in \ell^2 : x_{k+1} = x_{k+2} = \cdots = 0\}$. The unit sphere in ℓ^2 is denoted by S^{∞}. By *projection* we mean orthogonal projection. If L is a plane in ℓ^2, then $\mathfrak{p}_L : \ell^2 \to L$ denotes the orthogonal projection onto L. The closure of a set A in ℓ^2 is denoted by $\overline{A}$. The interior of a set A in ℓ^2 is denoted by $\operatorname{int} A$.

DEFINITION 1. Let L be a plane in ℓ^2. A plane $H \subset L$ is called a *k-hyperplane* in L if $\dim(H^{\perp} \cap L) = k$. In other words, a k-hyperplane is a plane with codimension k in the ambient space. A *hyperplane* H of L is a plane of L of codimension 1. A *shadow* of a set A is a projection of A onto a hyperplane. The two components of $L \setminus H$ are called the *sides* of the hyperplane H. We say that H *cuts* a subset A of L if A contains points on both sides of H. A subset V of L is called a *halfspace* of L if it is the union of a hyperplane and one of its sides. If L is a k-plane, $k \in \mathbb{N}$, then V is called a *k-halfplane* in ℓ^2. A 1-halfplane is called a *halfline* or a *ray*.

Definition 2. Let A be a nonempty subset of ℓ^2. We denote the convex hull of A by $\langle A\rangle$. The *affine hull* $\operatorname{aff} A$ of A is the intersection of all planes of ℓ^2 that contain A. Note that $\operatorname{codim} A = \operatorname{codim}(\operatorname{aff} A)$. Let ∂A stand for the boundary of A with respect to $\operatorname{aff} A$ and let $A^\circ = A \setminus \partial A$.

If A is a finite-dimensional convex set, then $A^\circ \neq \emptyset$. For infinite-dimensional convex sets this is not true (see Example 1).

Lemma 3. *Let B be a convex set in ℓ^2 with $B^\circ = \emptyset$. If A is a subset of B with finite codimension in ℓ^2, then $A^\circ = \emptyset$.*

Proof. Striving for a contradiction, we assume that there is an $X \subset B$ such that $X^\circ \neq \emptyset$ and $\operatorname{codim} X < \infty$. Now, let n be the minimum integer with the following property:

- There is a set $A \subset B$ such that $A^\circ \neq \emptyset$ and $\operatorname{codim} A = n$.

Put $H = \operatorname{aff} A$ and $F = B \cap H$ so $H = \operatorname{aff} F$ and $F^\circ \neq \emptyset$. Note that $F \neq B$ and select an $x \in B \setminus H$. Consider $H' = \operatorname{aff}(F \cup \{x\})$. Note that $\{x + t(y - x) : 0 < t < 1,\ y \in F^\circ\}$ is a nonempty open subset of H' that is contained in B. So we have $(H' \cap B)^\circ \neq \emptyset$. Clearly, $\operatorname{codim}(H' \cap B) = \operatorname{codim} H' = n - 1$ in violation of the minimality of n. ■

Definition 3. Let B be a closed convex set in ℓ^2. A nonempty subset F of B is called a *face* of B if there is a hyperplane H of $\operatorname{aff} B$ that does not cut B with the property $F = B \cap H$. Note that F is also closed and convex, and $\operatorname{codim} F > \operatorname{codim} B$ whenever $\operatorname{codim} B$ is finite. If F is a face of B we write $F \prec B$. We say that a subset F of B is a *derived face* of B if $F = B$ or there exists a sequence $F = F_1 \prec \cdots \prec F_m = B$ for some m.

Remark 1. Let $F \prec B$ and assume that $m = \operatorname{codim} F$ is finite. Put $H_m = \operatorname{aff} F$, $k = \operatorname{codim} B$, and $H_k = \operatorname{aff} B$. There is a hyperplane H_{k+1} of H_k that does not cut B and has the property $F = B \cap H_{k+1}$. If $H_{k+1} \neq \operatorname{aff} F$ then $m > k+1$ and we can fill in the missing dimensions and construct a sequence $H_m \subset H_{m-1} \subset \cdots \subset H_k$ of affine spaces such that $\operatorname{codim} H_i = i$ for $i \in \{k, \ldots, m\}$. Note that if $k + 1 < i \le m$ then

$$B \cap H_{i-1} \subset B \cap H_{k+1} = F \subset H_m \subset H_i$$

and hence H_i is a hyperplane H_{i-1} that does not cut $B \cap H_{i-1}$.

Observe now that if F is a derived face of B and $m \le \operatorname{codim} F$, then we can find a sequence of affine spaces $H_m \subset H_{m-1} \subset \cdots \subset H_0$ such that $\operatorname{codim} H_i = i$ for each i, $\operatorname{aff} F \subset H_m$, and H_i is a hyperplane in H_{i-1} that does not cut $B \cap H_{i-1}$ for $i \in \{1, \ldots, m\}$.

Remark 2. We list a few facts concerning closed convex sets and hyperplanes. Note that if $F \prec B$ then $F \subset \partial B$. Let B be a closed convex set in ℓ^2 with $B^\circ \neq \emptyset$. Since $\overline{\operatorname{int} B} = B$ (see [4, p. TVS II.14]), a hyperplane H cuts

B if and only if H meets the interior of B. According to the Hahn–Banach theorem (see [10, p. 197]) every point in ∂B is contained in a hyperplane H of $\operatorname{aff} B$ that does not cut B. In other words, ∂B equals the union of the faces of B. However, if $B^\circ = \emptyset$ then $\partial B = B$ may not equal the union of all its faces as the following example shows.

EXAMPLE 1. Consider the convex compactum

$$B = \{x \in \ell^2 : x_n \in [-2^{-n}, 2^{-n}] \text{ for all } n \in \mathbb{N}\}.$$

Assume that H is a hyperplane through the origin. Then H can be represented as $H = \{v \in \ell^2 : v \cdot u = 0\}$ for some $u \in S^\infty$. Thus, there is a $k \in \mathbb{N}$ such that $u_k \neq 0$. Let $v_1 = (0, \dots, 0, 2^{-k}, 0, \dots)$ and $v_2 = (0, \dots, 0, -2^{-k}, 0, \dots)$. Then v_1 and v_2 are on different sides of H because $u \cdot v_1$ and $u \cdot v_2$ have opposite signs. Consequently, H cuts B and hence $\mathbf{0}$ is contained in no face of B. This also means that B is contained in no hyperplane and hence $\operatorname{aff} B = \ell^2$ and $B^\circ = \emptyset$ because B is compact.

However, the union of the faces is always dense in ∂B.

LEMMA 4. *Let B be a closed convex set in ℓ^2 with $B^\circ = \emptyset$. Then the set $\bigcup\{F : F$ is a face of $B\}$ is dense in B.*

Proof. Let $x \in B = \partial B$ and $\varepsilon > 0$. Pick $z \in B_\varepsilon(x) \cap (\operatorname{aff} B \setminus B)$. According to [10, p. 347] there is a unique point $y \in B$ with minimal distance to z. By the Hahn–Banach theorem there is a hyperplane H in $\operatorname{aff} B$ separating B and $B_\delta(z)$, where $\delta = \|z - y\|$. Observe that $y \in H$. Hence y is a point of the face $H \cap B$. Also,

$$\|x - y\| \le \varepsilon + \delta \le 2\varepsilon.$$

Since ε is arbitrary, this completes the proof. ∎

DEFINITION 4. Let B be a closed convex set in ℓ^2 and let $k \in \mathbb{N}$. We define $\mathcal{E}^k(B)$ as the closure of

$$\bigcup\{F : F \text{ is a derived face of } B \text{ with } \operatorname{codim} F > k\}.$$

LEMMA 5. *Let B be a closed convex set in ℓ^2 with $B^\circ = \emptyset$. Then $\mathcal{E}^k(B) = B$ for every $k \in \mathbb{N}$.*

Proof. Assume that $\mathcal{E}^k(B) \neq B$ and consider the collection

$$\mathcal{F} = \{F : F \text{ is a derived face of } B \text{ such that } F \setminus \mathcal{E}^k(B) \neq \emptyset\}.$$

Since B is a derived face of itself we have $B \in \mathcal{F}$. By the definition of $\mathcal{E}^k(B)$, every $F \in \mathcal{F}$ has $\operatorname{codim} F \le k$. So we can select an F in $\mathcal{F}$ with maximal codimension. By Lemma 3 we have $F^\circ = \emptyset$. Since $F \setminus \mathcal{E}^k(B)$ is a nonempty open subset of F, Lemma 4 shows that there is a face G of F such that $G \setminus \mathcal{E}^k(B) \neq \emptyset$ and hence $G \in \mathcal{F}$. Since $\operatorname{codim} G > \operatorname{codim} F$, we have a contradiction with the maximality of $\operatorname{codim} F$. ∎

Definition 5. Let B be a closed convex set in ℓ^2. We define the *characteristic cone* of B by

$$\operatorname{cc} B = \{y \in \ell^2 : \text{there is an } x \text{ with } x + \alpha y \in B \text{ for all } \alpha \geq 0\}.$$

The *characteristic linear space* of B is defined by $\mathcal{L}_B = \operatorname{cc} B \cap - \operatorname{cc} B$. The *cross section* of B is the set $\operatorname{cs} B = B \cap \mathcal{L}_B^{\perp}$.

Remark 3. If $B \subset \ell^2$ is closed and convex, then we have the following facts (see [8, §2.5] and [2, p. 93]). The cone $\operatorname{cc} B$ is closed and convex. If x is any fixed element of B, then

$$\operatorname{cc} B = \{y \in \ell^2 : x + \alpha y \in B \text{ for all } \alpha \geq 0\}.$$

$\mathcal{L}_B$ is a closed linear space: the unique maximal linear subspace of ℓ^2 such that $B = B + \mathcal{L}_B$.

The following lemma is analogous to [1, Lemma 4] with a virtually identical proof.

Lemma 6. *Let B be a closed convex set in ℓ^2. If F is a* (*derived*) *face of B, then $\mathcal{L}_F = \mathcal{L}_B$ and $\operatorname{cs} F$ is a* (*derived*) *face of $\operatorname{cs} B$. If, on the other hand, F is a* (*derived*) *face of $\operatorname{cs} B$, then $\mathcal{L}_B + F$ is a* (*derived*) *face of B.*

Remark 4. We will need information about the topology of boundaries of *convex bodies B* in ℓ^2, i.e. closed convex sets with nonempty interior. According to [2, Proposition III.6.1] the boundary of a convex body is either empty or homeomorphic to ℓ^2 or $S^n \times \ell^2$ for some n-sphere S^n. Thus ∂B is either empty or it contains closed copies of ℓ^2.

3. Projecting onto k-hyperplanes. The main purpose of this section is to establish Theorem 1. We shall need the following result from [1], known as the Tipping Lemma.

Lemma 7. *Let B be a closed convex set in $\mathbb{R}^m$ for $m \geq 2$, let C be a closed subset of B, and let H be a hyperplane of $\mathbb{R}^m$ that does not cut B. If V is a halfspace of H such that $V \cap C = \emptyset$ and $V \cap B$ is nonempty and bounded, then there exists a halfspace V' of $\mathbb{R}^m$ such that $V \subset V'$, $V' \cap C = \emptyset$, and $V' \cap B$ is bounded.*

Before getting to the main theorems we need one more lemma.

Lemma 8. *Let C be a subset of ℓ^2 and let $D = \overline{\langle C \rangle}$. If p is a projection onto a plane such that $\overline{p(C)}$ is convex, then $\overline{p(C)} = \overline{p(D)}$. If B is a closed convex set such that $\mathfrak{p}_\ell(C) \subset \overline{\mathfrak{p}_\ell(B)}$ for every line ℓ in ℓ^2, then $D \subset B$.*

Proof. If $\overline{p(C)}$ is convex then

$$\overline{p(C)} \subset \overline{p(D)} = \overline{p(\overline{\langle C \rangle})} \subset \overline{p(\langle C \rangle)} = \overline{\langle p(C) \rangle} \subset \overline{\langle \overline{p(C)} \rangle} = \overline{\overline{p(C)}} = \overline{p(C)},$$

thus $\overline{p(C)} = \overline{p(D)}$.

For the second part, assume that there is an $x \in D \setminus B$. Since B is closed and convex we may assume that $x \in C \setminus B$. Then by [10, p. 191] there is a hyperplane H such that x and B are on different sides of H. Let ℓ be the line $H^{\perp}$ and note that $\mathfrak{p}_\ell(x)$ and $\mathfrak{p}_\ell(B)$ are separated in ℓ by the point of intersection of ℓ and H. Thus $\mathfrak{p}_\ell(x) \notin \overline{\mathfrak{p}_\ell(B)}$, which contradicts the premise that $\mathfrak{p}_\ell(C) \subset \overline{\mathfrak{p}_\ell(B)}$. We conclude that $D \subset B$. ∎

The following theorem is analogous to [1, Theorem 3]. It tells us which points are "extremal" with respect to projections onto k-hyperplanes.

THEOREM 9. *Let $k \in \mathbb{N}$, let B be a closed convex subset of ℓ^2, and let C be a closed set in ℓ^2. If $\overline{p(C)} = \overline{p(B)}$ for every projection p of ℓ^2 onto a k-hyperplane, then $\mathcal{E}^k(B) \subset C \subset B$.*

Proof. We first verify that $C \subset B$. Let ℓ be a line in ℓ^2. Select a k-hyperplane H that contains ℓ. Then

$$\mathfrak{p}_\ell(C) = \mathfrak{p}_\ell(\mathfrak{p}_H(C)) \subset \mathfrak{p}_\ell(\overline{\mathfrak{p}_H(B)}) \subset \overline{\mathfrak{p}_\ell(\mathfrak{p}_H(B))} = \overline{\mathfrak{p}_\ell(B)},$$

thus $C \subset \overline{\langle C \rangle} \subset B$ by Lemma 8.

In order to prove that $\mathcal{E}^k(B) \subset C$ it suffices to show that every derived face of B with codimension greater than k is contained in C. So assume that F is a derived face of B with codimension $m > k$ (m could be ∞) and $F \setminus C \neq \emptyset$. Choose a rectangular coordinate system for ℓ^2 such that $\mathbf{0} \in F \setminus C$. By Remark 1 we can find a sequence of affine spaces $\ell^2 = H_0 \supset H_1 \supset \cdots \supset H_{k+1}$ such that $\operatorname{codim} H_i = i$ for each i, $\operatorname{aff} F \subset H_{k+1}$, and H_i is a hyperplane in H_{i-1} that does not cut $B \cap H_{i-1}$ for $i \in \{1, \ldots, k+1\}$.

We construct by induction a sequence $\mathbf{0} \in V_1 \subset \cdots \subset V_{k+1}$ such that for $1 \le i \le k+1$:

(1) V_i is an i-halfplane in H_{k+1-i},
(2) $V_i \cap C = \emptyset$,
(3) $V_i \cap B$ is bounded.

Let V_1 be a ray in H_k that emanates from $\mathbf{0}$ into the side of H_{k+1} that is disjoint from B. Note that $V_1 \cap B = \{\mathbf{0}\}$ and hence $V_1 \cap C = \emptyset$ so the induction hypotheses are satisfied.

Now let $1 \le i \le k$ and assume that V_i has been found. Let $\ell \subset H_{k-i}$ be the line through $\mathbf{0}$ that is perpendicular to H_{k-i+1} and let $M \subset H_{k-i}$ be the $(i+1)$-plane $\ell + \operatorname{aff} V_i$. Put $H = H_{k-i+1} \cap M$, $C' = C \cap M$ and $B' = B \cap M$. Apply Lemma 7 to M, H, C', B', and V_i. We obtain a halfspace V_{i+1} of M such that $V_i \subset V_{i+1}$, $V_{i+1} \cap C' = V_{i+1} \cap C = \emptyset$ and $V_{i+1} \cap B' = V_{i+1} \cap B$ is bounded. This completes the induction.

Since $\mathbf{0}$ is an element of the $(k+1)$-halfplane V_{k+1}, there is a (unique) k-plane N such that $\mathbf{0} \in N \subset V_{k+1}$. Of course, $N \cap C = \emptyset$ and $N \cap B$ is bounded.

Next we prove that $d(N,C) > 0$. Since $N \cap B$ is bounded we can find an $a > 0$ such that the sphere $S = \{x \in N : \|x\| = a\}$ is disjoint from B. By compactness of S and of $K = \{x \in N : \|x\| \le a\}$ we have $\varepsilon = \min\{d(S,B), d(K,C)\} > 0$. Let $x \in C$ and $y \in N$ be such that $\|x - y\| < \varepsilon$. Then $x \in B$ and $b = \|y\| > a$. By convexity of B and $\mathbf{0} \in B$ we have $x' = (a/b)x \in B$. Put $y' = (a/b)y$ and note that $y' \in S$ and $\|x' - y'\| = (a/b)\|x - y\| < \varepsilon$. So we have a contradiction with $d(S,B) \ge \varepsilon$ and we may conclude that $d(N,C) \ge \varepsilon$.

Let O be the subspace of ℓ^2 that is the orthocomplement of N. Clearly, the codimension of O is k. Then $d(\mathbf{0}, \mathfrak{p}_O(C)) = d(N,C) > 0$ and hence $\mathbf{0} \notin \overline{\mathfrak{p}_O(C)} = \overline{\mathfrak{p}_O(B)}$, which contradicts the fact $\mathbf{0} = \mathfrak{p}_O(\mathbf{0}) \in \mathfrak{p}_O(B)$. We may conclude that $F \subset C$. ∎

Corollary 10. *Let $k \in \mathbb{N}$ and let B be a closed and convex set in ℓ^2 with $B^\circ = \emptyset$. If C is a closed set such that $\overline{\mathfrak{p}_H(B)} = \overline{\mathfrak{p}_H(C)}$ for every k-hyperplane H in ℓ^2, then $B = C$.*

Proof. This follows directly from Lemma 5 and Theorem 9. ∎

The following theorem is analogous to [1, Theorem 4] with an important distinction: the essential property (cf. Remark 4) that $F^\circ \neq \emptyset$ is gratis in finite-dimensional spaces whereas here it is based on Lemma 4.

Theorem 11. *Let $k \in \mathbb{N}$, let B be a closed convex set in ℓ^2, and let C be a closed subset of ℓ^2 such that $C \neq B$ and $\overline{p(C)} = \overline{p(B)}$ for every projection p of ℓ^2 onto a k-hyperplane. Then there exists a derived face F of B such that $F^\circ \neq \emptyset$, $\operatorname{codim} F \le k$ and $\partial F \subset C$.*

Proof. Consider the collection $\mathcal{D}$ of all derived faces of B that are not contained in C. By Theorem 9, $C \subset B$ so $B \neq C$ implies that $B \in \mathcal{D}$. Also by Theorem 9 every element of $\mathcal{D}$ has codimension at most k so there is an $F \in \mathcal{D}$ with maximal codimension in ℓ^2. Every face of F has a higher codimension than F so all the faces of F are subsets of C. If $F^\circ = \emptyset$ then by Lemma 4 the union of its faces is dense in F and hence $F \subset C$ because C is closed. This result contradicts $F \in \mathcal{D}$ so we have $F^\circ \neq \emptyset$. Then by Remark 2 every point of ∂F is contained in some face of F and hence $\partial F \subset C$. ∎

The zero-dimensional closed sets $\mathfrak{Z}_\varepsilon$ of Theorem 19 below have convex projections onto every hyperplane. However, they have the property that the boundary of every derived face of $\overline{\langle \mathfrak{Z}_\varepsilon \rangle}$ is empty, making Theorem 11 void when applied to such a set. Now the properness of certain projections in the premise of Theorem 1 comes into play.

Proof of Theorem 1. Let k, B and C be as in the statement. Assume that $\overline{p(C)} = \overline{p(B)}$ for every projection p of ℓ^2 onto a k-hyperplane and there exists a basis $\mathcal{B}$ for ℓ^2 such that $\overline{\mathfrak{p}_{L^\perp}(C)} \neq L^\perp$ for every linear space L generated by k elements of $\mathcal{B}$.

By Theorem 11 we can find a derived face F of B such that $\operatorname{codim} F \leq k$, $\partial F \subset C$, and $F^\circ \neq \emptyset$. Let H be a k-hyperplane contained in $\operatorname{aff} F$ and meeting F°. If we put $G = F \cap H$ then $\operatorname{codim} G = k$ and $G^\circ \neq \emptyset$. Thus G is a convex body in $\operatorname{aff} G = H$, a space isomorphic to ℓ^2. Clearly, ∂G is a subset of ∂F and C. By Remark 4, ∂G is empty or it contains a closed copy of Hilbert space. So, we only need to show that $\partial G \neq \emptyset$.

Striving for a contradiction, assume that $\partial G = \emptyset$. Hence $G = H$. We will show that there exists a k-hyperplane L such that $L^\perp$ is generated by elements of $\mathcal{B}$ and $\mathfrak{p}_L(H) = L$. Let ψ stand for the projection $\mathfrak{p}_{H^\perp}$ and note that since $\mathcal{B}$ is a basis for ℓ^2, the set $\{\psi(v) : v \in \mathcal{B}\}$ contains a basis for $H^\perp$, say $\{\psi(v^1), \ldots, \psi(v^k)\}$. Let M be the k-dimensional linear space spanned by $\{v^1, \ldots, v^k\}$ and put $L = M^\perp$. Pick an arbitrary $y \in L$ and let us show that there is an $x \in H$ such that $\mathfrak{p}_L(x) = y$. Indeed,

$$\psi(y) = \sum_{i=1}^{k} \alpha_i \psi(v^i).$$

Set

$$x = y - \sum_{i=1}^{k} \alpha_i v^i.$$

Then $x \in H$ since $\psi(x) = \mathbf{0}$. Moreover, $\mathfrak{p}_L(x) = y$ since $y = x + \sum_{i=1}^k \alpha_i v^i$ with $x \in H$ and $\sum_{i=1}^k \alpha_i v^i \in M = L^\perp$. That completes the proof of the theorem. ∎

The following result is a reformulation of Theorem 1 without reference to the convex set B.

THEOREM 12. *Let $k \in \mathbb{N}$ and let C be a closed nonconvex subset of ℓ^2. Assume that $\overline{p(C)}$ is convex for every projection p of ℓ^2 onto a k-hyperplane. If there exists a basis $\mathcal{B}$ for ℓ^2 such that $\overline{\mathfrak{p}_{L^\perp}(C)} \neq L^\perp$ for all k-subspaces L that have a subset of $\mathcal{B}$ as basis, then C contains a closed copy of ℓ^2.*

Proof. Put $B = \overline{\langle C \rangle}$. Since C is nonconvex we have $C \neq B$. According to Lemma 8, $\overline{p(C)} = \overline{p(B)}$ for every projection p onto a k-hyperplane. Now apply Theorem 1. ∎

4. Projecting onto finite-dimensional planes. Consider a compact set C in ℓ^2 such that all projections onto k-hyperplanes are convex. Put $B = \overline{\langle C \rangle}$ and note that B is also compact (see [10, p. 244]). If $B^\circ = \emptyset$ then according to Lemma 8 and Corollary 10, C is convex because $C = B$. If $B^\circ \neq \emptyset$ then $\operatorname{aff} B$ is finite-dimensional and we can find a k-hyperplane

H that contains $\operatorname{aff} B$. Thus $C = \mathfrak{p}_H(C)$ and C is convex. The following theorem improves on this observation.

THEOREM 13. *If C is a compactum in ℓ^2 such that $\mathfrak{p}_{\mathbb{R}^n}(C)$ is convex for each $n \in \mathbb{N}$, then C is convex. Thus C is either an n-cell or the Hilbert cube.*

Proof. Let $B = \overline{\langle C\rangle}$. Striving for a contradiction, assume that $B \neq C$, that is, there exists an $x \in B \setminus C$. Observe that by Lemma 8,

$$\mathfrak{p}_{\mathbb{R}^n}(B) = \mathfrak{p}_{\mathbb{R}^n}(C) \quad \text{for every } n \in \mathbb{N}.$$

Thus, for every $n \in \mathbb{N}$ we can pick a $y^n \in C$ such that

$$y_i^n = x_i \quad \text{for } 1 \le i \le n.$$

Consequently, the sequence $(y^n)_n$ converges coordinatewise to x. On the other hand, since C is compact we can find a subsequence $(y^{n_i})_i$ of $(y^n)_n$ converging with respect to the norm topology to a point, say z, in C. That implies that $(y^{n_i})_i$ converges coordinatewise to z. Hence $z = x$. We have arrived at a contradiction with $x \in B \setminus C$ and hence $B = C$. Therefore C is convex and by Keller's theorem [9] it is homeomorphic either to some n-cell or to the Hilbert cube. ∎

As an immediate consequence of Theorem 13 we get the following corollary.

COROLLARY 14. *Let C be a compactum in ℓ^2 all of whose projections onto finite-dimensional planes are convex. Then C must be convex.*

EXAMPLE 2. Consider the unit sphere S^∞ in ℓ^2. It is a bounded, closed, nonconvex set all of whose shadows are convex.

We finish this section with a more interesting example.

EXAMPLE 3. Let K be a Cantor set in $[1/2, 1]$, and for each $n \in \mathbb{N}$, let $f_n : K \to J^n$ be a continuous surjection, where $J = [-1, 1]$. Define

$$C_n = \{(f_n(c), c, 0, 0, \ldots) : c \in K\} \subset J^{n+1} \subset \ell^2.$$

Basically, the C_n's are the graphs of the f_n's, so each C_n is homeomorphic to K. Put

$$\mathfrak{C} = \bigcup_{n\in\mathbb{N}} C_n \quad \text{and} \quad \mathfrak{B} = \{x \in \ell^2 : x_n \in J \text{ for all } n \in \mathbb{N}\}.$$

Note that $\mathfrak{C}$ is zero-dimensional by [7, Theorem 1.5.3] and that $\mathfrak{B}$ is a closed convex set that contains $\mathfrak{C}$ and has a nonempty interior. Clearly, $J^n = \mathfrak{p}_{\mathbb{R}^n}(C_n) \subset \mathfrak{p}_{\mathbb{R}^n}(\mathfrak{C})$ and on the other hand $\mathfrak{p}_{\mathbb{R}^n}(\mathfrak{C}) \subset \mathfrak{p}_{\mathbb{R}^n}(\mathfrak{B}) = J^n$.

To prove that $\mathfrak{C}$ is closed it suffices to show that $\{C_n : n \in \mathbb{N}\}$ is a locally finite family. Indeed, fix $x \in \ell^2$. There exists an $m \in \mathbb{N}$ such that $\sum_{i=m}^{\infty} x_i^2 < 1/16$ and hence $|x_i| \le 1/4$ for each $i \ge m$. Now, if $z \in C_n$ with $n \ge m$, then

$$\|z - x\| \geq z_{n+1} - x_{n+1} \geq \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

Consequently, $z \notin B_{1/4}(x)$ and $C_n \cap B_{1/4}(x) = \emptyset$ for all $n \geq m$.

To summarize:

CLAIM 15. *$\mathfrak{C}$ is a closed zero-dimensional σ-compactum such that*

$$\mathfrak{p}_{\mathbb{R}^n}(\mathfrak{C}) = \mathfrak{p}_{\mathbb{R}^n}(\mathfrak{B}) = J^n \quad \textit{for all } n \in \mathbb{N}.$$

The following result shows that in Theorem 1 we cannot replace projections onto k-hyperplanes with projections onto finite-dimensional planes.

CLAIM 16. *For every finite-dimensional plane L in ℓ^2 we have*

$$\overline{\mathfrak{p}_L(\mathfrak{C})} = \overline{\mathfrak{p}_L(\mathfrak{B})}.$$

Proof. Let L be a finite-dimensional plane in ℓ^2. Since we may assume that $\mathbf{0} \in L$ we can choose an orthonormal basis $\{u^1, \ldots, u^k\}$ for L. Pick arbitrary $v \in \mathfrak{B}$ and $\varepsilon \in (0,1)$ and choose an $n \in \mathbb{N}$ such that

$$\sum_{j=n+1}^{\infty} v_j^2 < \frac{\varepsilon^2}{4k^2} \quad \text{and} \quad \sum_{j=n+1}^{\infty} (u_j^i)^2 < \frac{\varepsilon^2}{4k^2} \quad \text{for } i = 1, \ldots, k.$$

Observe that there is a $c \in K$ such that $w = (v_1, \ldots, v_n, c, 0, 0, \ldots)$ is in $\mathfrak{C}$. Then

$$\begin{aligned}\|\mathfrak{p}_L(v) - \mathfrak{p}_L(w)\| &= \|\mathfrak{p}_L(v - w)\| = \Big\| \sum_{i=1}^{k} ((v - w) \cdot u^i) u^i \Big\| \\ &\leq \sum_{i=1}^{k} |(v - w) \cdot u^i| = \sum_{i=1}^{k} \Big| -c u_{n+1}^i + \sum_{j=n+1}^{\infty} v_j u_j^i \Big| \\ &< \sum_{i=1}^{k} \left(\frac{\varepsilon}{2k} + \frac{\varepsilon^2}{4k^2} \right) < \varepsilon.\end{aligned}$$

Thus $\mathfrak{p}_L(\mathfrak{C})$ is dense in $\mathfrak{p}_L(\mathfrak{B})$ and the proof is complete. ■

We do not know whether Theorem 1 remains true if we consider projections onto planes of infinite codimension instead of k-hyperplanes. However, the example $\mathfrak{C}$ is well behaved with respect to such planes provided that they are associated with the standard basis. If $A \subset \mathbb{N}$ then we define the plane

$$M(A) = \{x \in \ell^2 : x_i = 0 \text{ for every } i \in A\}.$$

CLAIM 17. *For every infinite subset A of $\mathbb{N}$ we have*

$$\overline{\mathfrak{p}_{M(A)}(\mathfrak{C})} = \overline{\mathfrak{p}_{M(A)}(\mathfrak{B})}.$$

Proof. Let $A \subset \mathbb{N}$ be infinite. Pick arbitrary $v \in \mathfrak{B}$ and $0 < \varepsilon < 1$ and choose an $n \in A$ such that $\sum_{j=n+1}^{\infty} v_j^2 < \varepsilon^2$. Observe that there is a $c \in K$

such that $w = (v_1, \ldots, v_{n-1}, c, 0, 0, \ldots)$ is in $\mathfrak{C}$. Then

$$\|\mathfrak{p}_{M(A)}(v) - \mathfrak{p}_{M(A)}(w)\| = \|\mathfrak{p}_{M(A)}(v - w)\| = \Big\| \sum_{i \in \mathbb{N} \setminus A} (v_i - w_i) e^i \Big\|$$

$$= \Big\| \sum_{\substack{i \in \mathbb{N} \setminus A \\ i > n}} v_i e^i \Big\| \le \sqrt{\sum_{i=n+1}^{\infty} v_i^2} < \varepsilon.$$

Thus $\mathfrak{p}_{M(A)}(\mathfrak{C})$ is dense in $\mathfrak{p}_{M(A)}(\mathfrak{B})$ and the proof is complete. ∎

5. Zero-dimensional screens

Definition 6. Let $A, B \subset \ell^2$. We say that B *is a screen for* A if every line in ℓ^2 that meets A also meets B, or equivalently, every shadow of B contains the corresponding shadow of A.

Borsuk [3] has shown that there are Cantor sets in $\mathbb{R}^n$ that act as screens for ε-balls (see [6] for a simple proof). Since in infinite-dimensional vector spaces compacta are nowhere dense, we need a different approach to find zero-dimensional screens in ℓ^2.

Definition 7. Let $x \in \ell^2$ and let $a, b \in \mathbb{R}$ with $0 \le a < b$. Define the open set $\mathrm{sh}_a^b(x) = \{y : a < \|y - x\| < b\}$. We call any set of this form a *shell* with thickness $b - a$.

Let λ stand for the Lebesgue measure on $\mathbb{R}$. We extend the use of λ to lines ℓ in Hilbert space as follows: If A is a measurable set in $\mathbb{R}$, $x \in \ell$, and u is a unit vector parallel to ℓ, then $\lambda(\{x + tu : t \in A\}) = \lambda(A)$.

Lemma 18. *If $0 \le a < b$, $p \in \ell^2$, and ℓ is a line in ℓ^2, then $\lambda(\ell \cap \mathrm{sh}_a^b(p)) \le 2\sqrt{b^2 - a^2}$.*

Proof. We may assume that $p = \mathbf{0}$. Let x be the point on ℓ that is closest to $\mathbf{0}$ and note that

$$\lambda(\ell \cap \mathrm{sh}_a^b(\mathbf{0})) = \lambda(\{t \in \mathbb{R} : a^2 - \|x\|^2 < t^2 < b^2 - \|x\|^2\}).$$

If $\|x\| > b$ then $\ell \cap \mathrm{sh}_a^b(\mathbf{0}) = \emptyset$. If $a \le \|x\| \le b$ then

$$\lambda(\ell \cap \mathrm{sh}_a^b(\mathbf{0})) \le 2\sqrt{b^2 - \|x\|^2} \le 2\sqrt{b^2 - a^2}.$$

If $\|x\| < a$ then

$$\lambda(\ell \cap \mathrm{sh}_a^b(\mathbf{0})) \le 2(\sqrt{b^2 - \|x\|^2} - \sqrt{a^2 - \|x\|^2}) \le 2\sqrt{b^2 - a^2},$$

where we have used the fact $\sqrt{t + s} \le \sqrt{t} + \sqrt{s}$. ∎

Theorem 19. *For every $\varepsilon > 0$ there exists a zero-dimensional closed set $\mathfrak{Z}_\varepsilon$ in ℓ^2 such that $\lambda(\ell \setminus \mathfrak{Z}_\varepsilon) < \varepsilon$ for every line ℓ in ℓ^2, and hence $\mathfrak{Z}_\varepsilon$ is a screen for ℓ^2.*

Proof. Let $\varepsilon > 0$ and select a countable base $\mathcal{B} = \{B_{\gamma_n}(v_n) : n \in \mathbb{N}\}$ of neighbourhoods for ℓ^2. We may assume that $\gamma_n < 1$ for all n. Define $\delta_n = (1 + \varepsilon^2 4^{-n-1})^{1/4} > 1$ for every $n \in \mathbb{N}$ and put

$$\mathfrak{Z}_\varepsilon = \ell^2 \setminus \bigcup_{n=1}^{\infty} \mathrm{sh}_{\gamma_n/\delta_n}^{\gamma_n\delta_n}(v_n).$$

Since the complement of $\mathfrak{Z}_\varepsilon$ contains the boundary of every element of the base $\mathcal{B}$, the closed set $\mathfrak{Z}_\varepsilon$ is zero-dimensional. Let ℓ be an arbitrary line in ℓ^2 and note that by Lemma 18 we have

$$\lambda(\ell \setminus \mathfrak{Z}_\varepsilon) \leq \sum_{n=1}^{\infty} 2\sqrt{\gamma_n^2\delta_n^2 - \gamma_n^2\delta_n^{-2}} = \sum_{n=1}^{\infty} \frac{\gamma_n}{\delta_n}\,\varepsilon 2^{-n} < \sum_{n=1}^{\infty} \varepsilon 2^{-n} = \varepsilon. \ \blacksquare$$

COROLLARY 20. *If F is a closed subset of ℓ^2 and U is an open neighbourhood of F, then there exists a zero-dimensional closed screen for F in ℓ^2 that is contained in U.*

Proof. Let $G = \ell^2 \setminus U$. We define the following closed sets:

$$F_0 = \{x \in \ell^2 : d(x, G) \geq 1/8\}$$

and for $n \in \mathbb{N}$,

$$F_n = \{x \in \ell^2 : d(x, F) \leq 2^{-n} \text{ and } 2^{-n-3} \leq d(x, G) \leq 2^{-n}\}.$$

Invoking Theorem 19 we define

$$Z = \bigcup_{n=0}^{\infty} (F_n \cap \mathfrak{Z}_{2^{-n-2}}).$$

Since every F_n is disjoint from G we have $Z \subset U$. We prove that Z is closed by showing that $\{F_n \cap Z_{2^{-n-2}} : n \in \{0\} \cup \mathbb{N}\}$ is a locally finite family. If $x \in \ell^2$ then $d(x, F) > 0$ or $d(x, G) > 0$. If $\alpha = d(x, F) > 0$ then $B_{\alpha/2}(x)$ will miss every F_n with $2^{-n+1} < \alpha$ and $n > 0$. Likewise, if $\beta = d(x, G) > 0$ then $B_{\beta/2}(x)$ will miss every F_n with $2^{-n+1} < \beta$ and $n > 0$.

It remains to show that Z is a screen for F. Consider an arbitrary line ℓ such that $x \in \ell \cap F$. We need to show that $\ell \cap Z \neq \emptyset$. Note that $d(x, G) > 0$. If $d(x, G) > 1/4$ then $B_{1/8}(x) \subset F_0$, which means that $\lambda(\ell \cap F_0) \geq \lambda(\ell \cap B_{1/8}(x)) = 1/4$. Since $\lambda(\ell \setminus \mathfrak{Z}_{1/4}) < 1/4$ we have $\ell \cap F_0 \cap \mathfrak{Z}_{1/4} \neq \emptyset$ and hence $\ell \cap Z \neq \emptyset$. So we may assume that $d(x, G) \leq 1/4$. Then there is an $n \in \mathbb{N}$ such that $2^{-n-2} \leq d(x, G) \leq 2^{-n-1}$. Thus, $B_{2^{-n-3}}(x) \subset F_n$ and hence $\lambda(\ell \cap F_n) \geq \lambda(\ell \cap B_{2^{-n-3}}(x)) = 2^{-n-2}$. Since $\lambda(\ell \setminus \mathfrak{Z}_{2^{-n-2}}) < 2^{-n-2}$ we have $\ell \cap F_n \cap \mathfrak{Z}_{2^{-n-2}} \neq \emptyset$ and hence $\ell \cap Z \neq \emptyset$.

Finally, observe that Z is zero-dimensional as a countable union of zero-dimensional closed sets. $\blacksquare$

The next corollary explains why we consider only projections of *closed* sets in this paper.

Corollary 21. *If U is an open set in ℓ^2, then there is a zero-dimensional closed subset Z of U such that $\mathfrak{p}_H(Z) = \mathfrak{p}_H(U)$ for every hyperplane H in ℓ^2.*

Proof. Take a locally finite open cover $\mathcal{U} = \{U_i : i \in \mathbb{N}\}$ of the space U such that $\overline{U}_i \subset U$ for every $i \in \mathbb{N}$. Shrink $\mathcal{U}$ to a closed cover $\{V_i : i \in \mathbb{N}\}$ of U. Note that every V_i is closed in ℓ^2. Now, for each pair (V_i, U_i) we apply Corollary 20 to find a zero-dimensional closed set $Z_i \subset U_i$ that is a screen for V_i. The set $Z = \bigcup_{i=1}^{\infty} Z_i$ is as required. ∎

6. Imitating arbitrary closed convex sets. Suppose B is a closed convex set in ℓ^2. If C is a closed set such that B and C have the same projections onto k-hyperplanes, then Theorem 9 implies that C contains at least the set $\mathcal{E}^k(B)$. We show that for every B there exist "minimal" examples of such "imitations" C of B, in the sense that $\dim(C \setminus \mathcal{E}^k(B)) \le 0$. This was proved for closed sets B in $\mathbb{R}^n$ in [1, Theorem 6]. Our starting point is the construction given in [1] but again some of the details are more complicated when dealing with sets in Hilbert space.

Definition 8. If A is a nonempty set in ℓ^2, then $\operatorname{star} A = \{tx : 0 \le t \le 1$ and $x \in A\}$, that is, the union of all line segments that connect the origin to points of A. If A is closed and convex, $\mathbf{0} \in A^\circ$, and $k \in \mathbb{N}$, then we define $\mathcal{K}^k(A) = \operatorname{star}(\mathcal{E}^k(A)) \cup \operatorname{cc} A$. Note that $\mathcal{K}^k(A)$ is a closed subset of A and that $\partial A \cap \mathcal{K}^k(A) = \mathcal{E}^k(A)$.

The following lemma extends [1, Lemma 3] to sets in ℓ^2.

Lemma 22. *If $k \in \mathbb{N}$ and B is a closed convex set with $\mathbf{0} \in B^\circ$, then $\mathcal{K}^k(B)$ and B have identical projections onto all k-hyperplanes H such that $\mathcal{L}_B \cap H^\perp = \{\mathbf{0}\}$.*

Proof. Let H be a k-hyperplane such that $\mathfrak{p}_H(\mathcal{K}^k(B)) \ne \mathfrak{p}_H(B)$ and $\mathcal{L}_B \cap H^\perp = \{\mathbf{0}\}$. Since $\mathcal{K}^k(B) \subset B$ there is a $w \in B$ such that the k-plane $M = w + H^\perp$ is disjoint from $\mathcal{K}^k(B)$. Consider the collection

$$\mathcal{F} = \{F : F \text{ a derived face of } B \text{ with } F \cap M \ne \emptyset\}.$$

Note that $B \in \mathcal{F}$ because B is a derived face of itself and $w \in B \cap M$. Since $\mathcal{E}^k(B) \subset \mathcal{K}^k(B) \subset \ell^2 \setminus M$ it follows that $\operatorname{codim} F \le k$ for each $F \in \mathcal{F}$. Thus we can choose a derived face $F \in \mathcal{F}$ with maximal codimension in ℓ^2. If $F^\circ = \emptyset$ then Lemma 5 yields $F = \mathcal{E}^k(F) \subset \mathcal{E}^k(B) \subset \mathcal{K}^k(B)$ and we have a contradiction with $M \cap \mathcal{K}^k(B) = \emptyset$ and $F \cap M \ne \emptyset$. Thus $F^\circ \ne \emptyset$. Let $u \in F \cap M$.

Assume that $\dim(M \cap \operatorname{aff} F) \ge 1$ and so there is a line ℓ through u such that $\ell \subset M \cap \operatorname{aff} F$. If ℓ contains points outside F, then ℓ will have to meet ∂F. Since $F^\circ \ne \emptyset$ Remark 2 implies that ℓ meets some face G of F. Consequently,

$\operatorname{codim} G > \operatorname{codim} F$ and $G \in \mathcal{F}$, in violation of the maximality of $\operatorname{codim} F$. Thus ℓ is contained in F and hence $\ell - u \subset \mathcal{L}_F \subset \mathcal{L}_B$. On the other hand, $\ell - u \subset M - u = H^\perp$, which violates the assumption $\mathcal{L}_B \cap H^\perp = \{\mathbf{0}\}$.

Thus $M \cap \operatorname{aff} F = \{u\}$ and the two planes are transverse. We have $k \geq \operatorname{codim} F = \operatorname{codim}(\operatorname{aff} F) \geq \dim M = k$ and hence $\operatorname{codim} F = k$, which means that M and $\operatorname{aff} F$ are complementary planes. This allows us to define a (not necessarily orthogonal) projection $\psi : \ell^2 \to \operatorname{aff} F$ by the rule $\{\psi(x)\} = (x - u + M) \cap \operatorname{aff} F$. If $\psi(\mathbf{0}) = u$ then $\mathbf{0} \in M \cap \mathcal{K}^k(B)$ and we are done. So assume that $\psi(\mathbf{0}) \neq u$ and consider the ray $R = \{t(u - \psi(\mathbf{0})) : t \geq 0\}$ that emanates from the origin. Observe that since $u \in F$ and $\psi(\mathbf{0}) \in \operatorname{aff} F$ the ray $u + R$ is contained in $\operatorname{aff} F$. We first consider the case that $u + R \subset F$. Then $R \subset \operatorname{cc} B \subset \mathcal{K}^k(B)$. Note that $u = \psi(u - \psi(\mathbf{0})) \in \psi(R)$ so $M = \psi^{-1}(u)$ intersects R and $\mathcal{K}^k(B)$. If, on the other hand, the ray $u + R$ is not contained in F, then it intersects ∂F in some point $v = u + t(u - \psi(\mathbf{0}))$. Note that since $F^\circ \neq \emptyset$, v is contained in some face G of F and $\operatorname{codim} G > \operatorname{codim} F = k$, so $v \in \mathcal{E}^k(B)$. The line segment σ that connects $\mathbf{0}$ to v is contained in $\mathcal{K}^k(B)$. Note that $\psi(\sigma)$ is a line segment that connects $\psi(\mathbf{0})$ to v and hence it contains the point $u = (v + t\psi(\mathbf{0}))/(1 + t)$. Consequently, M intersects σ and hence it intersects $\mathcal{K}^k(B)$. ∎

Proof of Theorem 2. Let B be a closed convex set in ℓ^2 such that $\operatorname{codim} B \neq k \in \mathbb{N}$. If $B^\circ = \emptyset$ then according to Lemma 5, $\mathcal{E}^k(B) = B$ and there is nothing to prove. If $k < \operatorname{codim} B$ then also $\mathcal{E}^k(B) = B$. So we may assume that $B^\circ \neq \emptyset$ and $k > \operatorname{codim} B$. Choose a coordinate system such that $\mathbf{0} \in B^\circ$. We first prove the assertion for the case $\operatorname{aff} B = \ell^2$.

The k-imitation C will have the form $\mathcal{E}^k(B) \cup Z_1 \cup Z_2$, where both Z_1 and Z_2 are zero-dimensional sets. We first construct Z_1. Consider the open subset $D = B \setminus \mathcal{E}^k(B)$ of B and the set $K = \mathcal{K}^k(B) \setminus \mathcal{E}^k(B)$, which is closed in D. Choose a locally finite open cover $\{U_i : i \in \mathbb{N}\}$ for K in D consisting of sets whose closures are in D. Since $K \cap \partial B = \emptyset$ we may assume that the sets U_i are open in $\operatorname{aff} B = \ell^2$. Shrink this cover to a cover $\{V_i : i \in \mathbb{N}\}$ of K by closed sets such that $V_i \subset U_i$ for every $i \in \mathbb{N}$. According to Corollary 20 we can find for each $i \in \mathbb{N}$ a zero-dimensional screen F_i for V_i that is contained in U_i and hence every line in ℓ^2 that intersects V_i also meets F_i. Put $Z_1 = \bigcup_{i=1}^\infty F_i$ and note that every line that intersects K also meets Z_1. Then, since $\{F_i : i \in \mathbb{N}\}$ is a locally finite collection of closed zero-dimensional sets in D, it follows that Z_1 is closed in D and zero-dimensional. Consequently, $\mathcal{E}^k(B) \cup Z_1$ is closed in B and in ℓ^2. For Z_2 we simply take $B \cap \mathfrak{Z}_1$ (see Theorem 19).

It remains to show that C and B have identical projections onto k-hyperplanes. Let H be an arbitrary k-hyperplane and let $u \in B$. Let M be the k-dimensional plane $u + H^\perp = \mathfrak{p}_H^{-1}(\mathfrak{p}_H(u))$. First consider the case $\mathcal{L}_B \cap H^\perp \neq \{\mathbf{0}\}$. Then $\mathcal{L}_B \cap H^\perp$ contains a line ℓ through $\mathbf{0}$. Thus $u + \ell \subset$

$B \cap M$ and we see that $u + \ell \cap \mathfrak{Z}_1 \neq \emptyset$ and hence $u + \ell \cap Z_2 \neq \emptyset$. Thus M intersects C. If $\mathcal{L}_B \cap H^\perp = \{\mathbf{0}\}$ then Lemma 22 shows that M meets $\mathcal{K}^k(B)$ and therefore it meets $\mathcal{E}^k(B)$ or Z_1 by the construction of Z_1. Consequently, M intersects C and the theorem is proved for $\operatorname{codim} B = 0$.

Finally, consider the remaining case $k > \operatorname{codim} B > 0$. Put $n = k - \operatorname{codim} B$ and note that $H = \operatorname{aff} B$ is a copy of ℓ^2 that can be used as the ambient space. Applying what we have just proved to $B \subset H$ and the integer n we find a closed set $C \subset B$ such that every n-plane in H that meets B also meets C and $\dim(C \backslash \mathcal{E}^n_H(B)) \leq 0$, where $\mathcal{E}^n_H$ is determined with H as ambient space. Note that $\mathcal{E}^n_H(B) = \mathcal{E}^k(B)$. If M is a k-plane in ℓ^2 that meets B, then $\dim(M \cap H) \geq n$ so $M \cap H$ must meet C. The proof is complete. ∎

Remark 5. If $\operatorname{codim} B = k$, then the conclusion of Theorem 2 becomes invalid. Consider a B with $B^\circ \neq \emptyset$ and $\operatorname{codim} B = k$. According to Remark 2, ∂B is the union of the faces of B so $\mathcal{E}^k(B) = \partial B$ and $B \setminus \mathcal{E}^k(B)$ is homeomorphic to ℓ^2. However, if we project onto $\operatorname{aff} B$ we find that B is the only k-imitation of B.

Let x be an arbitrary vector in ℓ^2. Note that in the proof of Theorem 2 the zero-dimensional set $Z_1 \cup Z_2 = C \setminus \mathcal{E}^k(B)$ is constructed as a subset of $\bigcup_{\varepsilon>0} \mathfrak{Z}_\varepsilon$ (see also the proof of Corollary 20). If in the proof of Theorem 19 we use a fixed base $\mathfrak{B}$ such that x is contained in the boundary of one of the basic neighbourhoods, then we find that $x \notin C \setminus \mathcal{E}^k(B)$. Combining this observation with Theorem 9 we find:

Corollary 23. *If B is a closed convex set with* $\operatorname{codim} B \neq k$, *then $\mathcal{E}^k(B)$ equals the intersection of all closed k-imitations of B.*

We conclude with an example that shows that in Theorem 1 not even one of the directions in which the projections are proper can be missed. So the theorem is sharp in that respect.

Example 4. Consider the standard basis $\mathcal{B} = \{e^1, e^2, \ldots\}$ in ℓ^2. Fix $k \in \mathbb{N}$ and let $B = \{x \in \ell^2 : \sum_{i=1}^k x_i^2 \leq 1\}$. Note that $\mathcal{L}_B = \{x \in \ell^2 : x_1 = \cdots = x_k = 0\}$ is a k-hyperplane. According to Lemma 6 we have $\mathcal{E}^k(B) = \emptyset$ so by Theorem 2 the convex body B has a zero-dimensional closed k-imitation C. Consider the set $\mathcal{H}$ of all k-hyperplanes H such that $\mathbf{0} \in H$ and $H^\perp$ has a subset of $\mathcal{B}$ as a basis. Then we have the following:

$$\mathcal{L}_B \in \mathcal{H}, \quad \mathfrak{p}_{\mathcal{L}_B}(B) = \mathcal{L}_B,$$
$$\text{if } H \neq \mathcal{L}_B,\ H \in \mathcal{H}, \text{ then } \overline{\mathfrak{p}_H(B)} \neq H.$$

So, in other words, the closure of the projections onto all elements of $\mathcal{H}$ but one are proper and the k-imitation C, being zero-dimensional, does not contain a copy of ℓ^2.

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