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Algebraic properties of quasi-finite complexes

by

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Abstract. A countable CW complex K is quasi-finite (as defined by A. Karasev) if for every finite subcomplex M of K there is a finite subcomplex $e(M)$ such that any map $f : A \to M$, where A is closed in a separable metric space X satisfying $X\tau K$, has an extension $g : X \to e(M)$. Levin's results imply that none of the Eilenberg–MacLane spaces $K(G, 2)$ is quasi-finite if $G \neq 0$. In this paper we discuss quasi-finiteness of all Eilenberg–MacLane spaces. More generally, we deal with CW complexes with finitely many nonzero Postnikov invariants.

Here are the main results of the paper:

Theorem 0.1. *Suppose K is a countable CW complex with finitely many nonzero Postnikov invariants. If $\pi_1(K)$ is a locally finite group and K is quasi-finite, then K is acyclic.*

Theorem 0.2. *Suppose K is a countable non-contractible CW complex with finitely many nonzero Postnikov invariants. If $\pi_1(K)$ is nilpotent and K is quasi-finite, then K is extensionally equivalent to S^1.*

1. Introduction. The notation $K \in AE(X)$ or $X\tau K$ means that any map $f : A \to K$, with A closed in X, extends over X.

Theorem 1.1 (Chigogidze). *For each countable simplicial complex P the following conditions are equivalent*:

1. *$P \in AE(X)$ implies $P \in AE(\beta(X))$ for any normal space X.*
2. *There exists a P-invertible map $p : X \to I^\omega$ of a metrizable compactum X with $P \in AE(X)$ onto the Hilbert cube.*

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Karasev [15] gave an intrinsic characterization of countable complexes P satisfying the conditions of Theorem 1.1 and called them *quasi-finite complexes.*

DEFINITION 1.2. A CW complex K is called *quasi-finite* if there is a function e from the family of all finite subcomplexes of K to the same family satisfying the following property: For every separable metric space X such that K is an absolute extensor of X and for every map $f : A \to M$ with A closed in X and M a finite subcomplex of K, f extends to $g : X \to e(M)$.

For subsequent generalizations of quasi-finiteness see [16] and [2]. In particular, it is shown in [2] that a countable CW complex K is quasi-finite if and only if $X\tau K$ implies $\beta(X)\tau K$ for all separable metric spaces X. That is an improvement of Theorem 1.1.

The first example of a non-quasi-finite CW complex was given by Dranishnikov [5] who showed that $K(\mathbb{Z}, 4)$ admits a separable metric space X satisfying $X\tau K(\mathbb{Z}, 4)$ but not $\beta(X)\tau K(\mathbb{Z}, 4)$ (see [11] for other examples of such X). In [9] it was shown that for all $n \geq 3$ and $G \neq 0$, there is a separable metric space X so that $\dim_G(X) = n$ but $\dim_G(\beta(X)) > n$ (see also [17] for related results). Finally, Levin [19] established a result implying the same fact for all $G \neq 0$ and $n \geq 2$. The only remaining case among Eilenberg–MacLane spaces are complexes $K(G, 1)$.

PROBLEM 1.3. Characterize groups G such that $K(G, 1)$ is quasi-finite. What are the properties of the class of groups G such that $K(G, 1)$ is quasi-finite?

Problem 1.3 was the main motivation of this paper. More generally, we discuss quasi-finiteness of complexes with finitely many nontrivial Postnikov invariants.

2. Truncated cohomology. One of the main tools of this paper is truncated cohomology used for the first time by Dydak and Walsh [12] in their construction of an infinite-dimensional compactum X of integral dimension 2.

Given a pointed CW complex L and a pointed space X we define $h^k_L(X)$ as the $(-k)$th homotopy group of $\mathrm{Map}_*(X, L)$, the space of base-point preserving maps whose base-point is the constant map. Since we are interested in Abelian groups, k ranges from $-\infty$ to -2. Also, spaces X of interest in this paper are countable CW complexes.

CW complexes L for which trunctated cohomology h^*_L is of most use are those with finite homotopy groups. In that case h^*_L is *continuous* in the sense that any map $f : K \to \Omega^k L$ that is phantom (meaning that the restrictions $f|_M$ are homotopically trivial for all finite subcomplexes M of K) must be homotopically trivial if K is a countable CW complex. In the case of L having finite homotopy groups, Levin [18, Proposition 2.1] proved that h^*_L is *strongly continuous*: for any map $f : N \to \Omega^k L$, N a subcomplex of K, that cannot be extended over a countable CW complex K, there is a finite subcomplex M of K such that $f|_{M\cap N}$ cannot be extended over M.

Since we are interested in vanishing of truncated cohomology h^*_L, the remainder of this section is devoted to weak contractibility of mapping spaces.

We first recall a result known in the literature as the Zabrodsky lemma (see Miller [21, Proposition 9.5], and Bousfield [1, Theorem 4.6 and Corollary 4.8]).

LEMMA 2.1. *Let $F \to E \to B$ be a fibration where B has the homotopy type of a connected CW complex. Let X be a space. If* $\mathrm{Map}_*(F, X)$ *is weakly contractible*, *then the induced map* $\mathrm{Map}_*(B, X) \to \mathrm{Map}_*(E, X)$ *is a weak homotopy equivalence.* ∎

DEFINITION 2.2. Let $\mathcal{P}$ be a set of primes. By a *$\mathcal{P}$-complex* we mean a finite CW complex K that is simply connected and all its homotopy groups are *$\mathcal{P}$-groups*, that is, finite groups such that the order of each element is a product of primes belonging to $\mathcal{P}$.

A CW complex K is a *co-$\mathcal{P}$-complex* if for some k the mapping space $\mathrm{Map}_*(\Sigma^k K, L)$ is weakly contractible for all $\mathcal{P}$-complexes L.

LEMMA 2.3. *If K is one of the following*:

(1) *the classifying space BG of a Lie group G with a finite number of path components*,
(2) *a connected infinite loop space whose fundamental group is a torsion group*,
(3) *a simply connected CW complex with finitely many nontrivial homotopy groups*,

then $\mathrm{Map}_*(K, L)$ *is weakly contractible for all nilpotent finite complexes L with finite homotopy groups.*

Proof. Let L be a finite nilpotent complex with finite homotopy groups. Then L is complete with respect to Sullivan's finite completion (see [23]). Thus case (1) follows from Friedlander and Mislin [13, Theorem 3.1], while case (2) follows from McGibbon [20, Theorem 3]. Case (3) follows from Lemma 2.1 and (2) by induction over the number of nontrivial homotopy groups of K. See more details in the proof of Lemma 2.8. ∎

PROPOSITION 2.4. *A finite product* (*or a finite wedge*) *of co-$\mathcal{P}$-complexes is a co-$\mathcal{P}$-complex.*

Proof. For a finite wedge the proof is quite simple as $\mathrm{Map}_*(K \vee P, L)$ is the product of $\mathrm{Map}_*(K, L)$ and $\mathrm{Map}_*(P, L)$. For a finite product one can use induction plus the observation that Lemma 2.1 can be applied to a fibration $F \to K \to P$ to show that K is a co-$\mathcal{P}$-complex if both F and P are. ∎

PROPOSITION 2.5. *Let $\mathcal{P}$ be a set of primes. Suppose K_s, $s \in S$, is a family of CW complexes. If there is a natural number k so that all function spaces* $\mathrm{Map}_*(\Sigma^k K_s, L)$ *are weakly contractible for all $\mathcal{P}$-complexes L, then the wedge $K = \bigvee_{s\in S} K_s$ is a co-$\mathcal{P}$-complex. Moreover, if S is countable and each K_s is countable, then the weak product $\prod_{s\in S} K_s$ is a co-$\mathcal{P}$-complex.*

Proof. The case of the wedge is left to the reader. If S is countable, then each finite product $K_T = \prod_{s\in T} K_s$ has the property that $\mathrm{Map}_*(K_T, \Omega^k L)$ is weakly contractible for any $\mathcal{P}$-complex L as in the proof of Proposition 2.4. Using the fact that truncated cohomology with respect to $\Omega^k L$ is continuous, one finds that $K' = \prod_{s\in S} K_s$, being the direct limit of K_T, also has $\mathrm{Map}_*(K', \Omega^k L)$ weakly contractible. ∎

DEFINITION 2.6. Let $\mathcal{P}$ be a set of primes and let G be a group. Then G is called a *co-$\mathcal{P}$-group* if $K(G,1)$ is a co-$\mathcal{P}$-complex.

By Miller's theorem, all locally finite groups are co-$\mathcal{AP}$-groups, where $\mathcal{AP}$ is the set of all primes. Another example would consist of all acyclic groups. Divisible groups would serve as well. Note that by the Zabrodsky Lemma 2.1, a group extension $N \rightarrowtail G \twoheadrightarrow Q$ implies that under the assumption that N is a co-$\mathcal{P}$-group, G is a co-$\mathcal{P}$-group if and only if Q is.

DEFINITION 2.7. Let K be a connected CW complex. We say that K has *finitely many unstable Postnikov invariants* if for some $k \geq 0$, the k-connected cover $K\langle k\rangle$ of K is an infinite loop space. As usual, $K\langle k\rangle$ is the (homotopy) fibre of the kth Postnikov approximation $K \to P_k K$.

Note that infinite loop spaces (in particular infinite symmetric products) and Postnikov pieces are special cases.

LEMMA 2.8. *Suppose $\mathcal{P}$ is a set of primes. Let K be a connected CW complex with finitely many unstable Postnikov invariants. Then K is a co-$\mathcal{P}$-complex if and only if $G = \pi_1(K)$ is a co-$\mathcal{P}$-group.*

Proof. Let L be a $\mathcal{P}$-complex. Let $\widetilde{K}$ be the universal cover of K. If K is itself an infinite loop space, then so is $\widetilde{K}$, and therefore the space $\mathrm{Map}_*(\widetilde{K}, L)$ is weakly contractible by Theorem 3 of McGibbon [20]. Otherwise for some $i \geq 1$ the i-connected cover $\widetilde{K}\langle i\rangle$ of $\widetilde{K}$ is an infinite loop space. Consider the fibration sequence $\widetilde{K}\langle i\rangle \to \widetilde{K} \to P_i\widetilde{K}$ where $P_i\widetilde{K}$ is the ith Postnikov approximation of $\widetilde{K}$. The space $\mathrm{Map}_*(\widetilde{K}\langle i\rangle, L)$ is weakly contractible [20, Theorem 3]. It follows essentially from Zabrodsky [24, Theorem D], and the fact that L is Sullivan-complete, that $\mathrm{Map}_*(P_i\widetilde{K}, L)$ is weakly contractible (see also [20, Theorem 2]). Thus by Lemma 2.1, also $\mathrm{Map}_*(\widetilde{K}, L)$ is weakly contractible. The space $\widetilde{K}$ sits in the fibration sequence $\widetilde{K} \to K \to K(G,1)$ and another application of Lemma 2.1 shows that $\mathrm{Map}_*(K(G,1),L)$ and $\mathrm{Map}_*(K,L)$ are weakly equivalent. ∎

LEMMA 2.9. *Let $\mathcal{P}$ be a nonempty set of primes. If G is a nilpotent group that is local away from $\mathcal{P}$, then it is a co-$\mathcal{P}$-group.*

Proof. Let $\mathcal{P}'$ denote the set of primes not in $\mathcal{P}$. The hypotheses on G render $K(G,1)$ a $\mathcal{P}'$-local space. By the fundamental theorem on localization of nilpotent spaces, the homology of $K(G,1)$ is also $\mathcal{P}'$-local. Let $\cdots \to L_3 \to L_2 \to L_1 \to L_0$ denote the refined Postnikov tower for L. That is, L_0 is a point and for each i, the fibration $L_i \to L_{i-1}$ is principal with fibre $K(G_i,k_i)$ where G_i is p-torsion abelian. Note that L is weakly equivalent to the inverse limit $\lim_i L_i$, and since $K(G,1)$ is a CW complex it suffices to show that $\mathrm{Map}_*(K(G,1),\lim_i L_i)$ is weakly contractible. This latter space is homeomorphic to the inverse limit $\lim_i \mathrm{Map}_*(K(G,1),L_i)$. Since the fibrations are principal, the Puppe sequence shows that we only need to consider reduced cohomology $\widetilde{H}^*(K(G,1);G_i)$ with coefficients in G_i. Since $H_*(G)$ is local away from $\mathcal{P}$ it follows by the universal coefficient theorem that $\widetilde{H}^*(K(G,1);G_i)$ is trivial. ∎

COROLLARY 2.10. *Suppose $\mathcal{P}$ is a set of primes and G is a nilpotent group with Abelianization* $\mathrm{Ab}(G)$. *If* $\mathrm{Ab}(G)/\mathrm{Tor}(\mathrm{Ab}(G))$ *is $\mathcal{P}$-divisible, then G is a co-$\mathcal{P}$-group.*

Proof. By Corollary 6.4, $\mathrm{Ab}(G)/\mathrm{Tor}(\mathrm{Ab}(G))$ is $\mathcal{P}$-divisible if and only if G is local away from $\mathcal{P}$. ∎

3. Homology and cohomology of quasi-finite CW complexes.

In this section we deal with (co)homological properties of quasi-finite complexes. First, we need a generalization of Theorem II of [10].

THEOREM 3.1. *Suppose K is a countable CW complex and h_* is a generalized reduced homology theory such that $h_*(K) = 0$. For any CW complex P and any $\alpha \in h_*(P) \setminus \{0\}$ there is a compactum X and a map $f : A \to P$ from a closed subset A of X such that $X\tau K$, $\alpha = f_*(\gamma)$ for some $\gamma \in \check{h}_*(A)$ and γ is 0 in $\check{h}_*(X)$.*

Proof. Replacing P by the carrier of α we may assume P is finite. The compactum X is built as in Theorem II of [10]. We start with $X_1 = \mathrm{Cone}(P)$, $A_1 = P$ and build an inverse sequence (X_n, A_n) of compact polyhedra so that for every $g : B \to K$ with B closed in X_n, there is $m > n$ and a map $G : X_m \to K$ extending $g \circ p^m_n : B' \to K$, where $p^m_n : X_m \to X_n$ is the bonding map and $B' = (p^m_n)^{-1}(B)$. For each n we have $\gamma_n \in h_*(A_n)$ which vanishes in $h_*(X_n)$.

In the inductive step we pick $g : B \to K$, B closed in X_n, with an extension $G : X_n \to \mathrm{Cone}(K)$, and consider the pull-back E of the projection $K \times I \to \mathrm{Cone}(K)$ under G. The fibres of the projection $p : E \to X_n$ are either homeomorphic to K or single points. Therefore $h_*(p)$ is an isomorphism and one can pick a finite subpolyhedron A_{n+1} of E carrying $\gamma_{n+1} \in h_*(A_{n+1})$ which gets mapped to γ_n under $h_*(p)$. As γ_{n+1} vanishes in $h_*(E)$, it vanishes in a finite subpolyhedron X_{n+1} of E containing A_{n+1}. Since there are only countably many extension problems to be solved (see [4] or [6]), that process produces an inverse sequence whose inverse limit (X, A) satisfies $X\tau K$ and one has $\gamma \in \check{h}_*(A)$ that vanishes in $\check{h}_*(X)$ and $f_*(\gamma) = \alpha$, where $f : A \to P = A_1$ is the projection. ∎

THEOREM 3.2. *Suppose K is a countable CW complex and h^* is a strongly continuous truncated cohomology theory such that $h^*(K) = 0$. For any countable CW complex P and any $\alpha \in h^*(P) \setminus \{0\}$ there is a compactum X and a map $f : A \to P$ from a closed subset A of X such that $X\tau K$ and there is no $\gamma \in \check{h}^*(X)$ satisfying $\gamma|_A = f^*(\alpha)$.*

Proof. We can reduce the proof to the case of P being a finite polyhedron as there is a finite subcomplex M of P so that $\alpha|_M \neq 0$ and M can be used instead of P. The compactum X is built as in the proof of Theorem 3.1. We start with $X_1 = \mathrm{Cone}(P)$, $A_1 = P$ and built an inverse sequence (X_n, A_n) of compact polyhedra so that for every $g : B \to K$ with B closed in X_n, there is $m > n$ and a map $G : X_m \to K$ extending $g \circ p^m_n : B' \to K$, where $p^m_n : X_m \to X_n$ is the bonding map and $B' = (p^m_n)^{-1}(B)$. Also, for each n the pullback α_n of α under $A_n \to A_1$ does not extend over X_n.

In the inductive step we pick $g : B \to K$ with B closed in X_n, find $G : X_n \to \mathrm{Cone}(K)$, and consider the pull-back E of the projection $K \times I \to \mathrm{Cone}(K)$ under G. The fibres of the projection $p : E \to X_n$ are either homeomorphic to K or single points. Therefore $p^* = h^*(p)$ is an isomorphism. Since $p^*(\alpha_n)$ does not extend over E, there is a finite subpolyhedron X_{n+1} of E such that $p^*(\alpha_n)$ restricted to $A_{n+1} = X_{n+1} \cap p^{-1}(A_n)$ does not extend over X_{n+1}. As there are only countably many extension problems to be solved (see [4] or [6]), that process produces an inverse sequence whose inverse limit (X, A) satisfies $X\tau K$ and the projection $f : A \to P = A_1$ has the property that there is no $\gamma \in \check{h}^*(X)$ satisfying $\gamma|_A = f^*(\alpha)$. ∎

Recall that, given a map $i : M \to N$, $X\tau i$ means that for any map $f : A \to M$ with A closed in X, there is a map $g : X \to N$ extending $i \circ f$.

THEOREM 3.3. *Suppose K is a countable CW complex and $i : M \to N$ is a map of CW complexes such that $X\tau K$ implies $X\tau i$ for all compacta X.*

(1) *If h_* is a generalized reduced homology theory such that the induced homomorphism $h_*(M) \to h_*(N)$ is not trivial, then $h_*(K) \neq 0$.*

(2) *If h^* is a truncated strongly continuous cohomology theory such that the induced homomorphism $h^*(N) \to h^*(M)$ is not trivial and M is countable*, *then $h^*(K) \neq 0$.*

Proof. We may assume i is an inclusion.

(1) Suppose $\alpha \in h_*(M)$ does not become 0 in $h_*(N)$ and $h_*(K) = 0$. By Theorem 3.1 pick a map $f : A \to M$ of a closed subset of a compactum X so that $X\tau K$ and γ equals 0 in $\check{h}_*(X)$ for some $\gamma \in \check{h}_*(A)$ satisfying $f_*(\gamma) = \alpha$. If f extends to $g : X \to N$, then $\alpha = f_*(\gamma)$ becomes 0 in $h_*(N)$, a contradiction.

(2) Suppose $\alpha \in h^*(N)$ and $\alpha|_M \neq 0$. We may reduce this case to M finite by switching to a finite subcomplex L of M with $\alpha|_L \neq 0$. By Theorem 3.2 pick a map $f : A \to M$ of a closed subset of a compactum X so that $X\tau K$ and $f^*(\alpha|_M)$ does not extend over X. If $f : A \to M$ extends to $g : X \to N$, then $g^*(\alpha) \in \check{h}^*(X)$ extends $f^*(\alpha|_M)$, a contradiction. ∎

THEOREM 3.4. *Suppose $\mathcal{P}$ is a set of primes. Let K be a connected countable co-$\mathcal{P}$-complex. If K is quasi-finite*, *then it is $\mathbb{Z}_{(\mathcal{P})}$-acyclic.*

Proof. Assume K is quasi-finite and not $\mathbb{Z}_{(\mathcal{P})}$-acyclic. Replace K with ΣK (using [2]) if necessary to ensure $H_k(K;\mathbb{Z}_{(\mathcal{P})}) \neq 0$ for some $k \geq 2$. Let $\alpha_K \in H_k(K;\mathbb{Z}_{(\mathcal{P})})$ be nonzero. Since K is the colimit of its finite subcomplexes, α_K is the image of $\alpha_M \in H_k(M;\mathbb{Z}_{(\mathcal{P})})$ for some finite subcomplex M of K. Certainly the image of α_M under $H_k(M;\mathbb{Z}_{(\mathcal{P})}) \to H_k(e(M);\mathbb{Z}_{(\mathcal{P})})$ is nontrivial. Thus Lemma 7.2 yields a $\mathcal{P}$-complex L with the restriction morphism $[e(M), \Omega^2 L] \to [M, \Omega^2 L]$ nontrivial. This says that $h^*(e(M)) \to h^*(M)$ is nontrivial where h^* is the truncated cohomology theory defined by means of $\Omega^2 L$. The hypotheses on L ensure strong continuity of h^*. Thus the nontriviality of $h^*(e(M)) \to h^*(M)$ contradicts (2) of Theorem 3.3. ∎

COROLLARY 3.5. *Let K be a countable CW complex with finitely many nontrivial homotopy groups and $G = \pi_1(K)$ nilpotent. Suppose that G is not torsion. If K is quasi-finite*, *then the group $FG = G/\mathrm{Tor}(G)$* (*and thus also* $\mathrm{Ab}(G)/\mathrm{Tor}(\mathrm{Ab}(G))$) *is not divisible by any prime p.*

Proof. Suppose that, on the contrary, FG is divisible by a prime p, hence local away from p. Since G is not torsion and is nilpotent, also $\mathrm{Ab}(G)$ is not torsion, hence certainly $H_1(K) \otimes \mathbb{Z}_{(p)}$ is nontrivial. Thus Theorem 3.4 yields a contradiction. ∎

Theorem 3.4 and Lemma 2.8 imply the following.

COROLLARY 3.6. *Let K be a simply connected countable CW complex with at least one and at most finitely many nontrivial homotopy groups. Then K is not quasi-finite.* ∎

Corollary 3.7. *Suppose G is a locally finite countable group. If $K(G,1)$ is quasi-finite, then G is acyclic.* ■

However, there are also some countable acyclic groups G for which $K(G,1)$ is not quasi-finite. Cencelj and Repovš [3, §5], using results of Dranishnikov and Repovš [8], showed that the minimal grope M^* which is $K(\pi_1(M^*),1)$ is not quasi-finite. This also holds for the fundamental group of any grope: For a grope M let $\gamma(m)$ denote the maximal number of handles on the discs with handles used in the construction of the m-stage of M. Modify the inverse limit construction of the example of [3] by replacing every simplex in the triangulation of the kth element of the inverse system by the nth stage of the grope which has every generator replaced by a disc with $\gamma(kn)$ handles.

4. Ljubljana complexes

Definition 4.1. A connected CW complex L is called a *Ljubljana complex* (L-complex for short) if there is a co-$\mathcal{AP}$-complex K, $\mathcal{AP}$ being the set of all primes, such that, for any compactum X, the conditions $X\tau L$ and $X\tau K(H_1(K),1)$ imply $X\tau K$.

Lemma 4.2. *Suppose $F\to E\to B$ is a fibration of connected CW complexes. If F is a co-$\mathcal{AP}$-complex and B is an L-complex, then E is an L-complex.*

Proof. Notice that $\pi_1(E)\to\pi_1(B)$ is an epimorphism (use the long exact sequence of a fibration), which implies $H_1(E)\to H_1(B)$ is an epimorphism.

Pick a co-$\mathcal{AP}$-complex K such that $X\tau K$ and $X\tau K(H_1(B),1)$ imply $X\tau B$ for all compacta X. Let M be the wedge of F, K, $K(\mathbb{Q},1)$, and suppose $X\tau K(\mathbb{Z}/p^\infty,1)$ for all primes p. By Proposition 2.5 and the Miller theorem, M is a co-$\mathcal{AP}$-complex. Suppose X is a compactum such that $X\tau M$ and $X\tau K(H_1(E),1)$. By Lemma 6.5 one gets $X\tau K(H_1(B),1)$, which, together with $X\tau K$, implies $X\tau B$. Since $X\tau F$ and $X\tau B$, we infer $X\tau E$. ■

Corollary 4.3. *Let L be a connected CW complex with nilpotent fundamental group. If L has finitely many unstable Postnikov invariants, then L is an L-complex.*

Proof. Notice that the universal cover $\widetilde{L}$ of L is a co-$\mathcal{AP}$-complex by Lemma 2.8. We infer that $K(\pi_1(L),1)$ is an L-complex by Corollary 6.6. The fibration $\widetilde{L}\to L\to K(\pi_1(L),1)$ implies L is an L-complex. ■

Definition 4.4. A connected CW complex L is called *extensionally Abelian* if $X\tau K(H_n(L),n)$ for all $n\ge 1$ implies $X\tau L$ for all compacta X.

Proposition 4.5. *Each extensionally Abelian complex L is an L-complex.*

Proof. Let K be the weak product of $K(H_n(L), n)$, $n \geq 2$. By Lemma 2.3(2), K is a co-$\mathcal{AP}$-complex. Clearly, $X\tau K$ and $X\tau K(H_1(L), 1)$ imply $X\tau K(H_n(L), n)$ for all $n \geq 1$. Thus $X\tau L$. ■

Proposition 4.6. *A finite wedge* (*or finite product*) *of L-complexes is an L-complex.*

Proof. Let L be the wedge (or product) of Ljubljana complexes L_s, $s \in S$, where S is finite. For each $s \in S$ choose a co-$\mathcal{AP}$-complex K_s such that for any compactum X the conditions $X\tau K_s$ and $X\tau K(H_1(L_s), 1)$ imply $X\tau L_s$. Let K be the wedge of all K_s. By Proposition 2.4 it is a co-$\mathcal{AP}$-complex. Notice that $H_1(L_s)$ is a retract of $H_1(L)$ for each $s \in S$. Therefore any compactum X satisfying

(a) $X\tau K(H_1(L), 1)$,
(b) $X\tau K$,

also satisfies $X\tau K(H_1(L_s), 1)$ for each $s \in S$. Hence $X\tau L_s$ for each $s \in S$, which implies $X\tau L$. ■

There is a connection between Ljubljana complexes and co-$\mathcal{P}$-complexes.

Proposition 4.7. *Suppose K is a countable L-complex. If $\mathcal{P}$ is a set of primes such that $H_1(K)/\mathrm{Tor}(H_1(K))$ is $\mathcal{P}$-divisible, then K is a co-$\mathcal{P}$-complex.*

Proof. Choose a co-$\mathcal{AP}$-complex L such that, for any compactum X, the conditions $X\tau L$ and $X\tau K(H_1(K), 1)$ imply $X\tau K$. Let $\mathcal{P}'$ be the complement of $\mathcal{P}$ in the set of all primes. Consider K', the wedge of L, $K(\mathbb{Z}_{(\mathcal{P}')}, 1)$, $K(\mathbb{Q}, 1)$, and all $K(\mathbb{Z}/p, 1)$ (p ranging through all primes). By Corollary 2.10 and Proposition 2.5, K' is a co-$\mathcal{P}$-complex. Since $X\tau K'$ implies $X\tau K$ for all compacta, Theorem 3.3 implies that there is $k \geq 0$ such that the truncated cohomology of K with respect to $\Omega^k L$, L any $\mathcal{P}$-complex, is trivial. Thus K is a co-$\mathcal{P}$-complex. ■

Theorem 4.8. *Suppose K is a countable L-complex such that $\Sigma^m K$ is equivalent* (*over the class of compacta*) *to a quasi-finite countable complex L for some $m \geq 0$. If K is not acyclic*, *then it is equivalent to S^1.*

Proof. We may assume L is simply connected as $\Sigma^{m+1}K$ is equivalent to ΣL (see [7]) and ΣL is quasi-finite by [2].

Suppose K is not equivalent to S^1. Choose a co-$\mathcal{AP}$-complex P such that the conditions $X\tau P$ and $X\tau K(H_1(K), 1)$ imply $X\tau K$. Let $k \geq 2$ be such that all maps $\Sigma^n P \to R$ are null-homotopic whenever R is an $\mathcal{AP}$-complex and $n \geq k$.

Step 1. L is not contractible as otherwise $\Sigma^m K$ would have to be contractible, implying K being acyclic.

STEP 2. L is not acyclic as it is not contractible, by Step 1.

STEP 3. Since $X\tau K$ implies $X\tau K(H_1(K),1)$, the group $H_1(K)$ has $H_1(K)/\mathrm{Tor}(H_1(K))$ divisible by some prime p. Indeed, if $H_1(K)/\mathrm{Tor}(H_1(K))$ is not divisible by any prime, then the Bockstein basis of $H_1(K)$ consists of all Bockstein groups and $X\tau K(H_1(K),1)$ implies $X\tau S^1$ by the Bockstein First Theorem. Since $X\tau K$ implies $X\tau K(H_1(K),1)$ and $X\tau S^1$ implies $X\tau K$ for any compactum X, K is equivalent to S^1 over compacta.

Let e be the e-function of L.

CASE 1: $H_*(K)$ is a torsion group. By Step 2 there is M such that $H_*(M) \to H_*(e(M))$ is not trivial. By Lemma 7.1, there is a map $f : \Sigma^k(e(M)) \to J$ such that $f|_{\Sigma^k M}$ is not trivial, J is simply connected, and all homotopy groups of J are finite. Consider the wedge N of P and $K(\bigoplus_q \mathbb{Z}/q,1)$. Notice $X\tau N$ implies $X\tau K(H_1(K),1)$. Therefore $X\tau N$ implies $X\tau K$, which in turn implies $X\tau L$ and $X\tau i_M$, where $i_M : M \to e(M)$. Since $\mathrm{Map}_*(N,\Omega^k J)$ is weakly contractible, Theorem 3.3 implies homotopy triviality of $f|_{\Sigma^k M}$, a contradiction.

CASE 2: $H_*(K)$ is not a torsion group. Then $H_*(L)$ is not torsion either. Indeed, if it were, we could find a finite-dimensional compactum Y of high rational dimension but with all torsion dimensions 1. Such a compactum satisfies $Y\tau L$ but $Y\tau\Sigma^m K$ fails as it implies the rational dimension of Y to be at most $m+n$, where $H_n(K)$ is not torsion. There is M such that the image of $H_*(M) \to H_*(e(M))$ is not torsion. Therefore there is $n > 0$ such that $H_n(M;\mathbb{Z}_{(p)}) \to H_n(e(M);\mathbb{Z}_{(p)})$ is not trivial. By Lemma 7.2, there is a map $f : \Sigma^k(e(M)) \to J$ such that $f|_{\Sigma^k(M)}$ is not trivial, J is simply connected, and all homotopy groups of J are finite p-groups. Consider the wedge N of P and $K(\mathbb{Z}[1/p] \oplus \mathbb{Z}/p,1)$. Corollary 6.7 (for $G = H_1(K)$) and Lemma 6.5 show that $X\tau N$ implies $X\tau K$, which in turn implies $X\tau L$ and $X\tau i_M$, where $i_M : M \to e(M)$. Since $\mathrm{Map}_*(N,\Omega^k J)$ is weakly contractible, Theorem 3.3 implies homotopy triviality of $f|_{\Sigma^k M}$, a contradiction. ∎

COROLLARY 4.9. *Suppose G is a nontrivial nilpotent group. If $K(G,1)$ is quasi-finite, then it is equivalent, over the class of paracompact spaces, to S^1.*

5. Application to cohomological dimension theory

THEOREM 5.1. *Suppose $G \neq 1$ is a countable group such that $\dim_G(\beta(X)) = 1$ for every separable metric space X satisfying $\dim_G(X) = 1$. If G is nilpotent, then $\dim_G(X) \le 1$ implies $\dim(X) \le 1$ for all paracompact spaces X.*

Proof. By an improvement of Chigogidze's Theorem 1.1 of [2], $K(G,1)$ is quasi-finite. Therefore Corollary 4.9, says that $K(G,1)$ is equivalent to

S^1 over compacta. A result in [2] says that $K(G,1)$ is equivalent to S^1 over paracompact spaces, which completes the proof. ■

6. Appendix A. In this section we discuss results relating to groups that are needed in the paper.

LEMMA 6.1. *Let p be a natural number and let $\mathcal{D}_p$ be the class of groups G such that* $\mathrm{Ab}(G)/\mathrm{Tor}(\mathrm{Ab}(G))$ *is p-divisible, where* $\mathrm{Ab}(G)$ *is the Abelianization of G. If $f : G \to H$ is an epimorphism and $G \in \mathcal{D}_p$, then $H \in \mathcal{D}_p$.*

Proof. Notice that $G \in \mathcal{D}_p$ if and only if for each $a \in G$ there are $b \in G$ and $k \geq 1$ such that $(a \cdot b^{-p})^k$ belongs to the commutator subgroup $[G,G]$ of G. Suppose $a \in H$. Pick $b \in G$ with $a = f(b)$. There are $c \in G$ and $k \geq 1$ such that $(b \cdot c^{-p})^k \in [G,G]$. Now $(a \cdot f(c)^{-p})^k \in [H,H]$ and $H \in \mathcal{D}_p$. ■

LEMMA 6.2. *Let p be a natural number and let $\mathcal{D}_p$ be as in Lemma* 6.1. *If G, H are Abelian and $G \in \mathcal{D}_p$, then $G \otimes H \in \mathcal{D}_p$.*

Proof. It suffices to show that for each element a of $G \otimes H$ there is $b \in G \otimes H$ and an integer $k \neq 0$ such that $k \cdot a + kp \cdot b = 0$. That in turn can be reduced to generators of $G \otimes H$ of the form $g \otimes h$. Pick $u \in G$ and an integer $k \neq 0$ such that $k \cdot g + kp \cdot u = 0$. Now $k \cdot (g \otimes h) + kp \cdot (u \otimes h) = 0$. ■

We recall a result of Robinson (see [22, 5.2.6]) on the relation between a nilpotent group and its abelianization.

PROPOSITION 6.3 (Robinson). *Let $\mathcal{N}$ denote the category of nilpotent groups. Let $\mathcal{C}$ be a class of groups in $\mathcal{N}$ with the following properties.*

(1) *For A and B abelian, $B \in \mathcal{C}$, any quotient of $A \otimes B$ belongs to $\mathcal{C}$.*
(2) *For $K, Q \in \mathcal{C}$, an extension $1 \to K \to G \to Q \to 1$ in $\mathcal{N}$ implies $G \in \mathcal{C}$.*

Suppose that $G \in \mathcal{N}$. If $\mathrm{Ab}(G)$ *belongs to $\mathcal{C}$, so does G.* ■

We note the following corollary.

COROLLARY 6.4. *Let G be a nilpotent group. If* $\mathrm{Ab}(G)/\mathrm{Tor}(\mathrm{Ab}(G))$ *is p-divisible, then so is* $G/\mathrm{Tor}(G)$.

Proof. Define the class $\mathcal{C}_p$ by letting a nilpotent group G belong to $\mathcal{C}_p$ if and only if $F_p(G) = G/\mathrm{Tor}_p(G)$ is p-divisible where $\mathrm{Tor}_p(G)$ denotes the p-torsion subgroup of G. Note that $F_p(G)$ is p-divisible if and only if $G/\mathrm{Tor}(G)$ is, hence it suffices to check properties (1) and (2) of Proposition 6.3.

As for (1) it follows from Lemmas 6.1 and 6.2.

For (2), note that F_p is a functor $\mathcal{N} \to \mathcal{N}$. Let $1 \to K \to G \to Q \to 1$ be an extension in $\mathcal{N}$. We apply F_p. Since $\mathrm{Tor}_p(K) = K \cap \mathrm{Tor}_p(G)$, the morphism $F_p(K) \to F_p(G)$ is injective. Evidently, $q\colon F_p(G) \to F_p(Q)$ is surjective. Moreover, $F_p(K)$ is a subset of the kernel of q. Assume that K

belongs to $\mathcal{C}_p$. If $q(\xi) = 1$ for some $\xi \in F_p(G)$, then $\xi^{p^i} \in F_p(K)$ for i large enough. By assumption on K, the group $F_p(K)$ is p-divisible, hence $\xi^{p^i} = \eta^{p^i}$ for some $\eta \in F_p(K)$. But $F_p(G)$ is free of p-torsion (and nilpotent), so the equality $\xi^{p^i} = \eta^{p^i}$ in $F_p(G)$ implies $\xi = \eta$ (see for example Hilton, Mislin, and Roitberg [14, Corollary 2.3]). Therefore in fact $\xi \in F_p(K)$, i.e. $\ker q = F_p(K)$. This says that $1 \to F_p(K) \to F_p(G) \to F_p(Q) \to 1$ is an extension in $\mathcal{N}$. If, in addition, Q belongs to $\mathcal{C}_p$ then $F_p(K)$ and $F_p(Q)$ are p-divisible and free of p-torsion, and hence local away from p. Therefore so is $F_p(G)$, by Corollary 2.5 of [14]. ∎

Lemma 6.5. *Let $f : G \to H$ be an epimorphism of Abelian groups and let X be a compactum. If*

(a) $X \tau K(G, 1)$,
(b) $X \tau K(\mathbb{Q}, 1)$,
(c) $X \tau K(\mathbb{Z}/p^\infty, 1)$ *for all primes p*,

then $X \tau K(H, 1)$.

Proof. Suppose $X \tau K(H, 1)$ fails to hold. This can only happen if there is a Bockstein group F in the Bockstein basis $\sigma(H)$ such that $\dim_F(X) > 1$. That group must be either $\mathbb{Z}_{(p)}$ or $\mathbb{Z}/p$ for some p. $\mathbb{Z}_{(p)}$ belongs to $\sigma(H)$ if and only if $H/\mathrm{Tor}(H)$ is not divisible by p, in which case $\mathbb{Z}_{(p)}$ belongs to $\sigma(G)$ by Lemma 6.1 and $\dim_F(X) \le 1$ by Bockstein's First Theorem. Therefore $F = \mathbb{Z}/p$, which means that $\mathrm{Tor}(H)$ is not divisible by p. Now, either G is not divisible by p or its torsion group is not divisible by p, implying $\dim_F(X) \le 1$, a contradiction. ∎

Corollary 6.6. *Let G be a nilpotent group and let X be a compactum. If*

(a) $X \tau K(\mathrm{Ab}(G), 1)$,
(b) $X \tau K(\mathbb{Q}, 1)$,
(c) $X \tau K(\mathbb{Z}/p^\infty, 1)$ *for all primes p*,

then $X \tau K(G, 1)$.

Proof. Consider the lower central series of G: $G = \Gamma^1 G \supset \Gamma^2 G \supset \cdots \supset \Gamma^i G \supset \cdots$. Let $F_i = \Gamma^i G / \Gamma^{i+1} G$. Since there is an epimorphism from $F_i \otimes \mathrm{Ab}(G)$ to F_{i+1}, we have $X \tau K(F_i, 1)$ for all i by Lemma 6.5. We show by induction on $c - i$ (c being the nilpotency class of G) that $X \tau K(\Gamma^i G, 1)$. If $c - i = 0$, then $\Gamma^i G = F_i$ and we are done. Since the sequence $1 \to \Gamma^{i+1} G \to \Gamma^i G \to F_i \to 1$ is exact, one uses a fibration $K(\Gamma^{i+1} G, 1) \to K(\Gamma^i G, 1) \to K(F_i, 1)$ to conclude that $X \tau K(\Gamma^i G, 1)$ given $X \tau K(\Gamma^{i+1} G, 1)$. That constitutes the inductive step and, as $\Gamma^1 G = G$, we get $X \tau K(G, 1)$. ∎

Corollary 6.7. *Let p be a natural number and let $\mathcal{D}_p$ be as in Lemma* 6.1. *If $G \in \mathcal{D}_p$ is nilpotent and $X\tau K(\mathbb{Z}[1/p] \oplus \mathbb{Z}/p, 1)$, then $X\tau K(G,1)$ for any compactum X.*

Corollary 6.8. *Let G be a nilpotent group. If* $\mathrm{Ab}(G)$ *of G is a torsion group and $X\tau K(\bigoplus_p \mathbb{Z}/p, 1)$, then $X\tau K(G,1)$ for any compactum X.*

7. Appendix B. In this appendix we prove results allowing us to detect homology via maps to finite complexes with finite homotopy groups.

Lemma 7.1. *Let A be a finite CW complex and $\alpha \in H_k(A)$ a nontrivial element where $k \geq 2$. There exists a finite $(k-1)$-connected CW complex B with finite homotopy groups and a map $f\colon A \to B$ with $\beta = f_*(\alpha)$ nontrivial. Furthermore, if α is of infinite order in $H_k(A)$, then β may be assumed to be of order r for any given natural $r \geq 2$.*

Proof. Except the statements about the connectedness and order, this is precisely Lemma 2.1 of Levin [19]. In the course of proving the cited lemma, Levin constructs a $(k-1)$-connected complex L, and he makes β of order 2 if α has infinite order. The generalization to arbitrary r is trivial. ∎

Lemma 7.2. *Let M be a finite CW complex, and let $\mathcal{P}$ be a nonempty set of primes. Let $\alpha \in H_k(M; \mathbb{Z}_{(\mathcal{P})})$ be a nontrivial element for some $k \geq 2$. Then there exists a finite $(k-1)$-connected CW complex N with $\mathcal{P}$-torsion homotopy groups and a map $f\colon M \to N$ with $f_*(\alpha)$ nontrivial.*

Proof. The assumption is that there exists an element $\alpha \in H_k(M)$ which is either $\mathcal{P}$-torsion or of infinite order. We can apply Lemma 7.1 to obtain a $(k-1)$-connected finite complex N' with finite homotopy groups and a map $f'\colon M \to N'$ with $\beta' = f'_*(\alpha)$ nontrivial of an order whose prime divisors all belong to $\mathcal{P}$. Let $N' \to N$ be localization at the set $\mathcal{P}$. Then β' will map to nontrivial β under localization $\widetilde{H}_*(N') \to \widetilde{H}_*(N) = \widetilde{H}_*(N') \otimes \mathbb{Z}_{(\mathcal{P})}$ and N is (homotopy equivalent to) the finite complex as in the statement of the lemma. ∎

References

[1] A. K. Bousfield, *Localization and periodicity in unstable homotopy theory*, J. Amer. Math. Soc. 7 (1994), 831–873.

[2] M. Cencelj, J. Dydak, J. Smrekar, A. Vavpetič, and Ž. Virk, *Compact maps and quasi-finite complexes*, Topology Appl., to appear.

[3] M. Cencelj and D. Repovš, *On compacta of cohomological dimension one over non-abelian groups*, Houston J. Math. 26 (2000), 527–536.

[4] A. N. Dranishnikov, *On extension theory of compact spaces*, to appear.

[5] —, *Cohomological dimension is not preserved by Stone–Čech compactification*, C. R. Acad. Bulgare Sci. 41 (1988), 9–10 (in Russian).

[6] A. N. Dranishnikov, *Cohomological Dimension Theory of Compact Metric Spaces*, Topology Atlas, 1999.
[7] A. N. Dranishnikov and J. Dydak, *Extension dimension and extension types*, Proc. Steklov Math. Inst. 212 (1996), 55–88.
[8] A. Dranishnikov and D. Repovš, *Cohomological dimension with respect to perfect groups*, Topology Appl. 74 (1996), 123–140.
[9] J. Dydak, *Compactifications and cohomological dimension*, ibid. 50 (1993), 1–10.
[10] —, *Realizing dimension functions via homology*, ibid. 64 (1995), 1–7.
[11] J. Dydak and J. J. Walsh, *Spaces without cohomological dimension preserving compactifications*, Proc. Amer. Math. Soc. 113 (1991), 1155–1162.
[12] —, —, *Infinite dimensional compacta having cohomological dimension two*: *An application of the Sullivan Conjecture*, Topology 32 (1993), 93–104.
[13] E. M. Friedlander and G. Mislin, *Locally finite approximation of Lie groups. I*, Invent. Math. 83 (1986), 425–436.
[14] P. Hilton, G. Mislin and J. Roitberg, *Localization of Nilpotent Groups and Spaces*, North-Holland, Amsterdam, and American Elsevier, New York, 1975.
[15] A. Karasev, *On two problems in extension theory*, arXiv:math.GT/0312269 v1 12 Dec 2003.
[16] A. Karasev and V. Valov, *Extension dimension and quasi-finite CW complexes*, preprint, 2004.
[17] A. I. Karinskiĭ, *On cohomological dimension of the Stone–Čech compactification*, Vestnik Moskov. Univ. 1991, no. 4, 9–11 (in Russian).
[18] M. Levin, *Constructing compacta of different extensional dimensions*, Canad. Math. Bull. 44 (2001), 80–86.
[19] —, *Some examples in cohomological dimension theory*, Pacific J. Math. 202 (2002), 371–378.
[20] C. A. McGibbon, *A note on Miller's theorem about maps out of classifying spaces*, Proc. Amer. Math. Soc. 124 (1996), 3241–3245.
[21] H. Miller, *The Sullivan conjecture on maps from classifying spaces*, Ann. of Math. (2) 120 (1984), 39–87.
[22] D. J. S. Robinson, *A Course in the Theory of Groups*, Springer, New York, 1993.
[23] D. Sullivan, *Genetics of homotopy theory and the Adams conjecture*, Ann. of Math. (2) 100 (1974), 1–79.
[24] A. Zabrodsky, *On phantom maps and a theorem of H. Miller*, Israel J. Math. 58 (1987), 129–143.

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