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Borsuk's quasi-equivalence is not transitive

by

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Abstract. Borsuk's quasi-equivalence relation on the class of all compacta is considered. The open problem concerning transitivity of this relation is solved in the negative. Namely, three continua X, Y and Z lying in $\mathbb{R}^3$ are constructed such that X is quasi-equivalent to Y and Y is quasi-equivalent to Z, while X is not quasi-equivalent to Z.

1. Introduction. In [2] K. Borsuk defined a certain relation on the class $c\mathcal{M}$ of all (metrizable) compacta, called quasi-equivalence and denoted by $\overset{q}{\simeq}$. Let us recall its definition (in the original terms of fundamental sequences; see [1]).

Consider any two compacta X and Y lying in AR-spaces M and N respectively, and a neighbourhood V of Y in N. Two fundamental sequences $\underline{f} = \{f_k, X, Y\}_{M,N}$, $\underline{f}' = \{f'_k, X, Y\}_{M,N}$ are said to be *V-homotopic* (notation: $\underline{f} \underset{V}{\simeq} \underline{f}'$) if there exists a neighbourhood U_0 of X in M such that $f_k|U_0 \simeq f'_k|U_0$ in V for almost all k. (If V is open, then the condition reduces to $f_k|X \simeq f'_k|X$ in V for almost all k.)

Let, in addition, U be a neighbourhood of X in M. Then X and Y are said to be *(U,V)-equivalent in M, N* (notation: $X \underset{(U,V)}{\simeq} Y$) if there exist two fundamental sequences $\underline{f} = \{f_k, X, Y\}_{M,N}$, $\underline{g} = \{g_k, Y, X\}_{N,M}$ such that $\underline{g}\underline{f} \underset{U}{\simeq} \underline{i}_{X,M}$ and $\underline{f}\underline{g} \underset{V}{\simeq} \underline{i}_{Y,N}$, where $\underline{i}_{X,M}$ (resp. $\underline{i}_{Y,N}$) is the fundamental identity sequence for X in M (resp. Y in N).

Further, Borsuk defined X and Y to be *quasi-equivalent in M,N* (notation: $X \overset{q}{\simeq} Y$ in M,N) if $X \underset{(U,V)}{\simeq} Y$ for every neighbourhood U of X in M and every neighbourhood V of Y in N. After proving that the choice of

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the ambient AR-spaces M and N is immaterial, he defined X and Y to be *quasi-equivalent* (notation: $X \overset{q}{\simeq} Y$) if $X \overset{q}{\simeq} Y$ in some M, N.

Borsuk proved that quasi-equivalence is a shape invariant relation and that it is strictly coarser than shape type, i.e.

$$(X \overset{q}{\simeq} Y) \wedge (h(X) = h(X')) \wedge (\mathrm{Sh}(Y) = \mathrm{Sh}(Y')) \Rightarrow X' \overset{q}{\simeq} Y';$$

$$\mathrm{Sh}(X) = \mathrm{Sh}(Y) \Rightarrow X \overset{q}{\simeq} Y;$$

$$(\exists X, Y)\ (X \overset{q}{\simeq} Y) \wedge (\mathrm{Sh}(X) \neq \mathrm{Sh}(Y)).$$

For instance, all infinite 0-dimensional compacta are quasi-equivalent ([2, Theorem (6.3)]), while their shape types coincide with the topological types ([4, Theorem 20]). Further, in the case of compact ANR's, quasi-equivalence reduces to shape type, and hence to homotopy type. One should also mention that quasi-equivalence preserves some important shape invariants (Betti numbers, movability; [2, Theorems (10.3), (11.1)]). However, it has remained unknown whether quasi-equivalence is indeed an equivalence relation. Specifically, Borsuk stated the following question ([2, Problem (7.13)]): "Is the relation of quasi-equivalence transitive?".

A few months ago, the third named author found by chance an old unpublished manuscript of the first named author, containing a certain example intended to show that quasi-equivalence is not transitive. Unfortunately, an analysis by the second named author showed that the proof was incorrect. However, there was a strong feeling that the example might be appropriate. In this paper we provide a correct proof by using the same example (only the notation is slightly changed).

Thus, Borsuk's quasi-equivalence relation is *not transitive* because there exist continua X, Y and Z, lying in the Euclidian space $\mathbb{R}^3$, such that X is quasi-equivalent to Y and Y is quasi-equivalent to Z, while X is not quasi-equivalent to Z.

2. Preliminaries. The preliminary step in our considerations is to characterize $\overset{q}{\simeq}$ in terms of the Mardešić–Segal shape category (see [8]).

Recall the inv-category $\mathrm{HTop}^{\mathbb{N}}$ (see [6]). The objects are all inverse sequences $\boldsymbol{X} = (X_i, [p_{ii'}])$, $\boldsymbol{Y} = (Y_j, [q_{jj'}]), \ldots$ of topological spaces with the homotopy classes of mappings as bonding arrows, while the morphisms $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ are of the form $\boldsymbol{f} = (f, [f_j])$, where $f : \mathbb{N} \to \mathbb{N}$ and $f_j : X_{f(j)} \to Y_j$, $j \in \mathbb{N}$, are such that for every pair $j \leq j'$ there exists an $i \geq f(j), f(j')$ satisfying

$$[f_j][p_{f(j)i}] = [q_{jj'}][f_{j'}][p_{f(j')i}].$$

The composition of $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ and $\boldsymbol{g} = (g, [g_k]) : \boldsymbol{Y} \to \boldsymbol{Z}$ is the morphism

$$\boldsymbol{h} \equiv \boldsymbol{g}\boldsymbol{f} = (fg, [g_k f_{g(k)}]) : \boldsymbol{X} \to \boldsymbol{Z},$$

while the identity morphism on $\boldsymbol{X}$ is $1_{\boldsymbol{X}} = (1_{\mathbb{N}}, [1_{X_i}])$. With the natural equivalence relation $\boldsymbol{f} \simeq \boldsymbol{f}'$, i.e. for every j there exists an $i \geq f(j), f'(j)$ such that

$$[f_j][p_{f(j)i}] = [f'_j][p_{f'(j)i}],$$

one obtains the corresponding quotient category $\mathrm{HTop}^{\mathbb{N}}/\simeq$, i.e. the pro-category tow-HTop. The class of a morphism $\boldsymbol{f}$ is denoted by $[\boldsymbol{f}]$. Recall that every class $[\boldsymbol{f}]$ admits a *special* representative $\boldsymbol{f}'$, which means that for every pair $j \leq j'$,

$$[f'_j][p_{f'(j)f'(j')}] = [q_{jj'}][f'_{j'}].$$

The quotient (sub)category $\mathrm{HcANR}^{\mathbb{N}}/\simeq$ is the full subcategory tow-HcANR of tow-HTop (the terms X_i, Y_j, ... of its inverse sequences are compact ANR's). It represents the Mardešić–Segal shape category $\mathcal{S}h$ of compact metrizable spaces (see [6, Chap. I]). Namely,

$$\mathrm{Ob}(\mathcal{S}h) = \mathrm{Ob}(c\mathcal{M}), \qquad \mathrm{Sh}(X, Y) \approx \text{tow-HcANR}(\boldsymbol{X}, \boldsymbol{Y}),$$

where $\boldsymbol{X}$, $\boldsymbol{Y}$ are any compact ANR-sequences associated with X, Y respectively, i.e. $\boldsymbol{X} = H\underline{X}$ and $\boldsymbol{Y} = H\underline{Y}$, where $\lim \underline{X} = X$ and $\lim \underline{Y} = Y$, and H denotes the passage from an inverse sequence to the inverse sequence consisting of the same terms and of the homotopy classes of the given bonding mappings. For such a pair $\boldsymbol{X}, \boldsymbol{Y}$, the set tow-HcANR$(\boldsymbol{X}, \boldsymbol{Y})$ represents $\mathcal{S}h(X, Y)$.

It is a well-known fact ([5]; [1, Chap. IX]) that the Borsuk and Mardešić–Segal shape theories for compacta are equivalent. The following definitions and facts can be found in [8].

Definition 1. Let $\boldsymbol{f} = (f, [f_j]), \boldsymbol{f}' = (f', [f'_j]) : \boldsymbol{X} \to \boldsymbol{Y}$ be morphisms of inverse sequences, and let $s \in \mathbb{N}$. Then $\boldsymbol{f}$ is said to be *s-homotopic* to $\boldsymbol{f}'$, denoted by $\boldsymbol{f} \simeq_s \boldsymbol{f}'$, provided for every $j \in [1, s]_{\mathbb{N}}$ there exists an $i_j \geq f(j), f'(j)$ such that

$$[f_j][p_{f(j)i_j}] = [f'_j][p_{f'(j)i_j}].$$

Observe that $\boldsymbol{f} \simeq \boldsymbol{f}'$ if and only if $\boldsymbol{f} \simeq_s \boldsymbol{f}'$ for every $s \in \mathbb{N}$.

Lemma 1.

(i) *For every $s \in \mathbb{N}$, the relation $\simeq_s$ is an equivalence relation on each set* $\mathrm{HTop}^{\mathbb{N}}(\boldsymbol{X}, \boldsymbol{Y})$.

(ii) *For every pair $s \leq s'$, $\boldsymbol{f} \simeq_{s'} \boldsymbol{f}'$ implies $\boldsymbol{f} \simeq_s \boldsymbol{f}'$. Moreover, for every $s \in \mathbb{N}$, the relation $\simeq_s$ is natural from the right in the category* $\mathrm{HTop}^{\mathbb{N}}$, *i.e. for every $\boldsymbol{h} : \boldsymbol{W} \to \boldsymbol{X}$, $\boldsymbol{f} \simeq_s \boldsymbol{f}'$ implies $\boldsymbol{f}\boldsymbol{h} \simeq_s \boldsymbol{f}'\boldsymbol{h}$. On the other hand, if $\boldsymbol{g} : \boldsymbol{Y} \to \boldsymbol{Z}$, then $\boldsymbol{f} \simeq_s \boldsymbol{f}'$ implies $\boldsymbol{g}\boldsymbol{f} \simeq_t \boldsymbol{g}\boldsymbol{f}'$ whenever $g[[1, t]_{\mathbb{N}}] \subseteq [1, s]_{\mathbb{N}}$.*

Definition 2. Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be compact ANR-sequences. Then $\boldsymbol{X}$ is said to be *quasi-equivalent* to $\boldsymbol{Y}$, denoted by $\boldsymbol{X} \overset{q}{\simeq} \boldsymbol{Y}$, provided, for every $n \in \mathbb{N}$, there exist morphisms $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ and $\boldsymbol{g} : \boldsymbol{Y} \to \boldsymbol{X}$ such that $\boldsymbol{g}\boldsymbol{f} \simeq_n 1_{\boldsymbol{X}}$ and $\boldsymbol{f}\boldsymbol{g} \simeq_n 1_{\boldsymbol{Y}}$.

Lemma 2. *The relation* $\overset{q}{\simeq}$ *is isomorphism* (*i.e. shape*) *invariant in* tow-HcANR.

Proof. Let $\boldsymbol{X} \overset{q}{\simeq} \boldsymbol{Y}$ and let $\boldsymbol{X} \cong \boldsymbol{X}'$ in tow-HcANR. By definition, there exist sequences of maps $\boldsymbol{f}^n : \boldsymbol{X} \to \boldsymbol{Y}$ and $\boldsymbol{g}^n : \boldsymbol{Y} \to \boldsymbol{X}$ satisfying $\boldsymbol{g}^n\boldsymbol{f}^n \simeq_n 1_{\boldsymbol{X}}$ and $\boldsymbol{f}^n\boldsymbol{g}^n \simeq_n 1_{\boldsymbol{Y}}$, $n \in \mathbb{N}$. Further, there exist morphisms $\boldsymbol{u} : \boldsymbol{X} \to \boldsymbol{X}'$ and $\boldsymbol{v} : \boldsymbol{X}' \to \boldsymbol{X}$ such that $\boldsymbol{v}\boldsymbol{u} \simeq 1_{\boldsymbol{X}}$ and $\boldsymbol{u}\boldsymbol{v} \simeq 1_{\boldsymbol{X}'}$. Notice that

$$(\forall m \in \mathbb{N})(\exists s_m \geq m) \quad u[[1,m]_{\mathbb{N}}] \subseteq [1,s_m]_{\mathbb{N}}.$$

For each m, let

$$\boldsymbol{v}^m \equiv \boldsymbol{f}^{s_m}\boldsymbol{v} : \boldsymbol{X}' \to \boldsymbol{Y} \quad \text{and} \quad \boldsymbol{u}^m \equiv \boldsymbol{u}\boldsymbol{g}^{s_m} : \boldsymbol{Y} \to \boldsymbol{X}'.$$

Now, according to Lemma 1,

$$\begin{aligned}
&\boldsymbol{g}^{s_m}\boldsymbol{f}^{s_m} \simeq_{s_m} 1_{\boldsymbol{X}} \ \Rightarrow\ \boldsymbol{g}^{s_m}\boldsymbol{f}^{s_m}\boldsymbol{v} \simeq_{s_m} \boldsymbol{v} \ \Rightarrow\\
&\boldsymbol{u}\boldsymbol{g}^{s_m}\boldsymbol{f}^{s_m}\boldsymbol{v} \simeq_m \boldsymbol{u}\boldsymbol{v} \simeq 1_{\boldsymbol{X}'} \ \Rightarrow\ \boldsymbol{u}^m\boldsymbol{v}^m \simeq_m 1_{\boldsymbol{X}'};\\
&\boldsymbol{v}\boldsymbol{u} \simeq 1_{\boldsymbol{X}} \ \Rightarrow\ \boldsymbol{v}\boldsymbol{u}\boldsymbol{g}^{s_m} \simeq \boldsymbol{g}^{s_m} \ \Rightarrow\\
&\boldsymbol{f}^{s_m}\boldsymbol{v}\boldsymbol{u}\boldsymbol{g}^{s_m} \simeq \boldsymbol{f}^{s_m}\boldsymbol{g}^{s_m} \simeq_{s_m} 1_{\boldsymbol{Y}} \ \Rightarrow\ \boldsymbol{v}^m\boldsymbol{u}^m \simeq_m 1_{\boldsymbol{Y}}.
\end{aligned}$$

Thus, $\boldsymbol{X}' \overset{q}{\simeq} \boldsymbol{Y}$. In the same way one proves that $\boldsymbol{X} \overset{q}{\simeq} \boldsymbol{Y}$ and $\boldsymbol{Y} \cong \boldsymbol{Y}'$ imply $\boldsymbol{X} \overset{q}{\simeq} \boldsymbol{Y}'$. Therefore, $\overset{q}{\simeq}$ is an isomorphism invariant relation in the category tow-HcANR. ■

In order to compare Borsuk's quasi-equivalence on compacta to the new relation $\overset{q}{\simeq}$ on Ob(tow-HcANR), we shall prove the following lemma:

Lemma 3. *Let X and Y be compacta in the Hilbert cube Q, and let $\underline{X} = (X_i, p_{ii'})$ and $\underline{Y} = (Y_j, q_{jj'})$ be any associated inclusion compact ANR-sequences respectively. Let $\underline{g} = \{g_k, X, Y\}$ and $\underline{g}' = \{g'_k, X, Y\}$ be fundamental sequences* (*in Q*) *and let $\boldsymbol{f} = (f, [f_j])$ and $\boldsymbol{f}' = (f', [f'_j])$ be morphisms of* $\mathrm{HcANR}^{\mathbb{N}}(\boldsymbol{X}, \boldsymbol{Y})$, *where $\boldsymbol{X} = H\underline{X}$ and $\boldsymbol{Y} = H\underline{Y}$. If $\underline{g} = \{g_k, X, Y\}$ and $\boldsymbol{f} = (f, [f_j])$ as well as $\underline{g}' = \{g'_k, X, Y\}$ and $\boldsymbol{f}' = (f', [f'_j])$ are related, then*

- (i) *for every $n \in \mathbb{N}$ there exists a neighbourhood V of Y in Q such that $\underline{g} \underset{V}{\simeq} \underline{g}'$ implies $\boldsymbol{f} \simeq_n \boldsymbol{f}'$;*
- (ii) *for every neighbourhood V of Y in Q there exists an $n \in \mathbb{N}$ such that $\boldsymbol{f} \simeq_n \boldsymbol{f}'$ implies $\underline{g} \underset{V}{\simeq} \underline{g}'$.*

Proof. In this case "to be related" means (see [5] or [1, IX.4])

$$f_j = g_{f(j)}|X_{f(j)} : X_{f(j)} \to Y_j, \quad j \in \mathbb{N},$$
$$g_i|X_{f(j)} \simeq g_{f(j)}|X_{f(j)} \quad \text{in } Y_j,\ i \geq f(j),$$

and, similarly,

$$f'_j = g'_{f'(j)}|X_{f'(j)} : X_{f'(j)} \to Y_j, \quad j \in \mathbb{N},$$
$$g'_i|X_{f'(j)} \simeq g'_{f'(j)}|X_{f'(j)} \quad \text{in } Y_j,\ i \geq f'(j).$$

Moreover, we may assume that the index functions f and f' are increasing. Now, for (i), if an $n \in \mathbb{N}$ is given, choose $V = Y_n$. Then choose a $U_0 \supseteq X$ in Q coming from $\underline{g} \underset{V}{\simeq} \underline{g}'$, and an $i_0 \in \mathbb{N}$ such that $X_{i_0} \subseteq U_0$. Let $i_n = \max\{f(n), f'(n), i_0\}$. By choosing $i_j = i_n$ for every $j \in [1, n]_{\mathbb{N}}$, the relation $\boldsymbol{f} \simeq_n \boldsymbol{f}'$ is established. Further, for (ii), if a $V \supseteq Y$ in Q is given, choose the minimal $n \in \mathbb{N}$ such that $Y_n \subseteq V$. Let $i_0 \in \mathbb{N}$ be the maximum of all i_j coming from $\boldsymbol{f} \simeq_n \boldsymbol{f}'$. Then $\underline{g} \underset{V}{\simeq} \underline{g}'$ is realized via $U_0 = X_{i_0}$. ∎

THEOREM 1. *Let X and Y be compacta and let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be compact ANR-sequences associated with X and Y respectively. Then*

$$X \overset{q}{\simeq} Y \ \Leftrightarrow\ \boldsymbol{X} \overset{q}{\simeq} \boldsymbol{Y}.$$

Consequently, X and Y are quasi-equivalent, $X \overset{q}{\simeq} Y$, if and only if, for every $n \in \mathbb{N}$, there exist morphisms $\boldsymbol{f}^n : \boldsymbol{X} \to \boldsymbol{Y}$ and $\boldsymbol{g}^n : \boldsymbol{Y} \to \boldsymbol{X}$ such that $\boldsymbol{g}^n \boldsymbol{f}^n \simeq_n 1_{\boldsymbol{X}}$ and $\boldsymbol{f}^n \boldsymbol{g}^n \simeq_n 1_{\boldsymbol{Y}}$.

Proof. Recall that every compact metrizable space is, up to homeomorphism, the intersection of a decreasing sequence of compact ANR-neighbourhoods in the Hilbert cube. Further, recall (see [5] or [1, IX.4]) that every fundamental sequence $\underline{g} = \{g_k, X, Y\}$ admits a related morphism $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ and vice versa. According to Lemma 3, since Borsuk's quasi-equivalence is shape invariant, $X \overset{q}{\simeq} Y$ implies that there exist countable families $(\boldsymbol{f}^{(n,n')})$ and $(\boldsymbol{g}^{(n,n')})$, $(n, n') \in \mathbb{N}\times\mathbb{N}$, of morphisms $\boldsymbol{f}^{(n,n')} : \boldsymbol{X} \to \boldsymbol{Y}$ and $\boldsymbol{g}^{(n,n')} : \boldsymbol{Y} \to \boldsymbol{X}$ such that

$$\boldsymbol{g}^{(n,n')}\boldsymbol{f}^{(n,n')} \simeq_{n'} 1_{\boldsymbol{X}} \quad \text{and} \quad \boldsymbol{f}^{(n,n')}\boldsymbol{g}^{(n,n')} \simeq_n 1_{\boldsymbol{Y}}.$$

Clearly, by Lemma 1, both homotopies hold up to $\min\{n, n'\}$. Thus, by Definition 2 and Lemma 2, the necessity part follows. Conversely, let $\boldsymbol{X} \overset{q}{\simeq} \boldsymbol{Y}$, i.e. let there exist morphisms $\boldsymbol{f}^n : \boldsymbol{X} \to \boldsymbol{Y}$ and $\boldsymbol{g}^n : \boldsymbol{Y} \to \boldsymbol{X}$, $n \in \mathbb{N}$, such that

$$\boldsymbol{g}^n \boldsymbol{f}^n \simeq_n 1_{\boldsymbol{X}} \quad \text{and} \quad \boldsymbol{f}^n \boldsymbol{g}^n \simeq_n 1_{\boldsymbol{Y}}.$$

Given an ordered pair $(n, n') \in \mathbb{N} \times \mathbb{N}$, put

$$\boldsymbol{f}^{(n,n')} = \boldsymbol{f}^m \quad \text{and} \quad \boldsymbol{g}^{(n,n')} = \boldsymbol{g}^m, \qquad \text{where } m = \max\{n, n'\}.$$

Then $X \overset{q}{\simeq} Y$ according to Lemmata 3 and 1. ∎

REMARK 1. We may assume, without loss of generality, that all the morphisms realizing the relations $\boldsymbol{X} \overset{q}{\simeq} \boldsymbol{Y}$ are special with (strictly) increasing index functions. We may also assume that $n' \geq n$ implies $f^{n'} \geq f^n$, and similarly for all other index functions. Further, the conditions $\boldsymbol{g}^n \boldsymbol{f}^n \simeq_n 1_{\boldsymbol{X}}$ etc. may be relaxed to $\boldsymbol{g}^n \boldsymbol{f}^n \simeq_{s_n} 1_{\boldsymbol{X}}$ etc., where (s_n) is an unbounded sequence in $\mathbb{N} \cup \{0\}$.

To end this section we give a useful sufficient condition for a pair of compacta to be quasi-equivalent; it was formulated and proved earlier in the above mentioned manuscript.

LEMMA 4. *Let X, Y be a pair of compacta satisfying the following condition*: *For every $\varepsilon > 0$ there exist mappings $f : X \to Y$ and $g : Y \to X$ such that*

$$(\forall x \in X)\ d_X(gf(x), x) < \varepsilon \quad \text{and} \quad (\forall y \in Y)\ d_Y(fg(y), y) < \varepsilon.$$

Then X and Y are quasi-equivalent.

Proof. Without loss of generality, we may assume that X and Y lie in the Hilbert cube Q. Let U, V be any pair of neighbourhoods of X, Y in Q respectively. There exist compact ANR's U', V' such that $X \subseteq U' \subseteq \operatorname{Int} U$ and $Y \subseteq V' \subseteq \operatorname{Int} V$. Let $i : X \hookrightarrow U'$ and $j : Y \hookrightarrow V'$ be the inclusion mappings. It is well known that there exists an $\varepsilon > 0$ such that each pair of ε-near mappings of a metrizable space into U' (or into V') is homotopic. By assumption, there exist mappings $f : X \to Y$ and $g : Y \to X$ such that gf and 1_X as well as fg and 1_Y are ε-near. Consequently, $gfi, i : X \hookrightarrow U'$ as well as $fgj, j : Y \hookrightarrow V'$ are ε-near. Therefore, $gfi \simeq i$ and $fgj \simeq j$. This means $gf \simeq 1_X$ in $U' \subseteq \operatorname{Int} U$ and $fg \simeq 1_Y$ in $V' \subseteq \operatorname{Int} V$. Let $\underline{f} = \{f_k, X, Y\}$ and $\underline{g} = \{g_k, Y, X\}$ be fundamental sequences generated by f and g respectively. Now, apply the following fact (mentioned in the introduction):

Let A and B be compacta lying in Q, let W be an open neighbourhood of B in Q and let $\underline{h} = \{h_k, A, B\}$, $\underline{h}' = \{h'_k, A, B\}$ be fundamental sequences. Then $\underline{h} \underset{W}{\simeq} \underline{h}'$ if and only if $h_k|A \simeq h'_k|A$ in W for almost all $k \in \mathbb{N}$.

Consequently, $\underline{g}\underline{f} \underset{U}{\simeq} \underline{i}_X$ and $\underline{f}\underline{g} \underset{V}{\simeq} \underline{i}_Y$, where $\underline{i}_X$ and $\underline{i}_Y$ are the identity fundamental sequences for X in Q and Y in Q respectively. Therefore, $X \overset{q}{\simeq} Y$. ∎

3. The example. Let X be an infinite countable one-point union of pointed tori converging to the limit torus. Further, let Y be an infinite countable one-point union of pointed tori converging to the base point. Finally, let Z be the one-point union of X and a pointed circle. An explicit construction is given below.

For every $k \in \mathbb{N}$, let $A_k = S^1_k \cup \Sigma^1_k \subseteq \mathbb{R}^3$, where S^1_k and Σ^1_k are the following circles:

$$S^1_k = \left\{(\xi,\eta,0) \;\middle|\; \left(\xi - \frac{2k+3}{2k+2}\right)^2 + \eta^2 = \left(\frac{2k+3}{2k+2}\right)^2\right\},$$
$$\Sigma^1_k = \left\{(\xi,\eta,0) \;\middle|\; \left(\xi - \frac{2k+3}{2k+2}\right)^2 + \eta^2 = \left(\frac{6k+1}{8k+8}\right)^2\right\}.$$

Further, let

$$S^1_\infty = \{(\xi,\eta,0) \mid (\xi-1)^2 + \eta^2 = 1\} \subseteq \mathbb{R}^3,$$
$$\Sigma^1_\infty = \{(\xi,\eta,0) \mid (\xi-1)^2 + \eta^2 = 9/16\} \subseteq \mathbb{R}^3.$$

Notice that $A_k \cap A_{k'} = S^1_k \cap S^1_{k'} = \{(0,0,0)\}$ whenever $k \neq k' \in \mathbb{N} \cup \{\infty\}$, $\lim(S^1_k) = S^1_\infty$ and $\lim(\Sigma^1_k) = \Sigma^1_\infty$. For every $k \in \mathbb{N} \cup \{\infty\}$, let $T_k \subseteq \mathbb{R}^3$ be a torus, symmetric with the respect to the (ξ,η)-plane $\mathbb{R}^2$, such that $T_k \cap (\mathbb{R}^2 \times \{0\}) = A_k$. One can easily achieve that $T_k \cap T_{k'} = \{(0,0,0)\}$ whenever $k \neq k' \in \mathbb{N} \cup \{\infty\}$, and $\lim(T_k) = T_\infty$. Let

$$X = \bigcup_{k \in \mathbb{N} \cup \{\infty\}} T_k.$$

Similarly, for every $k \in \mathbb{N}$, let $A'_k = S'^1_k \cup \Sigma'^1_k \subseteq \mathbb{R}^3$, where

$$S'^1_k = \left\{(\xi,\eta,0) \;\middle|\; \left(\xi - \frac{1}{2^{3k-3}}\right)^2 + \eta^2 = \left(\frac{1}{2^{3k-3}}\right)^2\right\},$$
$$\Sigma'^1_k = \left\{(\xi,\eta,0) \;\middle|\; \left(\xi - \frac{1}{2^{3k-3}}\right)^2 + \eta^2 = \left(\frac{1}{2^{3k-2}}\right)^2\right\}.$$

Notice that $A'_k \cap A'_{k'} = S'^1_k \cap S'^1_{k'} = \{(0,0,0)\}$ whenever $k \neq k' \in \mathbb{N}$, and $\lim(S'^1_k) = \lim(\Sigma'^1_k) = \{(0,0,0)\}$. For every $k \in \mathbb{N}$, let $T'_k \subseteq \mathbb{R}^3$ be a torus, symmetric with respect to the (ξ,η)-plane $\mathbb{R}^2$, such that $T'_k \cap (\mathbb{R}^2 \times \{0\}) = A'_k$, $T'_k \cap T'_{k'} = \{(0,0,0)\}$ whenever $k \neq k'$, and $\lim(T'_k) = \{(0,0,0)\}$. Let

$$Y = \bigcup_{k \in \mathbb{N}} T'_k.$$

Finally, let

$$Z = X \cup S^1, \quad \text{where} \quad S^1 = \{(\xi,\eta,0) \mid (\xi+1)^2 + \eta^2 = 1\}.$$

Clearly, the subspaces X, Y and Z of $\mathbb{R}^3$ are compact and path connected.

THEOREM 2. *Borsuk's quasi-equivalence relation is not transitive.*

Proof. Consider the continua X, Y and Z defined above. It suffices to prove that $X \overset{q}{\simeq} Y$ and $Y \overset{q}{\simeq} Z$, and that X is not quasi-equivalent to Z. Fix $\varepsilon > 0$. Then there exists an $n'_\varepsilon \in \mathbb{N}$ such that $9/(8(n'_\varepsilon + 1)) < \varepsilon$. Given any

$n' \geq \max\{8, n'_\varepsilon\}$, set $X_{n'} = \bigcup_{k \leq n'} T_k \subseteq X$, which is a closed subspace. Let $r_{n'} : X \to X_{n'}$ be defined by

$$r_{n'}(x) = \begin{cases} x, & x \in X_{n'}, \\ \varrho_{n'}(x), & x \in X \setminus X_{n'}, \end{cases}$$

where $\varrho_{n'} : \bigcup_{k>n'} T_k \to T_{n'}$ is the radial mapping ("blowing up") from the circle passing through the middle of the bounded component of $\mathbb{R}^3 \setminus T_\infty$. Clearly, for every $k \in \{n'+1, n'+2, \ldots\} \cup \{\infty\}$, $\varrho_{n'}|T_k : T_k \to T_{n'}$ is a homeomorphism, $\varrho_{n'}[S_k] = S_{n'}$, $\varrho_{n'}[\Sigma_k] = \Sigma_{n'}$ and $\varrho_{n'}(0,0,0) = (0,0,0)$. Observe that $r_{n'}$ is a retraction. Further, the distance between $r_{n'}(x)$ and x reaches its maximum for some $x = (\xi, \eta, \zeta) \in \{(1/4, 0, 0), (7/4, 0, 0), (2, 0, 0)\} \subseteq A_0$. Since

$$\varrho_{n'}\left(\frac{1}{4}, 0, 0\right) = \frac{9}{8(n'+1)}, \quad \varrho_{n'}\left(\frac{7}{4}, 0, 0\right) = \frac{1}{8(n'+1)}, \quad \varrho_{n'}(2,0,0) = \frac{1}{n'},$$

and $n' \geq 8$, the maximal distance is $9/(8(n'+1))$. Thus, for every $x \in X$,

$$d(r_{n'}(x), x) \leq \frac{9}{8(n'+1)} \leq \frac{9}{8(n'_\varepsilon + 1)} < \varepsilon$$

whenever $n' \geq \max\{8, n'_\varepsilon\}$.

Similarly, there exists an $n''_\varepsilon \in \mathbb{N}$ such that $T'_k \subseteq B((0,0,0), \varepsilon)$ for every $k \geq n''_\varepsilon$, where $B((0,0,0),\varepsilon)$ is the ε-ball at the origin in $\mathbb{R}^3$. Given any $n'' \geq n''_\varepsilon$, set $Y_{n''} = \bigcup_{k \leq n''} T'_k \subseteq Y$, which is a closed subspace. Let $s_{n''} : Y \to Y_{n''}$ be defined by

$$s_{n''}(y) = \begin{cases} y, & y \in Y_{n''}, \\ (0,0,0), & y \in Y \setminus Y_{n''}. \end{cases}$$

It is obvious that $s_{n''}$ is a retraction and that

$$d(s_{n''}(y), y) < \varepsilon \quad \text{holds for every } y \in Y.$$

Consider now an $n \geq \max\{8, n'_\varepsilon, n''_\varepsilon\}$ and observe that the subspaces X_n and Y_n are homeomorphic. Let $h : X_n \to Y_n$ be a homeomorphism, and let $r : X \to X_n$ and $s : Y \to Y_n$ be defined as above, i.e. $r = r_n$ and $s = s_n$. Put

$$f = jhr : X \to Y \quad \text{and} \quad g = ih^{-1}s : Y \to X,$$

where $i : X_n \hookrightarrow X$ and $j : Y_n \hookrightarrow Y$ are the inclusion mappings. Let $x \in X$. If $x \in X_n$, then $r(x) = x$ and $hr(x) = h(x) \in Y_n$, and thus $jhr(x) = h(x)$ and $sjhr(x) = sh(x) = h(x)$. Therefore,

$$gf(x) = ih^{-1}sjhr(x) = ih^{-1}h(x) = x = r(x).$$

If $x \in X \setminus X_n$, then $r(x) = \varrho(x)$ and $hr(x) = h\varrho(x) \in Y_n$, and thus $jhr(x) = h\varrho(x)$ and $sjhr(x) = sh\varrho(x) = h\varrho(x)$. Therefore,

$$gf(x) = ih^{-1}sjhr(x) = ih^{-1}h\varrho(x) = i\varrho(x) = \varrho(x) = r(x).$$

Consequently,

$$d(gf(x),x) = d(r(x),x) < \varepsilon \quad \text{for every } x \in X.$$

In a similar way one can verify that

$$d(fg(y),y) = d(s(y),y) < \varepsilon \quad \text{for every } y \in Y.$$

According to Lemma 4, X is quasi-equivalent to Y.

Let us now prove that Y is quasi-equivalent to Z. Fix $\varepsilon > 0$. Choose an $n \geq \max\{8, n'_\varepsilon, n''_\varepsilon\}$, where n'_ε and n''_ε are as in the first part of the proof. Observe that $Y_n \cup S'^1_{n+1} \subseteq Y$ is a closed subspace. Define $s' : Y \to Y_n \cup S'^1_{n+1}$ by putting

$$s'(y) = \begin{cases} y, & y \in Y_n, \\ \varrho'(y), & y \in T'_{n+1}, \\ (0,0,0), & y \in Y \setminus Y'_{n+1}, \end{cases}$$

where $\varrho' : T'_{n+1} \to S'^1_{n+1}$ is a retraction of the torus onto the circle. It is clear that s' is a retraction satisfying

$$d(s'(y),y) < \varepsilon \quad \text{for every } y \in Y.$$

Further, $X_n \cup S^1 \subseteq Z = X \cup S^1$ is a closed subspace. Define $r' : Z \to X_n \cup S^1$ by putting

$$r'(z) = \begin{cases} r(z), & z \in X, \\ z, & z \in S^1, \end{cases}$$

where $r : X \to X_n$ is the retraction defined in the first part of the proof. Consequently, r' is a retraction satisfying

$$d(r'(z),z) < \varepsilon \quad \text{for every } z \in Z.$$

Observe that Y_n, S^1_{n+1} and $Y_n \cup S^1_{n+1}$ are homeomorphic to X_n, S^1 and $X_n \cup S^1$ respectively, and that $Y_n \cap S^1_{n+1} = \{(0,0,0)\} = X_n \cap S^1$. Let

$$h' : Y_n \cup S^1_{n+1} \to X_n \cup S^1$$

be a homeomorphism (also on each summand, and keeping $(0,0,0)$ fixed), and let

$$j' : Y_n \cup S^1_{n+1} \hookrightarrow Y \quad \text{and} \quad i' : X_n \cup S^1 \hookrightarrow Z$$

be the inclusion mappings. Put

$$f' = i'h's' : Y \to Z \quad \text{and} \quad g' = j'h'^{-1}r' : Z \to Y.$$

Let $y \in Y$. If $y \in Y_n$, then $s'(y) = y$ and $h's'(y) = h'(y) \in X_n$, and thus $i'h's'(y) = h'(y)$ and $r'i'h's'(y) = r'h'(y) = h'(y)$. Therefore,

$$g'f'(y) = j'h'^{-1}r'i'h's'(y) = j'h'^{-1}h'(y) = j'(y) = y = s'(y).$$

If $y \in T'_{n+1}$, then $s'(y) = \varrho'(y) \in S^1_{n+1}$ and $h's'(y) = h'\varrho'(y) \in S^1$, and thus $i'h's'(y) = h'\varrho'(y) \in S^1$ and $r'i'h's'(y) = h'\varrho'(y)$. Therefore,

$$g'f'(y) = j'h'^{-1}r'i'h's'(y) = j'h'^{-1}h'\varrho'(y) = j'\varrho'(y) = \varrho'(y) = s'(y).$$

If $y \in Y \setminus Y_{n+1}$, then $s'(y) = (0,0,0)$ and $h's'(y) = (0,0,0)$, and thus $i'h's'(y) = (0,0,0)$ and $r'i'h's'(y) = (0,0,0)$. Therefore,

$$g'f'(y) = j'h'^{-1}r'i'h's'(y) = j'h'^{-1}(0,0,0) = j(0,0,0) = (0,0,0) = s'(y).$$

Consequently,

$$d(g'f'(y), y) = d(s'(y), y) < \varepsilon \quad \text{for every } y \in Y.$$

In a similar way one can verify that

$$d(f'g'(z), z) = d(r'(z), z) < \varepsilon \quad \text{for every } z \in Z.$$

By Lemma 4, Y is quasi-equivalent to Z.

It remains to prove that X is not quasi-equivalent to Z. For every $i \in \mathbb{N}$ and every $j \in \mathbb{N}$, let (T_i^j, t_i) be a copy of a pointed torus $(\mathbb{T}, *)$. For every $i \in \mathbb{N}$, let

$$(X_i', x_i') = (T_i^1, t_i) \vee \cdots \vee (T_i^i, t_i).$$

We may assume that $(X_{i+1}', x_{i+1}') = (X_i', x_i') \vee (T_{i+1}^{i+1}, t_{i+1})$, $i \in \mathbb{N}$. Let

$$p_{i,i+1} : X_{i+1}' \to X_i', \quad i \in \mathbb{N},$$

be defined by requiring that the restrictions

$$p_{i,i+1}|X_i' : X_i' \to X_i' \quad \text{and} \quad p_{i,i+1}|T_{i+1}^{i+1} : T_{i+1}^{i+1} \to T_i^i \subseteq X_i'$$

be the identities. Notice that $p_{i,i+1} : (X_{i+1}', x_{i+1}') \to (X_i', x_i')$ is a base point preserving map.

Consider the pointed (compact ANR) inverse sequence $(\underline{X}, \underline{*}) = ((X_i', x_i'), p_{i,i+1})$ and its limit $\underline{p}_* = (p_i) : (X', *) = \lim(\underline{X}, \underline{*}) \to (\underline{X}, \underline{*})$. Further, for every $j \in \mathbb{N}$, let

$$(Z_j', z_j') = (X_j', x_j') \vee (Z_j, z_j),$$

where (Z_j, z_j) is a copy of a pointed circle $(\mathbb{S}^1, *)$. Let

$$q_{j,j+1} : (Z_{j+1}', z_{j+1}') \to (Z_j', z_j'), \quad j \in \mathbb{N},$$

be $p_{j,j+1}$ on X_{j+1}' and the identity on the copy of $\mathbb{S}^1$. Consider the pointed (compact ANR) inverse sequence $(\underline{Z}, \underline{*}) = ((Z_j', z_j'), q_{j,j+1})$ and its limit $\underline{q}_* = (q_j) : (Z', *) = \lim(\underline{Z}, \underline{*}) \to (\underline{Z}, \underline{*})$.

Claim. *$(X', *)$ is homeomorphic to $(X, (0,0,0))$, and $(Z', *)$ is homeomorphic to $(Z, (0,0,0))$.*

By construction,

$$Z = X \cup S^1,\ (Z, (0,0,0)) \approx (X, (0,0,0)) \vee (\mathbb{S}^1, *),\ (Z', *) \approx (X', *) \vee (\mathbb{S}^1, *).$$

Further, all the mappings preserve base points. Thus, it suffices to prove that $X' \approx X$. Let $\underline{p}' = (p_i') : X \to \underline{X}$ be defined by

$$p_i' = h_i r_i : X \to X_i', \quad i \in \mathbb{N},$$

where h_i is determined by the obvious homeomorphisms on the corresponding tori. Then $\underline{p}'$ distinguishes points and every p'_i is surjective. By applying Theorem 6 of [6, I.5.2], we infer that $\underline{p}' : X \to \underline{X}$ is the limit. Therefore, X and X' are homeomorphic.

Set

$$H\underline{X} = (X'_i, [p_{i,i+1}]) \equiv \boldsymbol{X} \quad \text{and} \quad H\underline{Z} = (Z'_j, [q_{j,j+1}]) \equiv \boldsymbol{Z}.$$

According to Theorem 1, it remains to prove that $\boldsymbol{X}$ is not quasi-equivalent to $\boldsymbol{Z}$. Suppose, on the contrary, that $\boldsymbol{X} \overset{q}{\simeq} \boldsymbol{Z}$. Then, by Definition 2 and Remark 1, for every $n \in \mathbb{N}$, there exist special morphisms $\boldsymbol{f}^n : \boldsymbol{X} \to \boldsymbol{Z}$ and $\boldsymbol{g}^n : \boldsymbol{Z} \to \boldsymbol{X}$ such that $\boldsymbol{g}^n\boldsymbol{f}^n \simeq_n 1_{\boldsymbol{X}}$ and $\boldsymbol{f}^n\boldsymbol{g}^n \simeq_n 1_{\boldsymbol{Z}}$. Let $n = 1$, and write $\boldsymbol{f}^1 \equiv \boldsymbol{f} = (f, [f_j])$ and $\boldsymbol{g}^1 \equiv \boldsymbol{g} = (g, [g_i])$. Since all X'_i and Z'_j are ANR-continua, one may assume that all the mappings f_j and g_i preserve the base points. Since $\boldsymbol{g}\boldsymbol{f} \simeq_1 1_{\boldsymbol{X}}$, the diagram

$$(\star)\qquad \begin{array}{ccccccccc} X'_1 & \longleftarrow & \cdots & \longleftarrow & X'_{fg(1)} & \longleftarrow & \cdots & \longleftarrow & X'_{f(j)} \\ & \nwarrow^{g_1} & & \swarrow_{f_{g(1)}} & & & & \swarrow_{f_j} & \\ & & Z'_{g(1)} & \longleftarrow & \cdots & \longleftarrow & Z'_j & & \end{array}$$

commutes up to homotopy. Let us apply the fundamental group functor π_1 to the left triangle of $(\star)$ (the choice of base points is irrelevant):

$$\begin{array}{ccc} \pi_1(T_1^1) & \longleftarrow & \pi_1(T^1_{fg(1)}) * \cdots * \pi_1(T^{fg(1)}_{fg(1)}) \\ \uparrow{\scriptstyle g_{1\#}} & \swarrow{\scriptstyle f_{g(1)\#}} & \\ \pi_1(T^1_{g(1)}) * \cdots * \pi_1(T^{g(1)}_{g(1)}) * \pi_1(S^1) & & \end{array}$$

Recall that the fundamental group of a finite wedge is the corresponding free product (by van Kampen's theorem, [3, Theorem 3.1, p. 122]), and that the fundamental groups of a circle and of a torus are $\mathbb{Z}$ and $\mathbb{Z} \times \mathbb{Z} \equiv \mathbb{Z}^2$ respectively. Observe that $f_{g(1)\#}|\pi_1(T^i_{fg(1)})$ is a monomorphism of $\mathbb{Z}^2$ into $\mathbb{Z}^2 * \cdots * \mathbb{Z}^2 * \mathbb{Z}$, for every $i = 1, \ldots, fg(1)$. Thus,

$$\mathbb{Z}^2 \cong (f_{g(1)\#}|\pi_1(T^i_{fg(1)}))(\mathbb{Z}^2) \le \mathbb{Z}^2 * \cdots * \mathbb{Z}^2 * \mathbb{Z}.$$

From the Kurosh subgroup theorem ([7, Theorem 1.10, p. 178]) it follows that if $H \le G = G_1 * \cdots * G_n$, then $H \cong F * H_1^{\sigma_1} * \cdots * H_n^{\sigma_n}$, where every H_i is a subgroup of some G_j, every $\sigma_i \in G$ and F is a free group.

Recall also that $\mathbb{Z}^2$ is not decomposable into a free product (see [7, Proposition 15.14, p. 107]). Since $\mathbb{Z}^2$ is not a free group, the Kurosh subgroup theorem implies that $\mathbb{Z}^2 \cong (f_{g(1)\#}|\pi_1(T^i_{fg(1)}))(\mathbb{Z}^2) \cong H_i^{\sigma_i}$, $H_i \le \pi_1(T^j_{g(1)}) \cong \mathbb{Z}^2$, for some $j \in \{1, \ldots, g(1)\}$, and $\sigma_i \in \pi_1(T^1_{g(1)}) * \cdots * \pi_1(T^{g(1)}_{g(1)}) * \pi_1(S^1) \cong$

$\mathbb{Z}^2 * \cdots * \mathbb{Z}^2 * \mathbb{Z}$. Let $a \in \pi_1(T^i_{fg(1)})$, $i \in \{1, \ldots, fg(1)\}$. Then

$$f_{g(1)\#}(a) = \sigma b \sigma^{-1}, \quad \sigma \in \mathbb{Z}^2 * \cdots * \mathbb{Z}^2 * \mathbb{Z},\ b \in \pi_1(T^j_{g(1)}),$$

for some $j \in \{1, \ldots, g(1)\}$. Since $f_{g(1)\#}|\pi_1(T^i_{fg(1)})$ is a monomorphism, its image in $\mathbb{Z}^2 * \cdots * \mathbb{Z}^2 * \mathbb{Z}$ must be isomorphic to $\pi_1(T^j_{g(1)}) \cong \mathbb{Z}^2$ for some $j \in \{1, \ldots, g(1)\}$. Consequently, if $a_1 \cdots a_m \in \pi_1(T^1_{fg(1)}) * \cdots * \pi_1(T^{fg(1)}_{fg(1)})$, where $a_k \in \pi_1(T^{i_k}_{fg(1)})$, then

$$f_{g(1)\#}(a_1 \cdots a_m) = \sigma_1 b_1 \sigma_1^{-1} \cdots \sigma_m b_m \sigma_m^{-1},$$

for some $\sigma_k \in \mathbb{Z}^2 * \cdots * \mathbb{Z}^2 * \mathbb{Z}$ and $b_k \in \pi_1(T^{j_k}_{g(1)})$, $k = 1, \ldots, m$, $j_k \in \{1, \ldots, g(1)\}$.

Further, the right rectangle of $(\star)$ yields the commutative diagram

$$\begin{array}{ccc}
\pi_1(T^1_{fg(1)}) * \cdots * \pi_1(T^{fg(1)}_{fg(1)}) & \longleftarrow & \pi_1(T^1_{f(j)}) * \cdots * \pi_1(T^{f(j)}_{f(j)}) \\
\big\downarrow f_{g(1)\#} & & \big\downarrow f_{j\#} \\
\pi_1(T^1_{g(1)}) * \cdots * \pi_1(T^{g(1)}_{g(1)}) * \pi_1(S^1) & \longleftarrow & \pi_1(T^1_j) * \cdots * \pi_1(T^j_j) * \pi_1(S^1)
\end{array}$$

which means $f_{g(1)\#} p_{fg(1)f(j)\#} = q_{g(1)j\#} f_{j\#}$. Since $p_{ii'}$ and $q_{jj'}$ are defined in a special way (by the identity mappings on the corresponding copies), one readily sees that, for every $j \geq g(1)$, the restriction $f_{j\#}|\pi_1(T^i_{f(j)})$ is also a monomorphism. Therefore, by following the same arguments, one can find that $f_{j\#}$ acts via a formula analogous to that for $f_{g(1)\#}$.

Consider now the relation $\boldsymbol{fg} \simeq_1 1_{\boldsymbol{Z}}$ inducing the retraction

$$r : Z'_1 = T^1_1 \vee S^1 \to S^1, \quad r[T^1_1] = \{*\},$$

i.e. the following diagram:

$$\begin{array}{ccccc}
 & & X'_{f(1)} & \longleftarrow \cdots \longleftarrow & X'_{fgf(1)} \\
 & & \big\downarrow f_1 & \nwarrow g_{f(1)} & \big\downarrow f_{gf(1)} \\
S^1 & \xleftarrow{\ r\ } & Z'_1 & \longleftarrow & Z'_{gf(1)}
\end{array}$$

(Caution: The right triangle might not homotopy commute, though the rectangle and the left triangle must homotopy commute!) Applying π_1 to the left triangle and to the rectangle yields the commutative diagrams

$$\begin{array}{ccccc}
 & & & & \pi_1(T^1_{f(1)}) * \cdots * \pi_1(T^{f(1)}_{f(1)}) \\
 & & \swarrow f_{1\#} & & \big\uparrow g_{f(1)\#} \\
\pi_1(S^1) & \xleftarrow{r_\#} & \pi_1(T^1_1) * \pi_1(S^1) & \longleftarrow & \pi_1(T^1_{gf(1)}) * \cdots * \pi_1(T^{gf(1)}_{gf(1)}) * \pi_1(S^1)
\end{array}$$

$$\begin{array}{ccccc} & & \pi_1(T^1_{f(1)}) * \cdots * \pi_1(T^{f(1)}_{f(1)}) & \longleftarrow & \pi_1(T^1_{fgf(1)}) * \cdots * \pi_1(T^{fgf(1)}_{fgf(1)}) \\ & & \downarrow{\scriptstyle f_{1\#}} & & \downarrow{\scriptstyle f_{gf(1)\#}} \\ \pi_1(S^1) & \xleftarrow{r_\#} & \pi_1(T^1_1) * \pi_1(S^1) & \longleftarrow & \pi_1(T^1_{gf(1)}) * \cdots * \pi_1(T^{gf(1)}_{gf(1)}) * \pi_1(S^1) \end{array}$$

Now, the composition

$$r_\# f_{1\#} g_{f(1)\#} : \mathbb{Z}^2 * \cdots * \mathbb{Z}^2 * \mathbb{Z} \to \mathbb{Z}$$

is the trivial homomorphism because $r_\# f_{1\#}$ is trivial. Namely, the restrictions of the bonding homomorphisms are the identities on the corresponding copies, $f_{1\#} p_{f(1) fgf(1)\#} = q_{1gf(1)\#} f_{gf(1)\#}$, $gf(1) \geq g(1)$ and we have already proved how $f_{gf(1)\#}$ acts. Thus, for every $a \in \pi_1(T^i_{f(1)})$, $i \in \{1, \ldots, f(1)\}$, we have $f_{1\#}(a) = \sigma b \sigma^{-1}$ for some $b \in \pi_1(T^1_1)$, $\sigma \in \pi_1(T^1_1) * \pi_1(S^1)$. Therefore, $r_\# f_{1\#}(a) = r_\#(\sigma b \sigma^{-1}) = r_\#(\sigma) r_\#(\sigma^{-1})$, and hence $r_\# f_{1\#}$ must be trivial. On the other hand, by the definitions of the relevant mappings, the composition

$$r_\# q_{1gf(1)\#} : \mathbb{Z}^2 * \cdots * \mathbb{Z}^2 * \mathbb{Z} \to \mathbb{Z}$$

preserves the free factor $\pi_1(S^1) \cong \mathbb{Z}$, so it is not trivial. Therefore, the two displayed compositions cannot be equal. ■

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