

Polyhedra with virtually polycyclic fundamental groups have finite depth

by

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Abstract. The notions of capacity and depth of compacta were introduced by K. Borsuk in the seventies together with some open questions. In a previous paper, in connection with one of them, we proved that there exist polyhedra with polycyclic fundamental groups and infinite capacity, i.e. dominating infinitely many different homotopy types (or equivalently, shapes). In this paper we show that every polyhedron with virtually polycyclic fundamental group has finite depth, i.e., there is a bound on the lengths of all descending sequences of different homotopy types (or shapes) dominated by this polyhedron. As a corollary, we deduce that for two ANR's with virtually polycyclic fundamental groups the so-called index of h-proximity, introduced by K. Borsuk in his monograph on retract theory, is finite. We also obtain an answer to some question of K. Borsuk concerning homotopy (or shape) decompositions of polyhedra into simple constituents.

1. Introduction. In 1979, at the International Topological Conference in Moscow, K. Borsuk introduced the *capacity* and *depth* in the shape category of compacta, together with some relevant questions (see [B1]). (The basic notions and results of shape theory can be found in [B4], [DS], [MS].)

Recall that a *domination* in a given category \mathcal{C} is a morphism $f : X \rightarrow Y$, $X, Y \in \text{Ob } \mathcal{C}$, for which there exists a morphism $g : Y \rightarrow X$ of \mathcal{C} such that $fg = \text{id}_Y$. Then we say that Y is dominated by X , and we write $Y \leq X$ or $X \geq Y$; moreover, $X < Y$ will denote that $X \leq Y$ holds but $Y \leq X$ fails.

In the following, \mathcal{C} is the homotopy category of CW-complexes and homotopy classes of cellular maps between them or the shape category of compacta (pointed or unpointed).

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Following K. Borsuk (cf. [B1]), define the *capacity* $C(A)$ of an $A \in \text{Ob}\mathcal{C}$ as the cardinality of the class of isomorphism classes of all the $X \in \text{Ob}\mathcal{C}$ such that $X \leq A$.

A system $X_1 < \cdots < X_k \leq A$, where $X_i \in \text{Ob}\mathcal{C}$ for $i = 1, \dots, k$, is called a *chain of length k* for $A \in \text{Ob}\mathcal{C}$. The *depth* $D(A)$ of A is the least upper bound of the lengths of all chains for A . If this upper bound is infinite, we write $D(A) = \aleph_0$ (cf. [B1]).

In other words, $A \in \text{Ob}\mathcal{C}$ has finite depth if and only if there exists an integer k such that each sequence $\cdots \leq X_i \leq \cdots \leq X_1 \leq A$ contains at most k different objects (up to isomorphism).

It is clear that $D(A) \leq C(A)$ for each $A \in \text{Ob}\mathcal{C}$.

In the following, we will assume (without loss of generality) that every polyhedron, ANR, and CW-complex considered is connected.

Note that for a polyhedron, the capacity and depth in the shape category of compacta are the same as in the homotopy category of CW-complexes and homotopy classes of cellular maps between them. Indeed, there is a 1-1 functorial correspondence between the shapes of compacta shape dominated by a given polyhedron and the homotopy types of CW-complexes homotopy dominated by it (see [DS, Theorem 2.2.6], [HaHe1] or [HaHe2], [EG]). Similarly, the capacity (and depth) of a given polyhedron is the same in both pointed and unpointed cases. This follows from [D, Theorem 5.1].

In 1968 at the Topological Conference in Herceg-Novi K. Borsuk stated the question: *Is the capacity of each polyhedron finite?* (see, for example, [B1, Question 4]).

By the classical results of shape theory, this was known to be the case for 1-dimensional polyhedra (see [Tr] and [B4, Theorem 7.1, p. 221]). In general the answer to Borsuk's question is negative: there exist polyhedra (even of dimension 2) homotopy dominating infinitely many different homotopy types, and equivalently, shapes ([K6], see also [K7]). Furthermore, there exist polyhedra with polycyclic (and even nilpotent) fundamental groups dominating infinitely many different homotopy types or shapes ([K1]).

On the other hand, all the simply-connected polyhedra, polyhedra with finite fundamental groups and nilpotent polyhedra have finite capacity, hence also finite depth ([K5], [K4], [K3]).

The main result of this paper is that although there exist polyhedra with polycyclic fundamental groups with infinite capacity, every polyhedron with virtually polycyclic fundamental group has finite depth.

This result, together with the previous ones from [K1], will enable us to give a positive answer to a question of Borsuk from [B2]: *Does there exist a polyhedron with infinitely many simple constituents?* Recall that X and Y are said to be *constituents* of the wedge (one-point union) $X \vee Y$ (cf. [B2]).

X is called *simple* if each of its constituents is either trivial or coincides with X in \mathcal{C} .

In his monograph [B3] on retract theory, K. Borsuk introduced the notions of h -neighbors and the index of h -proximity of two ANR's.

The homotopy type of X is a *left h -neighbor* of the homotopy type of Y if $X < Y$ and for every Z such that $X \leq Z \leq Y$, $X \simeq Z$ or $Y \simeq Z$ (see [B3, p. 349]). In the same manner we define a *right h -neighbor* of the homotopy type of Y .

We say that the *index of h -proximity* of two ANR's P and Q , $h(P, Q)$, is equal to n if there exists a sequence $P = X_0, X_1, \dots, X_n = Q$ such that X_i and X_{i+1} are h -neighbors. If no such finite sequence exists then we say that $h(P, Q)$ is infinite (see [B3, Ch. IX, 12]).

Concerning these notions, Borsuk asked (1967, [B3, Ch. IX, Problem (12.8)]): *Do there exist two ANR's, P and Q , for which $h(P, Q)$ is infinite?*

This question remains open. In the second part of this paper we will show, as another corollary to the main result, that the index of h -proximity of two ANR's with virtually polycyclic fundamental groups is finite.

2. Algebraic preliminaries

DEFINITION. Recall that a group G is *polycyclic* if it has a finite series of subgroups $G = G_0 \triangleright G_1 \triangleright \dots \triangleright G_l = 1$ for which each factor G_{i-1}/G_i (where $i = 1, \dots, l$) is finite cyclic or infinite cyclic (see, for example, [S, p. 2]).

We say that G is a *poly- \mathbb{Z} -group* if it has such a series with factors \mathbb{Z} .

DEFINITION. A group G is called *virtually polycyclic* if there exists a polycyclic group $H \triangleleft G$ such that G/H is finite.

DEFINITION. The number $h(G)$ of infinite cyclic factors in a series $G = G_0 \triangleright G_1 \triangleright \dots \triangleright G_l = 1$ with cyclic or finite factors is an invariant of a virtually polycyclic group G (independent of the series), known as the *Hirsch number* of G (see [S, p. 16]).

DEFINITION. Suppose that \mathcal{P} is a property of groups. A group G is called *poly- \mathcal{P}* if it has a finite series of subgroups $G = G_0 \triangleright G_1 \triangleright \dots \triangleright G_l = 1$ for which each factor G_{i-1}/G_i (where $i = 1, \dots, l$) has the property \mathcal{P} ([S, p. 2]).

DEFINITION. Let R be a ring. A right R -module M is said to be *noetherian* if every R -submodule of M is finitely generated, equivalently, M satisfies the *ascending chain condition*, i.e., every ascending sequence $M_1 \subsetneq M_2 \subsetneq \dots$ of submodules of M is finite (cf. [Pa, p. 419], or [La]).

We call a ring R *noetherian* if it is a noetherian module as a module over itself.

DEFINITION. Let R be a ring and M be a right R -module. The *Krull dimension* of M , denoted by $\text{Kdim } M$, is defined by transfinite recursion as

follows: if $M = 0$, then $\text{Kdim } M = -1$; if α is an ordinal and $\text{Kdim } M \not\leq \alpha$, then $\text{Kdim } M = \alpha$ provided there is no infinite descending chain $M = M_0 \supset M_1 \supset \dots$ of submodules M_i such that $\text{Kdim } M_{i-1}/M_i \not\leq \alpha$ for $i = 1, 2, \dots$. If there is no ordinal α such that $\text{Kdim } M = \alpha$, we say that M has no Krull dimension (see [GR, p. 5] or [Br]).

DEFINITION. If a module contains no infinite direct sum of nonzero submodules, then there is a fixed bound on the number of summands in finite sums it can contain; and the module is said to have *finite uniform dimension* (see [GR, p. 7]).

DEFINITION. Recall that in the category of groups or modules with homomorphisms, a homomorphism $f : A \rightarrow B$ is said to be an *r-homomorphism* if there exists a homomorphism $g : B \rightarrow A$ such that $fg = \text{id}_B$ (see, for example, [B3, Ch. 2]).

3. Polyhedra with virtually polycyclic fundamental groups have finite depth. To prove that every polyhedron with virtually polycyclic fundamental group has finite depth, we will use properties of finitely generated noetherian modules over the integral group rings $\mathbb{Z}G$, where G is virtually polycyclic, and the concept of Krull dimension of modules. We begin with some algebraic lemmas.

LEMMA 1 (cf. Lemma 1 from [K1]). *Let G be a virtually polycyclic group. There exists an integer k_G such that any sequence $G = G_0 \supseteq G_1 \supseteq \dots \supseteq G_l$ of subgroups of G with r -homomorphisms $r_i : G_{i-1} \rightarrow G_i$ for $i = 1, \dots, l$ contains no more than k_G distinct subgroups.*

Proof. Note that every subgroup of a virtually polycyclic group is also virtually polycyclic (cf. [S, Ex. 3, p. 2]). If $h(G)$ is the Hirsch number of the virtually polycyclic group G , $H \subseteq G$ and $N \triangleleft G$ is such that $H \cong G/N$, then $h(G) = h(N) + h(H)$ (see, for example, [S, Ex. 8, p. 16]).

Let N_i be the kernel of $r_i : G_{i-1} \rightarrow G_i$. Then N_i is a normal subgroup of G_{i-1} which satisfies $G_{i-1} = G_i N_i$ and $G_i \cap N_i = 1$ for $i = 1, \dots, l$. It follows that $h(G_{i-1}) = h(N_i) + h(G_i)$ for $i = 1, \dots, l$. Therefore $h(G_{i-1}) = h(G_i)$, hence $h(N_i) = 0$, for all i except at most $h(G)$.

Assume, without loss of generality, that $h(N_i) = 0$ for all $i \geq 2$. It follows that the N_i are all finite. Indeed, if a virtually polycyclic group has Hirsch number 0, then it is finite (for example by [S, p. 16], $h(H) = h(K)$ if and only if $|K : H| < \infty$).

There are only at most l_G , where l_G is an integer depending only on G , possible different finite subgroups N_i in G . This follows from the fact that for each integer $l \geq 2$, $N_l \cdot N_{l-1} \cdot \dots \cdot N_2$ is a finite subgroup of G (apply [Hu, Ch. I, Theorem 5.3, p. 42]) of rank $|N_l| \cdot |N_{l-1}| \cdot \dots \cdot |N_2|$ (by [Hu, Ch. I, Theorem 4.7, p. 39]). On the other hand, a given virtually polycyclic group has only

finitely many finite subgroups up to isomorphism (hence the ranks of finite subgroups are bounded). This can be deduced, for example, from the fact that a group is virtually polycyclic if and only if it is (poly- \mathbb{Z})-by-finite (see [S, Proposition 2, p. 2]).

Thus there exists an integer k_G such that $G_i = G_{i-1}$ for all i except at most k_G . This finishes the proof.

The next lemma is an analog of Lemma 1 for finitely generated modules over the integral group ring $\mathbb{Z}G$ of a virtually polycyclic group G .

LEMMA 2. *Let M be a finitely generated module over the integral group ring $\mathbb{Z}G$, where G is a virtually polycyclic group. Then there exists an integer k_M such that any sequence $M = M_0 \supseteq M_1 \supseteq \dots \supseteq M_l$ of $\mathbb{Z}G$ -submodules of M with r -homomorphisms $R_i : M_{i-1} \rightarrow M_i$ for $i = 1, \dots, l$ contains no more than k_M distinct modules.*

Proof. If G is a virtually polycyclic group then the group ring $\mathbb{Z}G$ is right noetherian, by the result of P. Hall (see [R, (15.3.3), p. 446]). Therefore, every finitely generated module over $\mathbb{Z}G$ is noetherian (see [La, Ch. VI] and Introduction). Thus M is noetherian.

Consider a sequence $M = M_0 \supseteq M_1 \supseteq \dots \supseteq M_l$ of $\mathbb{Z}G$ -modules with r -homomorphisms $R_i : M_{i-1} \rightarrow M_i$ for $i = 1, \dots, l$.

Let $N_i = \ker R_i$. We have $M_{i-1} \cong M_i \oplus N_i$ for $i = 1, \dots, l$. Thus

$$M_0 \cong N_1 \oplus \dots \oplus N_l \oplus M_l.$$

It was proved by Gabriel ([Ga], see also [GR, Proposition 1.3, p. 7]) that every noetherian module has Krull dimension. It was also shown by Krause [Kr] and Michler [Mi] that a module with Krull dimension has finite uniform dimension (see [GR, Proposition 1.4, p. 7]). Therefore, there is a fixed bound on the number of possible direct summands of $M = M_0$. It follows that there exists an integer k_M such that $N_i = 0$ for all except no more than k_M integers i . This means that $M_{i-1} = M_i$ for such i , and the proof is finished.

In this paper \tilde{X} denotes, as usual, the universal covering space of X .

Let $f : X \rightarrow Y$ be a cellular map of CW-complexes such that $f(x) = y$ for some vertices $x \in X, y \in Y$. Choose $\tilde{x} \in p^{-1}(x), \tilde{y} \in p^{-1}(y)$ (where p denotes the covering projections). There exists a unique map $\tilde{f} : \tilde{X} \rightarrow \tilde{Y}$ such that $p\tilde{f} = f p$ and $\tilde{f}(\tilde{x}) = \tilde{y}$. The map \tilde{f} induces homomorphisms $\tilde{f}_r : H_r(\tilde{X}) \rightarrow H_r(\tilde{Y})$ for all $r \geq 1$. We then say that \tilde{f}_r is induced by f (see [HL, p. 107]).

Applying the above algebraic results we obtain (cf. [K2, Theorem 1])

THEOREM 1. *Let P be a polyhedron with virtually polycyclic fundamental group. There exists an integer k_P such that each sequence*

$$P \geq X_1 \geq \dots \geq X_l$$

of spaces contains at most k_P homotopy dominations which are not homotopy equivalences.

Proof. Recall that by the classical result of J. H. C. Whitehead, each space homotopy dominated by a finite CW-complex has the homotopy type of a CW-complex, not necessarily finite (see also [Wa]). Thus assume X_i for $i = 1, \dots, l$ to be CW-complexes.

We will prove that if P is a polyhedron with virtually polycyclic fundamental group, then there exists an integer k_P such that for each sequence $P = X_0 \geq X_1 \geq \dots \geq X_l$ of CW-complexes, with dominations $d_i : X_{i-1} \rightarrow X_i$ and inverse maps $u_i : X_i \rightarrow X_{i-1}$ (i.e. $d_i u_i \simeq \text{id}_{X_i}$) for $i = 1, \dots, l$, all except at most k_P dominations (and their inverses) are homotopy equivalences.

Let $\pi_1(P) \cong G$. By Lemma 1, there exists an integer k_G such that for each sequence $P \geq X_1 \geq \dots \geq X_l$ all except at most k_G dominations $d_i : X_{i-1} \rightarrow X_i$ (and their inverses $u_i : X_i \rightarrow X_{i-1}$) induce isomorphisms of $\pi_1(X_i)$.

Let $\tilde{d}_i : \tilde{X}_{i-1} \rightarrow \tilde{X}_i$ and $\tilde{u}_i : \tilde{X}_i \rightarrow \tilde{X}_{i-1}$ be the liftings of d_i and u_i to the universal covers. Then \tilde{d}_i is a domination.

For each $2 \leq r \leq \dim P$, we will apply Lemma 2 to the $\mathbb{Z}G$ -module $M = H_r(\tilde{P})$.

From [Wa, Theorems A and B], if $\mathbb{Z}H$ is a noetherian ring, then for every CW-complex X with $\pi_1(X) \cong H$ which is dominated by a finite CW-complex, the $\mathbb{Z}H$ -modules $H_r(\tilde{X})$ for $r \geq 2$ are all finitely generated.

Since $\mathbb{Z}G$ is noetherian (cf. the proof of Lemma 2), M is a finitely generated $\mathbb{Z}G$ -module. Consider the sequence $M = M_0 \supseteq M_1 \supseteq \dots \supseteq M_l$ of $\mathbb{Z}G$ -modules in which

$$M_i = \tilde{u}_{1*} \tilde{u}_{2*} \dots \tilde{u}_{i*} (H_r(\tilde{X}_i)) \quad \text{for } i = 1, \dots, l,$$

where $\tilde{u}_{i*} : H_r(\tilde{X}_i) \rightarrow H_r(\tilde{X}_{i-1})$ is induced by $\tilde{u}_i : \tilde{X}_i \rightarrow \tilde{X}_{i-1}$ (with obvious r-homomorphisms between them).

By Lemma 2, there exists an integer k_r (depending only on P and r) such that $M_i = M_{i-1}$ for all except at most k_r integers i . It follows that in each sequence

$$H_r(\tilde{P}) = H_r(\tilde{X}_0) \rightarrow H_r(\tilde{X}_1) \rightarrow \dots \rightarrow H_r(\tilde{X}_l)$$

with r-homomorphisms $\tilde{d}_{i*} : H_r(\tilde{X}_{i-1}) \rightarrow H_r(\tilde{X}_i)$ induced by $\tilde{d}_i : \tilde{X}_{i-1} \rightarrow \tilde{X}_i$, the homomorphisms are all isomorphisms except at most k_r (and the same is true for $\tilde{u}_{i*} : H_r(\tilde{X}_i) \rightarrow H_r(\tilde{X}_{i-1})$ induced by $\tilde{u}_i : \tilde{X}_i \rightarrow \tilde{X}_{i-1}$).

Hence, for a given polyhedron P with virtually polycyclic fundamental group, there exists an integer k_P such that, for each sequence

$$P = X_0 \geq X_1 \geq \dots \geq X_l$$

of CW-complexes with dominations $d_i : X_{i-1} \rightarrow X_i$, d_i induce isomorphisms $\pi_1(X_{i-1}) \rightarrow \pi_1(X_i)$ and $H_r(\tilde{X}_{i-1}) \rightarrow H_r(\tilde{X}_i)$ (for all $r \geq 2$) for all $i = 1, \dots, l$ except at most k_P .

Recall that, by the Whitehead theorem, if X and Y are CW-complexes, then a map $f : X \rightarrow Y$ which induces an isomorphism $\pi_1(X) \rightarrow \pi_1(Y)$ and isomorphisms $H_r(\tilde{X}) \rightarrow H_r(\tilde{Y})$ for all $r \geq 2$ is a homotopy equivalence (see, for example, [H1, Theorem 3.1, p. 107]).

Thus, for a polyhedron P with virtually polycyclic fundamental group, there exists an integer k_P such that each sequence $P \geq X_1 \geq \dots \geq X_l$ of CW-complexes contains at most k_P homotopy dominations which are not homotopy equivalences, which is the desired conclusion.

THEOREM 2. *Every polyhedron with virtually polycyclic fundamental group has finite depth.*

Proof. This follows from Theorem 1.

COROLLARY 1. *Let P be a polyhedron with virtually polycyclic fundamental group. There exists an integer k_P such that each sequence $P \geq X_1 \geq X_2 \geq \dots$ contains at most k_P homotopy dominations which are not homotopy equivalences.*

COROLLARY 2. *Let P be a polyhedron with virtually polycyclic fundamental group. There exists an integer k_P such that each sequence $X_1 \leq X_2 \leq \dots$ with $X_i \leq P$ for each i contains at most k_P homotopy dominations which are not homotopy equivalences.*

4. The index of h-proximity of two ANR's with virtually polycyclic fundamental groups is finite. In [B3] K. Borsuk introduced the notions of h-neighbor, left h-neighbor, right h-neighbor, h-minorant, h-majorant, and the index of h-proximity of two ANR's. Some results concerning these notions can be found in [M]. In [K7] we gave an example of a polyhedron with infinitely many left neighbors (in the shape or homotopy category), which answers Borsuk's question from [B4, p. 349].

The following problem of K. Borsuk (1967) is still open:

PROBLEM [B3, Ch. IX, Problem (12.8)]. *Do there exist two ANR's, P and Q , for which the index of h-proximity $h(P, Q)$ is infinite?*

As a corollary to Theorem 1, we find that if the fundamental groups of two ANR's are virtually polycyclic, then their index of h-proximity is finite.

THEOREM 3. *Assume that $P, Q \in \text{ANR}$ have $\pi_1(P)$ and $\pi_1(Q)$ virtually polycyclic. Then $h(P, Q)$ is finite.*

Proof. Set $W = P \times Q \in \text{ANR}$. Obviously $P \leq W$ and $Q \leq W$.

Since a group is poly-(cyclic or finite) if and only if it is virtually polycyclic [S, Proposition 2, p. 2], $\pi_1(W) \cong \pi_1(P) \times \pi_1(Q)$ is virtually polycyclic.

Note that if $X, Y \leq W$, where W is a polyhedron and $\pi_1(W)$ is virtually polycyclic, then $X \geq Y$ and $Y \geq X$ imply that $X \simeq Y$ (by Corollary 1).

To prove the theorem, it suffices to show that there exist finite sequences $P = X_0 \leq X_1 \leq \dots \leq X_s = W$ and $Q = Y_0 \leq Y_1 \leq \dots \leq Y_t = W$ such that $X_i \leq Z \leq X_{i+1}$ implies $Z \simeq X_i$ or $Z \simeq X_{i+1}$, where $i = 0, \dots, s-1$, and $Y_i \leq Z \leq Y_{i+1}$ implies $Z \simeq Y_i$ or $Z \simeq Y_{i+1}$, where $i = 0, \dots, t-1$, respectively.

From Theorem 1 we infer that there exists an integer k_P such that each sequence $P \leq X_1 \leq \dots \leq X_s \leq W$ contains at most k_P different homotopy types. We also have an analog of this assertion for Q in place of P . This completes the proof.

5. There exist polyhedra with infinitely many simple constituents. In 1970 K. Borsuk asked:

PROBLEM ([B2, Problem 7]). *Is it true that for every $P \in \text{ANR}$, the shape $\text{Sh}(P, p)$ has only a finite number of simple constituents?*

Applying the main result of this paper, we can answer this question:

THEOREM 4. *There exists a polyhedron (with polycyclic fundamental group) with infinitely many simple constituents.*

The proof is based on the following corollary to Theorem 2:

LEMMA 3. *Let P be a polyhedron with virtually polycyclic fundamental group. Then there exists an integer l_P such that P cannot be decomposed into more than l_P constituents.*

Proof. Let $P \simeq Y_1 \vee \dots \vee Y_l$, where each Y_i has nontrivial homotopy type. Set $X_i = Y_1 \vee \dots \vee Y_i$ for $i = 1, \dots, l$. Then $X_1 \leq \dots \leq X_l$. By Theorem 1, there exists an integer k_P such that all except at most k_P dominations in this sequence are homotopy equivalences. On the other hand, $X_i \simeq X_{i+1}$ would imply that there exists a domination of X_i over itself which is not a homotopy equivalence. But this is impossible (see Corollary 1). Therefore $l_P = k_P + 1$, which ends the proof.

Proof of Theorem 4. In [K1] we proved that there exist polyhedra P with polycyclic fundamental groups such that $P \simeq K_i \vee S^3$ for infinitely many polyhedra K_i of different homotopy types.

By Lemma 3, there exists an integer l_P such that no K_i can be decomposed into more than l_P simple constituents. Since the K_i , for $i = 1, 2, \dots$, represent infinitely many different homotopy types, it is easily seen that P has infinitely many simple constituents with different homotopy types, and the proof is finished.

6. Final remarks. Let us state the following:

PROBLEM 1. *Does there exist a polyhedron with infinite depth?*

We suspect that the answer to this question is positive and that the example is possible even with soluble fundamental group. One may ask about other classes of polyhedra with finite depth. It is worth pointing out that the following questions of a similar nature, included in known lists of open problems, also remain unsolved:

PROBLEM 2 (Y. Rong, [RWZ]). *Let M and M_i be closed, orientable, aspherical 3-manifolds. Does there exist an integer N_M such that any sequence*

$$\pi_1(M) \rightarrow \pi_1(M_1) \rightarrow \pi_1(M_2) \rightarrow \cdots \rightarrow \pi_1(M_n)$$

of epimorphisms with $n \geq N_M$ contains an isomorphism?

PROBLEM 3 (Y. Rong, [Kb, Problem 3.100]). *Let M and M_i be closed, orientable 3-manifolds. Does there exist an integer N_M such that any sequence*

$$M \rightarrow M_1 \rightarrow M_2 \rightarrow \cdots \rightarrow M_n$$

of degree 1 maps with $n \geq N_M$ contains a homotopy equivalence?

PROBLEM 4 (J. Simon, [Kb, Problem 1.12]). *For a knot K in S^3 , let $G_K = \pi_1(S^3 \setminus K)$. Given K , does there exist an integer N_K such that any sequence*

$$G_K \rightarrow G_{K_1} \rightarrow G_{K_2} \rightarrow \cdots \rightarrow G_{K_n}$$

of epimorphisms of knot groups with $n \geq N_K$ contains an isomorphism?

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