FUNDAMENTA
MATHEMATICAE
170 (2001)

♣-like principles under CH

by

Winfried Just (Athens, OH)

Abstract. Some relatives of the Juhász Club Principle are introduced and studied in the presence of CH. In particular, it is shown that a slight strengthening of this principle implies the existence of a Suslin tree in the presence of CH.

Jensen's $\diamondsuit$-principle is an important strengthening of the Continuum Hypothesis that allows for a number of interesting constructions that cannot be carried out in ZFC + CH alone. Most notably, $\diamondsuit$ implies the existence of a Suslin tree. In [4], Ostaszewski introduced a weakening of $\diamondsuit$ known as $\clubsuit$ that suffices for many of the constructions made possible by $\diamondsuit$. Devlin showed that $\clubsuit$ + CH is equivalent to $\diamondsuit$ (see [4]). However, $\clubsuit$ does not imply CH and is thus strictly weaker than $\diamondsuit$ (for an elegant proof, see [5], page 43). In [3], Juhász introduced a weakening of $\clubsuit$ that will be denoted here by $\clubsuit_{\mathrm{J}}$. This principle is not equivalent to $\diamondsuit$ even under CH, and yet it retains some of the combinatorial power of the latter principle. Here we study some natural modifications of $\clubsuit_{\mathrm{J}}$. In particular, we show that some of these modifications, while still not equivalent to $\diamondsuit$ under CH, do imply the existence of a Suslin tree in the presence of CH.

Throughout this note, let $S_0 = \omega_1 \cap \mathbf{LIM} \setminus \{0\}$. The letter E will denote stationary subsets of S_0, and $\mathcal{S}(E)$ will denote the family of all stationary subsets of E. Instead of $\mathcal{S}(S_0)$ we will write $\mathcal{S}$. The family of all closed unbounded subsets of S_0 will be denoted by $\mathcal{C}$. A $\clubsuit(E)$*-sequence* is a sequence $\langle s_\alpha : \alpha \in S_0\rangle$ such that s_α is a cofinal subset of order type ω for all $\alpha \in S_0$, and for each $X \in [\omega_1]^{\aleph_1}$ there exists $\alpha \in E$ with $s_\alpha \subset X$. The $\clubsuit(E)$-*principle* asserts the existence of a $\clubsuit(E)$-sequence. Instead of $\clubsuit(S_0)$ we simply write $\clubsuit$.

2000 *Mathematics Subject Classification*: 03E65, 03E35.

Key words and phrases: $\diamondsuit$-like principles, Juhász Club Principle, Suslin tree, Continuum Hypothesis, pseudodiamond.

In order to better delineate where these principles fit in the general scheme of things, we first introduce a convenient notation for the study of $\diamondsuit$-like principles.

Definition 1. Let $\mathcal{G} \subseteq \mathcal{P}(S_0)$, $\mathcal{A} \subseteq \mathcal{P}(\omega_1)$, and let $\mathcal{F}$ be a family of sequences $\overline{F} = \langle F_\alpha : \alpha \in S_0 \rangle$ such that $F_\alpha \subseteq \mathcal{P}(\alpha)$ for every $\alpha \in S_0$. We say that $\overline{F}$ is a $\spadesuit(\mathcal{G}, \mathcal{A})$*-sequence* if for all $A \in \mathcal{A}$ there exists $G \in \mathcal{G}$ such that $A \cap \alpha \in F_\alpha$ for all $\alpha \in G$. We say that $\spadesuit(\mathcal{F}, \mathcal{G}, \mathcal{A})$ holds if there exists a $\spadesuit(\mathcal{G}, \mathcal{A})$-sequence $\overline{F}$ such that $\overline{F} \in \mathcal{F}$.

Example 1. (1) Let $\mathcal{F}_1$ be the family of all sequences $\overline{F} = \langle F_\alpha : \alpha \in S_0 \rangle$ such that $F_\alpha \subseteq \mathcal{P}(\alpha)$ and $|F_\alpha| = 1$ for each $\alpha \in S_0$, and let $\mathcal{F}_{\aleph_0}$ be the family of all sequences $\overline{F} = \langle F_\alpha : \alpha \in S_0 \rangle$ such that $F_\alpha \subseteq \mathcal{P}(\alpha)$ and $|F_\alpha| = \aleph_0$ for each $\alpha \in S_0$. Then $\spadesuit(\mathcal{F}_1, \mathcal{S}, \mathcal{P}(\omega_1))$ is the same as the ordinary diamond-principle $\diamondsuit$, $\spadesuit(\mathcal{F}_{\aleph_0}, \mathcal{S}, \mathcal{P}(\omega_1))$ is the same as Kunen's $\diamondsuit^-$, and $\spadesuit(\mathcal{F}_{\aleph_0}, \mathcal{C}, \mathcal{P}(\omega_1))$ is the same as $\diamondsuit^*$. Moreover, for a stationary $E \subseteq S_0$, the principle $\spadesuit(\mathcal{F}_1, \mathcal{S}(E), \mathcal{P}(\omega_1))$ is the same as $\diamondsuit(E)$.

(2) Let E be a stationary subset of S_0 and let $\mathcal{F}_\clubsuit$ be the family of all sequences $\overline{F} = \langle F_\alpha : \alpha \in S_0 \rangle$ such that for each $\alpha \in S_0$ there exists a set $s_\alpha \subseteq \alpha$ such that $\mathrm{ot}(s_\alpha) = \omega$, $\sup s_\alpha = \alpha$, and $F_\alpha = \{a \subseteq \alpha : s_\alpha \subseteq a\}$. Then $\spadesuit(\mathcal{F}_\clubsuit, \mathcal{P}(E) \setminus \{\emptyset\}, [\omega_1]^{\aleph_1})$ is the same as $\clubsuit(E)$.

(3) Let E be a stationary subset of S_0, and let $\mathcal{F}_{\clubsuit_{\mathrm{w}}}$ be the family of all sequences $\overline{F} = \langle F_\alpha : \alpha \in S_0 \rangle$ such that for each $\alpha \in S_0$ there exists a set $s_\alpha \subseteq \alpha$ such that $\mathrm{ot}(s_\alpha) = \omega$, $\sup s_\alpha = \alpha$, and $F_\alpha = \{a \subseteq \alpha : |s_\alpha \setminus a| < \aleph_0\}$. Then $\spadesuit(\mathcal{F}_{\clubsuit_{\mathrm{w}}}, \mathcal{P}(E) \setminus \{\emptyset\}, [\omega_1]^{\aleph_1})$ is the same as the principle $\clubsuit_{\mathrm{w}}(E)$ of [2] and $\clubsuit^1$ of [1].

(4) Let $\mathcal{F}_{\mathrm{filt}}$ be the family of all sequences $\overline{F} = \langle F_\alpha : \alpha \in S_0 \rangle$ such that for all $\alpha \in S_0$, F_α is a filter on α that consists of cofinal subsets of α. Then for every $E \subseteq S_0$ we have: $\clubsuit(E) \Rightarrow \clubsuit_w(E) \Rightarrow \spadesuit(\mathcal{F}_{\mathrm{filt}}, \mathcal{P}(E) \setminus \{\emptyset\}, [\omega_1]^{\aleph_1})$.

It is a historical accident that "official" definitions of $\diamondsuit$-principles take the form $\spadesuit(\mathcal{F}, \mathcal{S}(E), \mathcal{A})$ while definitions of $\clubsuit$-principles take the form $\spadesuit(\mathcal{F}, \mathcal{P}(E) \setminus \{\emptyset\}, \mathcal{A})$; in all cases of interest to us the two formulations are equivalent. This well known observation can be formalized in our terminology as follows:

Lemma 2. *Let $\mathcal{F}$ be a family of sequences such that for each $\langle F_\alpha : \alpha \in S_0 \rangle \in \mathcal{F}$ and each $\alpha \in S_0$, the family F_α consists of cofinal subsets of α, and if $a \subseteq b \subseteq \alpha$, $a \in F_\alpha$, then $b \in F_\alpha$. Let E be a stationary subset of S_0. Then $\spadesuit(\mathcal{F}, \mathcal{P}(E) \setminus \{\emptyset\}, [\omega_1]^{\aleph_1})$ implies $\spadesuit(\mathcal{F}, \mathcal{S}(E), [\omega_1]^{\aleph_1})$.*

A proof of the above lemma will be given in the appendix.

Example 2. The result of Devlin mentioned above can be generalized as follows: Let E be a stationary subset of S_0. Then $\spadesuit(\mathcal{F}_{\mathrm{filt}}, \mathcal{P}(E) \setminus \{\emptyset\}, [\omega_1]^{\aleph_1})$ + CH $\Rightarrow \diamondsuit(E)$. A proof will be given in the appendix.

Kunen has shown that for $E \subseteq S_0$, the principles $\diamondsuit^-(E)$ and $\diamondsuit(E)$ are equivalent. In our terminology this means that $\spadesuit(\mathcal{F}_1, \mathcal{S}(E), \mathcal{P}(\omega_1))$ is equivalent to $\spadesuit(\mathcal{F}_{\aleph_0}, \mathcal{S}(E), \mathcal{P}(\omega_1))$. It is natural to ask to what extent this result generalizes to $\clubsuit$-like principles. Formally, given a family $\mathcal{F}$ of S_0-sequences, let us define $\mathcal{F}^{(\omega)} = \{\langle H_\alpha : \alpha \in S\rangle : \exists \langle F_{\alpha,n} : \alpha \in S_0,\ n \in \omega\rangle$ $\forall n \in \omega\ \forall \alpha \in S_0\ (\langle F_{\alpha,n} : \alpha \in S_0\rangle \in \mathcal{F} \wedge H_\alpha = \bigcup_{n\in\omega} F_{\alpha,n})\}$. Note that, in particular, $\mathcal{F}_1^{(\omega)} = \mathcal{F}_{\aleph_0}$. The question now is for which $\mathcal{F}$ and $\mathcal{A}$ the implication $\spadesuit(\mathcal{F}^{(\omega)}, \mathcal{S}(E), \mathcal{A}) \Rightarrow \spadesuit(\mathcal{F}, \mathcal{S}(E), \mathcal{A})$ holds.

Example 3. (1) $\spadesuit(\mathcal{F}_{\clubsuit}^{(\omega)}, \mathcal{S}, [\omega_1]^{\aleph_1})$ does not imply $\spadesuit(\mathcal{F}_{\clubsuit}, \mathcal{S}, [\omega_1]^{\aleph_1})$. To see this, recall that in [1], it is shown that $\clubsuit^1$ (i.e., $\spadesuit(\mathcal{F}_{\clubsuit_w}, \mathcal{S}, [\omega_1]^{\aleph_1})$) does not imply $\clubsuit$. It is easy to see that $\spadesuit(\mathcal{F}_{\clubsuit_{\mathrm{w}}}, \mathcal{S}, [\omega_1]^{\aleph_1})$ implies $\spadesuit(\mathcal{F}_{\clubsuit}^{(\omega)}, \mathcal{S}, [\omega_1]^{\aleph_1})$.

(2) Under CH, an analogue of Kunen's Theorem for $\clubsuit$-like principles does hold: Let $\mathcal{F} \subseteq \mathcal{F}_{\mathrm{filt}}$ and let $E \subseteq S_0$. Then CH + $\spadesuit(\mathcal{F}^{(\omega)}, \mathcal{S}(E), [\omega_1]^{\aleph_1})$ implies $\spadesuit(\mathcal{F}, \mathcal{S}(E), [\omega_1]^{\aleph_1})$. A proof of this observation will be given in the appendix.

(3) A similar observation can be made for $\diamondsuit^*$: Let $\mathcal{F} \subseteq \mathcal{F}_{\mathrm{filt}}$ and let $E \subseteq S_0$. Then CH + $\spadesuit(\mathcal{F}^{(\omega)}, \mathcal{C}, [\omega_1]^{\aleph_1})$ implies $\diamondsuit^*$.

Naturally, the question arises whether Examples 2 and 3(2) have dual versions for ideals. Unfortunately, as the following examples show, the situation for ideals is more complex than the situation for filters.

Example 4. (1) Let $\mathcal{F}_{\mathrm{id}}$ denote the family of all sequences $\langle F_\alpha : \alpha \in S_0\rangle$ such that, for each α, $F_\alpha = I_\alpha^+$ $(= \{a \subseteq \alpha : a \notin I_\alpha\})$ for some ideal I_α of subsets of α that contains the ideal B_α of all bounded subsets of α. In particular, if we define $\overline{U} = \langle B_\alpha^+ : \alpha \in S_0\rangle$, then $\overline{U} \in \mathcal{F}_{\mathrm{id}}$ and $\overline{U}$ is a $\spadesuit(\mathcal{C}, [\omega_1]^{\aleph_1})$-sequence. It follows that $\spadesuit(\mathcal{F}_{\mathrm{id}}, \mathcal{S}, [\omega_1]^{\aleph_1})$ is a theorem of ZFC.

(2) Let $\mathcal{F}_{\mathrm{ult}}$ be the family of all sequences $\langle I_\alpha^+ : \alpha \in S_0\rangle$ such that for each α, I_α is a maximal ideal on α that extends B_α. Then $\mathcal{F}_{\mathrm{ult}} \subset \mathcal{F}$. On the other hand, for each α, I_α^+ is equal to the dual filter I_α^*, and thus $\mathcal{F}_{\mathrm{ult}} \subset \mathcal{F}_{\mathrm{filt}}$. Now Example 2 implies that CH + $\spadesuit(\mathcal{F}_{\mathrm{ult}}, \mathcal{S}, [\omega_1]^{\aleph_1})$ is equivalent to $\diamondsuit$.

Does Example 3(2) have an analogue for ideals? One could formalize this question by asking whether for all $\mathcal{F} \subseteq \mathcal{F}_{\mathrm{id}}$ and $E \subseteq S_0$, in the presence of CH, the statements CH + $\spadesuit(\mathcal{F}^{(\omega)}, \mathcal{S}(E), [\omega_1]^{\aleph_1})$ and CH + $\spadesuit(\mathcal{F}, \mathcal{S}(E), [\omega_1]^{\aleph_1})$ are equivalent. However, note that for any $\mathcal{F} \subseteq \mathcal{F}_{\mathrm{id}}$ we have $\mathcal{F}^{(\omega)} \subseteq \mathcal{F}_{\mathrm{id}}$, and while the above question may be of some interest, it does not have quite the same flavor as Example 3(2). The following approach is closer in spirit to Example 3(2). For a family $\mathcal{F}$ of sequences $\langle F_\alpha : \alpha \in S_0\rangle$, let ${}^{(\omega)}\mathcal{F} = \{\langle H_\alpha : \alpha \in S_0\rangle : \exists\{\overline{F}_n = \langle F_{\alpha,n} : \alpha \in S_0\rangle : n \in \omega\} \subseteq \mathcal{F}\ \forall \alpha \in S_0$ $(H_\alpha = \bigcap_{n\in\omega} F_{\alpha,n})\}$.

Now the question about the dual version of Example 3(2) can be formulated as follows.

Question 3. *Let $\mathcal{F} \subseteq \mathcal{F}_{\mathrm{id}}$, $E \subseteq S_0$. Are the principles* $\spadesuit(^{(\omega)}\mathcal{F}, \mathcal{S}(E), [\omega_1]^{\aleph_1})$ *and* $\spadesuit(\mathcal{F}, \mathcal{S}(E), [\omega_1]^{\aleph_1})$ *equivalent? Is* $\mathrm{CH} + \spadesuit(^{(\omega)}\mathcal{F}, \mathcal{S}(E), [\omega_1]^{\aleph_1})$ *equivalent to the statement* $\mathrm{CH} + \spadesuit(\mathcal{F}, \mathcal{S}(E), [\omega_1]^{\aleph_1})$*?*

To get a better appreciation of Question 3, let us consider a particularly interesting instance of it. In [3], Juhász introduced the following weakening of $\clubsuit$.

Definition 4. A $\clubsuit_{\mathrm{J}}$*-sequence* is a sequence $\langle s^k_\alpha : \alpha \in S_0,\ k \in \omega\rangle$ such that:

(a) $\forall \alpha \in S_0, k \in \omega$ (s^k_α is a cofinal subset of α of order type ω);
(b) $\forall \alpha \in S_0, k < m < \omega$ ($s^k_\alpha \cap s^m_\alpha = \emptyset$);
(c) $\forall X \in [\omega_1]^{\aleph_1}\ \exists \alpha \in S_0\ \forall k \in \omega\ (|s^k_\alpha \cap X| = \aleph_0)$.

We will abbreviate the statement "a $\clubsuit_{\mathrm{J}}$-sequence exists" by $\clubsuit_{\mathrm{J}}$ and call it the *Juhász Club Principle.*

Remark 5. The original definition of $\clubsuit_{\mathrm{J}}$ listed the following additional requirement: $\forall \alpha \in S_0$ $(\mathrm{ot}(\bigcup_{k\in\omega} s^k_\alpha) = \omega)$. This requirement can be safely dropped for our purposes. To see this, let $\langle s^k_\alpha : \alpha \in S_0,\ k \in \omega\rangle$ be a $\clubsuit_{\mathrm{J}}$-sequence. For each $\alpha \in S_0$, fix an increasing sequence $(\beta^k_\alpha)_{k\in\omega}$ of ordinals with limit α. Let $t^k_\alpha = s^k_\alpha \setminus \beta^k_\alpha$. Then $\mathrm{ot}(\bigcup_{k\in\omega} t^\alpha_k) = \omega$, as required in the original definition.

Definition 6. For $\alpha \in S_0$ and an unbounded set $s \subseteq \alpha$, let $J(s)$ denote the ideal $\{a \subseteq \alpha : \sup(a \cap s) < \alpha\}$. Now let $\mathcal{F}_{\mathrm{wJ}}$ denote the family of all sequences $\langle J(s_\alpha)^+ : \alpha \in S_0\rangle$ such that, for each α, s_α is a cofinal subset of α of order type ω. Instead of $\spadesuit(\mathcal{F}_{\mathrm{wJ}}, \mathcal{S}(E), [\omega_1]^{\aleph_1})$ we will write $\clubsuit_{\mathrm{wJ}}(E)$.

Clearly, $\mathcal{F}_{\mathrm{wJ}} \subset \mathcal{F}_{\mathrm{id}}$. Moreover, it follows from Lemma 2 that $\clubsuit_{\mathrm{J}}$ is the same as $\spadesuit(^{(\omega)}\mathcal{F}_{\mathrm{wJ}}, \mathcal{S}, [\omega_1]^{\aleph_1})$. Thus the following question is an instance of Question 3:

Question 7. *Can one show in* ZFC *or in* ZFC + CH *that the principles* $\clubsuit_{\mathrm{J}}$ *and* $\clubsuit_{\mathrm{wJ}}$ *are equivalent?*

Somewhat surprisingly, at least the CH part of the above question becomes much easier if one makes a seemingly very minor change to the definition of $\mathcal{F}_{\mathrm{wJ}}$.

Definition 8. Let $E \subseteq S_0$. A $\clubsuit_{\mathrm{wJ}^2}(E)$*-sequence* is a sequence $\langle s^n_\alpha : \alpha \in S_0, n \in \omega\rangle$ such that:

(a) $\forall \alpha \in S_0\ \forall n \in \omega\ (s^n_\alpha < s^{n+1}_\alpha \wedge \sup\{s^n_\alpha : n \in \omega\} = \alpha)$, and
(b) $\forall X \in [\omega_1]^{\aleph_1}\ (\{\alpha \in E : |\{n : \{s^n_\alpha, s^{n+1}_\alpha\} \subset X\}| = \aleph_0\} \in \mathcal{S}(E))$.

We will abbreviate the statement "a $\clubsuit_{\mathrm{wJ}^2}(E)$-sequence exists" by $\clubsuit_{\mathrm{wJ}^2}(E)$.

In order to appreciate the strength of $\clubsuit_{\mathrm{wJ}^2}$, consider the following:

THEOREM 9. $\mathrm{CH} + \clubsuit_{\mathrm{wJ}^2}$ *implies the existence of a Suslin tree.*

Proof. Assume CH and let $\{a_\alpha : \alpha \in S_0\}$ be an enumeration of all countable subsets of ω_1 such that every element of $[\omega_1]^{\le\aleph_0}$ is listed cofinally often. Let $\langle s_\alpha^n : \alpha \in S_0,\ n \in \omega\rangle$ be a $\clubsuit_{\mathrm{wJ}^2}$-sequence. We will construct a Suslin tree $T = \langle \omega_1, <_T\rangle$. The αth level of T will be denoted by $T(\alpha)$. Conversely, for $\xi \in \omega_1$, $L(\xi)$ will denote the unique α such that $\xi \in T(\alpha)$. Moreover, $\bigcup_{\beta<\alpha} T(\beta)$ will be denoted by $T_{(\alpha)}$.

We will construct the $T_{(\alpha)}$'s in such a way that for each $\alpha \in S_0$, the set of nodes of $T_{(\alpha)}$ is α, and so that $T_{(\alpha)}$ is *tall* and *splitting*, i.e., each node has at least two successors at each higher level.

Suppose $T_{(\alpha)}$ has been constructed. The important step in the construction of $T_{(\alpha+1)}$ is to decide which cofinal branches of $T_{(\alpha)}$ will be extended at level $T(\alpha)$; the rest of the construction is routine and will be left to the reader. For each node $\xi \in \alpha$ we need to put one node $n(\alpha,\xi)$ into $T(\alpha)$ such that $\xi <_T n(\alpha,\xi)$. Conversely, $T(\alpha)$ will precisely be the set of nodes $\{n(\alpha,\xi) : \xi \in \alpha\}$.

So pick $\xi \in \alpha$. We construct recursively a sequence $(\xi_n)_{n\in\omega}$ as follows:

- $\xi_0 = \xi$;
- if $s_\alpha^n < L(\xi_n)$, let $\xi_{n+1} = \xi_n$;
- if $L(\xi_n) \le s_\alpha^n$, pick $\xi_{n+1} \in T(s_\alpha^{n+1})$ in such a way that $\xi_n <_T \xi_{n+1}$ and, if possible, there exists $\eta \in a_{s_\alpha^{n+1}}$ such that $\eta <_T \xi_{n+1}$.

Finally, put $n(\alpha,\xi)$ on top of the branch of $T_{(\alpha)}$ generated by $\{\xi_n : n \in \omega\}$.

In order to prove that this construction works it suffices to show that there is no uncountable maximal antichain in T. Suppose towards a contradiction that A is an uncountable maximal antichain in T. Note that the set $C_A = \{\gamma : A \cap \gamma \text{ is maximal in } T\}$ is unbounded in ω_1. This allows us to construct a set $X = \{\beta_\eta : \eta < \omega_1\} \subseteq S_0$ as follows:

- $\beta_0 = \omega$;
- for limit ordinals $0 < \delta < \omega_1$, let $\beta_\delta = \sup\{\beta_\eta : \eta < \delta\}$;
- given β_η, let $\beta_{\eta+1} > \beta_\eta$ be such that for some $\gamma \in C_A$ with $\beta_\eta < \gamma < \beta_{\eta+1}$ we have $a_{\beta_{\eta+1}} = A \cap \gamma$.

Now let $\alpha \in S_0$ be such that for infinitely many $n \in \omega$ we have $s_\alpha^n, s_\alpha^{n+1} \in X$. Consider $n(\alpha,\xi) \in T(\alpha)$. By construction, there exist (infinitely many) $n \in \omega$ with $\xi <_T \xi_n <_T \xi_{n+1} <_T n(\alpha,\xi)$ and both $s_\alpha^n = L(\xi_n) \in X$ and $s_\alpha^{n+1} = L(\xi_{n+1}) \in X$. Now by the construction of X, $a_{s_\alpha^{n+1}}$ is a subset of A and is a maximal antichain in $T_{(\gamma)}$ for some γ with $s_\alpha^n < \gamma < s_\alpha^{n+1}$. Thus it is *possible* to choose ξ_{n+1} above some element of $a_{s_\alpha^{n+1}}$, and the construction rules of T force us to do so. It follows that every element of $T(\alpha)$ sits above some node of $A \cap \alpha$, and thus A cannot be an uncountable maximal antichain. ■

The author of this note does not know whether CH + $\clubsuit_{\mathrm{wJ}}$ implies the existence of a Suslin tree, and in particular, whether $\clubsuit_{\mathrm{wJ}}$ and $\clubsuit_{\mathrm{wJ}^2}$ are equivalent, even under CH.

While the definition of $\clubsuit_{\mathrm{wJ}^2}$ may look somewhat artificial, in the presence of CH we can find a rather elegant characterization.

Definition 10. Let $E \subseteq S_0'$ (where S_0' is the set of countable limits of countable limit ordinals). The *pseudodiamond* $\diamondsuit^{\mathrm{p}}(E)$ is the following statement: There exists a sequence $\langle a_\alpha^n : \alpha \in S_0',\ n \in \omega\rangle$ such that

(a) $\forall \alpha \in S_0'\ \forall n \in \omega\ (a_\alpha^n \subset \alpha)$;
(b) $\forall \alpha \in S_0'\ \forall n \in \omega\ (|(a_\alpha^{n+1} \setminus \sup a_\alpha^n)| = \aleph_0)$;
(c) $\forall \alpha \in S_0'\ (\sup \bigcup_{n\in\omega} a_\alpha^n = \alpha)$;
(d) $\forall X \in [\omega_1]^{\aleph_1}\ (\{\alpha \in S_0' : |\{n \in \omega : X \cap \sup a_\alpha^n = a_\alpha^n\}| = \aleph_0\} \in \mathcal{S}(E))$.

Remark 11. Note that if one omits condition (b) in the above definition, then the resulting principle is equivalent to CH. However, $\diamondsuit^{\mathrm{p}}$ is much stronger than that.

Theorem 12. *Let $E \subseteq S_0$. The following are equivalent*:

(a) $\diamondsuit^{\mathrm{p}}(E)$.
(b) $\mathrm{CH} + \clubsuit_{\mathrm{wJ}^2}(E)$.

Proof. Assume $\diamondsuit^{\mathrm{p}}(E)$. Then CH holds by Remark 11. Let $\langle a_\alpha^n : \alpha \in S_0',\ n \in \omega\rangle$ be a sequence that witnesses $\diamondsuit^{\mathrm{p}}(E)$. Let us construct a sequence $\langle s_\alpha^n : \alpha \in S_0,\ n \in \omega\rangle$ as follows: For each $\alpha \in S_0'$ and $n \in \omega$, we pick $s_\alpha^{2n}, s_\alpha^{2n+1} \in a_\alpha^{n+1}$ in such a way that $\sup a_\alpha^n < s_\alpha^{2n} < s_\alpha^{2n+1} \le \sup a_\alpha^{n+1}$. For $\alpha \in S_0 \setminus S_0'$ and $n \in \omega$, we pick s_α^n so as to satisfy condition (a) of Definition 8. Clearly, $\langle s_\alpha^n : \alpha \in S_0,\ n \in \omega\rangle$ is a $\clubsuit_{\mathrm{wJ}^2}(E)$-sequence.

Now assume CH + $\clubsuit_{\mathrm{wJ}^2}(E)$ holds, and let $\langle s_\alpha^n : \alpha \in S_0, n \in \omega\rangle$ be a $\clubsuit_{\mathrm{wJ}^2}(E)$-sequence. Fix an enumeration $\{b_\alpha : \alpha \in \omega_1\}$ of all countable subsets of ω_1 such that every element of $[\omega_1]^{\le\aleph_0}$ gets listed cofinally often. For $\alpha \in S_0' \setminus S_0''$, we define a_α^n in such a way that conditions (a) through (c) of Definition 10 hold. For $\alpha \in S_0''$ we proceed as follows: Suppose a_α^n has been defined and $\sup a_\alpha^n < s_\alpha^n + \omega^2$. If $b_{s_\alpha^{n+1}}$ is a subset of s_α^{n+1} such that $|b_{s_\alpha^{n+1}} \setminus \sup a_\alpha^n| = \aleph_0$, then let $a_\alpha^{n+1} = b_{s_\alpha^{n+1}}$. Otherwise, let $a_\alpha^{n+1} = \sup(a_\alpha^n) + \omega$. Note that, in either case, $\sup a_\alpha^{n+1} < s_\alpha^{n+1} + \omega^2$.

This construction yields a_α^n's that satisfy conditions (a)–(c) of Definition 10. In order to show that condition (d) also holds, let $X \in [\omega_1]^{\aleph_1}$. Construct a set $C = \{c_\eta : \eta \in \omega_1\} \subseteq \omega_1$ as follows:

- $c_0 = \omega$;
- $c_\delta = \sup\{c_\eta : \eta < \delta\}$ for countable limit ordinals δ;
- given c_η, let $c_{\eta+1}$ be such that $b_{c_{\eta+1}} \subset c_{\eta+1}$, $b_{c_{\eta+1}} = X \cap \sup b_{c_{\eta+1}}$, and $|b_{c_{\eta+1}} \setminus (c_\eta + \omega^2)| = \aleph_0$.

Now it suffices to observe that C is closed unbounded, and if $\alpha \in C \cap S_0'' \cap E$ is such that for infinitely many $n \in \omega$, $\{s_\alpha^n, s_\alpha^{n+1}\} \subset C$, then $a_\alpha^n = X \cap \sup a_\alpha^n$ for infinitely many $n \in \omega$, as desired. ∎

To elucidate the connection between $\clubsuit_{\mathrm{wJ}^2}$ and principles $\spadesuit(\mathcal{F}, \mathcal{S}, [\omega_1]^{\aleph_1})$ for $\mathcal{F} \subseteq \mathcal{F}_{\mathrm{id}}$, let us generalize the definition of $\mathcal{F}_{\mathrm{wJ}}$.

Definition 13. Let I be an ideal of subsets of ω such that I contains the ideal Fin of finite subsets of ω. Let $s = \{s^n : n \in \omega\}$ be a set of ordinals of order type ω, enumerated in increasing order, and let $\alpha = \sup(s)$. Then we define an ideal $J(s, I)$ on α as follows: $J(s, I) = \{a \subseteq \alpha : \{n : s^n \in a\} \in I\}$. Moreover, $\mathcal{F}_{\mathrm{wJ},I}$ will denote the family of all sequences $\langle F_\alpha : \alpha \in S_0\rangle$ such that for each α there exists a set $s_\alpha \subseteq \alpha$ of order type ω, with $\sup(s_\alpha) = \alpha$, such that $F_\alpha = J(s_\alpha, I)^+$.

Remark 14. Note that $\mathcal{F}_{\mathrm{wJ}}$ is the same as $\mathcal{F}_{\mathrm{wJ,Fin}}$. Thus $\clubsuit_{\mathrm{wJ}}$ is the same as $\spadesuit(\mathcal{F}_{\mathrm{wJ,Fin}}, \mathcal{S}(E), [\omega_1]^{\aleph_1})$ and $\clubsuit_J$ is the same as $\spadesuit({}^{(\omega)}\mathcal{F}_{\mathrm{wJ,Fin}}, \mathcal{S}(E), [\omega_1]^{\aleph_1})$. Moreover, if $I \subseteq I'$, $E \subseteq S_0$, then $\spadesuit({}^{(\omega)}\mathcal{F}_{\mathrm{wJ},I'}, \mathcal{S}(E), [\omega_1]^{\aleph_1})$ implies both principles $\spadesuit({}^{(\omega)}\mathcal{F}_{\mathrm{wJ},I}, \mathcal{S}(E), [\omega_1]^{\aleph_1})$ and $\spadesuit(\mathcal{F}_{\mathrm{wJ},I'}, \mathcal{S}(E), [\omega_1]^{\aleph_1})$, each of which in turn implies $\spadesuit(\mathcal{F}_{\mathrm{wJ},I}, \mathcal{S}(E), [\omega_1]^{\aleph_1})$.

If subsets of ω are identified with their characteristic functions, then $\mathcal{P}(\omega)$ can be treated as the topological space ${}^{\omega}\{0,1\}$ with the product topology, and it makes sense to talk about subsets of $\mathcal{P}(\omega)$ (e.g., ideals) of the first Baire category.

Here is an important example: For $x \subseteq \omega$, define the *upper density* of x by $d(x) = \limsup_{n\in\omega} |x \cap n|/n$. Then the ideal $I_1 = \{x \subset \omega : d(x) = 0\}$ is an ideal on ω, called the *ideal of density* 0 *sets.* The following characterization, due to S. A. Jalali-Naini and M. Talagrand [6], implies that I_1 is of the first Baire category.

Fact 15. *Let $I \subseteq \mathcal{P}(\omega)$ be an ideal on ω such that* Fin $\subseteq I$. *Then I is of the first Baire category if and only if there exists a sequence $(z_n)_{n\in\omega}$ of pairwise disjoint finite subsets of ω such that for all $x \in I$ the set $\{n \in \omega : z_n \subseteq x\}$ is finite.*

It is not true, even under CH, that for any two proper ideals I, I' the principles $\spadesuit(\mathcal{F}_{\mathrm{wJ},I}, \mathcal{S}, [\omega_1]^{\aleph_1})$ and $\spadesuit(\mathcal{F}_{\mathrm{wJ},I'}, \mathcal{S}, [\omega_1]^{\aleph_1})$ are equivalent. In particular, if I is maximal, then $\spadesuit(\mathcal{F}_{\mathrm{wJ,Fin}}, \mathcal{S}, [\omega_1]^{\aleph_1})$ and $\spadesuit(\mathcal{F}_{\mathrm{wJ},I}, \mathcal{S}, [\omega_1]^{\aleph_1})$ are not equivalent. However, it seems possible that such equivalences may hold for many, or even all, ideals of the first Baire category.

Theorem 16. *Let $E \subseteq S_0$, and let I be an ideal on ω of the first Baire category such that* Fin $\subseteq I$. *Then $\diamondsuit^{\mathrm{p}}(E)$ implies $\spadesuit({}^{(\omega)}\mathcal{F}_{\mathrm{wJ},I}, \mathcal{S}(E), [\omega_1]^{\aleph_1})$.*

Proof. Let $\langle a_\alpha^n : \alpha \in S_0',\ n \in \omega\rangle$ be a sequence that witnesses $\diamondsuit^{\mathrm{p}}(E)$, let I be an ideal as in the assumptions, and let $(z_n)_{n\in\omega}$ be a sequence as in

Fact 15 for I. Now construct recursively s_α^k with $s_\alpha^k < s_\alpha^{k+1}$ for $\alpha \in S_0$, $k \in \omega$ such that for all $\alpha \in S_0'$, $n \in \omega$, and $k \in z_n$ we have $s_\alpha^k \in a_\alpha^{n+1} \setminus \sup a_\alpha^n$. This is possible by point (b) of Definition 10.

The verification that the resulting sequence $\langle s_\alpha^k : \alpha \in S_0,\ k \in \omega\rangle$ witnesses $\spadesuit({}^{(\omega)}\mathcal{F}_{\mathrm{wJ},I}, \mathcal{S}(E), [\omega_1]^{\aleph_1})$ is routine. ∎

Note that Remark 14 and Theorem 16 give the following.

COROLLARY 17. $\diamondsuit^{\mathrm{p}}$ *implies* $\clubsuit_{\mathrm{J}}$.

Let us conclude this note by showing that, similarly to the well known result of [3] for $\clubsuit_J$, the principle $\clubsuit_{\mathrm{wJ}^2}$ holds in any model obtained by adding a single Cohen real.

THEOREM 18. *Let V be a model of (a sufficiently large fragment of)* ZFC, *and let $\mathbb{P}$ be a forcing notion for adding a single Cohen real. Then* $V^{\mathbb{P}} \models \clubsuit_{\mathrm{wJ}^2}$.

Proof. In V, for each $\alpha \in S_0$, fix a bijection $f_\alpha : \omega \to \alpha$. Since all countable forcing notions are equivalent, we may assume that $\mathbb{P} = {}^{<\omega}\omega$ is the set of all functions from finite ordinals into ω, partially ordered by reverse inclusion. Let $g : \omega \to \omega$ be such that $\{g{\restriction}n : n \in \omega\}$ is a V-generic ultrafilter in $\mathbb{P}$, and let $\dot{g}$ be a $\mathbb{P}$-name for g. We can identify $V^{\mathbb{P}}$ with $V[g]$.

In $V[g]$ we define, for each $\alpha \in S_0$, a sequence $\langle \ell_\alpha(n) : n \in \omega\rangle$ as follows:

- $\ell_\alpha(0) = 0$;
- $\ell_\alpha(n+1) = \min\{\ell : f_\alpha(g(\ell)) > f_\alpha(g(\ell_\alpha(n)))\}$.

We let $s_\alpha^n = f_\alpha(g(\ell_\alpha(n)))$ for all $n \in \omega$. A simple genericity argument shows that $\ell_\alpha(n)$ is well defined for all $n \in \omega$ and that the sequence $(s_\alpha^n)_{n\in\omega}$ is increasing and cofinal in α.

Now suppose $X \in V[g]$ and $V[g] \models X \in [\omega_1]^{\aleph_1}$. If $\dot{X}$ is a $\mathbb{P}$-name for X, then $X = \bigcup_{n\in\omega}\{\xi : g{\restriction}n \Vdash \check{\xi} \in \dot{X}\}$, and hence there exists an uncountable set $Y \subseteq X$ such that $Y \in V$. Consider $\alpha \in S_0$ such that $\sup(Y \cap \alpha) = \alpha$. It is easy to see that for all $p \in \mathbb{P}$ and $m \in \omega$,

$$p \nVdash \neg\exists n > m\ (\{\dot{s}_\alpha^n, \dot{s}_\alpha^{n+1}\} \subset \check{Y}),$$

and it follows that

$$\mathbb{P} \Vdash |\{n \in \omega : \{\dot{s}_\alpha^n, \dot{s}_\alpha^{n+1}\} \subset \check{Y}\}| = \aleph_0.$$

Thus

$$\mathbb{P} \Vdash \text{“}\langle \dot{s}_\alpha^n : \alpha \in S_0,\ n \in \omega\rangle \text{ is a } \clubsuit_{\mathrm{wJ}^2}\text{-sequence.”} \ ∎$$

Since adding a single Cohen real to a model where $\diamondsuit$ fails does not make $\diamondsuit$ true, we get the following.

COROLLARY 19. *The pseudodiamond principle $\diamondsuit^{\mathrm{p}}$ is strictly weaker than $\diamondsuit$ and strictly stronger than* CH.

Appendix. For completeness, we include proofs of some of the examples presented earlier in this paper. These proofs are based on the standard arguments of the classical theorems generalized by these examples.

Proof of Lemma 2. Let $E, \mathcal{F}$ be as in the assumption, and suppose $\langle F_\alpha : \alpha \in S_0 \rangle \in \mathcal{F}$ witnesses that $\spadesuit(\mathcal{F}, \mathcal{P}(E) \setminus \{\emptyset\}, [\omega_1]^{\aleph_1})$ holds. Now suppose towards a contradiction that $X \in [\omega_1]^{\aleph_1}$ is such that the set $N = \{\alpha \in E : X \cap \alpha \in F_\alpha\}$ is nonstationary. Let $C \in \mathcal{C}$ be such that $C \cap N = \emptyset$. Let Y be an uncountable subset of X such that for all $\xi_0, \xi_1 \in Y$ with $\xi_0 < \xi_1$ there exists $\gamma \in C$ such that $\xi_0 < \gamma < \xi_1$. Then $Y \cap \alpha$ is bounded in α and hence $Y \cap \alpha \notin F_\alpha$ for all $\alpha \in E \setminus C$. But if $\alpha \in E \cap C$, then $Y \cap \alpha \subseteq X \cap \alpha \notin F_\alpha$, and by the monotonicity assumption on F_α, again we have $Y \cap \alpha \notin F_\alpha$. Thus the set $\{\alpha \in E : Y \cap \alpha \in F_\alpha\}$ is empty, which contradicts the choice of $\langle F_\alpha : \alpha \in S_0 \rangle$. ■

Proof of Example 2. Assume CH, let E be a stationary subset of S_0, and let $\overline{F} = \langle F_\alpha : \alpha \in S_0 \rangle \in \mathcal{F}_{\text{filt}}$ be a sequence witnessing $\spadesuit(\mathcal{F}_{\text{filt}}, \mathcal{P}(E) \setminus \{\emptyset\}, [\omega_1]^{\aleph_1})$. By Lemma 2, we may assume that $\overline{F}$ witnesses $\spadesuit(\mathcal{F}_{\text{filt}}, \mathcal{S}(E), [\omega_1]^{\aleph_1})$. For a filter F of subsets of a set A and an indexed family $X = \{x_a : a \in A\}$ we define $\liminf_F X = \{y : \{a : y \in x_a\} \in F\}$. Let $\langle a_\alpha : \alpha < \omega_1 \rangle$ be an enumeration of all countable subsets of ω_1 such that every $a \in [\omega_1]^{\le\aleph_0}$ is listed cofinally often. For $\alpha \in S_0$ we define $b_\alpha = \liminf_{F_\alpha} \{a_\beta : \beta < \alpha\}$.

The following claim completes the argument.

Claim 20. *$\langle b_\alpha : \alpha \in S_0 \rangle$ is a $\diamondsuit(E)$-sequence.*

Proof. Let $A \subseteq \omega_1$ and let $C \in \mathcal{C}$. We need to show that there exists $\alpha \in E \cap C$ such that $A \cap \alpha = b_\alpha$. Recursively construct an increasing transfinite sequence $\langle \delta_\beta : \beta < \omega_1 \rangle$ of countable ordinals such that for all $\beta < \omega_1$ we have:

(i) $a_{\delta_{\beta+1}} = A \cap \delta_\beta$, and
(ii) there exists $\gamma \in C$ with $\delta_\beta < \gamma < \delta_{\beta+1}$.

The set $B = \{\delta_{\beta+1} : \beta < \omega_1\}$ can be considered a code for the set A. By the choice of $\overline{F}$, there exists $\alpha \in E$ such that $B \cap \alpha \in F_\alpha$. Then $B \cap \alpha$ is cofinal in α, and (ii) implies that $\alpha \in C$. From (i) and the definition of b_α it follows that $b_\alpha \cap \delta_\beta = A \cap \delta_\beta$ for all $\delta_\beta < \alpha$. Since B is cofinal in α, the latter implies that $b_\alpha = A \cap \alpha$, which concludes the proof of the claim and hence of Example 2. ■

Proof of Example 3(2). Assume CH and let E, $\mathcal{F}$ be as in the assumptions. It suffices to show that $\spadesuit(\mathcal{F}^{(\omega)}, \mathcal{S}(E), [\omega_1]^{\aleph_1})$ implies $\diamondsuit^-(E)$. Let $\{\overline{F}_n : n \in \omega\} = \langle F_{\alpha,n} : \alpha \in S_0,\ n \in \omega \rangle$ be sequences that witness $\spadesuit(\mathcal{F}^{(\omega)}_{\text{filt}}, \mathcal{S}(E), [\omega_1]^{\aleph_1})$. Let $\langle a_\alpha : \alpha < \omega_1 \rangle$ be an enumeration of all countable subsets of ω_1 such that every $a \in [\omega_1]^{\le\aleph_0}$ is listed cofinally often. For

$\alpha \in S_0$ we define $b_\alpha = \{\liminf_{F_{\alpha,n}}\{a_\beta : \beta < \alpha\} : n \in \omega\}$. The proof that $\langle b_\alpha : \alpha \in S_0\rangle$ is a $\diamondsuit^-(E)$-sequence proceeds exactly as the proof of Claim 20. ■

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Department of Mathematics
Ohio University
Athens, OH 45701, U.S.A.
E-mail: just@math.ohiou.edu

Received 23 January 2000;
in revised form 10 April 2001