\textbf{-like principles under CH}

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\textbf{Abstract.} Some relatives of the Juhász Club Principle are introduced and studied in the presence of CH. In particular, it is shown that a slight strengthening of this principle implies the existence of a Suslin tree in the presence of CH.

Jensen’s $\Diamond$-principle is an important strengthening of the Continuum Hypothesis that allows for a number of interesting constructions that cannot be carried out in ZFC + CH alone. Most notably, $\Diamond$ implies the existence of a Suslin tree. In [4], Ostaszewski introduced a weakening of $\Diamond$ known as $\clubsuit$ that suffices for many of the constructions made possible by $\Diamond$. Devlin showed that $\clubsuit + CH$ is equivalent to $\Diamond$ (see [4]). However, $\clubsuit$ does not imply CH and is thus strictly weaker than $\Diamond$ (for an elegant proof, see [5], page 43). In [3], Juhász introduced a weakening of $\clubsuit$ that will be denoted here by $\spadesuit_1$. This principle is not equivalent to $\Diamond$ even under CH, and yet it retains some of the combinatorial power of the latter principle. Here we study some natural modifications of $\spadesuit_1$. In particular, we show that some of these modifications, while still not equivalent to $\Diamond$ under CH, do imply the existence of a Suslin tree in the presence of CH.

Throughout this note, let $S_0 = \omega_1 \cap \text{LIM} \setminus \{0\}$. The letter $E$ will denote stationary subsets of $S_0$, and $\mathcal{S}(E)$ will denote the family of all stationary subsets of $E$. Instead of $\mathcal{S}(S_0)$ we will write $\mathcal{S}$. The family of all closed unbounded subsets of $S_0$ will be denoted by $\mathcal{C}$. A $\heartsuit(E)$-sequence is a sequence $\langle s_\alpha : \alpha \in S_0 \rangle$ such that $s_\alpha$ is a cofinal subset of order type $\omega$ for all $\alpha \in S_0$, and for each $X \in [\omega_1]^{<\omega}$ there exists $\alpha \in E$ with $s_\alpha \subset X$. The $\heartsuit(E)$-principle asserts the existence of a $\heartsuit(E)$-sequence. Instead of $\heartsuit(S_0)$ we simply write $\heartsuit$.

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In order to better delineate where these principles fit in the general scheme of things, we first introduce a convenient notation for the study of \(\diamond\)-like principles.

**Definition 1.** Let \(G \subseteq \mathcal{P}(S_0), \ A \subseteq \mathcal{P}(\omega_1)\), and let \(\mathcal{F}\) be a family of sequences \(\mathcal{F} = \langle F_\alpha : \alpha \in S_0 \rangle\) such that \(F_\alpha \subseteq \mathcal{P}(\alpha)\) for every \(\alpha \in S_0\). We say that \(\mathcal{F}\) is a \(\clubsuit(G, A)\)-sequence if for all \(A \in \mathcal{A}\) there exists \(G \in \mathcal{G}\) such that \(A \cap \alpha \in F_\alpha\) for all \(\alpha \in G\). We say that \(\clubsuit(\mathcal{F}, G, \mathcal{A})\) holds if there exists a \(\clubsuit(G, A)\)-sequence \(\mathcal{F}\) such that \(\mathcal{F} \in \mathcal{F}\).

**Example 1.** (1) Let \(\mathcal{F}_1\) be the family of all sequences \(\mathcal{F} = \langle F_\alpha : \alpha \in S_0 \rangle\) such that \(F_\alpha \subseteq \mathcal{P}(\alpha)\) and \(|F_\alpha| = 1\) for each \(\alpha \in S_0\), and let \(\mathcal{F}_{S_0}\) be the family of all sequences \(\mathcal{F} = \langle F_\alpha : \alpha \in S_0 \rangle\) such that \(F_\alpha \subseteq \mathcal{P}(\alpha)\) and \(|F_\alpha| = \aleph_0\) for each \(\alpha \in S_0\). Then \(\clubsuit(\mathcal{F}_1, \mathcal{S}, \mathcal{P}(\omega_1))\) is the same as the ordinary diamond-principle \(\diamond\), \(\clubsuit(\mathcal{F}_{S_0}, \mathcal{S}, \mathcal{P}(\omega_1))\) is the same as Kunen’s \(\diamond^-\), and \(\clubsuit(\mathcal{F}_{S_0}, \mathcal{C}, \mathcal{P}(\omega_1))\) is the same as \(\diamond^*\). Moreover, for a stationary \(E \subseteq S_0\), the principle \(\clubsuit(\mathcal{F}_1, \mathcal{S}(E), \mathcal{P}(\omega_1))\) is the same as \(\diamond(E)\).

(2) Let \(E\) be a stationary subset of \(S_0\) and let \(\mathcal{F}_\clubsuit\) be the family of all sequences \(\mathcal{F} = \langle F_\alpha : \alpha \in S_0 \rangle\) such that for each \(\alpha \in S_0\) there exists a set \(s_\alpha \subseteq \alpha\) such that \(\text{ot}(s_\alpha) = \omega\), \(\text{sup} s_\alpha = \alpha\), and \(F_\alpha = \{a \subseteq \alpha : s_\alpha \subseteq a\}\). Then \(\clubsuit(\mathcal{F}_\clubsuit, \mathcal{P}(E) \setminus \{\emptyset\}, [\omega_1]^\aleph_1)\) is the same as \(\clubsuit(E)\).

(3) Let \(E\) be a stationary subset of \(S_0\), and let \(\mathcal{F}_{\clubsuit_\omega}\) be the family of all sequences \(\mathcal{F} = \langle F_\alpha : \alpha \in S_0 \rangle\) such that for each \(\alpha \in S_0\) there exists a set \(s_\alpha \subseteq \alpha\) such that \(\text{ot}(s_\alpha) = \omega\), \(\text{sup} s_\alpha = \alpha\), and \(F_\alpha = \{a \subseteq \alpha : |s_\alpha \setminus a| < \aleph_0\}\). Then \(\clubsuit(\mathcal{F}_{\clubsuit_\omega}, \mathcal{P}(E) \setminus \{\emptyset\}, [\omega_1]^\aleph_1)\) is the same as the principle \(\clubsuit_\omega(E)\) of [2] and \(\clubsuit^1\) of [1].

(4) Let \(\mathcal{F}_{\text{filt}}\) be the family of all sequences \(\mathcal{F} = \langle F_\alpha : \alpha \in S_0 \rangle\) such that for all \(\alpha \in S_0\), \(F_\alpha\) is a filter on \(\alpha\) that consists of cofinal subsets of \(\alpha\). Then for every \(E \subseteq S_0\) we have: \(\clubsuit(E) \Rightarrow \clubsuit_\omega(E) \Rightarrow \clubsuit(\mathcal{F}_{\text{filt}}, \mathcal{P}(E) \setminus \{\emptyset\}, [\omega_1]^\aleph_1)\).

It is a historical accident that “official” definitions of \(\diamond\)-principles take the form \(\clubsuit(\mathcal{F}, \mathcal{S}(E), \mathcal{A})\) while definitions of \(\clubsuit\)-principles take the form \(\clubsuit(\mathcal{F}, \mathcal{P}(E) \setminus \{\emptyset\}, \mathcal{A})\); in all cases of interest to us the two formulations are equivalent. This well known observation can be formalized in our terminology as follows:

**Lemma 2.** Let \(\mathcal{F}\) be a family of sequences such that for each \(\langle F_\alpha : \alpha \in S_0 \rangle\) \(\in \mathcal{F}\) and each \(\alpha \in S_0\), the family \(F_\alpha\) consists of cofinal subsets of \(\alpha\), and if \(a \subseteq b \subseteq \alpha, a \in F_\alpha\), then \(b \in F_\alpha\). Let \(E\) be a stationary subset of \(S_0\). Then \(\clubsuit(\mathcal{F}, \mathcal{P}(E) \setminus \{\emptyset\}, [\omega_1]^\aleph_1)\) implies \(\clubsuit(\mathcal{F}, \mathcal{S}(E), [\omega_1]^\aleph_1)\).

A proof of the above lemma will be given in the appendix.

**Example 2.** The result of Devlin mentioned above can be generalized as follows: Let \(E\) be a stationary subset of \(S_0\). Then \(\clubsuit(\mathcal{F}_{\text{filt}}, \mathcal{P}(E) \setminus \{\emptyset\}, [\omega_1]^\aleph_1) + \text{CH} \Rightarrow \diamond(E)\). A proof will be given in the appendix.
Kunen has shown that for \( E \subseteq S_0 \), the principles \( \Box^-(E) \) and \( \Box(E) \) are equivalent. In our terminology this means that \( \clubsuit(\mathcal{F}_1, \mathcal{S}(E), \mathcal{P}(\omega_1)) \) is equivalent to \( \clubsuit(\mathcal{F}_{\mathcal{R}_0}, \mathcal{S}(E), \mathcal{P}(\omega_1)) \). It is natural to ask to what extent this result generalizes to \( \clubsuit \)-like principles. Formally, given a family \( \mathcal{F} \) of \( S_0 \)-sequences, let us define \( \mathcal{F}(\omega) = \{ (H_\alpha : \alpha \in S) : \exists (F_\alpha, n : \alpha \in S_0, n \in \omega) \forall n \in \omega \forall \alpha \in S_0 (\langle F_\alpha, n : \alpha \in S_0 \rangle \in \mathcal{F} \wedge H_\alpha = \bigcup_{n \in \omega} F_\alpha(n) \} \). Note that, in particular, \( \mathcal{F}_1(\omega) = \mathcal{F}_{\mathcal{R}_0} \). The question now is for which \( \mathcal{F} \) and \( \mathcal{A} \) the implication \( \clubsuit(\mathcal{F}(\omega), \mathcal{S}(E), \mathcal{A}) \Rightarrow \clubsuit(\mathcal{F}, \mathcal{S}(E), \mathcal{A}) \) holds.

**Example 3.** (1) \( \clubsuit(\mathcal{F}_{\mathcal{E}_1}, \mathcal{S}, [\omega_1]^\mathbb{R}_1) \) does not imply \( \clubsuit(\mathcal{F}_{\mathcal{E}_1}, \mathcal{S}, [\omega_1]^\mathbb{R}_1) \). To see this, recall that in [1], it is shown that \( \clubsuit^1 \) (i.e., \( \clubsuit(\mathcal{F}_{\mathcal{E}_1}, \mathcal{S}, [\omega_1]^\mathbb{R}_1) \)) does not imply \( \clubsuit \). It is easy to see that \( \clubsuit(\mathcal{F}_{\mathcal{E}_1}, \mathcal{S}, [\omega_1]^\mathbb{R}_1) \) implies \( \clubsuit(\mathcal{F}_{\mathcal{E}_1}, \mathcal{S}, [\omega_1]^\mathbb{R}_1) \).

(2) Under CH, an analogue of Kunen’s Theorem for \( \clubsuit \)-like principles does hold: Let \( \mathcal{F} \subseteq \mathcal{F}_{\mathcal{F}_{\mathcal{E}_1}} \) and let \( E \subseteq S_0 \). Then \( \text{CH} + \clubsuit(\mathcal{F}(\omega), \mathcal{S}(E), [\omega_1]^\mathbb{R}_1) \) implies \( \clubsuit(\mathcal{F}, \mathcal{S}(E), [\omega_1]^\mathbb{R}_1) \). A proof of this observation will be given in the appendix.

(3) A similar observation can be made for \( \Box^* \): Let \( \mathcal{F} \subseteq \mathcal{F}_{\mathcal{F}_{\mathcal{E}_1}} \) and let \( E \subseteq S_0 \). Then \( \text{CH} + \clubsuit(\mathcal{F}(\omega), \mathcal{C}, [\omega_1]^\mathbb{R}_1) \) implies \( \Box^* \).

Naturally, the question arises whether Examples 2 and 3(2) have dual versions for ideals. Unfortunately, as the following examples show, the situation for ideals is more complex than the situation for filters.

**Example 4.** (1) Let \( \mathcal{F}_{\mathcal{I}_{\mathcal{F}_{\mathcal{F}_{\mathcal{E}_1}}}} \) denote the family of all sequences \( \langle F_\alpha : \alpha \in S_0 \rangle \) such that, for each \( \alpha \), \( F_\alpha = I_\alpha^+ (= \{ a \subseteq \alpha : a \notin I_\alpha \}) \) for some ideal \( I_\alpha \) of subsets of \( \alpha \) that contains the ideal \( B_\alpha \) of all bounded subsets of \( \alpha \). In particular, if we define \( U = \langle B_\alpha^+ : \alpha \in S_0 \rangle \), then \( U \in \mathcal{F}_{\mathcal{F}_{\mathcal{E}_1}} \) and \( U \) is a \( \clubsuit(\mathcal{C}, [\omega_1]^\mathbb{R}_1) \)-sequence. It follows that \( \clubsuit(\mathcal{F}_{\mathcal{I}_{\mathcal{F}_{\mathcal{F}_{\mathcal{E}_1}}}}, \mathcal{S}, [\omega_1]^\mathbb{R}_1) \) is a theorem of ZFC.

(2) Let \( \mathcal{F}_{\mathcal{F}_{\mathcal{F}_{\mathcal{F}_{\mathcal{E}_1}}}} \) be the family of all sequences \( \langle I_\alpha^+ : \alpha \in S_0 \rangle \) such that for each \( \alpha \), \( I_\alpha \) is a maximal ideal on \( \alpha \) that extends \( B_\alpha \). Then \( \mathcal{F}_{\mathcal{F}_{\mathcal{F}_{\mathcal{E}_1}}} \subseteq \mathcal{F}_{\mathcal{F}_{\mathcal{E}_1}} \). On the other hand, for each \( \alpha \), \( I_\alpha^+ \) is equal to the dual filter \( I_\alpha^+ \), and thus \( \mathcal{F}_{\mathcal{F}_{\mathcal{F}_{\mathcal{E}_1}}} \subseteq \mathcal{F}_{\mathcal{F}_{\mathcal{E}_1}} \). Now Example 2 implies that \( \text{CH} + \clubsuit(\mathcal{F}_{\mathcal{F}_{\mathcal{F}_{\mathcal{E}_1}}}, \mathcal{S}, [\omega_1]^\mathbb{R}_1) \) is equivalent to \( \Box^* \).

Does Example 3(2) have an analogue for ideals? One could formalize this question by asking whether for all \( \mathcal{F} \subseteq \mathcal{F}_{\mathcal{I}_{\mathcal{F}_{\mathcal{F}_{\mathcal{E}_1}}}} \) and \( E \subseteq S_0 \), in the presence of CH, the statements \( \text{CH} + \clubsuit(\mathcal{F}(\omega), \mathcal{S}(E), [\omega_1]^\mathbb{R}_1) \) and \( \text{CH} + \clubsuit(\mathcal{F}, \mathcal{S}(E), [\omega_1]^\mathbb{R}_1) \) are equivalent. However, note that for any \( \mathcal{F} \subseteq \mathcal{F}_{\mathcal{I}_{\mathcal{F}_{\mathcal{E}_1}}} \) we have \( \mathcal{F}(\omega) \subseteq \mathcal{F}_{\mathcal{I}_{\mathcal{F}_{\mathcal{E}_1}}} \), and while the above question may be of some interest, it does not have quite the same flavor as Example 3(2). The following approach is closer in spirit to Example 3(2). For a family \( \mathcal{F} \) of sequences \( \langle F_\alpha : \alpha \in S_0 \rangle \), let \( \langle \mathcal{F}_\alpha \rangle \mathcal{F} = \{ (H_\alpha : \alpha \in S_0) : \exists \{ F_n = (F_\alpha, n : \alpha \in S_0, n \in \omega) \subseteq \mathcal{F} \forall \alpha \in S_0 \} \} \).

Now the question about the dual version of Example 3(2) can be formulated as follows.
QUESTION 3. Let $\mathcal{F} \subseteq \mathcal{F}_{id}$. $E \subseteq S_0$. Are the principles $\diamondsuit((\omega)\mathcal{F}, \mathcal{S}(E), [\omega_1]^{\aleph_1})$ and $\diamondsuit(\mathcal{F}, \mathcal{S}(E), [\omega_1]^{\aleph_1})$ equivalent? Is $CH + \diamondsuit((\omega)\mathcal{F}, \mathcal{S}(E), [\omega_1]^{\aleph_1})$ equivalent to the statement $CH + \diamondsuit(\mathcal{F}, \mathcal{S}(E), [\omega_1]^{\aleph_1})$?

To get a better appreciation of Question 3, let us consider a particularly interesting instance of it. In [3], Juhasz introduced the following weakening of $\diamondsuit$.

DEFINITION 4. A $\diamondsuit_J$-sequence is a sequence $\langle s^k_\alpha : \alpha \in S_0, \, k \in \omega \rangle$ such that:

(a) $\forall \alpha \in S_0, k \in \omega$ ($s^k_\alpha$ is a cofinal subset of $\alpha$ of order type $\omega$);
(b) $\forall \alpha \in S_0, k < m < \omega$ ($s^k_\alpha \cap s^m_\alpha = \emptyset$);
(c) $\forall X \in [\omega_1]^{\aleph_1}$ $\exists \alpha \in S_0 \forall k \in \omega$ ($|s^k_\alpha \cap X| = \aleph_0$).

We will abbreviate the statement "a $\diamondsuit_J$-sequence exists" by $\diamondsuit_J$ and call it the Juhasz Club Principle.

REMARK 5. The original definition of $\diamondsuit_J$ listed the following additional requirement: $\forall \alpha \in S_0$ (ot($\bigcup_{k \in \omega} s^k_\alpha$) = $\omega$). This requirement can be safely dropped for our purposes. To see this, let $\langle s^k_\alpha : \alpha \in S_0, \, k \in \omega \rangle$ be a $\diamondsuit_J$-sequence. For each $\alpha \in S_0$, fix an increasing sequence $(\beta^k_\alpha)_{k \in \omega}$ of ordinals with limit $\alpha$. Let $t^k_\alpha = s^k_\alpha \setminus \beta^k_\alpha$. Then ot($\bigcup_{k \in \omega} t^k_\alpha$) = $\omega$, as required in the original definition.

DEFINITION 6. For $\alpha \in S_0$ and an unbounded set $s \subseteq \alpha$, let $J(s)$ denote the ideal $\{a \subseteq \alpha : \sup(a \cap s) < \alpha\}$. Now let $\mathcal{F}_{wJ}$ denote the family of all sequences $\langle J(s^*_\alpha) : \alpha \in S_0 \rangle$ such that, for each $\alpha$, $s^*_\alpha$ is a cofinal subset of $\alpha$ of order type $\omega$. Instead of $\diamondsuit(\mathcal{F}_{wJ}, \mathcal{S}(E), [\omega_1]^{\aleph_1})$ we will write $\diamondsuit_{wJ}(E)$.

Clearly, $\mathcal{F}_{wJ} \subset \mathcal{F}_{id}$. Moreover, it follows from Lemma 2 that $\diamondsuit_J$ is the same as $\diamondsuit((\omega)\mathcal{F}_{wJ}, \mathcal{S}, [\omega_1]^{\aleph_1})$. Thus the following question is an instance of Question 3:

QUESTION 7. Can one show in ZFC or in ZFC + CH that the principles $\diamondsuit_J$ and $\diamondsuit_{wJ}$ are equivalent?

Somewhat surprisingly, at least the CH part of the above question becomes much easier if one makes a seemingly very minor change to the definition of $\mathcal{F}_{wJ}$.

DEFINITION 8. Let $E \subseteq S_0$. A $\diamondsuit_{wJ^2}(E)$-sequence is a sequence $\langle s^n_\alpha : \alpha \in S_0, n \in \omega \rangle$ such that:

(a) $\forall \alpha \in S_0 \forall n \in \omega$ ($s^n_\alpha < s^n_{\alpha+1} \land \sup\{s^n_n : n \in \omega\} = \alpha$), and
(b) $\forall X \in [\omega_1]^{\aleph_1}$ $\{\alpha \in E : |\{n : \{s^n_\alpha, s^n_{\alpha+1}\} \subset X\}| = \aleph_0\} \in \mathcal{S}(E)$.

We will abbreviate the statement "a $\diamondsuit_{wJ^2}(E)$-sequence exists" by $\diamondsuit_{wJ^2}(E)$.

In order to appreciate the strength of $\diamondsuit_{wJ^2}$, consider the following:
Theorem 9. CH + \( \clubsuit_{w, \omega^2} \) implies the existence of a Suslin tree.

Proof. Assume CH and let \( \{a_\alpha : \alpha \in S_0\} \) be an enumeration of all countable subsets of \( \omega_1 \) such that every element of \([\omega_1]^{\leq \aleph_0}\) is listed cofinally often. Let \( \langle s^n_\alpha : \alpha \in S_0, n \in \omega \rangle \) be a \( \clubsuit_{w, \omega^2} \)-sequence. We will construct a Suslin tree \( T = \langle \omega_1, <_T \rangle \). The \( \alpha \)th level of \( T \) will be denoted by \( T(\alpha) \).

Conversely, for \( \xi \in \omega_1 \), \( L(\xi) \) will denote the unique \( \alpha \) such that \( \xi \in T(\alpha) \). Moreover, \( \bigcup_{\beta < \alpha} T(\beta) \) will be denoted by \( T(\alpha) \).

We will construct the \( T(\alpha) \)'s in such a way that for each \( \alpha \in S_0 \), the set of nodes of \( T(\alpha) \) is \( \alpha \), and so that \( T(\alpha) \) is tall and splitting, i.e., each node has at least two successors at each higher level.

Suppose \( T(\alpha) \) has been constructed. The important step in the construction of \( T(\alpha+1) \) is to decide which cofinal branches of \( T(\alpha) \) will be extended at level \( T(\alpha) \); the rest of the construction is routine and will be left to the reader. For each node \( \xi \in \alpha \) we need to put one node \( n(\alpha, \xi) \) into \( T(\alpha) \) such that \( \xi <_T n(\alpha, \xi) \). Conversely, \( T(\alpha) \) will precisely be the set of nodes \( \{n(\alpha, \xi) : \xi \in \alpha\} \).

So pick \( \xi \in \alpha \). We construct recursively a sequence \( \langle \xi_n \rangle_{n \in \omega} \) as follows:

- \( \xi_0 = \xi \);
- if \( s^n_\alpha < L(\xi_n) \), let \( \xi_{n+1} = \xi_n \);
- if \( L(\xi_n) \leq s^n_\alpha \), pick \( \xi_{n+1} \in T(s^{n+1}_\alpha) \) in such a way that \( \xi_n <_T \xi_{n+1} \) and, if possible, there exists \( \eta \in a_{s^{n+1}_\alpha} \) such that \( \eta <_T \xi_{n+1} \).

Finally, put \( n(\alpha, \xi) \) on top of the branch of \( T(\alpha) \) generated by \( \{\xi_n : n \in \omega\} \).

In order to prove that this construction works it suffices to show that there is no uncountable maximal antichain in \( T \). Suppose towards a contradiction that \( A \) is an uncountable maximal antichain in \( T \). Note that the set \( C_A = \{\gamma : A \cap \gamma \text{ is maximal in } T\} \) is unbounded in \( \omega_1 \). This allows us to construct a set \( X = \{\beta_\eta : \eta < \omega_1\} \subseteq S_0 \) as follows:

- \( \beta_0 = \omega \);
- for limit ordinals \( 0 < \delta < \omega_1 \), let \( \beta_\delta = \sup\{\beta_\eta : \eta < \delta\} \);
- given \( \beta_\eta \), let \( \beta_{\eta+1} > \beta_\eta \) be such that for some \( \gamma \in C_A \) with \( \beta_\eta < \gamma < \beta_{\eta+1} \) we have \( a_{s^{\eta+1}_\alpha} = A \cap \gamma \).

Now let \( \alpha \in S_0 \) be such that for infinitely many \( n \in \omega \) we have \( s^n_\alpha, s^{n+1}_\alpha \in X \). Consider \( n(\alpha, \xi) \in T(\alpha) \). By construction, there exist (infinitely many) \( n \in \omega \) with \( \xi <_T \xi_n <_T \xi_{n+1} <_T n(\alpha, \xi) \) and both \( s^n_\alpha = L(\xi_n) \in X \) and \( s^{n+1}_\alpha = L(\xi_{n+1}) \in X \). Now by the construction of \( X \), \( a_{s^{n+1}_\alpha} \) is a subset of \( A \) and is a maximal antichain in \( T(\gamma) \) for some \( \gamma \) with \( s^n_\alpha < \gamma < s^{n+1}_\alpha \). Thus it is possible to choose \( \xi_{n+1} \) above some element of \( a_{s^{n+1}_\alpha} \), and the construction rules of \( T \) force us to do so. It follows that every element of \( T(\alpha) \) sits above some node of \( A \cap \alpha \), and thus \( A \) cannot be an uncountable maximal antichain. \( \blacksquare \)
The author of this note does not know whether CH +  ♠_{w,J} implies the existence of a Suslin tree, and in particular, whether  ♠_{w,J} and  ♠_{w,J^2} are equivalent, even under CH.

While the definition of  ♠_{w,J^2} may look somewhat artificial, in the presence of CH we can find a rather elegant characterization.

**Definition 10.** Let  \( E \subseteq S'_0 \) (where  \( S'_0 \) is the set of countable limits of countable limit ordinals). The pseudodiamond  \( ♦^P(E) \) is the following statement: There exists a sequence  \( \langle a^n_\alpha : \alpha \in S'_0, n \in \omega \rangle \) such that

(a)  \( \forall \alpha \in S'_0 \ \forall n \in \omega \ (a^n_\alpha \subseteq \alpha) \);
(b)  \( \forall \alpha \in S'_0 \ \forall n \in \omega \ (|a^{n+1}_\alpha \setminus \sup a^n_\alpha| = \aleph_0) \);
(c)  \( \forall \alpha \in S'_0 \ (\sup \bigcup_{n \in \omega} a^n_\alpha = \alpha) \);
(d)  \( \forall X \in [\omega_1]^\aleph_1 \ (\{ \alpha \in S'_0 : |\{ n \in \omega : X \cap \sup a^n_\alpha = a^n_\alpha \}| = \aleph_0 \} \in S(E)) \).

**Remark 11.** Note that if one omits condition (b) in the above definition, then the resulting principle is equivalent to CH. However,  \( ♦^P \) is much stronger than that.

**Theorem 12.** Let  \( E \subseteq S_0 \). The following are equivalent:

(a)  \( ♦^P(E) \).
(b)  \( \text{CH} +  ♠_{w,J^2}(E) \).

_Proof._ Assume  \( ♦^P(E) \). Then CH holds by Remark 11. Let  \( \langle a^n_\alpha : \alpha \in S'_0, n \in \omega \rangle \) be a sequence that witnesses  \( ♦^P(E) \). Let us construct a sequence  \( \langle s^n_\alpha : \alpha \in S_0, n \in \omega \rangle \) as follows: For each  \( \alpha \in S'_0 \) and  \( n \in \omega \), we pick  \( s^{2n}_\alpha, s^{2n+1}_\alpha \in a^{n+1}_\alpha \) in such a way that  \( \sup a^n_\alpha < s^{2n}_\alpha < s^{2n+1}_\alpha \leq \sup a^{n+1}_\alpha \). For  \( \alpha \in S_0 \setminus S'_0 \) and  \( n \in \omega \), we pick  \( s^n_\alpha \) so as to satisfy condition (a) of Definition 8. Clearly,  \( \langle s^n_\alpha : \alpha \in S_0, n \in \omega \rangle \) is a  \(  ♠_{w,J^2}(E) \)-sequence.

Now assume  \( \text{CH} +  ♠_{w,J^2}(E) \) holds, and let  \( \langle s^n_\alpha : \alpha \in S_0, n \in \omega \rangle \) be a  \(  ♠_{w,J^2}(E) \)-sequence. Fix an enumeration  \( \{ b_\alpha : \alpha \in \omega_1 \} \) of all countable subsets of  \( \omega_1 \) such that every element of  \( [\omega_1]^\leq \aleph_0 \) gets listed cofinally often. For  \( \alpha \in S'_0 \setminus S''_0 \), we define  \( a^n_\alpha \) in such a way that conditions (a) through (c) of Definition 10 hold. For  \( \alpha \in S''_0 \) we proceed as follows: Suppose  \( a^{n_0}_\alpha \) has been defined and  \( \sup a^{n_0}_\alpha < s^{n_0}_\alpha + \omega^2 \). If  \( b_{s^{n_0+1}_\alpha} \) is a subset of  \( s^{n_0+1}_\alpha \) such that  \( |b_{s^{n_0+1}_\alpha} \setminus \sup a^{n_0}_\alpha| = \aleph_0 \), then let  \( a^{n_0+1}_\alpha = b_{s^{n_0+1}_\alpha} \). Otherwise, let  \( a^{n_0+1}_\alpha = \sup(a^{n_0}_\alpha) + \omega \). Note that, in either case,  \( \sup a^{n_0+1}_\alpha < s^{n_0+1}_\alpha + \omega^2 \).

This construction yields  \( a^n_\alpha \)'s that satisfy conditions (a)–(c) of Definition 10. In order to show that condition (d) also holds, let  \( X \in [\omega_1]^\aleph_1 \). Construct a set  \( C = \{ c_\eta : \eta \in \omega_1 \} \subseteq \omega_1 \) as follows:

-  \( c_0 = \omega \);
-  \( c_\delta = \sup\{ c_\eta : \eta < \delta \} \) for countable limit ordinals  \( \delta \);
-  given  \( c_\eta \), let  \( c_{\eta+1} \) be such that  \( b_{c_{\eta+1}} \subseteq c_{\eta+1}, b_{c_{\eta+1}} = X \cap \sup b_{c_{\eta+1}}, \) and  \( |b_{c_{\eta+1}} \setminus \langle c_\eta + \omega^2 \rangle| = \aleph_0 \).
Now it suffices to observe that $C$ is closed unbounded, and if $\alpha \in C \cap S''_0 \cap E$ is such that for infinitely many $n \in \omega$, $\{s^n_\alpha, s^{n+1}_\alpha\} \subseteq C$, then $a^n_\alpha = X \cap \sup a^n_\alpha$ for infinitely many $n \in \omega$, as desired.

To elucidate the connection between $\bullet_{w,1^2}$ and principles $\bullet(\mathcal{F}, \mathcal{S}, [\omega]^{|\mathcal{R}_1|})$ for $\mathcal{F} \subseteq \mathcal{F}_{id}$, let us generalize the definition of $\mathcal{F}_{w,J}$.

**Definition 13.** Let $I$ be an ideal of subsets of $\omega$ such that $I$ contains the ideal $\text{Fin}$ of finite subsets of $\omega$. Let $s = \{s^n : n \in \omega\}$ be a set of ordinals of order type $\omega$, enumerated in increasing order, and let $\alpha = \sup(s)$. Then we define an ideal $J(s, I)$ on $\alpha$ as follows: $J(s, I) = \{a \subseteq \alpha : \{n : s^n \in a\} \subseteq I\}$. Moreover, $\mathcal{F}_{w,J}$ will denote the family of all sequences $(F_\alpha : \alpha \in S_0)$ such that for each $\alpha$ there exists a set $s_\alpha \subseteq \alpha$ of order type $\omega$, with $\sup(s_\alpha) = \alpha$, such that $F_\alpha = J(s_\alpha, I)^+$. 

**Remark 14.** Note that $\mathcal{F}_{w,J}$ is the same as $\mathcal{F}_{w,J,\text{Fin}}$. Thus $\bullet_{w,J}$ is the same as $\bullet(\mathcal{F}_{w,J,\text{Fin}}, \mathcal{S}(E), [\omega]^{|\mathcal{R}_1|})$ and $\bullet_J$ is the same as $\bullet(\omega \mathcal{F}_{w,J,\text{Fin}}, \mathcal{S}(E), [\omega]^{|\mathcal{R}_1|})$. Moreover, if $I \subseteq I'$, $E \subseteq S_0$, then $\bullet(\omega \mathcal{F}_{w,J,\text{Fin}}, \mathcal{S}(E), [\omega]^{|\mathcal{R}_1|})$ implies both principles $\bullet(\omega \mathcal{F}_{w,J}, \mathcal{S}(E), [\omega]^{|\mathcal{R}_1|})$ and $\bullet(\mathcal{F}_{w,J,I'}, \mathcal{S}(E), [\omega]^{|\mathcal{R}_1|})$, each of which in turn implies $\bullet(\mathcal{F}_{w,J,I}, \mathcal{S}(E), [\omega]^{|\mathcal{R}_1|})$.

If subsets of $\omega$ are identified with their characteristic functions, then $\mathcal{P}(\omega)$ can be treated as the topological space $\{0, 1\}$ with the product topology, and it makes sense to talk about subsets of $\mathcal{P}(\omega)$ (e.g., ideals) of the first Baire category.

Here is an important example: For $x \subseteq \omega$, define the upper density of $x$ by $d(x) = \limsup_{n \in \omega} |x \cap n|/n$. Then the ideal $I_1 = \{x \subseteq \omega : d(x) = 0\}$ is an ideal on $\omega$, called the ideal of density 0 sets. The following characterization, due to S. A. Jalali-Naini and M. Talagrand [6], implies that $I_1$ is of the first Baire category.

**Fact 15.** Let $I \subseteq \mathcal{P}(\omega)$ be an ideal on $\omega$ such that $	ext{Fin} \subseteq I$. Then $I$ is of the first Baire category if and only if there exists a sequence $(z_n)_{n \in \omega}$ of pairwise disjoint finite subsets of $\omega$ such that for all $x \in I$ the set $\{n \in \omega : z_n \subseteq x\}$ is finite.

It is not true, even under CH, that for any two proper ideals $I, I'$ the principles $\bullet(\mathcal{F}_{w,J}, \mathcal{S}, [\omega]^{|\mathcal{R}_1|})$ and $\bullet(\mathcal{F}_{w,J,I'}, \mathcal{S}, [\omega]^{|\mathcal{R}_1|})$ are equivalent. In particular, if $I$ is maximal, then $\bullet(\mathcal{F}_{w,J,\text{Fin}}, \mathcal{S}, [\omega]^{|\mathcal{R}_1|})$ and $\bullet(\mathcal{F}_{w,J,I}, \mathcal{S}, [\omega]^{|\mathcal{R}_1|})$ are not equivalent. However, it seems possible that such equivalences may hold for many, or even all, ideals of the first Baire category.

**Theorem 16.** Let $E \subseteq S_0$, and let $I$ be an ideal on $\omega$ of the first Baire category such that $\text{Fin} \subseteq I$. Then $\Diamond^P(E)$ implies $\bullet(\omega \mathcal{F}_{w,J}, \mathcal{S}(E), [\omega]^{|\mathcal{R}_1|})$.

**Proof.** Let $(a^n_\alpha : \alpha \in S'_0, n \in \omega)$ be a sequence that witnesses $\Diamond^P(E)$, let $I$ be an ideal as in the assumptions, and let $(z_n)_{n \in \omega}$ be a sequence as in
Fact 15 for $I$. Now construct recursively $s^k_\alpha$ with $s^k_\alpha < s^{k+1}_\alpha$ for $\alpha \in S_0$, $k \in \omega$ such that for all $\alpha \in S'_0$, $n \in \omega$, and $k \in z_n$ we have $s^k_\alpha \in a^{n+1}_\alpha \setminus \sup a^n_\alpha$. This is possible by point (b) of Definition 10.

The verification that the resulting sequence $\langle s^k_\alpha : \alpha \in S_0, k \in \omega \rangle$ witnesses $\clubsuit(\omega)F_{w,I,S(E),[\omega_1]^{|\mathbb{R}|}}$ is routine. 

Note that Remark 14 and Theorem 16 give the following.

**Corollary 17.** $\Diamond^\mathbb{P}$ implies $\clubsuit_j$.

Let us conclude this note by showing that, similarly to the well known result of [3] for $\clubsuit_j$, the principle $\clubsuit_{w_2}$ holds in any model obtained by adding a single Cohen real.

**Theorem 18.** Let $V$ be a model of (a sufficiently large fragment of) ZFC, and let $\mathbb{P}$ be a forcing notion for adding a single Cohen real. Then $V^\mathbb{P} \models \clubsuit_{w_2}$.

**Proof.** In $V$, for each $\alpha \in S_0$, fix a bijection $f_\alpha : \omega \to \alpha$. Since all countable forcing notions are equivalent, we may assume that $\mathbb{P} = \mathbb{P} = \mathbb{P}$ is the set of all functions from finite ordinals into $\omega$, partially ordered by reverse inclusion. Let $g : \omega \to \omega$ be such that $\{g(n) : n \in \omega\}$ is a $V$-generic ultrafilter in $\mathbb{P}$, and let $\dot{g}$ be a $\mathbb{P}$-name for $g$. We can identify $V^\mathbb{P}$ with $V[g]$.

In $V[g]$ we define, for each $\alpha \in S_0$, a sequence $\langle \ell_\alpha(n) : n \in \omega \rangle$ as follows:

- $\ell_\alpha(0) = 0$;
- $\ell_\alpha(n + 1) = \min\{\ell : f_\alpha(g(\ell)) > f_\alpha(g(\ell_\alpha(n)))\}$.

We let $s^n_\alpha = f_\alpha(g(\ell_\alpha(n)))$ for all $n \in \omega$. A simple genericity argument shows that $\ell_\alpha(n)$ is well defined for all $n \in \omega$ and that the sequence $(s^n_\alpha)_{n \in \omega}$ is increasing and cofinal in $\alpha$.

Now suppose $X \in V[g]$ and $V[g] \models X \in [\omega_1]^{|\mathbb{R}|}$. If $\dot{X}$ is a $\mathbb{P}$-name for $X$, then $X = \bigcup_{n \in \omega} \{\xi : g(n) \forces \xi \in \dot{X}\}$, and hence there exists an uncountable set $Y \subseteq X$ such that $Y \in V$. Consider $\alpha \in S_0$ such that $\sup(Y \cap \alpha) = \alpha$. It is easy to see that for all $p \in \mathbb{P}$ and $m \in \omega$,

$p \not\models \exists n > m \left( \{s^n_\alpha, s^{n+1}_\alpha \} \subseteq \dot{Y} \right),$

and it follows that

$\mathbb{P} \models \exists \{n \in \omega : \{s^n_\alpha, s^{n+1}_\alpha \} \subseteq \dot{Y} \} = \aleph_0.$

Thus

$\mathbb{P} \models \langle s^n_\alpha : \alpha \in S_0, n \in \omega \rangle$ is a $\clubsuit_{w_2}$-sequence. 

Since adding a single Cohen real to a model where $\Diamond$ fails does not make $\Diamond$ true, we get the following.

**Corollary 19.** The pseudodiamond principle $\Diamond^\mathbb{P}$ is strictly weaker than $\Diamond$ and strictly stronger than $\text{CH}$. 
\textbf{Appendix.} For completeness, we include proofs of some of the examples presented earlier in this paper. These proofs are based on the standard arguments of the classical theorems generalized by these examples.

\textbf{Proof of Lemma 2.} Let $E, F$ be as in the assumption, and suppose $\langle F_\alpha : \alpha \in S_0 \rangle \in F$ witnesses that $\clubsuit(F, \mathcal{P}(E) \setminus \{\emptyset\}, [\omega_1]^{\aleph_1})$ holds. Now suppose towards a contradiction that $X \in [\omega_1]^{\aleph_1}$ is such that the set $N = \{\alpha \in E : X \cap \alpha \in F_\alpha\}$ is nonstationary. Let $C \subseteq C$ be such that $C \cap N = \emptyset$. Let $Y$ be an uncountable subset of $X$ such that for all $\xi_0, \xi_1 \in Y$ with $\xi_0 < \xi_1$ there exists $\gamma \in C$ such that $\xi_0 < \gamma < \xi_1$. Then $Y \cap \alpha$ is bounded in $\alpha$ and hence $Y \cap \alpha \notin F_\alpha$ for all $\alpha \in E \setminus C$. But if $\alpha \in E \cap C$, then $Y \cap \alpha \subseteq X \cap \alpha \notin F_\alpha$, and by the monotonicity assumption on $F_\alpha$, again we have $Y \cap \alpha \notin F_\alpha$. Thus the set $\{\alpha \in E : Y \cap \alpha \in F_\alpha\}$ is empty, which contradicts the choice of $\langle F_\alpha : \alpha \in S_0 \rangle$.

\textbf{Proof of Example 2.} Assume CH, let $E$ be a stationary subset of $S_0$, and let $F = \langle F_\alpha : \alpha \in S_0 \rangle \in F_{\text{filt}}$ be a sequence witnessing $\clubsuit(F_{\text{filt}}, \mathcal{P}(E) \setminus \{\emptyset\}, [\omega_1]^{\aleph_1})$. By Lemma 2, we may assume that $F$ witnesses $\clubsuit(F_{\text{filt}}, S(E), [\omega_1]^{\aleph_1})$. For a filter $F$ of subsets of a set $A$ and an indexed family $X = \{x_\alpha : a \in A\}$ we define $\liminf_F X = \{y : \{a : y \in x_a\} \in F\}$. Let $\langle a_\alpha : \alpha < \omega_1 \rangle$ be an enumeration of all countable subsets of $\omega_1$ such that every $a \in [\omega_1]^{\leq \aleph_0}$ is listed cofinally often. For $\alpha \in S_0$ we define $b_\alpha = \liminf_F \{a_\beta : \beta < \alpha\}$.

The following claim completes the argument.

\textbf{CLAIM 20.} $\langle b_\alpha : \alpha \in S_0 \rangle$ is a $\diamondsuit(E)$-sequence.

\textbf{Proof.} Let $A \subseteq \omega_1$ and let $C \subseteq C$. We need to show that there exists $\alpha \in E \cap C$ such that $A \cap \alpha = b_\alpha$. Recursively construct an increasing transfinite sequence $\langle \delta_\beta : \beta < \omega_1 \rangle$ of countable ordinals such that for all $\beta < \omega_1$ we have:

(i) $a_{\delta_{\beta + 1}} = A \cap \delta_\beta$, and

(ii) there exists $\gamma \in C$ with $\delta_\beta < \gamma < \delta_{\beta + 1}$.

The set $B = \{\delta_{\beta + 1} : \beta < \omega_1\}$ can be considered a code for the set $A$. By the choice of $F$, there exists $\alpha \in E$ such that $B \cap \alpha \in F_\alpha$. Then $B \cap \alpha$ is cofinal in $\alpha$, and (ii) implies that $\alpha \in C$. From (i) and the definition of $b_\alpha$ it follows that $b_\alpha \cap \delta_\beta = A \cap \delta_\beta$ for all $\delta_\beta < \alpha$. Since $B$ is cofinal in $\alpha$, the latter implies that $b_\alpha = A \cap \alpha$, which concludes the proof of the claim and hence of Example 2.

\textbf{Proof of Example 3(2).} Assume CH and let $E, F$ be as in the assumptions. It suffices to show that $\clubsuit(F^{(\omega)}(E), S(E), [\omega_1]^{\aleph_1})$ implies $\diamondsuit^-(E)$. Let $\langle F_n : n \in \omega \rangle = \langle F_{\alpha, n} : \alpha \in S_0, n \in \omega \rangle$ be sequences that witness $\clubsuit(F_{\text{filt}}^{(\omega)}(E), S(E), [\omega_1]^{\aleph_1})$. Let $\langle a_\alpha : \alpha < \omega_1 \rangle$ be an enumeration of all countable subsets of $\omega_1$ such that every $a \in [\omega_1]^{\leq \aleph_0}$ is listed cofinally often. For
\(\alpha \in S_0\) we define \(b_\alpha = \{\liminf_{F_{\alpha,n}} a_\beta : \beta < \alpha\} : n \in \omega\). The proof that \(\langle b_\alpha : \alpha \in S_0\rangle\) is a \(\Diamond^- (E)\)-sequence proceeds exactly as the proof of Claim 20.

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