FUNDAMENTA
MATHEMATICAE
170 (2001)

The power set of ω
Elementary submodels and weakenings of CH

by

István Juhász (Budapest) and **Kenneth Kunen** (Madison, WI)

Abstract. We define a new principle, SEP, which is true in all Cohen extensions of models of CH, and explore the relationship between SEP and other such principles. SEP is implied by each of CH^*, the weak Freeze–Nation property of $\mathcal{P}(\omega)$, and the $(\aleph_1, \aleph_0)$-ideal property. SEP implies the principle $\mathrm{C}^{\mathrm{s}}_2(\omega_2)$, but does not follow from $\mathrm{C}^{\mathrm{s}}_2(\omega_2)$, or even $\mathrm{C}^{\mathrm{s}}(\omega_2)$.

1. Introduction. There are many consequences of CH which are independent of ZFC, but are still true in Cohen models—that is, models of the form $V[G]$, where $V \vDash \mathrm{GCH}$ and $V[G]$ is a forcing extension of V obtained by adding some number (possibly 0) of Cohen reals; see [1, 2, 5, 7, 8]. Roughly, these consequences fall into two classes. One type are *elementary submodel* axioms, saying that for all suitably large regular λ, there are many elementary submodels $N \prec H(\lambda)$ such that $|N| = \aleph_1$ and $N \cap \mathcal{P}(\omega)$ "captures" in some way all of $\mathcal{P}(\omega)$; these are trivial under CH, where we could take $N \cap \mathcal{P}(\omega) = \mathcal{P}(\omega)$. The other are *homogeneity* axioms, saying that given a sequence of reals, $\langle r_\alpha : \alpha < \omega_2\rangle$, there are ω_2 of them which "look alike"; again, this is trivial under CH.

In this paper, we define a new axiom, SEP, of the elementary submodel type, and explore its connection with known axioms of both types.

A large number of applications of such axioms may be found in [2, 4, 7, 8].

2. Some principles true in Cohen models. We begin with a remark on elementary submodels. Under CH, one can easily find $N \prec H(\lambda)$ such that $|N| = \omega_1$ and N is *countably closed*; that is, $[N]^\omega \subseteq N$. Without CH,

---

2000 *Mathematics Subject Classification*: Primary 03E50, 03E35.

The first author supported by NSF Grant DMS-9704477 and OTKA grant 25745.
The second author supported by NSF Grant DMS-9704520.

this is clearly impossible, but one can still find such N which are *ω-covering*; this means that $\forall T \in [N]^\omega\, \exists S \in N \cap [N]^\omega\ [T \subseteq S]$, or $N \cap [N]^\omega$ is cofinal in $[N]^\omega$.

LEMMA 2.1. *$\{N \prec H(\lambda) : |N| = \omega_1$ and $N \cap [N]^\omega$ is cofinal in $[N]^\omega\}$ is cofinal in $[H(\lambda)]^{\omega_1}$ for any λ.*

See, e.g., [2] for a proof. Various weakenings of CH involve the existence of N such that $B = N \cap \mathcal{P}(\omega)$ "captures" $\mathcal{P}(\omega)$ in one of the following senses:

DEFINITION 2.2. If $B \subseteq \mathcal{P}(\omega)$ then we write:

(i) $B \leq_\sigma \mathcal{P}(\omega)$ iff for all $a \in \mathcal{P}(\omega)$, there is a countable $C \subseteq B \cap \mathcal{P}(a)$ such that for all $b \in B \cap \mathcal{P}(a)$ there is a $c \in C$ with $b \subseteq c \subseteq a$;

(ii) $B \leq_{\omega_1} \mathcal{P}(\omega)$ iff for all $K \in [B]^{\omega_1}$, there is an $L \in [K]^{\omega_1}$ such that $\bigcup L \in B$;

(iii) $B \leq_{\mathrm{sep}} \mathcal{P}(\omega)$ iff for all $a \in \mathcal{P}(\omega)$ and $K \in [B \cap \mathcal{P}(a)]^{\omega_1}$, there is a set $b \in B \cap \mathcal{P}(a)$ such that $|K \cap \mathcal{P}(b)| = \omega_1$.

It is obvious that both $B \leq_\sigma \mathcal{P}(\omega)$ and $B \leq_{\omega_1} \mathcal{P}(\omega)$ imply $B \leq_{\mathrm{sep}} \mathcal{P}(\omega)$, and that all three hold in the case of $B = \mathcal{P}(\omega)$.

$\leq_\sigma$ is relevant to axioms of the wFN (weak Freeze–Nation) type:

DEFINITION 2.3. wFN($\mathcal{P}(\omega)$) asserts that for all suitably large regular λ: for all $N \prec H(\lambda)$ with $\omega_1 \subset N$, we have $N \cap \mathcal{P}(\omega) \leq_\sigma \mathcal{P}(\omega)$.

DEFINITION 2.4. $\mathcal{P}(\omega)$ has the $(\aleph_1, \aleph_0)$-*ideal property* iff for all suitably large regular λ: for every $N \prec H(\lambda)$ such that $|N| = \omega_1$ and $N \cap [N]^\omega$ is cofinal in $[N]^\omega$, we have $N \cap \mathcal{P}(\omega) \leq_\sigma \mathcal{P}(\omega)$.

Clearly, wFN($\mathcal{P}(\omega)$) implies that $\mathcal{P}(\omega)$ has the $(\aleph_1, \aleph_0)$-ideal property. Definition 2.4 is from [2]. The usual definition of wFN($\mathcal{P}(\omega)$) is in terms of wFN maps from $\mathcal{P}(\omega)$ to $[\mathcal{P}(\omega)]^{\leq\omega}$, but this definition was shown in [5] to be equivalent to Definition 2.3.

In [8], a different kind of elementary submodel axiom, called CH*, was considered:

DEFINITION 2.5. $\mathcal{N}_\lambda$ consists of those $N \prec H(\lambda)$ with $|N| = \omega_1$ that satisfy both

(i) $N \cap [N]^\omega$ is cofinal in $[N]^\omega$, and
(ii) for every $K \in [N \cap ON]^{\omega_1}$, there is a $B \in [K]^{\omega_1}$ which has an *N-cover* $\widetilde{B}$, that is:
  (a) $B \subseteq \widetilde{B} \subseteq N$;
  (b) $[\widetilde{B}]^\omega \cap N$ is cofinal in $[\widetilde{B}]^\omega$;
  (c) if $S \in N \cap [\widetilde{B}]^\omega$ then $|S \cap B| = \omega$.

DEFINITION 2.6. CH* asserts that for each large enough regular cardinal λ, $\mathcal{N}_\lambda$ is cofinal in $[H(\lambda)]^{\omega_1}$.

The property $N \in \mathcal{N}_\lambda$ is a weakening of N being countably closed; N cannot really be countably closed unless CH is true, in which case CH* holds trivially.

The following result shows that CH* yields a property of $\mathcal{P}(\omega)$ of the wFN type, but replacing $\le_\sigma$ by $\le_{\omega_1}$.

THEOREM 2.7. *If $N \in \mathcal{N}_\lambda$, where $\lambda > 2^\omega$, then $N \cap \mathcal{P}(\omega) \le_{\omega_1} \mathcal{P}(\omega)$.*

Proof. Suppose that $K \subseteq N \cap \mathcal{P}(\omega)$ and $|K| = \omega_1$. Using $N \in \mathcal{N}_\lambda$ (and a bijection in N between $\mathcal{P}(\omega)$ and the ordinal $\mathfrak{c}$), we may fix $B \in [K]^{\omega_1}$ such that that B has an N-cover $\widetilde{B}$. Now let

$$a = \{n \in \omega : |\{b \in B : n \in b\}| = \omega_1\}.$$

Then $T_0 = \{b \in B : b \not\subseteq a\}$ is countable, so there is some $S_0 \in N \cap [\widetilde{B}]^\omega$ with $T_0 \subseteq S_0$. Let $L = B \setminus S_0$. Since $\bigcup L = a$, it will suffice to show that $a \in N$.

To see this, first choose $T \in [L]^\omega$ that satisfies $|\{b \in T : n \in b\}| = \omega$ for every $n \in a$, and then choose $S \in N \cap [\widetilde{B}]^\omega$ such that $T \subseteq S$. We may assume that $S \cap S_0 = \emptyset$, since $S_0 \in N$. Let

$$d = \{n \in \omega : |\{b \in S : n \in b\}| = \omega\}.$$

Then $d \in N$, and we show that $a = d$. First, $a \subseteq d$ because $T \subseteq S$. To see that $d \subseteq a$, fix $n \in d$. Let $W = \{b \in S : n \in b\}$. We have $W \in N$, so $W \cap B \neq \emptyset$ by property (c) in Definition 2.5. Hence, $W \cap L \neq \emptyset$ (since $S \cap S_0 = \emptyset$), so $n \in \bigcup L = a$. ∎

Since $\le_{\mathrm{sep}}$ is weaker than both $\le_\sigma$ and $\le_{\omega_1}$, we arrive at the following principle SEP that is consequently implied by both the $(\aleph_1, \aleph_0)$-ideal property (hence also by the wFN property) of $\mathcal{P}(\omega)$, and by CH*:

DEFINITION 2.8. $\mathcal{M}_\lambda$ consists of those $N \prec H(\lambda)$ with $|N| = \omega_1$ that satisfy both

(1) $N \cap [N]^\omega$ is cofinal in $[N]^\omega$, and
(2) $N \cap \mathcal{P}(\omega) \le_{\mathrm{sep}} \mathcal{P}(\omega)$.

DEFINITION 2.9. SEP denotes the statement that for all large enough regular cardinals λ, the family $\mathcal{M}_\lambda$ is cofinal in $[H(\lambda)]^{\omega_1}$.

Geschke [6] has shown that $B \le_{\mathrm{sep}} \mathcal{P}(\omega)$ and $B \le_\sigma \mathcal{P}(\omega)$ are equivalent when $|B| = \omega_1$, but that nevertheless it is consistent to have SEP hold while the $(\aleph_1, \aleph_0)$-ideal property fails for $\mathcal{P}(\omega)$. Note that SEP only requires that $\mathcal{M}_\lambda$ be cofinal, whereas the $(\aleph_1, \aleph_0)$-ideal property requires that $\mathcal{M}_\lambda$ contain all N with $N \cap [N]^\omega$ cofinal in $[N]^\omega$.

In a completely different direction, we have homogeneity properties such as $\mathrm{C^s}(\kappa)$ and $\mathrm{HP}(\kappa)$ [1, 7]. The $\mathrm{C^s}$ principles are defined as follows:

Definition 2.10. Let $\{A(\alpha, n) : \alpha < \kappa \,\&\, n < \omega\}$ be a matrix of subsets of ω, $T \subseteq \omega^{<\omega}$, and $S \subseteq \kappa$. Then $A\restriction(S \times \omega)$ is *T-adic* iff for all $m \in \omega$ and all $t \in T$ with $\mathrm{lh}(t) = m$, and all *distinct* $\alpha_0, \dots, \alpha_{m-1} \in S$: $A(\alpha_0, t_0) \cap \dots \cap A(\alpha_{m-1}, t_{m-1}) \neq \emptyset$.

Definition 2.11. $\mathrm{C^s}(\kappa)$ states: For any matrix $\{A(\alpha, n) : \alpha < \kappa \,\&\, n < \omega\}$ of subsets of ω and any $T \subseteq \omega^{<\omega}$, either

(1) there is a stationary $S \subseteq \kappa$ such that $A\restriction(S \times \omega)$ is T-adic, or

(2) there are m, t, and stationary $S_k \subseteq \kappa$ for $k < m$, with $t \in \omega^m \cap T$, such that for all $\beta_0, \dots, \beta_{m-1}$, with each $\beta_k \in S_k$, we have $\bigcap_{k<m} A(\beta_k, t_k) = \emptyset$.

$\mathrm{C}^{\mathrm{s}}_m(\kappa)$ is $\mathrm{C^s}(\kappa)$ restricted to $T \subseteq \omega^m$.

We remark that in (2), without loss of generality the S_k are disjoint, so that we get an equivalent statement if we require the β_k to be distinct, as in [1, 7]. As in most partition theorems, (1) and (2) are not necessarily mutually exclusive, in that (1) might hold on S while (2) holds for some S_k disjoint from S.

A strengthening of the $\mathrm{C^s}$ principles, called $\mathrm{HP}(\kappa)$ and $\mathrm{HP}_m(\kappa)$, is described in [1]. The principle $\mathrm{C^s}(\kappa)$ does not imply $\mathrm{HP}(\kappa)$, or even $\mathrm{HP}_2(\kappa)$ (see Theorem 3.9 below). We do not state HP here, since all we shall need is the consequence of it stated in (1) of the next lemma (proved in [1]). Part (2) is from [7].

Lemma 2.12. (1) $\mathrm{HP}_2(\kappa)$ *implies that if R is any relation on $\mathcal{P}(\omega)$ which is first-order definable over $H(\omega_1)$, then there is no $X \subseteq \mathcal{P}(\omega)$ such that $(X; R)$ is isomorphic to $(\kappa; <)$.*

(2) $\mathrm{C}^{\mathrm{s}}_2(\kappa)$ *implies the special case of* (1) *where R is $\subset^*$.*

$\mathrm{C}^{\mathrm{s}}_2(\kappa)$ has many other interesting consequences (see [7]); for example, every first countable separable T_2 space of size κ contains two disjoint open sets of size κ ([7], Theorem 4.14).

In [1], it was shown that $\mathrm{wFN}(\mathcal{P}(\omega))$ implies that $C^{\mathrm{s}}_2(\kappa)$ holds for every regular cardinal $\kappa > \omega_1$. Our next result shows that, at least for $\kappa = \omega_2$, the same conclusion follows already from the much weaker assumption SEP. It will be clear from the proof that for any regular $\kappa > \omega_1$ we could formulate a κ-version SEP_κ of SEP (with $\mathrm{SEP}_{\omega_2} = \mathrm{SEP}$), which also follows from the wFN property of $\mathcal{P}(\omega)$ and which implies $C^{\mathrm{s}}_2(\kappa)$.

Theorem 2.13. SEP *implies $C^{\mathrm{s}}_2(\omega_2)$.*

Proof. Fix $\mathcal{A} = \langle A(\alpha, n) : \langle \alpha, n\rangle \in \omega_2 \times \omega\rangle$, a matrix of subsets of ω, and $T \subseteq \omega^2$. Assume that for every stationary $S \subseteq \omega_2$ the submatrix $\mathcal{A}\restriction(S \times \omega)$ is not T-adic.

For every set $X \subseteq \omega_2$, define $H(X) \subseteq X$ recursively by

$$\gamma \in H(X) \;\Leftrightarrow\; \gamma \in X \text{ and } \mathcal{A}\restriction[[\{\gamma\} \cup (\gamma \cap H(X))] \times \omega] \text{ is } T\text{-adic.}$$

Note that then $\mathcal{A}\restriction(H(X)\times\omega)$ will be T-adic, hence by our assumption, $H(X)$ is always non-stationary in ω_2. We may (and shall) assume that $T = T^{-1}$, so that if $\gamma \in X \setminus H(X)$, there is a $\beta \in H(X) \cap \gamma$ and a $t \in T$ such that

$$A(\beta, t_0) \cap A(\gamma, t_1) = \emptyset.$$

By SEP, fix an $N \in \mathcal{M}_\lambda$ with $\mathcal{A}, T \in N$. Let $\mathcal{C}(\omega_2)$ denote the family of club subsets of ω_2. Since $N \cap [N]^\omega$ is cofinal in $[N]^\omega$ (Definition 2.8(1)), we may choose an ω_1-sequence $\{C_\xi : \xi \in \omega_1\} \subseteq N \cap \mathcal{C}(\omega_2)$ such that $\xi < \eta$ implies $C_\eta \subseteq C_\xi$, and for every $C \in N \cap \mathcal{C}(\omega_2)$ there is some $\xi < \omega_1$ with $C_\xi \subseteq C$.

Next, for every $\xi \in \omega_1$ let $S_\xi = H(C_\xi)$. Then $S_\xi \in N$ because $C_\xi \in N$, and S_ξ is non-stationary.

Definition 2.8(1) also implies that $\delta := N \cap \omega_2$ is an ordinal. It is easy to see that δ belongs to every $C \in N \cap \mathcal{C}(\omega_2)$; hence $\delta \notin S_\xi$ for each $\xi \in \omega_1$. Applying $\delta \in C_\xi \setminus H(C_\xi)$, we may choose a $\beta^\xi \in S_\xi \cap \delta$ and a $t^\xi \in T$ such that

$$A(\beta^\xi, t_0^\xi) \cap A(\delta, t_1^\xi) = \emptyset.$$

Now, fix a $t \in T$ and an uncountable set $Q \subseteq \omega_1$ such that $t^\xi = t$ for all $\xi \in Q$. Then for every $\xi \in Q$, we have

$$A(\beta^\xi, t_0) \subseteq \omega \setminus A(\delta, t_1).$$

Since $\beta^\xi < \delta$, each $A(\beta^\xi, t_0) \in N$, so by Definition 2.8(2), there is some set $b \in N$ such that $b \subseteq \omega \setminus A(\delta, t_1)$ and $R := \{\xi \in Q : A(\beta^\xi, t_0) \subseteq b\}$ is uncountable. Since $b \in N$, so also are the sets

$$D = \{\beta \in \omega_2 : A(\beta, t_0) \subseteq b\} \quad \text{and} \quad E = \{\beta \in \omega_2 : A(\beta, t_1) \cap b = \emptyset\}.$$

We claim that both D and E are stationary. For this, however, it suffices to show that they meet every $C \in N \cap \mathcal{C}(\omega_2)$. Fix such a C, and then fix $\xi \in R$ with $C_\xi \subseteq C$. Then $\beta^\xi \in C_\xi \cap D$, so $C \cap D \neq \emptyset$, and $\delta \in C_\xi \cap E$, so $C \cap E \neq \emptyset$.

Finally, we obviously have $A(\beta, t_0) \cap A(\gamma, t_1) = \emptyset$ whenever $\beta \in D$ and $\gamma \in E$, and this completes the proof of $\mathrm{C}_2^{\mathrm{s}}(\omega_2)$. ∎

We do not know if SEP (or even any of the stronger assumptions $\mathrm{wFN}(\mathcal{P}(\omega))$ or CH^*) implies $\mathrm{C}^{\mathrm{s}}(\omega_2)$ or just $\mathrm{C}_3^{\mathrm{s}}(\omega_2)$, but by Theorem 3.8 below, $\mathrm{C}^{\mathrm{s}}(\omega_2)$, and in fact $\mathrm{C}^{\mathrm{s}}(\kappa)$ for "most" regular $\kappa > \omega_1$, does not imply SEP.

3. Some independence results. As usual in forcing (see, e.g., [9]), a *partial order* $\mathbb{P}$ really denotes a triple, $(\mathbb{P}, \leq, \mathbb{1})$, where $\leq$ is a transitive reflexive relation on $\mathbb{P}$ and $\mathbb{1}$ is a largest element of $\mathbb{P}$. Then $\prod_{i\in I} \mathbb{P}_i$ denotes the product of the $\mathbb{P}_i$, with the natural product order. Elements $\vec{p} \in \prod_{i\in I} \mathbb{P}_i$ are I-sequences, with each $p_i \in \mathbb{P}_i$. The *finite support product* is given by:

Definition 3.1. If $\vec{p} \in \prod_{i\in I} \mathbb{P}_i$, then the *support* of $\vec{p}$, $\mathrm{supt}(\vec{p})$, is $\{i \in I : p_i \neq \mathbb{1}\}$; and $\prod^{\mathrm{fin}}_{i\in I} \mathbb{P}_i = \{\vec{p} \in \prod_{i\in I} \mathbb{P}_i : |\mathrm{supt}(\vec{p})| < \aleph_0\}$.

The principle $\mathrm{C}^{\mathrm{s}}(\kappa)$ was first stated in [7], where it was proved to hold in Cohen extensions (i.e., using some $\mathrm{Fn}(I,2)$) over a model in which κ is $\aleph_0$-*inaccessible* (that is, κ is regular, and $\theta^{\aleph_0} < \kappa$ whenever $\theta < \kappa$). The following result generalizes this:

Theorem 3.2. *Suppose, in V: κ is $\aleph_0$-inaccessible and $\mathbb{P} = \prod^{\mathrm{fin}}_{i\in I} \mathbb{P}_i$, where $\mathbb{P}$ is ccc and each $|\mathbb{P}_i| \le 2^{\aleph_0}$. Then $\mathrm{C}^{\mathrm{s}}(\kappa)$ holds in $V[G]$ whenever G is $\mathbb{P}$-generic over V.*

We remark that each $\mathbb{P}_i$ could be the trivial (1-element) order, so $V[G] = V$; that is, as pointed out in [7], $\mathrm{C}^{\mathrm{s}}(\kappa)$ holds whenever κ is $\aleph_0$-inaccessible.

In the case when all the $\mathbb{P}_i$ are the same, this theorem is due to [1]. In fact, in this case, [1] proves that the stronger property $\mathrm{HP}(\kappa)$ holds in $V[G]$; this can fail when the $\mathbb{P}_i$ are different (see Theorem 3.9 below). Here, as in [1, 7], we use a Δ-system argument (in V), applying the following lemma, due to Erdős and Rado (see [7] for a proof):

Lemma 3.3. *If κ is $\aleph_0$-inaccessible, and K_α is a countable set for each $\alpha < \kappa$, then there is a stationary $S \subseteq \kappa$ such that $\{K_\alpha : \alpha \in S\}$ forms a Δ-system.*

In [1, 7], this is used to show that given a κ-sequence of reals in $V[G]$, we can find κ of them which are disjointly supported. Then, in [1], one finds κ of these which "look alike", proving $\mathrm{HP}(\kappa)$ in $V[G]$. That cannot work here when $\kappa \le 2^{2^{\aleph_0}}$, since there are $2^{2^{\aleph_0}}$ possibilities for the $\mathbb{P}_i$. Instead, we use the fact that $\mathrm{C}^{\mathrm{s}}(\kappa)$ explicitly involves empty intersections, together with a separation lemma (Lemma 3.5 below), which reduces empty intersections in $V[G]$ to empty intersections in V. First, we need some further notation for product orders:

Definition 3.4. Let $\mathbb{P} = \prod^{\mathrm{fin}}_{i\in I} \mathbb{P}_i$. For $J \subseteq I$, let $\mathbb{P}\restriction J = \prod^{\mathrm{fin}}_{j\in J} \mathbb{P}_j$, and let $\varphi_J : P\restriction J \to \mathbb{P}$ be the natural injection: $\varphi_j(\vec{q})$ is the $\vec{p} \in \mathbb{P}$ such that $\vec{p}\restriction J = \vec{q}$ and $p_i = \mathbb{1}$ for $i \notin J$. If τ is a $\mathbb{P}\restriction J$-name, we also use $\varphi_J(\tau)$ for the corresponding $\mathbb{P}$-name. If τ is a $\mathbb{P}$-name, then the *support* of τ, $\mathrm{supt}(\tau)$, is the minimal $J \subseteq I$ such that $\tau = \varphi_J(\tau')$ for some $\mathbb{P}\restriction J$-name τ'. If $G \subseteq \mathbb{P}$, let $G\restriction J = \varphi_J^{-1}(G)$.

If one uses Shoenfield-style names, as in [9], then $\mathrm{supt}(\tau)$ may be computed inductively; if $\tau = \{(\sigma_\xi, p_\xi) : \xi < \alpha\}$, then $\mathrm{supt}(\tau) = \bigcup\{\mathrm{supt}(\sigma_\xi) \cup \mathrm{supt}(p_\xi) : \xi < \alpha\}$. By the usual iteration lemma for product forcing, if $\mathbb{P} \in V$ and G is $\mathbb{P}$-generic over V, and $J \subseteq I$ with $J \in V$, then $V[G] = V[G\restriction J][G\restriction(I\backslash J)]$, where $G\restriction J$ is $\mathbb{P}\restriction J$-generic over V and $G\restriction(I\backslash J)$ is $\mathbb{P}\restriction(I\backslash J)$-generic over $V[G\restriction J]$.

LEMMA 3.5. *Assume that $\mathbb{P} = \prod_{i\in I}^{\mathrm{fin}} \mathbb{P}_i \in V$ and G is $\mathbb{P}$-generic over V. In $V[G]$, suppose that $A_k \subseteq \omega$ for $k < m$, where $m \in \omega$, and $\bigcap_{k<m} A_k = \emptyset$. Suppose that there are names $\dot{A}_k$ (for $k < m$) such that $A_k = (\dot{A}_k)_G$ and the* $\mathrm{supt}(\dot{A}_k)$, *for $k < m$, are pairwise disjoint. Then there are $X_k \in \mathcal{P}(\omega) \cap V$ (for $k < m$) such that $\bigcap_{k<m} X_k = \emptyset$ and each $A_k \subseteq X_k$.*

Proof. Fix $\vec{p} \in G$ such that $\vec{p} \Vdash \bigcap_{k<m} \dot{A}_k = \emptyset$. In V, let

$$X_k = \{\ell \in \omega : \exists \vec{q} \le \vec{p}\ [\vec{q} \Vdash \ell \in \dot{A}_k]\}.$$

Then $A_k \subseteq X_k$. Now, suppose $\ell \in \bigcap_{k<m} X_k$. For each $k < m$, choose $\vec{q}_k \le \vec{p}$ such that $\vec{q}_k \Vdash \ell \in \dot{A}_k$. We may assume that $(q_k)_i = p_i$ for $i \notin \mathrm{supt}(\dot{A}_k)$. But then, since the $\mathrm{supt}(\dot{A}_k)$ are disjoint, the $\vec{q}_k$ are all compatible, so they have a common extension $\vec{q}$. So, $\vec{q} \le \vec{p}$ and $\vec{q} \Vdash \ell \in \bigcap_{k<m} \dot{A}_k$, a contradiction. ∎

Proof of Theorem 3.2. In $V[G]$, suppose we have a matrix $\{A(\alpha, n) : \alpha < \kappa\ \&\ n < \omega\}$ where each $A(\alpha, n) \subseteq \omega$. So, actually, A is a function from $\kappa \times \omega$ into $\mathcal{P}(\omega)$. Then we have a name $\dot{A} \in V$ such that $(\dot{A})_G = A$. By a standard use of the maximal principle, we may assume that $\mathbb{1} \Vdash \dot{A} : \kappa \times \omega \to \mathcal{P}(\omega)$.

Now, in V: For each α, let $K_\alpha \subseteq I$ be countable, so that K_α is a *support* of $\{A(\alpha, n) : n < \omega\}$ in the following sense: for each n, there is a name $\dot{A}_{\alpha,n}$ such that $\mathrm{supt}(\dot{A}_{\alpha,n}) \subseteq K_\alpha$ and such that $\mathbb{1} \Vdash \dot{A}(\check{\alpha}, \check{n}) = \dot{A}_{\alpha,n}$. We may choose K_α to be countable because $\mathbb{P}$ is ccc. Then, apply Lemma 3.3 to fix a stationary $S \subseteq \kappa$ such that $\{K_\alpha : \alpha \in S\}$ is a Δ-system, with some root J.

Next, we may assume that $J = \emptyset$. If not, then $V \subseteq V[G{\restriction}J] \subseteq V[G]$, and we may view $V[G]$ as an extension of $V[G{\restriction}J]$ by $G{\restriction}(I \setminus J)$. If we regard $V[G{\restriction}J]$ as the ground model, the $A(\alpha, n)$, for $\alpha \in S$, are named by names with support contained in $K_\alpha \setminus J$. Note that κ remains $\aleph_0$-inaccessible in $V[G{\restriction}J]$ because $\mathbb{P}{\restriction}J$ is ccc and $|\mathbb{P}{\restriction}J| \le 2^{\aleph_0}$.

Now (with $J = \emptyset$), work in $V[G]$: Since κ is regular and $\kappa > |\mathcal{P}(\omega) \cap V|$, we may construct a stationary $S' \subseteq S$ such that for all $X \in \mathcal{P}(\omega) \cap V$ and all $n \in \omega$, $\{\delta \in S' : A(\delta, n) \subseteq X\}$ is either empty or stationary. So, to verify $\mathrm{C^s}(\kappa)$, suppose $T \subseteq \omega^{<\omega}$. If $A{\restriction}(S' \times \omega)$ is T-adic, we are done. Otherwise, fix $t \in T$ with $m = |t|$, and distinct $\alpha_0, \ldots, \alpha_{m-1} \in S'$ such that $A(\alpha_0, t_0) \cap \ldots \cap A(\alpha_{m-1}, t_{m-1}) = \emptyset$. Then, by Lemma 3.5, choose $X_k \in \mathcal{P}(\omega) \cap V$ for $k < m$ such that $\bigcap_{k<m} X_k = \emptyset$ and each $A(\alpha_k, t_k) \subseteq X_k$. Finally, for $k < m$, let $S_k = \{\delta \in S' : A(\delta, t_k) \subseteq X_k\}$; this is non-empty, and hence stationary. Whenever $\beta_0, \ldots, \beta_{m-1} < \kappa$, with each $\beta_k \in S_k$, we have $\bigcap_{k<m} A(\beta_k, t_k) = \emptyset$. ∎

To refute SEP and $\mathrm{HP}(\omega_2)$ in such models, we use trees of subsets of ω. As usual, we consider $2^{<\omega_1}$ to be a binary tree, with root the empty sequence, $\emptyset$, and tree order defined by $s \le t \leftrightarrow \exists \xi\ [t{\restriction}\xi = s]$.

Definition 3.6. An *embedded tree* in $\mathcal{P}(\omega)$ is a pair (B,ψ) such that:

(1) B is a sub-tree of the binary tree $2^{<\omega_1}$ of height ω_1;
(2) $\psi : B \to [\omega]^\omega$;
(3) $\psi(\emptyset) = \omega$;
(4) $\forall s,t \in B\ [s < t \to \psi(t) \subset^* \psi(s)]$;
(5) for all $s \in B$: $s^\frown 0, s^\frown 1 \in B$ and $\psi(s^\frown 0) \cap \psi(s^\frown 1)$ is finite.

Lemma 3.7. *There is an embedded tree (B,ψ) such $B = 2^{<\omega_1}$.*

Theorem 3.8. *It is consistent to have $\neg\,\mathrm{SEP}$ together with $\mathrm{C^s}(\kappa)$ for each regular $\kappa > \omega_1$ which is not a successor of an ω-limit.*

Proof. In V: Assume GCH. Let (B,ψ) be an embedded tree as in Lemma 3.7. Let $\{f_\alpha : \alpha \in \omega_2\} \subseteq 2^{\omega_1}$ list ω_2 distinct branches of B. Let $\mathbb{P}_\alpha$ be the usual σ-centered forcing order which adds an infinite $x_\alpha \subset \omega$ such that $x_\alpha \subset^* \psi(f_\alpha\restriction\xi)$ for every $\xi \in \omega_1$ (see [3], §§11, 14). Let $\mathbb{P} = \prod^{\mathrm{fin}}_{\alpha\in\omega_2} \mathbb{P}_\alpha$.

Let G be $\mathbb{P}$-generic over V, and work in $V[G]$: We have $\mathrm{C^s}(\kappa)$ for all appropriate regular $\kappa > \omega_1$ by Theorem 3.2. To prove that SEP fails, we show that $(B,\psi) \notin N$ whenever $N \in \mathcal{M}_\lambda$.

Still in $V[G]$: Assume, by contradiction, that $(B,\psi) \in N \in \mathcal{M}_\lambda$. For each $\alpha \in \omega_2$, choose $n = n_\alpha$ such that $E_\alpha := \{\xi : (x_\alpha \setminus n) \subseteq \psi(f_\alpha\restriction\xi)\}$ is uncountable. Applying the definition (2.2(iii)) of $N \cap \mathcal{P}(\omega) \le_{\mathrm{sep}} \mathcal{P}(\omega)$ to $a := n \cup (\omega \setminus x_\alpha)$ and $K := \{\omega \setminus \psi(f_\alpha\restriction\xi) : \xi \in E_\alpha\}$, we get a $y_\alpha \supseteq x_\alpha \setminus n$ such that $y_\alpha \in N$ and $\{\xi \in E_\alpha : y_\alpha \subseteq \psi(f_\alpha\restriction\xi)\}$ is uncountable. Then $y_\alpha \subset^* \psi(f_\alpha\restriction\xi)$ for every $\xi \in \omega_1$. But then the y_α, for $\alpha \in \omega_2$, are infinite and pairwise almost disjoint, so that $|N| \ge \omega_2$, a contradiction. ∎

We now show that $\mathrm{HP}(\kappa)$ can fail in such a model:

Theorem 3.9. *It is consistent to have $\neg\,\mathrm{HP}_2(\omega_2)$ together with $\mathrm{C^s}(\kappa)$ for each regular $\kappa > \omega_1$ which is not a successor of an ω-limit.*

Proof. In V: Assume $V = L$, and hence GCH. For $f,g \in 2^{\omega_1}$, define $f \le^* g$ iff $\exists\xi < \omega_1\ \forall\eta > \xi\ [f(\eta) \le g(\eta)]$. Define $f <^* g$ iff $f \le^* g$ but $g \not\le^* f$. Let (B,ψ), $\{f_\alpha : \alpha \in \omega_2\}$, and $\mathbb{P} = \prod^{\mathrm{fin}}_{\alpha\in\omega_2} \mathbb{P}_\alpha$ be exactly as in the proof of Theorem 3.8, but assume also that $f_\alpha <^* f_\beta$ whenever $\alpha < \beta < \omega_2$; that is, the f_α are the characteristic functions of an ω_2-chain of sets in $\mathcal{P}(\omega_1)/\text{countable}$.

In $V[G]$: We again have $x_\alpha \subset \omega$ such that $x_\alpha \subset^* \psi(f_\alpha\restriction\xi)$ for every $\xi \in \omega_1$. For $x,y \subseteq \omega$, define xRy iff

$$\exists\xi < \omega_1\ \forall\eta \ge \xi\ \forall s,t \in B$$
$$\big[[\mathrm{lh}(s) = \mathrm{lh}(t) > \eta \;\&\; x \subset^* \psi(s) \;\&\; y \subset^* \psi(t)] \Rightarrow s(\eta) \le t(\eta)\big].$$

Then $\{x_\alpha : \alpha < \omega_2\}$ is well-ordered by R in type ω_2. By Lemma 2.12(1), this refutes $\mathrm{HP}_2(\omega_2)$ if R is definable over $H(\omega_1)$.

In V: $B = 2^{<\omega_1}$ is certainly definable over $H(\omega_1)$. Applying $V = L$, we can make ψ definable as well.

Then, in $V[G]$: we can, by quantifying over $H(\omega_1)$, refer to $(H(\omega_1))^V$ as $L(\omega_1)$, so that B and ψ will remain definable over $H(\omega_1)$. Hence, R will be definable over $H(\omega_1)$. ■

References

[1] J. Brendle, S. Fuchino and L. Soukup, *Coloring ordinals by reals*, to appear.

[2] A. Dow and K. P. Hart, *Applications of another characterization of* $\beta\mathbb{N}\setminus\mathbb{N}$, preprint.

[3] D. H. Fremlin, *Consequences of Martin's Axiom*, Cambridge Univ. Press, 1984.

[4] S. Fuchino, S. Geschke and L. Soukup, *More combinatorics around the weak Freese–Nation property*, Arch. Math. Logic, to appear.

[5] S. Fuchino, S. Koppelberg and S. Shelah, *Partial orderings with the weak Freese–Nation property*, Ann. Pure Appl. Logic 80 (1996), 35–54.

[6] S. Geschke, *Remarks on a paper by Juhász and Kunen*, preprint.

[7] I. Juhász, L. Soukup and Z. Szentmiklóssy, *Combinatorial principles from adding Cohen reals*, in: Logic Colloquium '95 (Haifa), Springer, Berlin, 1998, 79–103.

[8] —, —, —, *What is left of CH after you add Cohen reals?* Topology Appl. 85 (1998), 165–174.

[9] K. Kunen, *Set Theory*, North-Holland, 1980.

Alfréd Rényi Institute of Mathematics
Hungarian Academy of Sciences
P.O. Box 127
H-1364 Budapest, Hungary
E-mail: juhasz@math-inst.hu

Department of Mathematics
University of Wisconsin
Madison, WI 53706, U.S.A.
E-mail: kunen@math.wisc.edu
URL: http://www.math.wisc.edu/~kunen

Received 8 March 2000;
in revised form 10 April 2001