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On the continuity of Hausdorff dimension of Julia sets and similarity between the Mandelbrot set and Julia sets

by

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Abstract. Given $d \geq 2$ consider the family of polynomials $P_c(z) = z^d + c$ for $c \in \mathbb{C}$. Denote by J_c the Julia set of P_c and let $\mathcal{M}_d = \{c \mid J_c \text{ is connected}\}$ be the connectedness locus; for $d = 2$ it is called the Mandelbrot set. We study semihyperbolic parameters $c_0 \in \partial\mathcal{M}_d$: those for which the critical point 0 is not recurrent by P_{c_0} and without parabolic cycles. The Hausdorff dimension of J_c, denoted by $\mathrm{HD}(J_c)$, does not depend continuously on c at such $c_0 \in \partial\mathcal{M}_d$; on the other hand the function $c \mapsto \mathrm{HD}(J_c)$ is analytic in $\mathbb{C} - \mathcal{M}_d$. Our first result asserts that there is still some continuity of the Hausdorff dimension if one approaches c_0 in a "good" way: there is $C = C(c_0) > 0$ such that for a sequence $c_n \to c_0$,

$$\text{if} \quad \mathrm{dist}(c_n, \mathcal{M}_d) \geq C|c_n - c_0|^{1+1/d}, \quad \text{then} \quad \mathrm{HD}(J_{c_n}) \to \mathrm{HD}(J_{c_0}).$$

To prove this we use the fact that $\mathcal{M}_d$ and J_{c_0} are similar near c_0. In fact we prove that the biholomorphism $\psi : \overline{\mathbb{C}} - J_{c_0} \to \overline{\mathbb{C}} - \mathcal{M}_d$ tangent to the identity at infinity is conformal at c_0: there is $\lambda \neq 0$ such that

$$\psi(w) = c_0 + \lambda(w - c_0) + \mathcal{O}(|w - c_0|^{1+1/d}) \quad \text{for } w \notin J_{c_0}.$$

This implies that the local structures of $\mathcal{M}_d$ and J_{c_0} at c_0 are similar. The fact that $\lambda \neq 0$ is related to a transversality phenomenon that is well known for Misiurewicz parameters and that we extend to the semihyperbolic case. We also prove that for some $C > 0$,

$$d_{\mathrm{H}}(J_c, J_{c_0}) \leq C|c - c_0|^{1/d} \quad \text{and} \quad d_{\mathrm{H}}(K_c, J_{c_0}) \leq C|c - c_0|^{1/d},$$

where d_{H} denotes the Hausdorff distance.

1. Introduction. Given $d \geq 2$ consider the family of monic polynomials $P_c(z) = z^d + c$, for $c \in \mathbb{C}$, whose unique finite critical point is 0. The set

$$K_c = \{z \in \mathbb{C} \mid \{P_c^n(z)\}_{n \geq 0} \text{ is bounded}\}$$

is called the *filled-in Julia set* of P_c and $J_c = \partial K_c$ is called the *Julia set* of P_c.

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We study dynamics of polynomials P_c such that the critical point 0 is not recurrent and $0 \in J_c$. These polynomials are *semihyperbolic* in the sense of [CJY]: a semihyperbolic polynomial P_{c_0} is either as above or it is *hyperbolic*. The latter means that either $0 \notin K_{c_0}$ or P_{c_0} has an attracting cycle. Dynamics of hyperbolic polynomials is well understood, so we only consider semihyperbolic polynomials that are not hyperbolic.

Examples of semihyperbolic polynomials are *Misiurewicz* polynomials: a polynomial P_{c_0} is said to be Misiurewicz if the critical point of P_{c_0} is strictly preperiodic. In this case P_{c_0} is not hyperbolic; see [CG]. The set of parameters $c \in \mathbb{C}$ for which P_c is Misiurewicz is countable; see [DH2]. The set of parameters $c \in \mathbb{C}$ for which P_c is semihyperbolic but not hyperbolic, is much larger: Shishikura proved in [Sh] that it has Hausdorff dimension two.

It follows from a theorem of Fatou that the Julia set J_c is connected if and only if $c \in K_c$; see [CG]. Consider the *connectedness locus*

$$\mathcal{M}_d = \{c \in \mathbb{C} \mid J_c \text{ is connected}\} = \{c \in \mathbb{C} \mid c \in K_c\}.$$

If $d = 2$ this set is also denoted by $\mathcal{M}$ and is called the *Mandelbrot set*. It is known that $\mathcal{M}_d$ is compact and connected and moreover $\mathbb{C} - \mathcal{M}_d$ is homeomorphic to $\{|z| > 1\}$; see [DH2].

If P_{c_0} is semihyperbolic, then $c_0 \in \partial\mathcal{M}_d$ if and only P_{c_0} is not hyperbolic; see [DH2].

1.1. *On the continuity of Hausdorff dimension.* Our first result is about the dependence of the Hausdorff dimension of Julia sets on the parameter; see also [McM], [DSZ], [GSm], [GSw] and [UZ]. We denote the Hausdorff dimension by HD.

The Hausdorff dimension of the Julia set varies in an analytic way in the exterior of $\mathcal{M}_d$; see [R]. On the other hand Shishikura proved in [Sh] that there is a residual (hence dense) set of parameters in $\partial\mathcal{M}_d$ for which $\mathrm{HD}(J_c) = 2$ and by [U] we have $\mathrm{HD}(J_{c_0}) < 2$ for any $c_0 \in \partial\mathcal{M}_d$ such that P_{c_0} is semihyperbolic. So the Hausdorff dimension of the Julia set does not vary continuously in the parameter at such c_0.

The following theorem asserts that there is some continuity of the Hausdorff dimension at $c_0 \in \partial\mathcal{M}_d$ such that P_{c_0} is semihyperbolic, if one approaches c_0 from the exterior of $\mathcal{M}_d$ in a "good" way.

Theorem A. *Let $c_0 \in \partial\mathcal{M}_d$ be such that P_{c_0} is semihyperbolic. Then there is some $C > 0$ such that if a sequence $c_n \to c_0$ is such that*

$$\mathrm{dist}(c_n, \mathcal{M}_d) \geq C|c_n - c_0|^{1+1/d},$$

then $\mathrm{HD}(J_{c_n}) \to \mathrm{HD}(J_{c_0})$.

As we will see below this is much stronger than radial continuity of the Hausdorff dimension; see Figure 1.

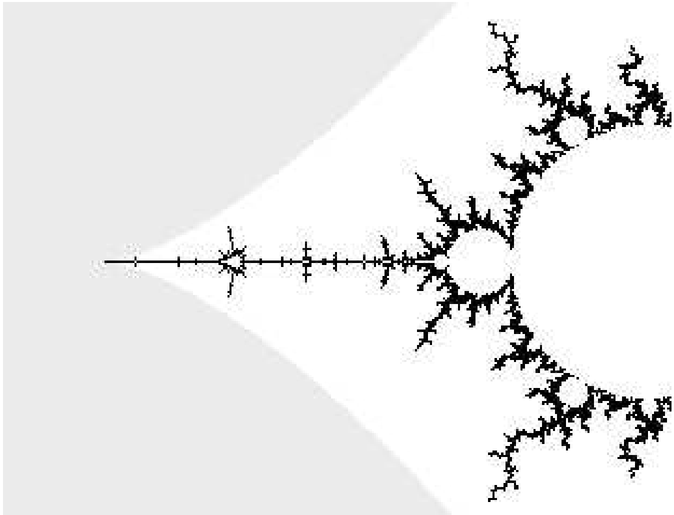

Fig. 1. Detail of the Mandelbrot set about the Misiurewicz parameter -2. The Hausdorff dimension of Julia sets corresponding to a sequence of parameters in the grey part converging to -2, should converge to $\mathrm{HD}(J_{-2}) = 1$.

1.2. *Similarities between the Mandelbrot set and Julia sets.* One of the main ingredients in the proof of Theorem A is the *similarity of* $\mathcal{M}_d$ *and* J_{c_0} at a parameter $c_0 \in \partial\mathcal{M}_d$ such that P_{c_0} is semihyperbolic.

There is an heuristic principle that the local structure in the dynamical plane should be similar to the structure in the parameter plane, at least at parameters with some expanding property. For example Shishikura made use of this principle in proving that the boundary of the Mandelbrot set has Hausdorff dimension two; see [Sh]. Also Wenstorm in [W] proved some remarkable similarities between the Mandelbrot set and the Fibonacci Julia set; see also [R-L1].

In [L] T. Lei proved that for a *Misiurewicz* parameter c_0, the sets $\mathcal{M}$ and J_{c_0} are *asymptotically similar* at c_0; see also [R-L1] and [R-L2]. To define this notion let us consider the following definitions. Given a compact subset X of $\mathbb{C}$ and $r > 0$, let

$$X_r = (\{r^{-1}w \mid w \in X\} \cap \mathbb{D}) \cup \partial\mathbb{D}.$$

That is, to construct X_r consider the intersection of X with the disc of radius r centered at 0, scale it to the unit disk and for technical reasons add $\partial\mathbb{D}$. Moreover, for $\lambda \in \mathbb{C} - \{0\}$ and $\zeta \in \mathbb{C}$ we define $\lambda X = \{\lambda w \mid w \in X\}$ and $X - \zeta = \{w - \zeta \mid w \in X\}$.

Theorem (T. Lei [L]). *Consider* $c_0 \in \mathcal{M}$ *such that* P_{c_0} *is Misiurewicz. Then there is* $\lambda \in \mathbb{C} - \{0\}$ *such that*

$$\lim_{r\to 0} d_{\mathrm{H}}((\mathcal{M}-c_0)_r,(\lambda(J_{c_0}-c_0))_r)=0,$$

where d_{H} *denotes the Hausdorff distance.*

See also [R-L1]. Whenever this holds, it is said that $\mathcal{M}_d$ and J_{c_0} are *asymptotically similar* at c_0. We generalize this theorem to semihyperbolic parameters as an easy corollary to the following theorem.

Theorem B ($C^{1+1/d}$-conformality of external maps). *Let* $c_0 \in \partial\mathcal{M}_d$ *be such that* P_{c_0} *is semihyperbolic, and let* $\varphi_{c_0} : \overline{\mathbb{C}} - J_{c_0} \to \overline{\mathbb{C}} - \overline{\mathbb{D}}$ *and* $\varphi_{\mathcal{M}_d} : \overline{\mathbb{C}} - \mathcal{M}_d \to \overline{\mathbb{C}} - \overline{\mathbb{D}}$ *be the canonical uniformizations* (*tangent to the identity at infinity*). *Then* $\lambda = \sum_{n\geq 0} 1/(P^n_{c_0})'(c_0) \in \mathbb{C} - \{0\}$ *and*

$$\varphi^{-1}_{c_0}\circ\varphi_{\mathcal{M}_d}(c) = c_0 + \lambda(c-c_0) + \mathcal{O}(|c-c_0|^{1+1/d}) \qquad \text{for } c \notin \mathcal{M}_d,$$
$$\varphi^{-1}_{\mathcal{M}_d}\circ\varphi_{c_0}(w) = c_0 + \lambda^{-1}(w-c_0) + \mathcal{O}(|w-c_0|^{1+1/d}) \qquad \text{for } w \notin J_{c_0}.$$

The conformality of these maps is a finer notion of similarity; see [R-L1]. For Misiurewicz parameters the *similarity factor*, given by the series above, is essentially a geometric series and therefore it can be calculated explicitly. For example -2 is a quadratic Misiurewicz parameter and $\lambda = 2/3$ in this case.

The following are corollaries of Theorem B, whose proofs are in Appendix 1. The following one improves T. Lei's theorem.

Corollary (Asymptotic similarity). *Let* $c_0 \in \partial\mathcal{M}_d$ *be such that* P_{c_0} *is semihyperbolic. Then there is a constant* $C > 0$ *such that for* $r > 0$ *small*,

$$d_{\mathrm{H}}((\mathcal{M}_d-c_0)_r,(\lambda(J_{c_0}-c_0))_r) \leq Cr^{1/d}.$$

In particular $\mathcal{M}_d$ *and* J_{c_0} *are asymptotically similar at* c_0.

Corollary. *Let* $c_0 \in \partial\mathcal{M}_d$ *be such that* P_{c_0} *is semihyperbolic and put* $D = \mathrm{HD}(J_{c_0})$. *Then there is a constant* $C > 0$ *such that*

$$\textit{Lebesgue-measure } (\mathcal{M}_d \cap B_r(c_0)) \leq Cr^{2+(2-D)/d}.$$

Recall that by [U], $D = \mathrm{HD}(J_{c_0}) < 2$ in this case, so these parameters are density points of the complement of $\mathcal{M}_d$. Combining Theorems A and B we obtain the following immediate corollary.

Corollary. *Let* $c_0 \in \partial\mathcal{M}_d$ *be such that* P_{c_0} *is semihyperbolic and let* $\lambda = \sum_{n\geq 0} 1/(P^n_{c_0})'(c_0) \in \mathbb{C} - \{0\}$ *be as in Theorem B. Then there is some* $C > 0$ *such that if* $z_n \to c_0$ *with* $\mathrm{dist}(z_n, J_{c_0}) \geq C|z_n - c_0|^{1+1/d}$, *then letting* $c_n = c_0 + \lambda(z_n - c_0)$ *we have*

$$\mathrm{HD}(J_{c_n}) \to \mathrm{HD}(J_{c_0}).$$

For example the polynomial z^2-2 is a quadratic Misiurewicz polynomial and its Julia set is the interval $[-2,2]$. So, by the previous corollary, there is a constant $C_0 > 0$ such that if a sequence $c_n \to c_0$ is at the left of the graph of the semicubical parabola $y = C_0(x+2)^{3/2}$, then $\mathrm{HD}(J_{c_n}) \to \mathrm{HD}(J_{c_0})$; see Figure 1.

It follows by [H] that if $\zeta \in \partial\mathbb{D}$ is such that the *ray* $\{\varphi_{\mathcal{M}_d}^{-1}(r\zeta) \mid r > 1\}$ accumulates at c_0, then $\varphi_{\mathcal{M}_d}^{-1}$ extends continuously to ζ, with $\varphi_{\mathcal{M}_d}^{-1}(\zeta) = c_0$. Recall that a *Stolz angle* in $\mathbb{C} - \overline{\mathbb{D}}$ at ζ is a set of the form

$$\{re^{i\theta}\zeta \mid r \geq 1 \text{ and } |\theta| \leq C(r-1)\} \quad \text{for some } C > 0.$$

The following corollary was obtained in [BR] for Misiurewicz parameters; see also [GSm] and [GSw]. It follows from the previous corollary together with the fact that $\mathbb{C} - J_{c_0}$ is a John domain; see [CJY] and Preliminaries.

Corollary (Angular convergence of Hausdorff dimension). *Let $c_0 \in \partial\mathcal{M}_d$ be such that P_{c_0} is semihyperbolic and let $\zeta \in \partial\mathbb{D}$ be such that $c_0 = \varphi_{\mathcal{M}_d}^{-1}(\zeta)$. Then the function*

$$w \mapsto \mathrm{HD}(J_{\varphi_{\mathcal{M}_d}^{-1}(w)})$$

is continuous in the closure of any Stolz angle in $\mathbb{C} - \overline{\mathbb{D}}$ at ζ.

1.3. *Stability of Julia sets.* Our final result is about the stability of Julia sets under perturbation.

Theorem C. *Let $c_0 \in \partial\mathcal{M}_d$ be such that P_{c_0} is semihyperbolic. Then there is $C > 0$ such that for $c \in \mathbb{C}$ close to c_0,*

$$d_{\mathrm{H}}(J_{c_0}, J_c) \leq C|c - c_0|^{1/d} \quad \text{and} \quad d_{\mathrm{H}}(J_{c_0}, K_c) \leq C|c - c_0|^{1/d}.$$

In particular the Julia set varies continuously with the parameter at semihyperbolic parameters. This is true for all parameters without parabolic or Siegel cycles; see [D].

We remark that this theorem is sharp, that is, $\mathcal{O}(|c - c_0|^{1/d})$ cannot be replaced by $o(|c - c_0|^{1/d})$; see end of Section 4.2.

1.4. *Organization of the paper.* It follows from a theorem of Mañé that for a semihyperbolic polynomial the set of accumulation points of the orbit of the critical point, denoted by $\omega(0)$, is a hyperbolic set. In Section 2 we construct a Markov partition for $\omega(0)$ with puzzles.

In Section 3 we prove that semihyperbolic polynomials have a property that we call *Almost Uniform Expansion*, which by [R-L3] is equivalent to the Collet–Eckmann condition. With this property we prove the Main Lemma about backward stability under perturbations (Section 3.1). This property is related to backward shadowing properties. *Section 3 is independent of Section 2.*

In Section 4 we prove Theorems B and C. In Section 4.1 we prove that some big sets in the dynamical plane can be extended in a holomorphic motion compatible with dynamics, in some neighborhoods of the parameter. As a consequence we obtain Theorem C. In Section 4.2 we prove that these holomorphic motions are close to a *non-degenerate* affine map near the critical value. This yields Theorem B about the conformality of the external maps. The non-degeneracy of this affine map is due to a transversality property proven in Appendix 2. We prove the sharpness of Theorem C at the end of Section 4.2.

In Section 5 we prove the *HD Lemma*, which is a criterion for the convergence of the Hausdorff dimension of Julia sets. Theorem A is an easy consequence of Theorem B and this lemma.

In Appendix 1 we prove two corollaries of Theorem B stated in the introduction.

In Appendix 2 we prove a *transversality* property. This property of semi-hyperbolic polynomials states that the graph of the dynamical continuation of the critical value is transversal to the diagonal. This generalizes a well known property of Misiurewicz maps; see for example [DH2]. For $d = 2$ this also follows from [vS], which was done independently.

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Preliminaries. The basic facts stated here can be found in [DH2], [CG] and [CJY].

For two numbers A and B, $A = \mathcal{O}(B)$ and $A \sim B$ mean $A < CB$ and $C^{-1}B < A < CB$, respectively, for some implicit constant C. Moreover $B_r(x)$ denotes the open ball of radius r and center x.

Throughout all this work we fix $d \geq 2$ and we let $P_c(z) = z^d + c$ for $c \in \mathbb{C}$. For $c \in \mathcal{M}_d$, the filled-in Julia set K_c is a compact connected set and $\overline{\mathbb{C}} - K_c$ is homeomorphic to $\overline{\mathbb{C}} - \overline{\mathbb{D}}$, where $\overline{\mathbb{C}}$ denotes the Riemann sphere and $\mathbb{D}$ denotes the unit disc. Moreover there is a unique conformal representation

$$\varphi_c : \overline{\mathbb{C}} - K_c \to \overline{\mathbb{C}} - \overline{\mathbb{D}}$$

which is tangent to the identity at infinity. This representation conjugates P_c in $\overline{\mathbb{C}} - J_c$ to $z \mapsto z^d$ in $\overline{\mathbb{C}} - \overline{\mathbb{D}}$. For $r > 1$ the set $\{z \mid \varphi_{c_0}(z) = r\}$ is an analytic Jordan curve called an *equipotential* and for $\theta \in \mathbb{R}$ the preimage of $\{re^{2\pi i\theta} \mid r > 1\}$ under φ_{c_0} is called the *ray of angle* θ. We say that the ray of angle θ *lands* at z if $\lim_{r\to 1} \varphi_{c_0}^{-1}(re^{2\pi i\theta}) = z$; in this case $z \in J_{c_0}$.

For $c \notin \mathcal{M}_d$, there is a map φ_c defined in a neighborhood of infinity that conjugates P_c to z^d near infinity. Moreover we may assume that φ_c is tangent to the identity at infinity. Then φ_c can be extended in a canonical way to c. It is rather surprising that the function $\varphi_{\mathcal{M}_d}$, defined by $\varphi_{\mathcal{M}_d}(c) = \varphi_c(c)$ for $c \notin \mathcal{M}_d$, is a conformal representation of $\mathbb{C} - \mathcal{M}_d$ to $\mathbb{C} - \overline{\mathbb{D}}$ that is tangent to the identity at infinity; see [DH1].

Consider $c_0 \in \partial\mathcal{M}_d$ so that P_{c_0} is semihyperbolic. It follows from a theorem of Mañé [Ma] that all (finite) periodic points of P_{c_0} are repelling. Moreover, the set $\omega(0)$ of accumulation points of the orbit of 0 is a hyperbolic set of P_{c_0}. Thus, by the expansive property, there is $l > 1$ such that $P_{c_0}^l(0) \in \omega(0)$. We suppose that $l > 1$ is the least integer with this property. We usually set $z_0 = P_{c_0}^l(0)$.

In [CJY] it is proved that there are constants $\varepsilon > 0$, $C > 0$ and $\theta \in (0,1)$ such that for all $x \in J_{c_0}$ and any connected component B of $P_{c_0}^{-n}(B_\varepsilon(x))$ for $n \geq 0$, the map

$$P_{c_0}^n : B \to B_\varepsilon(x)$$

has degree at most d and $\operatorname{diam}(B) < C\theta^n$.

Moreover the complement of J_{c_0} is a *John domain*; this means that J_{c_0} is locally connected and there is $\delta > 0$ such that if $z \in J_{c_0}$ and w belongs to a ray landing at z, then $B_{\delta|z-w|}(w) \cap J_{c_0} = \emptyset$. In particular, by Carathéodory's theorem, the map $\varphi_{c_0}^{-1}$, defined in $\overline{\mathbb{C}} - \overline{\mathbb{D}}$, extends continuously to $\partial\mathbb{D}$, so every ray lands at some point in J_{c_0}.

2. Markov partitions. Fix $c_0 \in \partial\mathcal{M}_d$ such that P_{c_0} is semihyperbolic. It follows by [Ma] that P_{c_0} is uniformly expanding in $\omega(0)$. In Section 2.1 we construct a Markov partition for $\omega(0)$ with puzzles; a *puzzle* is a set bounded by a finite number of (closures of) rays and an equipotential. Recall that by [CJY] all rays land at some point in J_{c_0}. Puzzles are homeomorphic to a disc.

PROPOSITION (Markov partitions). *There is a Markov partition for $\omega(0)$ with puzzles. That is, there is a finite collection of disjoint puzzles U_a, $a \in A$, that cover $\omega(0)$ so that P_{c_0} is univalent in U_a and such that if $a, b \in A$ are such that $U_a \cap P_{c_0}(U_b) \neq \emptyset$, then $U_a \subset P_{c_0}(U_b)$.*

The proof of this proposition is in Section 2.1. The main reason that we need a Markov partition with *puzzles*, instead of any other type of set, is to have the following property needed in the proof of Proposition 4.2. If $z \notin J_{c_0}$ belongs to the boundary of a puzzle, then the piece of ray from z to infinity is disjoint from that puzzle.

Consider the Markov partition U_a, $a \in A$, given by the Proposition. For $n \geq 0$, the preimages of the sets U_a under $P_{c_0}^n$ that intersect $\omega(0)$ are called the *nth step pieces* of the Markov partition. Note that for $n \geq 1$ the collection of all the nth step pieces is a Markov partition; we call it a *refinement* of the Markov partition U_a, $a \in A$.

In Section 2.2 we prove that, refining the Markov partition if necessary, we have the following important property.

Bounded Distortion Property. *For any $k \geq 0$ the distortion of $P_{c_0}^k$ in each of the kth step pieces of the Markov partition is bounded by some constant $K > 1$, independent of k.*

In Section 2.2 we also consider some holomorphic motions of the Markov partition and a uniform bonded distortion property.

2.1. *Construction of a Markov partition.* The idea to construct the Markov partition is the following. Consider a finite set $\mathcal{P} \subset J_{c_0}$ of preperiodic points, which is forward invariant under P_{c_0}. There are a finite number of rays landing at a given preperiodic point; see [DH1]. Consider the collection of puzzles U_b, $b \in B$, determined by some equipotential and all rays that land at some point in $\mathcal{P}$. This collection of puzzles has the Markov property: if $a, b \in B$ are such that $P(U_a) \cap U_b \neq \emptyset$, then $U_b \subset P_{c_0}(U_a)$. The difficulty is to find such a $\mathcal{P}$ disjoint from $\omega(0)$ (so that the puzzles U_b cover $\omega(0)$) and such that the puzzle containing 0 is disjoint from $\omega(0)$ (so that P_{c_0} is univalent in every U_b intersecting $\omega(0)$). Then the collection of puzzles intersecting $\omega(0)$ is the desired Markov partition.

We first reduce the situation to the non-renormalizable case; see [H] for references. Since all periodic points of P_{c_0} are repelling, P_{c_0} has d fixed points, $d-1$ of them are the landing points of the $d-1$ fixed rays; we denote by β the landing point of the ray of angle 0. The remaining fixed point, which we denote by α, is the landing point of $q \geq 2$ rays that are cyclically permuted by P_{c_0}.

Consider the q puzzles $P_1, \ldots, P_q$ determined by the q rays landing at α and an equipotential. Then there are two cases. Either the diameters of the successive preimages of these puzzles converge uniformly to 0, or P_{c_0} is *renormalizable*: this means that there is a pull-back P of P_i, for some $1 \leq i \leq q$, that contains 0 and such that for some $n \geq 1$ we have $P \subset P' = P_{c_0}^n(P)$ and $P_{c_0}^n : P \to P'$ is proper of degree d. In this case the puzzles P, $P_{c_0}(P), \ldots, P_{c_0}^{n-1}(P)$ are pairwise disjoint and the polynomial $P_{c_0}^n$ is conjugate to some polynomial P_{c_1} in $\overline{P}$. It follows that P_{c_1} is also semihyperbolic. The conjugacy maps the maximal invariant set $J_1 \subset J_{c_0}$ of $P_{c_0}^n$ in $\overline{P}$ to the Julia set J_{c_1} of P_{c_1}, which is connected.

In this case it is enough to find a finite set $\mathcal{P}_1 \subset J_{c_1}$ of preperiodic points of P_{c_1}, as above. In fact, if $\mathcal{P}_1' \subset J_1$ is the set corresponding to $\mathcal{P}_1$, then

$$\mathcal{P} = \mathcal{P}_1 \cup P_{c_0}(\mathcal{P}_1) \cup \ldots \cup P_{c_0}^{n-1}(\mathcal{P}_1)$$

is a finite set forward invariant under P_{c_0} that has the desired properties.

As before it may happen that P_{c_1} is renormalizable. Semihyperbolic polynomials are at most finitely renormalizable, so this process must end; see [H]. So by the above we may assume that P_{c_0} is not renormalizable.

Now we consider the tree structure of J_{c_0}; see [DH1] for references. Since J_{c_0} is a locally connected compact set with empty interior, it has a tree structure: given two different points $\delta, \gamma \in J_{c_0}$ there is a set $[\delta, \gamma] \subset J_{c_0}$, homeomorphic to a compact interval, which is the least connected subset of J_{c_0} containing δ and γ. We write $(\gamma, \delta] = [\delta, \gamma) = [\delta, \gamma] - \{\gamma\}$ and $(\gamma, \delta) = [\gamma, \delta) - \{\gamma\}$. Such sets are called *arcs*. Moreover for every $\delta_0, \delta_1, \delta_2 \in J_{c_0}$ there is $\gamma \in J_{c_0}$ so that $[\delta_0, \delta_1] \cap [\delta_0, \delta_2] = [\delta_0, \gamma]$.

It is easy to see that $\alpha \in [\beta, c_0]$ and that if P_{c_0} is not injective in an arc, then the critical point, 0, belongs to it. If $w \in J_{c_0}$ is such that there are γ and δ in J_{c_0} such that $w \in (\gamma, \delta)$, then w is the landing point of at least two rays. The following property is well known.

LEMMA 2.1. *Let $\gamma \in J_{c_0}$ be the landing point of at least two rays. Then there is k such that $P_{c_0}^k(\gamma) \in [\alpha, c_0]$.*

Proof. Let θ^+ and θ^- be different angles of rays landing at γ. By iterating if necessary, we may assume that there is $i \in \{1, \ldots, d-1\}$ such that θ^+ and θ^- lay in different connected components of $\mathbb{T} - \{i/d, 0\}$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$; equivalently there is a preimage β' of β so that $\gamma \in (\beta, \beta')$. Note that $0 \in (\beta, \beta')$ and therefore

$$P_{c_0}((\beta, 0)) = P_{c_0}((\beta', 0)) = (\beta, c_0).$$

Since $\alpha \in (\beta, c_0)$ there is a preimage α' of α in (β, α), so that

$$P_{c_0}((\alpha, 0)) = P_{c_0}((\alpha', 0)) = (\alpha, c_0),$$

and $P_{c_0}((\beta, \alpha')) = (\beta, \alpha)$. So for every $\delta \in (\beta, c_0)$ there is $n \geq 0$ so that $P_{c_0}^n(\delta) \in [\alpha, c_0]$. Since $P_{c_0}(\gamma) \in (\beta, c_0]$, the lemma follows. ∎

LEMMA 2.2. *Suppose that P_{c_0} is not renormalizable. Then*:

(i) *The preimages of α and the preimages of 0 are dense in $(0, \alpha)$.*

(ii) *For any $\gamma \in J_{c_0} - \{\alpha\}$ there are $\delta \in (\alpha, \gamma)$ and $k \geq 1$ such that $P_{c_0}^k$ is univalent in (α, δ) and $P_{c_0}^k((\alpha, \delta)) = (\alpha, 0)$.*

(iii) *The periodic points of P_{c_0} are dense in $(\alpha, 0)$.*

(iv) *If c_0 is not preperiodic, there is $k \geq 0$ such that for every $\gamma \in J_{c_0} - \{P_{c_0}^k(0)\}$ there is a periodic point of P_{c_0} in $(P_{c_0}^k(0), \gamma)$ that is not in $\omega(0)$.*

Proof. (i) Note that the boundary of a preimage of a puzzle P_i intersects J_{c_0} at preimages of α. Since the closures of the nth preimages of the P_i, for $1 \leq i \leq q$, cover J_{c_0} and have small diameter, it follows that for every subarc $I \subset (0, \alpha)$ there is a preimage P of some P_i intersecting I with diameter much smaller than that of I. So $\partial P \cap I \subset \partial P \cap J_{c_0}$ is not empty and therefore it contains a preimage of α. To prove that the preimages of 0 are dense in I, note that by the above, I contains at least two preimages of α. Therefore there is $k \geq 1$ so that $P_{c_0}^k$ is not injective in I, thus I contains a preimage of 0.

(ii) Iterating at most $q-1$ times we may suppose that γ belongs to the same connected component of $J_{c_0} - \{\alpha\}$ that contains 0; so $(\alpha, \gamma] \cap (\alpha, 0]$ is of the form $(a, \tilde{\gamma}]$ and we may suppose that γ belongs to $(\alpha, 0]$. By (i), (α, γ) contains a kth preimage δ of 0; we suppose that δ minimizes k, so that $P_{c_0}^k$ is univalent in (α, δ) and $P_{c_0}^k((\alpha, \delta)) = (\alpha, 0)$.

(iii) Consider a subarc I of $(\alpha, 0)$. By (i) there is a preimage γ of α in I; let k be such that $P_{c_0}^k(\gamma) = \alpha$ and $\gamma' \in I - \{\gamma\}$ so that $P_{c_0}^k$ is univalent in (γ, γ'). By (ii), taking γ' closer to γ if necessary, we may assume that there is l so that $P_{c_0}^l$ is univalent in $(P_{c_0}^k(\gamma), P_{c_0}^k(\gamma'))$ and $P_{c_0}^{k+l}((\gamma, \gamma')) = (\alpha, 0)$. Thus there is a periodic point of P_{c_0} in I.

(iv) If there are at least three rays landing at c_0, then c_0 is preperiodic; see [Th] and [K]. So there is nothing to prove in this case. So we assume that there are at most two rays landing at c_0.

If there is only one ray landing at c_0 then for every $\delta \in J_{c_0} - \{c_0\}$, the arc (c_0, δ) contains a subarc of (α, c_0). By (iii) the periodic points of P_{c_0} are dense in (α, c_0) so the assertion follows with $k = 1$, by considering that c_0 is not in $\omega(0)$.

If there are exactly two rays landing at c_0 and c_0 is not eventually mapped to α, then by Lemma 2.1 there is $l \geq 0$ so that $P_{c_0}^l(c_0) \in (\alpha, c_0)$. Then for every $\gamma \in J_{c_0} - \{P_{c_0}^l(0)\}$, the arc $(P_{c_0}^l(0), \gamma)$ contains a subarc of $(\alpha, 0)$. By (i) the preimages of 0 are dense in $(\alpha, 0)$ so $\omega(0)$ is nowhere dense in $(\alpha, 0)$. On the other hand, by (iii), the periodic points of P_{c_0} are dense in $(\alpha, 0)$ so the assertion follows with $k = l + 1 \geq 1$. ■

Proof of the Proposition. As already mentioned we may suppose that P_{c_0} is not renormalizable, so the previous lemma applies. If c_0 is preperiodic and not eventually mapped to α we can construct a Markov partition for the finite set $\omega(0)$ with preimages of the puzzles constructed with α. If c_0 is eventually mapped to α then clearly in the previous lemma property (iii) implies (iv). So we can assume that P_{c_0} satisfies (iv) of the previous lemma.

Let $k \geq 1$ be as in (iv) of the previous lemma and let $\delta > 0$ be such that $|P_{c_0}^k(0)| > \delta$. So by the previous lemma, for every $y \in \partial B_\delta(P_{c_0}^k(0)) \cap J_{c_0}$ there is a periodic point $p(y) \in (y, P_{c_0}^k(0))$ that is not in $\omega(0)$. Moreover we may suppose that $p(y) \in B_\delta(P_{c_0}^k(0))$.

Since $p(y)$ belongs to the arc $(y, P_{c_0}^k(0))$ there are at least two rays that land at $p(y)$. These rays divide $\mathbb{C}$ in at least two parts; let $U(y)$ be the one containing y. Since $\partial B_\delta(P_{c_0}^k(0)) \cap J_{c_0}$ is a compact set, we may choose points $y_1, \ldots, y_n$ in this set so that the $U(y_i)$ for $1 \leq i \leq n$ cover $\partial B_\delta(P_{c_0}^k(0)) \cap J_{c_0}$.

Consider the puzzle P containing $P_{c_0}^k(0)$ determined by all rays that land at the points $p(y_1), \ldots, p(y_n)$ and some equipotential. Choosing the equipotential with sufficiently small potential, we may assume that the puzzle P is contained in $B_\delta(P_{c_0}^k(0))$. So $\overline{P}$ does not contain 0.

Let $\mathcal{P}$ be the set of all points in the forward orbit of points in $P_{c_0}^{-k}(p(y_i))$ for $1 \leq i \leq n$. So $\mathcal{P}$ is a finite set of preperiodic points that is forward invariant under P_{c_0} and is disjoint from $\omega(0)$ by construction. Consider the collection U_b, $b \in B$, of the puzzles determined by the rays that land at points in $\mathcal{P}$ and some equipotential. By construction the puzzle containing the critical point is disjoint from $\omega(0)$. Then, as remarked above, the collection U_a, $a \in A \subset B$, of the puzzles that intersect $\omega(0)$ forms a Markov partition for $\omega(0)$. ∎

2.2. *Bounded distortion property and holomorphic motions.* Consider the Markov partition U_a, $a \in A$, for $\omega(0)$ given by the Proposition. Refining the Markov partition if necessary we prove that it has the property stated below. As in immediate consequence, together with the Koebe Distortion Theorem, we obtain the Bounded Distortion Property.

UNIVALENT EXTENSION PROPERTY. *Let W be an nth step piece of the Markov partition U_a, $a \in A$. Then the inverse of*

$$P_{c_0}^n : W \to U_a = P_{c_0}^n(W)$$

extends in a univalent way to a neighborhood of $\overline{U}_a$, only depending on $a \in A$.

Proof. We will prove that there is $m \geq 1$ such that all the mth step pieces of the Markov partition are compactly contained in some U_a. Then the Markov partition formed by the mth step pieces will be the desired Markov partition. Thus it is enough to prove that the diameters of the mth step pieces of the Markov partition converge uniformly to zero as $m \to \infty$.

Let $\varepsilon > 0$, $C > 0$ and $\theta \in (0,1)$ be as in [CJY]; see Preliminaries. Let $N \geq 1$ be such that we can partition each U_a, $a \in A$, in at most N connected sets of diameter less than $\varepsilon > 0$. Let W be an mth step piece of the Markov partition, so that $P_{c_0}^m$ is univalent in W. Then by [CJY] it follows that $\operatorname{diam}(W) \leq NC\theta^m$. ∎

Since P_{c_0} is uniformly expanding in $\omega(0)$ there is a holomorphic motion $j : B_\delta(c_0) \times \omega(0) \to \mathbb{C}$, for some $\delta > 0$, which is compatible with dynamics; see [Sh]. This means that for each $c \in B_\delta(c_0)$ the map $j_c : \omega(0) \to \mathbb{C}$ is injective and for each $z \in \omega(0)$ the function $c \mapsto j_c(z)$ is holomorphic. Being compatible with dynamics means that for every $c \in B_\delta(c_0)$ the map j_c conjugates P_{c_0} on $\omega(0)$ to P_c on $j_c(\omega(0))$.

Recall that $l > 1$ is the least integer such that $P_{c_0}^l(0) \in \omega(0)$; see Preliminaries. Reducing $\delta > 0$ if necessary we extend the holomorphic motion

j to c_0; thus the function $z(c) = j_c(c_0)$ is the *dynamical continuation* of the critical value c_0. This holomorphic function will be important for Theorem B. In fact the similarity factor is $\lambda = 1 - z'(c_0)$ and the Transversality property proven in Appendix 2 is that $\lambda = 1 - z'(c_0) \neq 0$. Note that the function z is defined in $B_\delta(c_0)$, it satisfies $z(c_0) = c_0$ and by definition $P_c^{l-1}(z(c)) = j_c(P_{c_0}^{l-1}(c_0))$.

PROPOSITION 2.3. *Consider a Markov partition U_a, $a \in A$, of $\omega(0)$ as in Section 2.1. Then there is $\delta > 0$ and a holomorphic motion $j : B_\delta(c_0) \times \bigcup_{a\in A} U_a \to \mathbb{C}$ compatible with dynamics. Moreover there is $R > 0$ such that $j(B_\delta(c_0) \times \bigcup_{a\in A} U_a) \subset B_R(0)$.*

Proof. Let $\mathcal{P} \subset J_{c_0}$ be the finite set of preperiodic points used in the construction of the Markov partition U_a, $a \in A$. By construction $\mathcal{P}$ is forward invariant and does not contain 0. By reducing $\delta > 0$ we may suppose that j is also defined in $\mathcal{P}$. Thus for any $z \in \mathcal{P}$ and any $c \in B_\delta(c_0)$, $j_c(z)$ is a preperiodic point of P_c. Moreover the rays of P_c landing at $j_c(z)$ have the same angles as those of P_{c_0} landing at z and we may extend the holomorphic motion j to the set of rays landing at points of $\mathcal{P}$. Extending j to the equipotential used in the construction of the Markov partition (reducing $\delta > 0$ if necessary) we may suppose that j is defined in $\bigcup_{a\in A} \partial U_a$. By [Sł] we may extend j to a fundamental domain of the Markov partition, and then to $\bigcup_{a\in A} U_a$ by dynamics.

Since for all $c \in B_\delta(c_0)$, $j_c(\bigcup_{a\in A} U_a)$ is contained in the set bounded by an equipotential with potential independent of c, it follows that there is $R > 0$ such that $j(B_\delta(c_0) \times \bigcup_{a\in A} U_a) \subset B_R(0)$. ■

We end this section with the following lemma, which is independent of dynamics and is used in the proof of Lemma 4.3.

LEMMA 2.4. *Let $j : \mathbb{D} \times X \to \mathbb{C}$ be a holomorphic motion such that $j(\mathbb{D} \times X) \subset B_R(0)$ for some $R > 0$. If $x, y \in X$ and $z \in \mathbb{D}$ are such that $|z|\big(\ln\big(\frac{1}{2R}|x-y|\big)\big)^{-1} \leq \frac{1}{4}$, then*

$$|j_z(x) - j_z(y) - (x-y)| \leq 4|z| \cdot |x-y| \left(\ln\left(\frac{1}{2R}|x-y|\right)\right)^{-1}.$$

Proof. Fix $y \in X$ and consider the holomorphic motion $i : \mathbb{D} \times X \to \mathbb{D}$ defined by $i(x) = (2R)^{-1}(j_z(x) - j_z(y))$, so for $x \neq y$, $i_z(x) \in \mathbb{D} - \{0\}$. Moreover fix $x \in X - \{y\}$ and put $x_z = i_z(x)$ for $z \in \mathbb{D}$. The map $w \mapsto x_0 e^{\frac{-2w}{1+w}\ln|x_0|}$ is a local isometry between $\mathbb{D}$ and $\mathbb{D} - \{0\}$ which maps 0 to $x_0 = x$. Therefore by Schwarz' Lemma,

$$x_z \in \{w \in \mathbb{D} - \{0\} \mid \varrho_{\mathbb{D}-\{0\}}(w, x_0) \leq \varrho_{\mathbb{D}}(0, z)\} = \{x_0 e^{\frac{-2w}{1+w}\ln|x_0|} \mid |w| \leq |z|\},$$

where $\varrho_{\mathbb{D}-\{0\}}$ and $\varrho_{\mathbb{D}}$ denote the hyperbolic distances in $\mathbb{D} - \{0\}$ and in $\mathbb{D}$

respectively. The hypothesis $|z|\ln(1/|x_0|) \le 1/4$ implies that

$$|e^{\frac{2|z|}{1+|z|}\ln\frac{1}{|x_0|}} - 1| \le 4|z|\ln\frac{1}{|x_0|},$$

so

$$|x_z - x_0| \le \sup_{|w|<|z|} |x_0 e^{\frac{-2w}{1+w}\ln|x_0|} - x_0| = |x_0| \cdot |e^{\frac{2|z|}{1+|z|}\ln\frac{1}{|x_0|}} - 1|$$
$$\le 4|z| \cdot |x_0| \ln\frac{1}{|x_0|}. \ \blacksquare$$

3. Expansion and backward stability. Fix $c_0 \in \partial\mathcal{M}_d$ such that P_{c_0} is semihyperbolic. We begin this section by proving that P_{c_0} has the Almost Uniform Expansion property stated below. As a consequence we prove the Main Lemma in Section 3.1, about backward stability of semihyperbolic polynomials.

Consider the constants $\varepsilon > 0$, $C > 0$ and $\theta \in (0,1)$ as in [CJY]; see Preliminaries.

Almost Uniform Expansion. *There is a constant $A > 0$ such that given $z \in \mathbb{C}$ and $k \ge 0$, if $0 \le j < k$ is such that $|P_{c_0}^j(z)| \le |P_{c_0}^i(z)|$ for all $0 \le i < k$, then*

$$|(P_{c_0}^j)'(z)| \ge A\theta^{-j} \quad \textit{and} \quad |(P_{c_0}^{k-j-1})'(P_{c_0}^{j+1}(z))| \ge A\theta^{-(k-j-1)}.$$

This property is equivalent to the Collet–Eckmann condition, which requires a positive Lyapunov exponent at the critical value; see [R-L3]. In the presence of more than one critical point this is no longer true.

The following is a distortion lemma for ramified maps, which is independent of dynamics.

Lemma 3.1. *Let $N \ge 1$ and $\gamma \in (0,1)$ be given. Then there exists $\kappa = \kappa(\gamma, N) \in (0,1)$ such that for any open and simply connected bounded set $U \subset \mathbb{C}$ and any ramified covering $R : U \to \mathbb{D}$ of degree N, we have*

$$\operatorname{diam}(U') \le \gamma \operatorname{diam}(U),$$

where U' is any connected component of $R^{-1}(B_\kappa(0))$.

Proof. It is enough to prove that the hyperbolic diameter of U' in U goes to zero as $\kappa \to 0$. So suppose that $U = \mathbb{D}$ and $R(0) = 0$; so there are $a_1, \ldots, a_{N-1} \in \mathbb{D}$ and $\lambda \in \partial\mathbb{D}$ such that

$$R(z) = \lambda z \prod_{i=1}^{N-1} \frac{z - a_i}{1 - \overline{a}_i z}.$$

Note that for any $\nu \in (0,1)$ we may choose $\frac{1}{2}\nu < \xi < \nu$ such that for any w satisfying $|w| = \xi$ we have $|w - a_i| \geq \nu/(4N)$ for $1 \leq i < N$. So

$$|R(w)| = \left|\lambda \prod_{i=1}^{N-1} \frac{w - a_i}{1 - \overline{a}_i w}\right| \geq \left(\frac{\nu}{8N}\right)^{N-1}.$$

Thus for $\kappa(N,\gamma) = (\nu/(8N))^{N-1}$ we have $U' \subset B_\nu(0)$. So the hyperbolic diameter of U' goes to 0 as $\kappa \to 0$. ∎

Let $\kappa \in (0,1)$ be as in Lemma 3.1 for $N = d$ and $\gamma = C^{-1\varepsilon/2}$, so that by [CJY] for any $z \in B_{\kappa\varepsilon/2}(J_{c_0})$, any $n \geq 0$ and any connected component W of $P_{c_0}^{-n}(B_{\kappa\varepsilon/2}(z))$, we have $\operatorname{diam}(W) < \varepsilon/2$.

Lemma 3.2. *Let $\kappa \in (0,1)$ be as above and let $\delta > 0$ be such that for all z and $k \geq 1$ such that $P_{c_0}^k(z) \in B_\delta(J_{c_0})$ we have $P_{c_0}^i(z) \in B_{\kappa\varepsilon/2}(J_{c_0})$ for $0 \leq i \leq k+1$. Then there is a constant $A_0 > 0$ such that for all z and k such that $P_{c_0}^k(z) \in B_{\varepsilon/2}(0)$ or $P_{c_0}^k(z) \notin B_\delta(J_{c_0})$ we have*

$$|(P_{c_0}^k)'(z)| \geq A_0 \max(\theta^{-k}, |z|^{-1}).$$

Proof. If $P_{c_0}^k(z) \in B_{\varepsilon/2}(0)$ then the pull-back of $B_\varepsilon(0)$ by $P_{c_0}^k$ to z is univalent, so by the Koebe $\frac{1}{4}$ Theorem $|(P_{c_0}^k)'(z)| \geq \frac{1}{4}(\varepsilon/2)|z|^{-1}$. By [CJY] and Schwarz' Lemma we have $|(P_{c_0}^k)'(z)| \geq (\varepsilon/2)C^{-1}\theta^{-k}$, so the lemma follows in this case with constant $A_1 = (\varepsilon/2)\min(1/4, C^{-1})$.

Suppose that $P_{c_0}^k(z) \notin B_\delta(J_{c_0})$, so we may assume that $k \geq 1$. Note that there is a constant $A_2 > 0$ such that for all $w \notin B_\delta(J_{c_0})$ and all $n \geq 0$ we have $|(P_{c_0}^n)'(z)| \geq A_2\theta^{-n}$. Thus we may assume that $P_{c_0}^{k-1}(z) \in B_\delta(J_{c_0})$, so $P_{c_0}^i(z) \in B_{\kappa\varepsilon/2}(J_{c_0})$ for $0 \leq i \leq k$.

If $P_{c_0}^m(z) \notin B_{\varepsilon/2}(0)$ for $0 \leq m < k$, then the pull-back of $B_{\kappa\varepsilon/2}(P_{c_0}^k(z))$ to z is univalent and the lemma follows as before in this case with constant κA_1.

Otherwise let $0 \leq m < k$ be the greatest integer such that $P_{c_0}^m(z) \in B_{\varepsilon/2}(0)$. As before the pull-back of $B_{\kappa\varepsilon/2}(P_{c_0}^k(z))$ by $P_{c_0}^{k-m}$ to $P_{c_0}^m(z)$ is univalent so $|(P_{c_0}^{k-m})'(z)| \geq \kappa A_1 \max(\theta^{-(k-m)}, |P_{c_0}^m(z)|^{-1})$; thus the assertion holds if $m = 0$. If $m > 0$ we have $|(P_{c_0}^m)'(z)| \geq \kappa A_1 \max(\theta^{-m}, |z|^{-1})$, so the lemma follows. ∎

Proof of the Almost Uniform Expansion property. Let $\kappa \in (0,1)$ be as in Lemma 3.1 for $N = d$ and $\gamma = C^{-1}\varepsilon/2$ as before and let $\delta > 0$ be as in Lemma 3.2. As in the proof of Lemma 3.2 we may assume that $P_{c_0}^k(z)$ belongs to $B_\delta(J_{c_0})$ so that $P_{c_0}^i(z) \in B_{\kappa\varepsilon/2}(J_{c_0})$ for $0 \leq i \leq k$. Consider the pull-back B_i of $B_k = B_{\kappa\varepsilon/2}(P_{c_0}^k(z))$ by $P_{c_0}^{k-i}$ to $P_{c_0}^i(z)$; so by definition of κ we have $\operatorname{diam}(B_i) < \varepsilon/2$.

If $|P_{c_0}^j(z)| \geq \varepsilon/2$ then $P_{c_0}^{k-j}$ is univalent in B_j so $|(P_{c_0}^{k-j})'(z)|^{-1} \geq (\varepsilon/2)C^{-1}\theta^{-(k-j)}$ by Schwarz' Lemma. In a similar way the pull-back of

$B_{\kappa\varepsilon/2}(P_{c_0}^j(z))$ by $P_{c_0}^j$ to z is univalent and $|(P_{c_0}^j)'(z)| \geq A_3\theta^{-j}$, where $A_3 = \kappa(\varepsilon/2)C^{-1}$.

If $|P_{c_0}^j(z)| < \varepsilon/2$ we see by Lemma 3.2 that $|(P_{c_0}^j)'(z)| \geq A_0\theta^{-j}$. If $P_{c_0}^{k-j}$ is univalent in B_j then $|(P_{c_0}^{k-j})'(z)| \geq A_3\theta^{-(k-j)}$ and we are done. Otherwise there is a unique $j \leq i < k$ so that $0 \in B_i$. In this case $B_i \subset B_{\varepsilon/2}(0)$, so by Lemma 3.2, $|(P_{c_0}^{j-i})'(z)| \geq A_0\theta^{-(j-i)}$ and on the other hand $P_{c_0}^{k-i-1}$ is univalent in B_{i+1} so $|(P_{c_0}^{k-i-1})'(P_{c_0}^{i+1}(z))| \geq A_3\theta^{-(k-i-1)}$. Considering that by hypothesis $|P_{c_0}'(P_{c_0}^j(z))|^{-1} \geq |P_{c_0}'(P_{c_0}^{i-1}(z))|^{-1}$, it follows that

$$|P_{c_0}^{i-j-1}(P_{c_0}^{j+1}(z))| \geq |P_{c_0}^{i-j-1}(P_{c_0}^j(z))| \geq A_3\theta^{-(i-j-1)}. \ \blacksquare$$

3.1. *Backward stability and shadowing.* In this section we prove the following lemma.

Main Lemma. *Let V be the neighborhood of J_{c_0} bounded by the equipotential with potential* 1. *Then there is a constant $M > 0$, only depending on P_{c_0}, such that for $\varrho > 0$ small there is a finite collection $\{D_i\}$ of open sets with the following properties*:

- diam$(D_i) \leq M\varrho^{1/d}$.
- *For all $z \in V$ there is D_i such that $B_\varrho(z) \subset D_i$.*
- *For all D_i and any connected component W of $P_{c_0}^{-1}(B_\varrho(D_i))$ there is D_j such that $W \subset D_j$.*

The proof of the Main Lemma is at the end of this section and is based on Lemmas 3.3 and 3.4 below. Note that this lemma implies that for $\varrho > 0$ small any ϱ backward pseudo-orbit in V is $M\varrho^{1/d}$-shadowed by a backward orbit of P_{c_0}.

Let us introduce some notation for the next lemmas. Fix $\alpha \in (1/d - 1/d^2, 1/d)$ only depending on d. For given $R > 0$ and $\varrho > 0$ small put $\eta = 1 + M_1R\varrho^{1/d-\alpha}$, for some $M_1 > 0$ to be chosen in Lemma 3.4. Let V be as in the Main Lemma and put $V' = P_{c_0}^{-1}(V) \subset V$. If ζ is a preimage of 0 or $\zeta \in V - J_{c_0}$ let r_ζ be defined by

$$r_\zeta = \begin{cases} R\varrho^{1/d} & \text{if } |\zeta| < \varrho^\alpha, \\ R\varrho & \text{if } \zeta \in V - V', \\ r_\zeta = \eta|P_{c_0}'(\zeta)|^{-1}(r_{P_{c_0}(\zeta)} + 4\varrho) & \text{otherwise.} \end{cases}$$

Lemma 3.3. *There is $M_2 > 0$ only depending on P_{c_0} such that if ϱ is small enough then for every preimage ζ of* 0 *and for $\zeta \in V - J_{c_0}$, we have $r_\zeta \leq M_2R\varrho^{1/d}$.*

Proof. Let ζ be a preimage of 0 or let $\zeta \in V - J_{c_0}$. So there is $n \geq 0$ such that $P_{c_0}^n(\zeta) \in B_{\varrho^\alpha}(0)$, resp. $P_{c_0}^n(\zeta) \in V - V'$; let n be minimal with this property. Then we have the following recursive formula for r_ζ:

$$r_\zeta = \eta^n |(P_{c0}^n)'(\zeta)|^{-1} r_{P_{c0}^n(\zeta)} + 4\varrho \sum_{m=1}^{n} \eta^m |(P_{c0}^m)'(\zeta)|^{-1},$$

where $r_{P_{c0}^n(z)} = R\varrho$ or $R\varrho^{1/d}$. By Lemma 3.2, $|(P_{c0}^n)'(\zeta)|^{-1} \geq A_0\theta^{-n}$. So if $\varrho > 0$ is small enough so that $\eta\theta < 1$ we have

$$\eta^n |(P_{c0}^n)'(\zeta)|^{-1} \leq A_0^{-1}(\eta\theta)^n \leq A_0^{-1}.$$

By the Almost Uniform Expansion property, for all $m \leq n$ we have $|(P_{c0}^m)'(\zeta)| \geq A^2\theta^{-(m-1)} d(\varrho^\alpha)^{d-1}$. Therefore there is a constant $M_3 > 0$ only depending on P_{c0} such that for ϱ small,

$$r_\zeta \leq M_3(R\varrho^{1/d} + \varrho^{1-(d-1)\alpha}) \leq 2M_3 R\varrho^{1/d},$$

the last considering that by definition $\alpha < 1/d$. ∎

LEMMA 3.4. *For appropriate choices of $M_1 > 0$ and $R > 0$, only depending on P_{c0}, if $\varrho > 0$ is small enough, then for any preimage $\zeta \neq 0$ of 0 and for any $\zeta \in V' - J_{c0}$, if W is the connected component of*

$$P_{c0}^{-1}(B_{r_{P_{c0}(\zeta)}+4\varrho}(P_{c0}(\zeta)))$$

containing ζ, then $W \subset B_{r_\zeta}(\zeta)$.

Proof. 1. Suppose $|\zeta| \geq \varrho^\alpha$ so by definition $r_\zeta = \eta|P_{c0}'(\zeta)|^{-1}(r_{P_{c0}(\zeta)} + 4\varrho)$. Note that P_{c0} is univalent in $B_{\mu_d\varrho^\alpha}(\zeta)$ where $\mu_d > 0$ is a constant that only depends in d (the degree of P_{c0}). By the previous lemma $r_\zeta \leq M_2 R\varrho^{1/d}$, so the distortion of P_{c0} in $B_{r_\zeta}(\zeta)$ is bounded by $1 + KM_2R\varrho^{1/d-\alpha}$ for some constant $K > 0$ given by the Koebe Distortion Theorem. Since M_2 only depends on P_{c0}, we may choose a priori $M_1 = KM_2$ so that the distortion of P_{c0} in $B_{r_\zeta}(\zeta)$ is bounded by $\eta = 1 + M_1R\varrho^{1/d-\alpha}$. Therefore $P_{c0}(B_{r_\zeta}(\zeta))$ contains the ball of radius $\eta^{-1}|P_{c0}'(\zeta)|r_\zeta = r_{P_{c0}} + 4\varrho$ centered at $P_{c0}(\zeta)$.

2. Now suppose that $|\zeta| < \varrho^\alpha$ and put $\zeta_1 = P_{c0}(\zeta)$. We will prove that there is a constant $K_1 > 0$ only depending on P_{c0}, and not on R, so that if ϱ is small enough then $r_{\zeta_1} \leq K_1\varrho$. Let $n > 1$ be the first integer such that $|P_{c0}^n(\zeta_1)| < \varrho^\alpha$, resp. $P_{c0}^n(\zeta_1) \in V' - V$. Thus $|P_{c0}'(\zeta)| \leq |P_{c0}'(P_{c0}^m(\zeta_1))|$ for $0 \leq m < n$, and by the Almost Uniform Expansion property applied to $z = \zeta$, we have $|(P_{c0}^m)'(\zeta_1)|^{-1} \leq A^{-1}\theta^m$, so

$$\sum_{m=1}^{n-1} \eta^m |(P_{c0}^m)'(\zeta_1)|^{-1} \leq K_2 = A^{-1}\frac{\eta\theta}{1-\theta\eta}.$$

On the other hand, by Lemma 3.2 we have

$$d|\zeta|^{d-1}|(P_{c0}^n)'(\zeta_1)| = |(P_{c0}^{n+1})'(\zeta)| \geq A_0 \max(\theta^{-n}, |\zeta|^{-1}), \quad \text{so}$$

$$|(P_{c0}^n)'(\zeta_1)| \geq A_0 d^{-1}|\zeta|^{-d} \geq A_0 d^{-1}\varrho^{-d\alpha},$$

and we may suppose $\varrho > 0$ small enough so that $|(P_{c_0}^n)'(\zeta_1)| \geq \theta^{-n}$. Thus $\eta^n = \theta^{-n\beta} \leq |(P_{c_0}^n)'(\zeta_1)|^\beta$, where $\beta = -\ln\theta/\ln\eta > 0$ goes to 0 as $\eta \to 1$, so $\beta \to 0$ as $\varrho \to 0$. Therefore,

$$\eta^n |(P_{c_0}^n)'(\zeta_1)|^{-1} \leq |(P_{c_0}^n)'(\zeta_1)|^{-(1-\beta)} \leq K_3 \varrho^{(1-\beta)d\alpha},$$

where $K_3 = (A_0 d^{-1})^{-(1-\beta)}$. By the recursive formula for r_{ζ_1}, as in the proof of the previous lemma we have

$$r_{\zeta_1} \leq K_3 \varrho^{(1-\beta)d\alpha} R \varrho^{1/d} + 4\varrho(K_2 + K_3 \varrho^{(1-\beta)d\alpha}).$$

Since by definition $d\alpha > 1 - 1/d$ we have $1/d + (1-\beta)d\alpha > 1$ if ϱ is small enough. Thus we may suppose that ϱ is close enough to 0 so that $r_{\zeta_1} \leq 8K_2\varrho$. This is the assertion with $K_1 = 8K_2$.

3. If $|\zeta_1 - c_0| \leq 2(K_1+4)\varrho$ then $B_{r_{\zeta_1}+4\varrho}(\zeta_1) \subset B_{4(K_1+4)\varrho}(c_0)$ so

$$P_{c_0}^{-1}(B_{r_{\zeta_1}+4\varrho}(\zeta_1)) \subset B_{(4(K_1+4)\varrho)^{1/d}}(0).$$

Moreover $|\zeta| \leq (2(K_1+4))^{1/d}$, thus if $R \geq 2(4(K_1+4))^{1/d}$ the lemma follows in this case.

If $|\zeta_1 - c_0| > 2(K_1+4)\varrho$ then the corresponding inverse branch of $P_{c_0}^{-1}$ defined in $B_{r_{\zeta_1}+4\varrho}(\zeta_1)$ is univalent and has distortion bounded by some constant $K_4 > 0$, given by the Koebe Distortion Theorem. Hence, if W is the connected component of $P_{c_0}^{-1}(B_{r_{\zeta_1}+4\varrho}(\zeta_1))$ that contains ζ then

$$W \subset B_{K_4|P'_{c_0}(\zeta)|^{-1}(r_{\zeta_1}+4\varrho)}(\zeta),$$

but

$$\begin{aligned} K_4|P'_{c_0}(\zeta)|^{-1}(r_{\zeta_1}+4\varrho) &\leq K_4 d(2(K_1+4)\varrho)^{-(d-1)/d}(K_1+4)\varrho \\ &= K_4 d 2^{-1}(2(K_1+4)\varrho)^{1/d}. \end{aligned}$$

Thus the lemma holds with

$$R = \max(K_4 d 2^{-1}(2(K_1+4))^{1/d}, 2(4(K_1+4))^{1/d}). \blacksquare$$

Proof of the Main Lemma. Let $\{z_i\}$ be a ϱ-dense set in V and let $r_i = \sup_{\zeta\in B_\varrho(z_i)} r_\zeta$ where the supremum is over all ζ for which r_ζ is defined. Put $D_i = B_{r_i+2\varrho}(z_i)$. By Lemma 3.3 there is a constant $M > 0$ only depending on P_{c_0} such that $\operatorname{diam}(D_i) \leq M\varrho^{1/d}$.

Since $\{z_i\}$ is ϱ-dense in V, for all $z \in J_{c_0}$ there is z_i at a distance at most ϱ from z and therefore $B_\varrho(z) \subset B_{2\varrho}(z_i) \subset D_i$.

Fix i and let ζ_k be a convergent sequence in $B_\varrho(z_i)$ such that $r_{\zeta_k} \to r_i$ as $k \to \infty$. Moreover choose a convergent sequence of preimages ζ'_k of ζ_k under P_{c_0}. Taking a subsequence if necessary we may assume that there is z_j such that $|\zeta'_k - z_j| \leq \varrho$, so $r_{\zeta'_k} \leq r_j$. By Lemma 3.4 if W_k is the connected component of $P_{c_0}^{-1}(B_{r_{\zeta_k}+4\varrho}(\zeta_k))$ that contains ζ'_k then

$$W_k \subset B_{r_{\zeta'_k}}(\zeta'_k) \subset D_j.$$

Since $B_\varrho(D_i) \subset \bigcup_k B_{r_{\zeta_k}+4\varrho}(\zeta_k)$, the lemma follows. $\blacksquare$

4. Similarities between dynamical and parameter planes. In this section we prove Theorems B and C. Fix a semihyperbolic parameter $c_0 \in \partial \mathcal{M}_d$ throughout this section. In this section the positive constants $C_0, C_1, \ldots$ and all implicit constants depend on P_{c_0} only.

Consider a Markov partition U_a, $a \in A$, for $\omega(0)$ as in Section 2. Recall that $l > 1$ is the least integer such that $P_{c_0}^l(0) \in \omega(0)$. Let $z_0 = P_{c_0}^l(0)$. We will denote by U_n the nth step piece of the Markov partition that contains z_0. By the Univalent Extension Property and Koebe Distortion Theorem it follows that $\operatorname{dist}(z_0, U_n) \sim \operatorname{diam}(U_n)$; see Section 2.2. Denote by V_n the pull-back of U_n to 0 by $P_{c_0}^l$. Then we have

$$\operatorname{dist}(\partial V_n, 0) \sim \operatorname{diam}(V_n) \sim (\operatorname{diam}(U_n))^{1/d}.$$

It follows by the Markov property and considering that the U_a are puzzles that if W is a pull-back of V_n, then either $W \cap V_n = \emptyset$ or $W \subset V_n$. We define

$$K_n = \{z \mid P_{c_0}^k(z) \notin V_n \text{ for } k \geq 0\},$$

which is a closed and forward invariant set. It follows that V_n is the connected component of $\mathbb{C} - K_n$ that contains 0 and for every connected component W of $\mathbb{C} - K_n$ there is k so that $P_{c_0}^k : W \to V_n$ is a biholomorphism. Let us consider the following easy lemma about the geometry of K_n.

LEMMA 4.1. *There is a constant $C_1 > 0$ only depending on P_{c_0} such that for n large and any connected component W of $\mathbb{C} - K_n$,*

(i) $\operatorname{diam}(W) \leq C_1 \operatorname{diam}(V_n)$.
(ii) $\operatorname{diam}(W) \leq C_1 \operatorname{dist}(W, c_0) \operatorname{diam}(V_n)$.

Proof. Consider the constants $\varepsilon > 0$, $\theta \in (0,1)$ and $C > 0$ given by [CJY]; see Preliminaries. Choose n large enough so that $V_n \subset B_{\varepsilon/2}(0)$. Thus, if $m \geq 0$ is such that $P_{c_0}^m : W \to V_n$ is a biholomorphism, the corresponding pull-back W_0 of $B_\varepsilon(0)$ is univalent and moreover $c_0 \notin W_0$. Denote by W' the corresponding pull-back of $B_{\varepsilon/2}(0)$. So by the Koebe Distortion Theorem there is a constant $D > 0$ so that $\operatorname{diam}(W) \leq D \operatorname{diam}(W') \operatorname{diam}(V_n) \leq DC\theta^m \operatorname{diam}(V_n)$.

Hence (i) follows for $C_1 \geq DC$. On the other hand, note that the modulus of the annulus $W_0 - \overline{W'}$ is equal to the modulus of the annulus $B_\varepsilon - \overline{B_{\varepsilon/2}(0)}$, which does not depend on ε. Since $c_0 \notin W_0$, there is a universal constant $K_0 > 0$ so that

$$\operatorname{diam}(W') \leq K_0 \operatorname{dist}(W', c_0) \leq K_0 \operatorname{dist}(W, c_0),$$

and so (ii) follows. ∎

4.1. *Holomorphic motions compatible with dynamics.* Recall that a *holomorphic motion* j of a set $X \subset \mathbb{C}$ defined in an open set $W \subset \mathbb{C}$ is a map

$j : W \times X \to \mathbb{C}$ so that for all $\lambda \in W$, the map $j_\lambda : X \to \mathbb{C}$ is injective and for every $x \in X$, $j_\lambda(x)$ depends holomorphically on λ.

PROPOSITION 4.2. *Let V be the neighborhood of J_{c_0} bounded by the equipotential with potential* 1, *as in the Main Lemma. Then there are constants $\nu > 0$ and $C_2 > 0$, only depending on P_{c_0}, such that for large n, there exists a holomorphic motion $i_n : B_n \times K_n \to \mathbb{C}$, where $B_n = B_{\nu\,\mathrm{diam}(U_n)}(c_0)$, such that*:

(i) *$(i_n)_{c_0}$ is the identity.*

(ii) *i_n is compatible with dynamics*, *that is*, *$(i_n)_c(P_{c_0}(z)) = P_c((i_n)_c(z))$ for all $(c, z) \in B_n \times K_n$.*

(iii) *For $(c, z) \in B_n \times (K_n \cap V)$ we have $|(i_n)_c(z) - z| \le C_2\,\mathrm{diam}(V_n)$.*

Moreover, *if $(c, z) \in B_n \times K_n$ is such that $c = (i_n)_c(z)$, then $c = \varphi_{\mathcal{M}_d}^{-1} \circ \varphi_{c_0}(z)$.*

Before proving this proposition let us deduce Theorem C from it.

Proof of Theorem C. We will prove that $d_{\mathrm{H}}(J_{c_0}, K_c) \le C_3|c - c_0|^{1/d}$ for some constant $C_3 > 0$ only depending on P_{c_0}. The assertion about $d_{\mathrm{H}}(J_{c_0}, J_c)$ follows in a similar way. Consider n large and let $c \in B_n - B_{n+1}$ be so that $|c - c_0| \sim \mathrm{diam}(U_n)$. By Proposition 4.2 we have

$$d_{\mathrm{H}}(J_{c_0}, (i_n)_c(K_n \cap J_{c_0})) \le C_4|c - c_0|^{1/d}.$$

Moreover by Lemma 4.1 every connected component of $\mathbb{C} - (i_n)_c(K_n)$ has diameter less than $C_5|c - c_0|^{1/d}$. Hence $(i_n)_c(K_n \cap J_{c_0}) \subset K_c$ is $C_5|c - c_0|^{1/d}$-dense in K_c. Therefore $d_{\mathrm{H}}(J_{c_0}, K_c) \le (C_4 + C_5)|c - c_0|^{1/d}$ as wanted. ∎

Proof of Proposition 4.2.

1. Put $V' = P_{c_0}^{-1}(V)$ and let $\delta > 0$ be so small that for all $c \in B_\delta(c_0)$, the function φ_c^{-1} is well defined in $\{\ln|\zeta| \ge 1/d\}$; so the holomorphic motion $i : B_\delta(c_0) \times (\mathbb{C} - V') \to \mathbb{C}$ defined by $i_c = \varphi_c^{-1} \circ \varphi_{c_0}$ has properties (i) and (ii).

2. Note that there is a constant $\mu_d > 0$ only depending on d so that if $D \subset \mathbb{C}$ satisfies $\mathrm{diam}(D) < \mu_d\,\mathrm{dist}(0, D)$, then P_{c_0} is univalent in D. Let n be large and recall that $\mathrm{dist}(0, \partial V_n) \sim \mathrm{diam}(V_n) \sim (\mathrm{diam}(U_n))^{1/d}$. Then there is $\nu_0 > 0$ only depending on P_{c_0} such that if we take $\varrho = \varrho_n = \nu_0\,\mathrm{diam}(U_n)$ in the Main Lemma, then the D_i (given by the Main Lemma) intersecting K_n satisfy $\mathrm{diam}(D_i) < \mu_d\,\mathrm{dist}(D_i, 0)$ and therefore P_{c_0} is injective in such D_i.

Note that for $w \in V - V'$ and $c \in B_\delta(c_0)$, $i_c(w)$ is bounded independently of c and w. Thus there is a constant $C_6 > 0$ only depending on P_{c_0} such that for every $w \in V - V'$ and $c \in B_\delta(c_0)$ we have $|i_c(w) - w| \le C_6\delta$. Hence, by Schwarz' Lemma, for all $w \in V - V'$ and all $c \in B_{\min(\varrho_n, \delta)}(0)$ we have $|i_c(w) - w| \le C_6\varrho_n$. So we may choose $\nu \in (0, \nu_0)$, only depending on P_{c_0}, so that $|i_c(w) - w| \le \varrho_n$ for all $c \in B_{\nu\,\mathrm{diam}(U_n)}(c_0) = B_n$ and $w \in V - V'$.

3. We define i_n to be equal to i in $B_n \times (\mathbb{C} - V')$. Let $z \in K_n - J_{c_0}$ be such that $z \in V'$ and let k be such that $P_{c_0}^k(z) \in V - V'$. Moreover let $c \in B_n = B_{\nu\,\mathrm{diam}(U_n)}(c_0)$.

By the above, $|i_c(P_{c_0}^k(z)) - P_{c_0}^k(z)| \le \varrho_n$. So by the Main Lemma there is D_i so that $i_c(P_{c_0}^k(z)) \in D_i$ and there is D_j containing the connected component W of $P_{c_0}^{-1}(B_{\varrho_n}(D_i))$ that contains $P_{c_0}^{k-1}(z)$. Moreover, since $c \in B_n \subset B_{\varrho_n}(c_0)$, we have $P_c(W) \subset B_{\varrho_n}(P_{c_0}(W))$ for $c \in B_n$. So it follows that there is a unique preimage of $i_c(P_{c_0}^k(z))$ under P_c in D_j.

Repeating this process we obtain a sequence i_j, for $0 \le j \le k$, such that $P_{c_0}^j(z) \in D_{i_j}$ and a uniquely determined orbit $z_0, \ldots, z_k = i_c(P_{c_0}^k(z))$ of P_c such that $z_j \in D_{i_j}$. We define $(i_n)_c(z) = z_0$, which clearly depends holomorphically on c and satisfies (i) and (ii).

Let us prove that $(i_n)_c$ is injective in $K_n - J_{c_0}$, so that i_n is a holomorphic motion of $K_n - J_{c_0}$ defined in B_n. If not we would have different $w_0, w_1 \in K_n$ such that for some $c \in B_n$, $(i_n)_c(w_0) = (i_n)_c(w_1)$. Considering that $(i_n)_{c_0} = \mathrm{id}$ we must have $i_c(w_0) = i_c(w_1) = 0$, since $(i_n)_c$ is compatible with dynamics. But this is not possible since $(i_n)_c(w_0)$ is contained is some D_i intersecting K_n and by hypothesis we have $\mathrm{dist}(D_i, 0) \ge \mu_d \,\mathrm{diam}(D_i) > 0$.

So $i_n : B_n \times (K_n - J_{c_0}) \to \mathbb{C}$ is a holomorphic motion. Since V_n is a puzzle we may suppose that for any $z \in K_n \cap J_{c_0}$ there is a ray contained in K_n landing at z. So $\overline{K_n - J_{c_0}} = K_n$ and by the λ-Lemma of [MSS], the holomorphic motion i_n extends to K_n.

4. It remains to prove that if $(c, z) \in B_n \times (K_n - J_{c_0})$ is such that $c = (i_n)_c(z)$, then $c = \varphi_{\mathcal{M}_d}^{-1} \circ \varphi_{c_0}(z)$. Let $\zeta = \varphi_{c_0}(z)$. Since V_n is a puzzle we may suppose that the piece of ray $\mathcal{R} = \{\varphi_{c_0}^{-1}(r\zeta) \mid r > 1\}$ is contained in K_n, so $(i_n)_c(\mathcal{R})$ is a piece of ray for P_c with the same angle as $\mathcal{R}$ and by construction the potential of $c = (i_n)_c(z)$ for P_c is the same as that of z for P_{c_0}. Thus $\varphi_{\mathcal{M}_d}(c) = \varphi_c(c) = \varphi_{c_0}(z)$. ∎

4.2. *Conformality of external maps.* In this section we prove Theorem B about the $C^{1+1/d}$-conformality of the map $\varphi_{\mathcal{M}_d}^{-1} \circ \varphi_{c_0}$ and its inverse at c_0. The proof depends on Lemma 4.3 below. We also prove the sharpness of Theorem C at the end of this section.

LEMMA 4.3. *Let n be large and let $B_n = B_{\nu\,\mathrm{diam}(U_n)}(c_0)$ be as in Proposition 4.2. Then there is a constant $C_7 > 0$ only depending on P_{c_0} such that for $S > 1$ given and $w \in K_n$ such that $S^{-1}\,\mathrm{diam}(U_n) \le |w - c_0| \le S\,\mathrm{diam}(U_n)$ we have, for $n \ge N_0(S)$ and $c \in B_n$,*

$$|(i_n)_c(w) - w - (z(c) - c_0)| \le C_7(\mathrm{diam}(U_n))^{1+1/d}.$$

Proof. 1. By the expansive property of hyperbolic sets there is $\varepsilon_0 > 0$ such that for any w close to c_0 there is $N = N(w) > 0$ such that

$$|P_{c_0}^N(w) - P_{c_0}^N(c_0)| > \varepsilon_0.$$

For fixed w, consider the smallest such N. Reducing ε_0 if necessary we may assume that

$$\varepsilon_0 < |P_{c_0}^N(w) - P_{c_0}^N(c_0)| < \varepsilon_0 D \ll \min_{a\in A} \operatorname{diam}(U_a),$$

where D is the supremum of $|P'_{c_0}(z)|$ over $z \in \bigcup_{a\in A} U_a$. By the Bounded Distortion Property,

$$|(P_{c_0}^N)'(c_0)| \cdot |w - c_0| \ge K^{-1}|P_{c_0}^N(w) - P_{c_0}^N(c_0)| \ge K^{-1}\varepsilon_0.$$

So $|(P_{c_0}^N)'(c_0)|^{-1} \le K\varepsilon_0^{-1}|w - c_0| \le K\varepsilon_0^{-1}S\varrho_n$, where $\varrho_n = \operatorname{diam}(U_n)$.

2. Consider the holomorphic motion $j : B_\delta(c_0) \times \bigcup_{a\in A} U_a \to \mathbb{C}$ given by Proposition 2.3 and let $R > 0$ be such that $j(B_\delta(c_0) \times \bigcup_{a\in A} U_a) \subset B_R(0)$; see Section 2.2.

Fix $c \in B_n$ and let w' be such that $j_c(w') = (i_n)_c(w)$, so $j_c(P_{c_0}^N(w')) = (i_n)_c(P_{c_0}^N(w))$. By Schwarz' Lemma applied to the function $\widehat{c} \mapsto j_{\widehat{c}}(P_{c_0}^N(w'))$ we have

$$|P_{c_0}^N(w') - (i_n)_c(P_{c_0}^N(w))| \le C_8|c - c_0| \le C_8\nu\varrho_n,$$

where $C_8 = 2R/\delta$. By Proposition 4.2,

$$|(i_n)_c(P_{c_0}^N(w)) - P_{c_0}^N(w)| \le C_9\varrho_n^{1/d},$$

so for large n we have $|P_{c_0}^N(w') - P_{c_0}^N(w)| \le C_9\varrho_n^{1/d} + C_8\nu\varrho_n \le 2C_9\varrho_n^{1/d}$ and by the Bounded Distortion Property,

$$|w' - w| \le K|(P_{c_0}^N)'(c_0)|^{-1}|P_{c_0}^N(w') - P_{c_0}^N(w)| \le K^2\varepsilon_0^{-1}2C_9S\varrho_n^{1+1/d}.$$

So, if n is larger than some N_0 (depending on S), we have $(2S)^{-1}\varrho_n \le |w' - c_0| \le 2S\varrho_n$.

3. By 2 we have

$$|c - c_0|\bigg(\ln\bigg(\frac{1}{2R}|w' - c_0|\bigg)\bigg)^{-1} \le C_{10}\varrho_n(\ln(S\varrho_n))^{-1} \to 0 \quad \text{as } n \to \infty.$$

Considering that $j_c(w') = (i_n)_c(w)$ we have, by Lemma 2.4,

$$\begin{aligned}|w' - j_c(w') - (c_0 - j_c(c_0))| &\le 4\,\frac{|c - c_0|}{\operatorname{diam}(B)}\,|w' - c_0|\bigg(\ln\bigg(\frac{1}{2R}|w' - c_0|\bigg)\bigg)^{-1}\\ &\le \frac{8}{\operatorname{diam}(B)}\,C_{10}S\varrho_n^2(\ln(S\varrho_n))^{-1}.\end{aligned}$$

Since $j_c(w') = (i_n)_c(w)$ and $j_c(c_0) = (i_n)_c(c_0)$, it follows that for n large,

$$\begin{aligned}&|w - (i_n)_c(w) - (c_0 - (i_n)_c(c_0))|\\ &\quad \le K^2\varepsilon_0^{-1}2C_9S\varrho_n^{1+1/d} + \frac{8C_{10}}{\operatorname{diam}(B)}\,S\varrho_n^2(\ln(S\varrho_n))^{-1} \le C_7S\varrho_n^{1+1/d}. \ \blacksquare\end{aligned}$$

Proof of Theorem B. We will prove the $C^{1+1/d}$-conformality of $\xi = \varphi_{\mathcal{M}_d}^{-1} \circ \varphi_{c_0}$ at c_0; the conformality of its inverse follows in the same way. Note that ξ is defined in $\mathbb{C} - J_{c_0}$.

Let $z(c) = j_c(c_0)$ be the dynamical continuation of the critical value c_0, so that $1 - z'(c_0) \neq 0$; see Section 2.2 and Appendix 2. Since $c - z(c) = \lambda(c - c_0) + \mathcal{O}(|c - c_0|^2)$, it is enough to prove that for $w \notin J_{c_0}$ close to c_0 we have

$$|w - c_0 - (\xi(w) - z(\xi(w)))| = \mathcal{O}(|w - c_0|^{1+1/d}).$$

Fix $w \notin J_{c_0}$ close to c_0 and let c' close to c_0 be such that $w = c_0 + c' - z(c')$, so $|w - c_0| = |c' - z(c')| \sim |c' - c_0|$. Let n be the greatest integer such that $c' \in B_{n+1}$, so $\operatorname{diam}(U_n) \sim |c' - c_0|$. Given $C > 0$, to be chosen below, let $\eta : [0,1] \to \mathbb{C}$ be defined by

$$\eta(\theta) = c' + C|c' - c_0|^{1+1/d} e^{2\pi i\theta}.$$

Then there are two cases.

Case 1: $w \in K_n$. By Lemma 4.3, for all $\widetilde{c} \in B_n$ we have

$$|(i_n)_{\widetilde{c}}(w) - w - (z(\widetilde{c}) - c_0)| = \mathcal{O}(|c' - c_0|^{1+1/d}).$$

We may suppose that w is close enough to c_0 so that $\eta(\theta) \in B_n$ for all $\theta \in [0,1]$. Note that $|w - c_0 - (\widetilde{c} - z(\widetilde{c}))| = |c' - z(c') - (\widetilde{c} - z(\widetilde{c}))| \sim |c' - \widetilde{c}|$. So we may choose $C > 0$ large enough so that for all $\theta \in [0,1]$,

$$|(i_n)_{\eta(\theta)}(w) - w - (z(\eta(\theta)) - c_0)| < |w - c_0 - (\eta(\theta) - z(\eta(\theta)))|.$$

Thus by the Rouché theorem there is $c \in B_n$ such that $(i_n)_c(w) = c$ and $|c - c'| \leq C|c' - c_0|^{1+1/d}$. It follows that $c = \xi(w)$ and

$$|w - c_0 - (c - z(c))| = |c' - z(c') - (c - z(c))| = \mathcal{O}(|w - c_0|^{1+1/d}).$$

Case 2: $w \notin K_n$. Let $y \in \partial K_n$ be the unique point which is in the same ray as w and let l be the piece of ray joining them. Let U be the connected component of $\mathbb{C} - K_n$ containing w. By Lemma 4.1, $\operatorname{diam}(U) = \mathcal{O}(|y - c_0|^{1+1/d})$, therefore $|y - w| = \mathcal{O}(|y - c_0|^{1+1/d})$ and $|y - c_0| \sim |w - c_0| \sim |c' - c_0|$. So $\operatorname{diam}(U) = \mathcal{O}(|c' - c_0|^{1+1/d})$.

By [Sł], i_n extends to a continuous function in $B_n \times \mathbb{C}$ such that for each $\widetilde{c} \in B_n$, the $(i_n)_{\widetilde{c}}$ is a homeomorphism of $\mathbb{C}$. Then it follows from the last observation and from Lemma 4.3 that, for $\widetilde{c} \in B_n$,

$$\operatorname{diam}((i_n)_{\widetilde{c}}(U)) = \mathcal{O}(|c' - c_0|^{1+1/d}).$$

Given $\widetilde{c} \in B_n$ and $w_0 \in \varphi_{\widetilde{c}}^{-1}(\varphi_{c_0}(w))$ consider the Green line $\widetilde{l}$ joining w_0 and $(i_n)_{\widetilde{c}}(y)$. Then $\widetilde{l} \cap K_n = \{i_{\widetilde{c}}(y)\}$. It follows that $\widetilde{l} \subset (i_n)_{\widetilde{c}}(U)$, so $w_0 \in i_{\widetilde{c}}(U)$ and $\varphi_c^{-1}(\varphi_{c_0}(w)) \subset i_{\widetilde{c}}(U)$.

Therefore, for all $\widetilde{c} \in B_n$ and $\zeta \in \varphi_{\widetilde{c}}^{-1}(\varphi_{c_0}(w))$ we have $|\zeta - i_{\widetilde{c}}(y)| \leq \operatorname{diam}(i_{\widetilde{c}}(U)) = \mathcal{O}(|c' - c_0|^{1+1/d})$. Thus, by the considerations of Case 1 applied to y instead of w, we may choose the constant $C > 0$ large enough so that

$$|\zeta - w - (z(\eta(\theta)) - c_0)| < |w - c_0 - (\eta(\theta) - z(\eta(\theta)))|$$

for all $\zeta \in \varphi_{\eta(\theta)}^{-1}(\varphi_{c_0}(w))$ and $\theta \in [0, 1]$. Then by the Rouché theorem there is c such that $\varphi_{\mathcal{M}_d}(c) = \varphi_c(c) = \varphi_{c_0}(w)$ and $|c - c'| \leq C|c' - c_0|^{1+1/d}$. It follows that $c = \xi(w)$ and

$$|w - c_0 - (c - z(c))| = |c' - z(c') - (c - z(c))| = \mathcal{O}(|w - c_0|^{1+1/d}).$$

Similarity factor. Recall that $\lambda = 1 - z'(c_0) \neq 0$; see Appendix 2. For $n \geq 0$ let $z_n(c) = P_c^n(z(c))$, which is equal to $j_c(P_c^n(z(c)))$ for $n \geq l$. Since the image of the holomorphic motion j is bounded, it follows by Schwarz' Lemma that there is $D > 0$ such that $|z_n'(c_0)| \leq D$ for $n \geq 0$. By the equation $z_{n+1}(c) = z_n(c)^d + c$ we have

$$z_{n+1}'(c) = dz_n(c)^{d-1} z_n'(c) + 1 = P_c'(z_n(c)) z_n'(c) + 1.$$

Thus, for $n \geq 0$ we have

$$z_0'(c) = -\left(\frac{1}{P_c'(z_0(c))} + \frac{1}{(P_c^2)'(z_0(c))} + \ldots + \frac{1}{(P_c^n)'(z_0(c))}\right) + \frac{z_n'(c)}{(P_c^n)'(z_0(c))}.$$

Considering that $z = z_0$, $z(c_0) = c_0$ and that $(P_{c_0}^n)'(c_0) \to \infty$ as $n \to \infty$, we have

$$\lambda = 1 - z'(c_0) = \sum_{n \geq 0} \frac{1}{(P_{c_0}^n)'(c_0)}. \ \blacksquare$$

Sharpness of Theorem C. Now we can prove that the estimate of Theorem C is sharp. We will use the fact that J_{c_0} is a John domain; see [CJY]. This means that there is $\delta > 0$ such that for any ray $\mathcal{R}$ landing at a point $z \in J_{c_0}$ and any $w \in \mathcal{R}$ the ball $B_{\delta|w-z|}(w)$ is disjoint from J_{c_0}.

Let $z(c) = j_c(c_0)$ be the dynamical continuation of the critical value c_0, so that $z'(c_0) \neq 0$; see Section 2.2 and Appendix 2. In particular, we have $|z(c) - c_0| \sim |c - c_0|$. Hence, if $w \in P_c^{-1}(z(c))$ then $|w| \sim |c - c_0|^{1/d}$. One may choose c such that w belongs to the ray of J_{c_0} landing at 0, so by the John property $\operatorname{dist}(w, J_{c_0}) \sim |c - c_0|^{1/d}$. But $z(c) \in J_c$ so $w \in J_c$, and therefore $d_{\mathrm{H}}(J_{c_0}, J_c) \sim |c - c_0|^{1/d}$.

5. Hausdorff dimension. In this section we prove Theorem A that follows easily from Theorem B and the HD Lemma stated below. This lemma is a criterion for the convergence of the Hausdorff dimension of Julia sets, and follows from a similar lemma in [BR].

In this section the positive constants $C_1, C_2, \ldots$ and all implicit constants depend on P_{c_0} only.

HD Lemma. *Let* $c_0 \in \partial\mathcal{M}_d$ *be such that* P_{c_0} *is semihyperbolic and consider a sequence* $c_n \to c_0$ *such that*

$$|c_n - c_0|^{d/(d-1)}(\operatorname{dist}(c_n, J_{c_n}))^{-1} \to 0.$$

Then $\operatorname{HD}(J_{c_n}) \to \operatorname{HD}(J_{c_0})$.

The proof of this lemma is in Section 5.1. The proof of Theorem A is based on the following lemma.

Lemma 5.1. *Let* $c_0 \in \partial\mathcal{M}_d$ *be such that* P_{c_0} *is semihyperbolic and let* c *be close to* c_0. *Then there is a constant* $C > 0$, *only depending on* P_{c_0}, *such that*

$$\text{if} \quad \operatorname{dist}(c, \mathcal{M}_d) \geq C|c - c_0|^{1+1/d}, \quad \text{then} \quad \operatorname{dist}(c, J_c) \geq |c - c_0|^{1+1/d}.$$

Proof. Let $c \in \mathbb{C}$ be such that $\operatorname{dist}(c, \mathcal{M}_d) > C|c - c_0|^{1+1/d}$ for some constant $C > 0$ to be determined later, and let $z = \varphi_{c_0}^{-1} \circ \varphi_{\mathcal{M}_d}(c)$. For $m \gg 1$ let $B_m = B_{\nu \operatorname{diam}(U_m)}(c_0)$ and $i_m : B_m \times K_m \to \mathbb{C}$ be as in Proposition 4.2. Let m be such that $c \in B_m - B_{m+1}$. By Theorem B there is $S > 1$, only depending on P_{c_0}, such that $S^{-1} \operatorname{diam}(U_m) < |z - c_0| < S \operatorname{diam}(U_m)$.

If $C > 0$ is large enough there is $C' > 0$ so that $\operatorname{dist}(z, \partial K_m) > C'|z - c_0|^{1+1/d}$ (cf. Lemma 4.1). Moreover we may choose $C' > 0$ arbitrarily large, by choosing C large.

Consider w such that $|w - z| = C'|z - c_0|^{1+1/d}$. Thus $w \in K_m - J_{c_0}$. Moreover, if c is close enough to c_0, then $(2S)^{-1} \operatorname{diam}(U_m) < |w - c_0| < 2S \operatorname{diam}(U_m)$. Let $C_7 > 0$ be the constant given by Lemma 4.3. Since $c = (i_m)_c(z)$, for any $\zeta \in B_{|c-c_0|^{1+1/d}}(c)$ we have

$$\begin{aligned}
|(i_m)_c(w) - \zeta| &\geq |(i_m)_c(w) - c| - |c - \zeta| \\
&\geq |w - z| - |c - c_0|^{1+1/d} \\
&\quad - |(i_m)_c(z) - z(c) - (z - c_0)| - |(i_m)_c(w) - z(c) - (w - c_0)|.
\end{aligned}$$

By Lemma 4.3 the last two quantities do not exceed $2C_7 S(\operatorname{diam}(U_m))^{1+1/d}$. Since $c \in B_m$, we have $|c - c_0| \leq \nu \operatorname{diam}(U_m)$. Thus, choosing $C > 0$ large enough so that $C' > 4C_7 S + \nu^{1+1/d}$, we have $|(i_m)_c(w) - \zeta| > 0$.

Since this holds for every w such that $|w - z| = C'|z - c_0|^{1+1/d}$, it follows by the Rouché theorem that there is w_ζ such that $|w_\zeta - z| < C'|z - c_0|^{1+1/d}$ and $(i_m)_c(w_\zeta) = \zeta$. As remarked above $w_\zeta \notin J_{c_0}$ in this case, so $\zeta = (i_m)_c(w_\zeta) \notin J_c$. Thus $B_{|c-c_0|^{1+1/d}}(c) \cap J_c = \emptyset$. ∎

Proof of Theorem A. Let $C > 0$ be as in the previous lemma and suppose that the sequence $c_n \to c$ is such that $\operatorname{dist}(c_n, \mathcal{M}_d) \geq C|c_n - c_0|^{1+1/d}$. By the previous lemma $\operatorname{dist}(c_n, J_{c_n}) \geq |c_n - c_0|^{1+1/d}$. Thus,

$$|c_n - c_0|^{d/(d-1)}(\operatorname{dist}(c_n, J_{c_n}))^{-1} \leq |c_n - c_0|^{1/(d(d-1))} \to 0$$

as $n \to \infty$. Hence, by the HD Lemma, $\operatorname{HD}(J_{c_n}) \to \operatorname{HD}(J_{c_0})$. ∎

5.1. *Conformal measures and atoms.* The proof of the HD Lemma is as follows. For $c \in \mathbb{C} - \mathcal{M}_d$ there is a unique conformal probability measure μ_c for P_c supported in J_c. Moreover μ_c has exponent $d_c = \mathrm{HD}(J_c)$; see [Su]. This means that for every measurable set U where P_c is injective, $\mu_c(P_c(U)) = \int_U |P_c'|^{d_c}\, d\mu_c$. Furthermore the μ_c measure of a point is zero, that is, μ_c is *not atomic*.

The unique conformal probability measure for P_{c_0}, supported in J_{c_0}, either has exponent $d_{c_0} = \mathrm{HD}(J_{c_0})$ or is atomic, supported in $\{P_{c_0}^{-n}(0)\}_{n\geq 0}$; see [DU] and [McM]. Thus to prove that

$$\lim_{n\to\infty} \mathrm{HD}(J_{c_n}) = \mathrm{HD}(J_{c_0}),$$

it is enough to prove that

$$\lim_{r\to 0}\lim_{n\to\infty} \mu_{c_n}(B_r(0)) = 0.$$

In fact, if μ_{c_0} is any weak limit of $\{\mu_{c_n}\}_{n\geq 1}$, then μ_{c_0} is a conformal probability measure supported in J_{c_0}. The previous limit implies that the measure μ_{c_0} is not atomic at 0, so it has exponent d_{c_0} and it follows that $d_{c_n} \to d_{c_0}$; see also [McM], [DSZ] and [UZ].

Consider a Markov partition U_a, $a \in A$, as in Section 2 and consider a holomorphic motion $j : B_\delta(c_0) \times \bigcup_{a\in A} U_a \to \mathbb{C}$ given by Proposition 2.3. Taking $\delta > 0$ smaller if necessary we may assume that there are constants $C_0 > 0$ and $\theta_0 \in (0,1)$ such that for all $m \geq 1$, all $c \in B_\delta(c_0)$ and all $w \in i_c(\omega(0))$, we have $|(P_c^m)'(w)|^{-1} \leq C_0\theta_0^m$. Moreover we may suppose that there is a uniform Bounded Distortion Property: *There is a constant* $K > 1$ *so that for every* $c \in B_\delta(c_0)$, *every* $k \geq 1$ *and every* k*th step piece* W *of the Markov partition* $j_c(U_a)$, $a \in A$, *the distortion of* P_c^k *in* W *is bounded by* K; cf. Section 2.2.

Recall that U_n is the nth step piece containing $P_{c_0}^l(0) \in \omega(0)$ and V_n is the pull-back of U_n by $P_{c_0}^l$ containing 0. Denote $j_c(U_n)$ by U_n^c and let V_n^c be the pull-back of U_n^c by P_c^l containing 0. It follows that for $r > 0$ small there is $n = n(r) \to \infty$ as $r \to 0$ so that $B_r(0) \subset V_n^c$ for all c sufficiently close c_0. Hence it is enough to prove that

$$\lim_{n\to\infty}\lim_{s\to\infty} \mu_{c_s}(V_n^{c_s}) = 0.$$

Proof of HD Lemma. 1. Let D be a disc containing 0, small enough so that for $c \in B_\delta(c_0)$, $P_c^l|_D$ is at most of degree d. Refining the Markov partition if necessary, suppose that $U_1^c \subset P_c^l(D)$ for all $c \in B_\delta(c_0)$.

Since for $c \notin \mathcal{M}_d$ the probability measure μ_c is not atomic, for all $n \geq 1$ we have

$$\mu_c(V_n^c) = \sum_{m\geq n} \mu_c(V_m^c - V_{m+1}^c).$$

Recall that $z(c) = j_c(c_0)$ is the dynamical continuation of the critical value c_0 and $z'(c_0) \neq 1$; see Appendix 2. For $c \in B_\delta(c_0)$ let $\zeta(c) = j_c(P_{c_0}^{l-1}(c_0)) = P_c^{l-1}(z(c))$ and put $q_c = P_c^l(0)$. Note that for $m \geq 1$ we have

$$\mu_c(V_m - V_m^c) \leq d\mu_c(U_m^c - U_{m+1}^c) \inf_{(V_m^c - V_{m+1}^c) \cap J_c} |(P_c^l)'(z)|^{-d_c}.$$

By the uniform Bounded Distortion Property and considering that μ_c is a probability measure, we have

$$\mu_c(U_m^c - U_{m+1}^c) \leq K^{d_c} |(P_c^m)'(\zeta(c))|^{-d_c}.$$

On the other hand there is $C_1 > 0$ such that for all $c \in B_\delta(c_0)$ and $z \in V_1^c$,

$$|(P_c^l)'(z)| > C_1 |P_c^l(z) - q_c|^{(d-1)/d}.$$

2. Let $k = k(c)$ be the greatest integer such that $q_c \in U_k^c$; see Figure 2. Let $m \geq 1$. Then there are three cases.

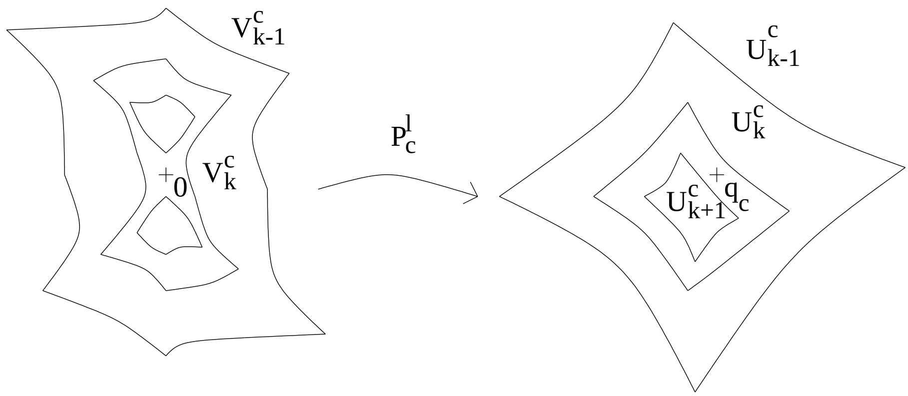

Fig. 2. Relative position of $q_c = i_c(P_{c_0}^l(0))$ in the Markov partition

Case 1: $k-1 \leq m \leq k+1$. By the uniform Bounded Distortion Property and by Transversality (Appendix 2), we have

$$|(P_c^m)'(\zeta(c))|^{-1} \sim |\zeta(c) - q_c| \sim |z(c) - c| \sim |c - c_0|,$$

with implicit constants independent of $c \in B_\delta(c_0)$. Hence $|(P_c^m)'(\zeta(c))|^{-1} \leq C_2 |c - c_0|$ for some $C_2 > 0$ independent of c. On the other hand

$$\operatorname{dist}(q_c, (U_m^c - U_{m+1}^c) \cap J_c) \geq \operatorname{dist}(q_c, J_c) \geq C_3 \operatorname{dist}(c, J_c).$$

So for all $z \in V_m^c - V_{m+1}^c \cap J_c$

$$|(P_c^l)'(z)| > C_1 C_3^{(d-1)/d} (\operatorname{dist}(c, J_c))^{(d-1)/d},$$

thus

$$\mu_c(V_m - V_{m+1}) \leq C_4 |c - c_0|^{d_c} (\operatorname{dist}(c, J_c))^{-d_c(d-1)/d},$$

where $C_4 = d(KC_2(C_1 C_3^{(d-1)/d})^{-1})^{d_c}$.

CASE 2: $m < k-1$. Note that

$$\operatorname{dist}(q_c, U^c_m - U^c_{m+1}) \geq \operatorname{dist}(\partial U^c_{m+1}, U^c_{m+2}),$$

thus by the uniform Bounded Distortion Property,

$$\operatorname{dist}(q_c, U^c_m - U^c_{m+1}) > C_5 |(P^m_c)'(\zeta(c))|^{-1}.$$

Hence, by 1 we have

$$\begin{aligned}|(P^l_c)'(z)| &> C_1(\operatorname{dist}(q_c, U^c_m - U^c_{m+1}))^{(d-1)/d}\\ &\geq C_1 C_5^{(d-1)/d} |(P^m_c)'(\zeta(c))|^{-(d-1)/d}.\end{aligned}$$

Therefore,

$$\mu_c(V^c_m - V^c_{m+1}) \leq dK^{d_c} |(P^m_c)'(\zeta(c))|^{-d_c} (C_1 C_5^{(d-1)/d})^{-d_c} |(P^m_c)'(\zeta(c))|^{d_c(d-1)/d}.$$

Thus $\mu_c(V^c_m - V^c_{m+1}) \leq C_6 \theta_0^{md_c/d}$, where $C_6 = dK^{d_c}(C_1 C_5^{(d-1)/d})^{-d_c} C_0^{d_c/d}$.

CASE 3: $m > k+1$. We have $\operatorname{dist}(q_c, U^c_m - U^c_{m+1}) \geq \operatorname{dist}(\partial U^c_{m-1}, U^c_m)$. Thus reducing $C_5 > 0$ if necessary, we have, as in Case 2,

$$\operatorname{dist}(q_c, U^c_m - U^c_{m+1}) > C_5 |(P^m_c)'(\zeta(c))|^{-1},$$

and $\mu_c(V^c_m - V^c_{m+1}) \leq C_6 \theta_0^{md_c/d}$.

3. By 1 and 2, for $n \geq 1$ we have

$$\mu_c(V^c_n) \leq 3C_4 |c - c_0|^{d_c} (\operatorname{dist}(c, J_c))^{-d_c(d-1)/d} + C_6 \sum_{m \geq n,\, m \neq k-1, k, k+1} \theta_0^{md_c/d}.$$

Since

$$\sum_{m \geq n} \theta_0^{md_c/d} = \frac{(\theta_0^{d_c}/d)^n}{1 - \theta^{d_c/d}}$$

and since by hypothesis $|c_s - c_0|(\operatorname{dist}(c_s, J_{c_s}))^{-(d-1)/d} \to 0$ as $s \to \infty$, we conclude that

$$\lim_{n\to\infty} \lim_{s\to\infty} \mu_{c_s}(V^{c_s}_n) = 0. \ \blacksquare$$

Appendix 1. Proof of corollaries of Theorem B

Proof of asymptotic similarity. We only use the conformality of $\varphi^{-1}_{\mathcal{M}_d} \circ \varphi_{c_0}$ at c_0 and that J_{c_0} has empty interior. Put $\xi = \varphi^{-1}_{\mathcal{M}_d} \circ \varphi_{c_0} : \overline{\mathbb{C}} - J_{c_0} \to \overline{\mathbb{C}} - \mathcal{M}_d$, which is a proper map.

Let $w \in J_{c_0}$ and put $r = |w - c_0|$. Consider a sequence $\{w_i\}$ disjoint from J_{c_0} such that $w_i \to w$. Since ξ is proper we may suppose that $c_i = \xi(w_i) \to c \in \mathcal{M}_d$. Thus, by Theorem B there is $C_1 > 0$ only depending on P_{c_0} such that $|w - c_0 - \lambda(c - c_0)| \leq C_1 r^{1+1/d}$.

Let $c \in \mathcal{M}_d$ be close to c_0 and put $r = |c - c_0|$. By the Rouché theorem and by Theorem B there is $C_2 > 0$, only depending on P_{c_0}, such that if w satisfies $|w - c_0| \sim r$ and $B_{C_2 r^{1+1/d}}(w) \cap J_{c_0} = \emptyset$, then $c' = c_0 + \lambda^{-1}(w - c_0) \notin$

$\mathcal{M}_d$. Thus, if $w = c_0 + \lambda(c - c_0)$, then $B_{C_2 r^{1+1/d}}(w) \cap J_{c_0} \neq \emptyset$. Let $w' \in B_{C_2 r^{1+1/d}}(w) \cap J_{c_0}$. By Theorem B there is $C_3 > 0$, only depending on P_{c_0}, such that $|c - c_0 - \lambda(w' - c_0)| \leq C_3 r^{1+1/d}$. Then the corollary follows. ∎

Proof of the second corollary. It follows by [Ma] that P_{c_0} is uniformly expanding in the set of accumulation points of the orbit of 0, denoted by $\omega(0)$. So the Bounded Distortion Property holds: for $\delta > 0$ small and $z \in \omega(0)$ there is $n \geq 0$ so that $P^n_{c_0}(B_r(z))$ is of unit size and the distortion of $P^n_{c_0}$ in $B_\delta(z)$ is bounded by some constant independent of z; see Section 2.

By [DU] for all $z \in J_{c_0}$ and $r > 0$ small $m_D(B_r(z)) \sim r^D$, where m_D denotes the restriction to J_{c_0} of the D-dimensional Hausdorff measure. Hence J_{c_0} can be covered with a collection of $\sim r^{-D/d}$ balls of radius $r^{1/d}$. By the Bounded Distortion Property $(J_{c_0} - c_0)_r$ can also be covered by such a collection of balls and by the previous corollary this also holds for $(\mathcal{M}_d - c_0)_r$. Hence the measure of $(\mathcal{M}_d - c_0)_r$ is $\mathcal{O}(r^{-D/d} r^{2/d})$, and therefore the measure of $\mathcal{M}_d \cap B_r(c_0)$ is $\mathcal{O}(r^{2+(2-D)/d})$. ∎

Appendix 2. Transversality. Fix $d \geq 2$ and consider a semihyperbolic polynomial $P_{c_0}(z) = z^d + c$ such that $c_0 \in \partial\mathcal{M}_d$. By [Ma], P_{c_0} is uniformly expanding in $\omega(0)$. So for some $\delta > 0$ there is a holomorphic motion $j : B_\delta(c_0) \times \omega(0) \to \mathbb{C}$ which is compatible with dynamics; see [Sh]. By the expansive property of hyperbolic sets there is $l > 1$ such that $P^l_{c_0}(0) \in \omega(0)$, so there is a holomorphic function $z(c)$ such that $z(c_0) = c_0$ and $P^{l-1}_c(z(c)) = j_c(P^{l-1}_c(c_0))$ for $c \in B_\delta(c_0)$.

The objective of this appendix is to prove the following property.

Transversality. *The graph of the function z is transversal to the diagonal at (c_0, c_0); that is, $z(c_0) = c_0$ and $z'(c_0) \neq 1$.*

In the Misiurewicz case, this property is well known and there are various different proofs of this fact. For example there is an algebraic proof by A. Gleason in [DH2] and there is a proof of A. Epstein from a more abstract result (in [E]) based on infinitesimal Thurston rigidity.

We prove Transversality in the more general semihyperbolic case, using Thurston rigidity. For $d = 2$ this also follows from [vS], which was done independently. The idea is to argue by contradiction. So if Transversality does not hold, then one can find two different parameters c_1 and c_2, close to c_0, such that their respective polynomials have the same dynamical properties and then we prove that in fact $c_1 = c_2$. More concretely it will be proved that the polynomials P_{c_1} and P_{c_2} are equivalent in the sense of Thurston; see [DH3]. This implies that P_{c_1} and P_{c_2} are holomorphically conjugate and since c_1 and c_2 are close to c_0 it follows that $c_1 = c_2$. One may also argue with external rays, but with Thurston rigidity the argument generalizes to rational functions. I am grateful to A. Douady who suggested to me to argue by contradiction.

Let us consider a Markov partition U_a, $a \in A$, as in Section 2 and note that as in Section 4.2 we may suppose that j is defined in $\bigcup_{a\in A} U_a$. Denote by K_{c_0} the maximal invariant set for the Markov partition. Enlarging it if necessary, we may suppose that $P^l_{c_0}(0)$ is an accumulation point of periodic points in K_{c_0}.

Proof of Transversality. Suppose that Transversality does not hold. By [DH2] we have $z(c) \not\equiv c$, therefore there is $m > 1$ such that $|z(c) - c| \sim |c - c_0|^m$. Let w be close to c_0 so that $P^{l-1}_{c_0}(w) \in K_{c_0}$ and let $c'_1 \neq c'_2$ be such that $c'_i - z(c'_i) + c_0 = w$ for $i = 1, 2$. Thus, if w is close enough to c_0 then

$$\frac{|c'_1 - c'_2|}{|c'_1 - c_0|} > \frac{1}{2}|e^{2\pi i/m} - 1|.$$

Applying the Rouché theorem as in the proof of Theorem B (with the help of Lemma 4.3) there are $c_1 \neq c_2$ such that $P^{l-1}_{c_i}(c_i) = j_{c_i}(P^{l-1}_{c_0}(w))$ with $c_i \in D$ close to c_0 for $i = 1, 2$; see also Lemma 5.1.

1. Put $U' = \bigcup_{a\in A} \overline{U}_a$ and let U'_∞ be a neighborhood of ∞ such that $\overline{U'}$ and $\overline{U'_\infty}$ are disjoint.

Let D be a small disc centered at c_0 and let $i : D \times (\overline{U'} \cup \overline{U'_\infty}) \to \mathbb{C}$ be the holomorphic motion that coincides with j in U' and is defined by $i_c(z) = \varphi_c^{-1} \circ \varphi_{c_0}(z)$ for $z \in \overline{U'_\infty}$. So i is compatible with dynamics.

By hypothesis $P^{l-1}_{c_0}(c_0)$ is an accumulation point of periodic points in K_{c_0}, so c_0 is the limit of a sequence of preperiodic points w with $P^{l-1}_{c_0}(w) \in K_{c_0}$. Extend i to $\bigcup_{0<m<l} P^m_{c_0}(w)$ so that i is compatible with dynamics. Let c_1 and c_2 be the parameters corresponding to w as above, so that $i_{c_i}(w) = c_i$.

Put $V' = U' \cup U'_\infty \cup \bigcup_{0<m<l} P^m_{c_0}(w)$ and $V = P^{-1}_{c_0}(V')$. Extend i to $\overline{V} - \{0\}$ by dynamics and let $i_c(0) \equiv 0$. Since D is conformally equivalent to a disc, one can extend i to $D \times \overline{\mathbb{C}}$; see [Sł].

2. For $c \in D$ consider the homeomorphism $\theta'_c = i_c \circ i^{-1}_{c_1}$ of $\overline{\mathbb{C}}$, so that $\theta'_{c_1} \equiv \mathrm{id}$. For $z \in \overline{V} - \{0\}$ let $\theta_c(z) = j_c^{-1} \circ j_{c_1}(z)$, so $P_c \circ \theta_c = \theta'_c \circ P_{c_1}(z)$ in $i_{c_1}(\overline{V})$ and θ_c coincides with θ'_c in $i_{c_1}(K_{c_0})$ and in $i_{c_1}(\bigcup_{0<m<l} P^m_{c_0}(w))$.

Choose $\delta > 0$ such that $\{|z - c_0| < \delta\} \cap i_c(\overline{V'}) = i_c(w)$ for $c \in D$; such a δ can be chosen before choosing w. For $c \in D$ let h_c be a homeomorphism of $\overline{\mathbb{C}}$ depending continuously on c so that h_c is the identity outside the ball $\{|z - c_0| < \delta\}$ and $h_c(c) = i_c(w)$. So $h_{c_i}(c_i) = c_i$ for $i = 1, 2$. Moreover we assume that h_{c_i} is the identity for $i = 1, 2$.

Then for $c \in D$ we see that $h_c \circ P_c : \overline{\mathbb{C}} - \overline{V} \to \overline{\mathbb{C}} - \overline{V'}$ is a d-fold covering map. Since $h_c \circ P_c \circ \theta_c \equiv \theta'_c \circ P_{c_1}$ in $i_{c_1}(\overline{V})$, there is a unique way to extend θ_c to a homeomorphism so that $h_c \circ P_c \circ \theta_c \equiv \theta'_c \circ P_{c_1}$.

3. Considering that h_{c_i} is the identity for $i = 1, 2$, we have $P_{c_2} \circ \theta_{c_2} \equiv \theta'_{c_2} \circ P_{c_1}$. Moreover $\theta_{c_1} = \theta'_{c_1} = \mathrm{id}$ and for all $c \in D$, the homeomorphism $\theta_c^{-1} \circ \theta'_c$ is the identity on $K_{c_0} \cup i_{c_1}(\bigcup_{0<m<l} P_{c_0}^m(w))$. So $\theta_c^{-1} \circ \theta'_c$ is the identity in the post-critical set of P_{c_1}.

Consider a path $\gamma : [0,1] \to D$ such that $\gamma(0) = c_1$ and $\gamma(1) = c_2$. Then $t \mapsto \theta_{\gamma(t)}^{-1} \circ \theta'_{\gamma(t)}$ is an isotopy between id and $\theta_{c_2}^{-1} \circ \theta'_{c_2}$, relative to the post-critical set of P_{c_1}. This means exactly that P_{c_1} and P_{c_2} are equivalent in the sense of Thurston. Therefore P_{c_1} and P_{c_2} are holomorphically conjugate; see [DH3]. Since c_1 and c_2 are close to c_0 we find that $c_1 = c_2$, which is a contradiction. ■

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