

On a universality property of some abelian Polish groups

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Abstract. We show that every abelian Polish group is the topological factor group of a closed subgroup of the full unitary group of a separable Hilbert space with the strong operator topology. It follows that all orbit equivalence relations induced by abelian Polish group actions are Borel reducible to some orbit equivalence relations induced by actions of the unitary group.

1. Introduction. For a class \mathcal{C} of topological groups, there are usually two competing notions of universality. In some context, a universal object is a topological group into which every group in the class \mathcal{C} can be isomorphically embedded as a topological subgroup. In a different sense, a universal object means a topological group of which every group in \mathcal{C} is a topological factor group, i.e., there is a continuous and open homomorphism from the universal group onto each group in \mathcal{C} . The notions are sometimes distinguished from each other by being respectively called *injective universality* and *projective universality*, but the terminology has not been standardized. In either of the two senses, it is of definite interest whether a universal object belongs to the class \mathcal{C} , although the mere existence of universal objects, no matter in \mathcal{C} or not, can be more important.

Here we consider a universality property that combines the above two senses. A topological group G is universal for \mathcal{C} in our sense if every group in \mathcal{C} is a topological factor group of a topological subgroup of G . This is a weaker notion than either one mentioned above. It is also easy to see that the relation “ H is a topological factor group of a topological subgroup of G ” is transitive.

The class \mathcal{C} we deal with in this paper is that of all abelian Polish groups. The existence of both injectively and projectively universal Polish groups is already known. However, those universal groups are of a special kind,

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not at all well understood, while the groups universal in our weaker sense count among them such familiar objects as the additive group of the Banach space ℓ_1 . Since the latter topological group embeds into the full unitary group U_∞ of the separable complex Hilbert space ℓ_2 , equipped with the strong operator topology, it follows that U_∞ is universal in our sense for the class of all abelian Polish groups.

Our investigation is motivated by questions in descriptive set theory of equivalence relations. Let us briefly review the main concepts of this theory. If a Polish group G acts in a Borel manner on a standard Borel space X (in which case X is called a *Borel G -space*), we denote the induced orbit equivalence relation by E_G^X . If E and F are equivalence relations on standard Borel spaces X and Y respectively, then we say that E is *Borel reducible to F* , denoted $E \leq_B F$, if there is a Borel function $f : X \rightarrow Y$ such that, for all $x, y \in X$,

$$xEy \Leftrightarrow f(x)Ff(y).$$

An important open problem in the theory of equivalence relations is: Is there an orbit equivalence relation induced by a Polish group action which is not Borel reducible to any orbit equivalence relation of an action of the unitary group? We provide a partial answer as follows.

THEOREM 4.4. *Let G be an abelian Polish group and X be a Borel G -space. Then there is a Borel U_∞ -space Y such that $E_G^X \leq_B E_{U_\infty}^Y$.*

The following interesting question seems to be open: Is every separable metrizable topological group a topological factor group of a suitable topological subgroup of U_∞ ? If the answer to this question is in the affirmative, then the above-mentioned open problem about orbit equivalence relations would be completely settled.

A by-product of our investigation is a new (and more elegant) proof of the known result from [16]: every separable metrizable abelian topological group embeds as a topological subgroup into a monothetic metrizable topological group.

The two main tools used in our paper are transportation distances and positive definite functions. Transportation distances have been independently discovered in different areas of mathematics and are thus known under numerous names. We give a survey of the theory in Section 2. Section 3 outlines the use of positive definite functions to construct strongly continuous unitary representations of some topological groups. In Section 4 the main Borel reducibility results are deduced. We have attempted to make the article relatively self-contained, collecting in it definitions and hints of proofs of known results for the reader's convenience.

2. Transportation distances. Transportation distances were initially introduced by Kantorovich in his 1942 paper [9] in order to study the classical mass transportation problem, and have since then found numerous applications in different areas of mathematics, in some of which they have been rediscovered independently and explored to varying degrees of depth and from various angles.

2.1. Free normed spaces. Let $X = (X, d, *)$ be a pointed metric space, that is, a triple where d is a metric on a set X and $*$ $\in X$ is a distinguished point. Denote by $L(X, *)$, or simply by $L(X)$, the real vector space having $X \setminus \{*\}$ as its Hamel basis and $*$ as zero.

There obviously exists a largest prenorm, p , on $L(X)$ with the property that the distance induced on X does not exceed d : for all $x, y \in X$, $p(x - y) \leq d(x, y)$.

Such a p is in fact a norm, and the restriction of the associated distance to X coincides with d . Indeed, these are equivalent to saying that every metric space isometrically embeds into a normed space as a linearly independent set. Here is such an embedding (described in [10] and, independently, [1]; cf. also [14]). Denote by $\text{Lip}(X, *)$ the Banach space of all Lipschitz functions $f : X \rightarrow \mathbb{R}$ with the property $f(*) = 0$, and with the norm $\|f\|$ being the smallest Lipschitz constant for f . For an $x \in X$, denote by \hat{x} the evaluation functional:

$$\text{Lip}(X, *) \ni f \mapsto f(x) \in \mathbb{R}.$$

The mapping

$$X \ni x \mapsto \hat{x} \in \text{Lip}(X, *)'$$

is an isometric embedding of X into the dual Banach space of $\text{Lip}(X, *)$ as a linearly independent subset (an easy check).

In fact, more is true: every element of $L(X, *)$, if considered as a finitely supported measure on $X \setminus \{*\}$, determines a bounded linear functional on $\text{Lip}(X, *)$, and thus $L(X, *)$ embeds into the dual Banach space $\text{Lip}(X, *)'$ as a normed subspace. The dual norm on $L(X)$ induced from $\text{Lip}(X, *)'$ is exactly the maximal prenorm that we are after. Notice also that X is closed in $L(X)$.

The normed space $L(X)$ has the following universal property, which provided the main motivation for the investigations of [1, 20, 4, 5].

THEOREM 2.1. *Let E be a normed space, and let $f : X \rightarrow E$ be a 1-Lipschitz map with $f(*) = 0$. Then there is a unique linear operator $\bar{f} : L(X) \rightarrow E$ of norm 1 extending f .*

Proof. The existence of a unique linear operator \bar{f} as above is clear. It remains to notice that the prenorm on $L(X)$ denoted by $q(x) = \|\bar{f}(x)\|_E$ has the property $q(x - y) \leq d(x, y)$ for all $x, y \in X$, and thus $q(z) \leq \|z\|$ for all $z \in L(X)$ and the statement follows. ■

The formula (2.1) that follows can be seen both as the definition of the transportation distance [9], and as an alternative description of the norm of the free normed space [1, 20, 4, 5] going back to Graev [8], where it appeared in the context of free (abelian) groups.

THEOREM 2.2. *Let $x \in L(X, *, d)$. Then*

$$(2.1) \quad \|x\| = \inf \left\{ \sum_{i=1}^n |\lambda_i| d(x_i, y_i) : n \in \mathbb{N}, \lambda_i \in \mathbb{R}, x_i, y_i \in X, \right. \\ \left. x = \sum_{i=1}^n \lambda_i x_i, 0 = \sum_{i=1}^n \lambda_i y_i \right\}.$$

Proof. Denote by $\|\cdot\|'$ the prenorm determined by the expression on the right hand side of (2.1), and let $\|\cdot\|$ stand for the norm of the free normed space $L(X)$. If $x \in L(X)$, then for any two decompositions of x and 0 as in (2.1) one has

$$\|x\| \leq \sum_{i=1}^n \|\lambda_i(x_i - y_i)\| = \sum_{i=1}^n |\lambda_i| d(x_i, y_i),$$

and consequently $\|x\| \leq \|x\|'$. Now let $x, y \in X$. Writing $x - y = 1 \cdot x + (-1)y$ and $0 = 1 \cdot x + (-1)x$, one concludes that

$$\|x - y\|' \leq 1 \cdot d(x, x) + 1 \cdot d(x, y) = d(x, y),$$

and consequently $\|x\|' \leq \|x\|$ for every $x \in L(X)$. ■

The Banach space completion of $L(X)$ is denoted by $B(X)$ and called the *free Banach space* on the pointed metric space $(X, d, *)$. It has a universal property similar to that in Theorem 2.1 with respect to all *Banach spaces* E .

EXAMPLE 2.3. If $X = \Gamma \cup \{*\}$ is a set equipped with a discrete ($\{0, 1\}$ -valued) metric, then the free Banach space $B(\Gamma \cup \{*\})$ (where $*$ is the distinguished point) is isometrically isomorphic to $\ell_1(\Gamma)$.

It is easy to see that the following three conditions are equivalent: (i) a metric space X is separable; (ii) the free normed space $L(X)$ is separable; (iii) the free Banach space $B(X)$ is separable.

On this occasion let us recall a well known and simple fact from classical Banach space theory (cf. e.g. p. 108 in [12]).

THEOREM 2.4. *Every separable Banach space E is a factor space of ℓ_1 .*

Proof. Let f be an arbitrary map from \mathbb{N}_+ onto an everywhere dense subset of the sphere of radius $1/2$ around zero in E . The map f is 1-Lipschitz with respect to the $\{0, 1\}$ -valued metric on \mathbb{N} , and thus extends to a linear operator \bar{f} of norm ≤ 1 (in fact, exactly 1) from $B(\mathbb{N}) \cong \ell_1$ to E . (Here $0 \in \mathbb{N}$ serves as the distinguished point.) Let $x \in E$ be arbitrary with $\|x\| = 1/2$.

It is possible to choose recursively a sequence of elements $x_n \in \mathbb{N}$ and non-negative scalars $\lambda_n \leq 2^{-n}$ in such a way that each element $\sum_{i=1}^n \lambda_i f(x_i)$ is at a distance $< 2^{-(n+1)}$ from x . Consequently, $z = \sum_{i=1}^{\infty} \lambda_i x_i$ is in ℓ_1 and $\bar{f}(z) = x$. Thus, the operator \bar{f} is onto, and the Open Mapping Theorem finishes the proof. ■

Let $a, b \in X$. The following fact, standard in the theory of free objects, is established by applying the universal property from Theorem 2.1 to the 1-Lipschitz mapping $X \ni x \mapsto x - a + b \in B(X, b)$.

PROPOSITION 2.5. *Let $X = (X, d)$ be a metric space. For all choices of the distinguished point $* \in X$ the resulting free Banach spaces $B(X, d, *)$ are isometrically isomorphic. ■*

Let us assume temporarily that (X, d) has diameter ≤ 1 . Denote by X^\dagger the metric space obtained from X by adding an extra point $\{\dagger\}$ at a distance 1 from every $x \in X$. Denote by ϕ the linear functional of norm 1 on $B(X^\dagger, \dagger)$ which takes X to $\{1\}$ and which exists by Theorem 2.1. Let $B(X)_0$ stand for the kernel of ϕ , and let $L(X)_0 = B(X)_0 \cap L(X)$.

PROPOSITION 2.6. *Assume that $\text{diam } X \leq 1$. For every $* \in X$, the free Banach space $B(X, d, *)$ (respectively the free normed space $L(X, d, *)$) is isometrically isomorphic to $B(X)_0$ (respectively, $L(X)_0$). ■*

Here, similarly to the proof of Proposition 2.5, the isomorphic embedding $B(X, d, *) \hookrightarrow B(X^\dagger, \dagger)$, when restricted to X , is of the form $x \mapsto x - *$.

Recall that if μ is a measure on the product of two standard Borel spaces X and Y , then the *marginals* of μ are the push-forward measures $\pi_{i,*}\mu$, $i = 1, 2$, along the coordinate projections. The (finitely supported) signed measures μ on $X \times X$ whose marginals are, respectively, x and 0, can be identified with a pair of representations of x and 0 as in (2.1).

Denote by \tilde{d} the distance determined by the free norm, $\tilde{d}(x, y) = \|x - y\|$. Theorem 2.2 and Proposition 2.6 lead to the following result.

THEOREM 2.7. *Let μ_1, μ_2 be finitely supported probability measures on X . Then*

$$(2.2) \quad \tilde{d}(\mu_1, \mu_2) = \inf \left\{ \int_{X \times X} d(x, y) d\nu : \pi_{i,*}(\nu) = \mu_i, \quad i = 1, 2 \right\}. \quad \blacksquare$$

The formula (2.2) makes sense for arbitrary Borel probability measures on a metric space, and the distance \tilde{d} is known in this and similar contexts as the *transportation distance*, *Monge–Kantorovich distance*, *Prokhorov distance* (in probability), *Wasserstein distance* (in ergodic theory), or else *Earth Mover’s distance* (in computer science). See the two-volume mono-

graph [19], largely devoted to the study of the transportation distance and containing a very large—though still not exhaustive—bibliography.

Here is the master result (cf. e.g. [19, Section 4.1]).

THEOREM 2.8 (Kantorovich optimality criterion). *Let $X = (X, d)$ be a metric space. A probability measure ν on $X \times X$ whose marginals are, respectively, μ_1 and μ_2 , achieves the infimum in (2.2) if and only if there exists a 1-Lipschitz function $f : X \rightarrow \mathbb{R}$ such that for all pairs $(x, y) \in \text{supp } \nu$ one has*

$$f(x) - f(y) = d(x, y). \blacksquare$$

The following is an immediate consequence (cf. [20, 4, 5] for direct proofs).

COROLLARY 2.9. *The infimum in (2.1) is achieved at some representations of x and 0 with $x_i, y_i \in \text{supp } x \cup \{0\}$. ■*

COROLLARY 2.10 (Integer Value Property). *If x is a linear combination with integer coefficients, then the infimum in (2.1) is achieved at some representations of x and 0 as linear combinations with integer coefficients.*

Proof. The Kantorovich criterion reduces the result to the following fact, established by an easy combinatorial argument. Suppose a matrix A with real entries is such that the entries in each column and in each row add up to an integer. Then all non-zero entries of A can be replaced with integers without altering the column sums and row sums of A . ■

In the language of optimization theory, Corollary 2.10 says that a transportation problem with integer supply and demand has an integer optimal solution. This is a classical result, to be found in textbooks such as [21] (Remark 10 on page 179).

For complex free normed spaces the above results starting with 2.2 are no longer true [4, 5].

2.2. Graev metrics on free abelian groups. The group envelope of X in $L(X, *)$ is just the free abelian group having $X \setminus \{*\}$ as the set of free generators. We will denote it by $A(X, *)$ or simply by $A(X)$. The restriction of the distance \tilde{d} , generated by the free norm, to $A(X, *)$, which we denote by \bar{d} , is a bi-invariant metric, and $\bar{d}|_X = d$. An analogue of Theorem 2.2 can be stated for \bar{d} , and together with Corollary 2.10 it implies the following.

COROLLARY 2.11. *The metric \bar{d} is the maximal among all bi-invariant metrics on $A(X)$ whose restriction to X is majorized by d . ■*

In the theory of topological groups, the metric \bar{d} is known as the *Graev metric* [8].

COROLLARY 2.12. *The metric group $A(X, d, *)$ is a (closed) metric subgroup of the normed space $L(X, d, *)$. ■*

The above two corollaries are just equivalent forms of the same result, first stated by Tkachenko [25], who offered a direct, albeit incomplete, proof. Uspenskii [27] later noted that the result in question follows from the Integer Value Property.

The metric space completion of the group $A(X)$ equipped with the metric \bar{d} is an abelian topological group, which we will denote by $\widehat{A}(X, d, *)$.

COROLLARY 2.13. *The complete metric group $\widehat{A}(X, d, *)$ is a (closed) metric subgroup of the Banach space $B(X, d, *)$. ■*

The following universal property of the metric group $A(X, d, *)$ is a standard result in the theory.

PROPOSITION 2.14 (Graev). *Let $X = (X, d)$ be a metric space, let G be an abelian group equipped with a bi-invariant metric ς , and let $f : G$ be a 1-Lipschitz map (with regard to the distances d on X and ς on G), taking $*$ to 0_G . Then there exists a unique 1-Lipschitz homomorphism $\bar{f} : A(X, d, *) \rightarrow (G, \varsigma)$ extending f from X .*

Proof. For purely algebraic reasons, there is only one group homomorphism $\bar{f} : A(X, d, *) \rightarrow (G, \varsigma)$ extending f . Let us show that \bar{f} is in fact 1-Lipschitz as well. Define a pseudometric ϱ on $A(X)$ as follows: for $x, y \in A(X)$,

$$\varrho(x, y) := \varsigma(\bar{f}(x), \bar{f}(y)).$$

This ϱ is a bi-invariant pseudometric, and the restriction of ϱ to X is majorized by d . We conclude by Corollary 2.11 that $\varrho \leq \bar{d}$. But this is another way of saying that $\bar{f} : (A(X), \bar{d}) \rightarrow (G, \varsigma)$ is 1-Lipschitz. ■

COROLLARY 2.15. *Let (G, ς) be an abelian group equipped with a complete bi-invariant metric. Let $f : (X, d) \rightarrow (G, \varsigma)$ be a 1-Lipschitz map taking $*$ to 0_G . Then f extends to a unique continuous homomorphism $\bar{f} : \widehat{A}(X) \rightarrow G$ which is moreover 1-Lipschitz. ■*

Here is another elementary and well known observation, again going back to [8].

PROPOSITION 2.16. *Every metrizable abelian topological group G is a topological factor group of a metrizable group of the form $A(X, d, *)$. If G is completely metrizable, it is a quotient group of a group of the form $\widehat{A}(X, d, *)$. If G is Polish, then the latter group can be assumed Polish as well.*

Proof. Let d be any bi-invariant metric on G generating the topology. Set $X = G$, $*$ = e_G , and consider the group $A(G, e_G, d)$. The identity map

from G to itself extends to a unique 1-Lipschitz epimorphism $i : A(X) \rightarrow G$. Every ε -neighborhood of identity, V_ε , in $A(X)$ contains the ε -neighborhood formed within X , and therefore the image $\bar{f}(V_\varepsilon)$ has a non-empty interior in G (as it contains $f(V_\varepsilon \cap X)$). It follows that \bar{f} is an open homomorphism. The remaining statements are obvious. ■

Proposition 2.16 and Corollary 2.12 together imply:

COROLLARY 2.17. *Every abelian metrizable group G is isomorphic to a topological factor group of a closed subgroup of the additive group of a normed space E . If G is complete metrizable, then E is a Banach space. If G is Polish, then E is separable Banach.* ■

The authors of the paper [15], where the above result appeared in print for the first time, ought to have mentioned that the corollary had in fact entered topological folklore shortly after the publication of Tkachenko's influential work [25].

Invoking Theorem 2.4, one obtains:

COROLLARY 2.18. *Every abelian Polish group G is isomorphic to a topological factor group of a closed topological subgroup of the additive group of ℓ_1 .*

Proof. Let $\pi : \bar{A}(X) \rightarrow G$ be a factor homomorphism, and let $T : \ell_1 \rightarrow B(X)$ be an open linear epimorphism. Then the complete preimage, E , of $\pi^{-1}(G)$ under T is a closed topological subgroup of ℓ_1 . The restriction of T to the complete preimage of a closed set is a quotient map. Consequently, the composition $\pi \circ (T|_E)$ is an open homomorphism of topological groups (as a composition of two open homomorphisms). ■

3. Positive definite functions and topological subgroups of U_∞ .

Recall that a complex-valued function f on a group G is *positive definite* if for every finite collection g_i , $i = 1, \dots, n$, of elements of G and any complex scalars λ_i , $i = 1, \dots, n$, $n \in \mathbb{N}$,

$$\sum_{i,j=1}^n f(g_i g_j^{-1}) \lambda_i \bar{\lambda}_j \geq 0.$$

It is a standard fact in representation theory that continuous positive definite functions on a topological group G are in one-one correspondence with strongly continuous cyclic representations of G having a (distinguished) cyclic vector (cf. e.g. [17, §30]). For a separable metrizable group G , its embeddability into U_∞ is thus closely related to the existence of topology-generating positive definite functions on G . The standard argument in fact gives the following finer result.

THEOREM 3.1. *Let G be a separable metrizable group and 1_G be its identity element. Then G is isomorphic to a topological subgroup of U_∞ iff there is a continuous positive definite function on G separating 1_G and closed subsets of G not containing 1_G .*

Proof. Let $f : G \rightarrow \mathbb{C}$ be a continuous positive definite function on G which separates 1_G and closed subsets not containing 1_G . Form the linear space X of all complex-valued functions on G with finite support. For $x, y \in X$, let

$$\langle x, y \rangle = \sum_{g, h \in G} f(h^{-1}g)x(g)\overline{y(h)}.$$

Let $N = \{x \in X : \langle x, x \rangle = 0\}$. Then N is a linear subspace of X and the sesquilinear form induces an inner product on X/N , making X/N a pre-Hilbert space. Let H be the completion of X/N under the induced norm metric. Then H is a separable complex Hilbert space. The standard representation of G in $U(H)$ defined by

$$T_g x(h) = x(g^{-1}h)$$

is easily checked to be a topological embedding of G into $U(H)$ with the strong operator topology.

Conversely, assume that G is a topological subgroup of $U(H)$ with the strong operator topology, where H is some separable complex Hilbert space. Note that for any $v \in H$, the function

$$f_v(g) = \langle g(v), v \rangle$$

is continuous and positive definite on G . Moreover the collection $\{f_v : v \in H\}$ generates the topology on G . By separability of G there is a countable subcollection which already generates the topology of G . Denote this subcollection by \mathcal{F}_0 . Then the set \mathcal{F}_1 of all finite products of elements of \mathcal{F}_0 is again a collection of positive definite functions on G , and this new set separates 1_G and closed subsets of G not containing 1_G . Let $\mathcal{F}_1 = \{f_n : n \in \mathbb{N}\}$. Without loss of generality we can assume that $f_n(1_G) \leq 1/2^n$ for each $n \in \mathbb{N}$. Finally, define

$$f(x) = \sum_{n \in \mathbb{N}} f_n(x).$$

Then f is a continuous positive definite function on G separating 1_G and closed subsets of G not containing 1_G . ■

In [24] Shoenberg proved that, for $1 \leq p \leq 2$, the function $e^{-\|x\|^p}$ is positive definite on ℓ_p . Since this function obviously separates the identity from closed subsets not containing the identity, one obtains the following.

PROPOSITION 3.2. *The additive group of each ℓ_p , $1 \leq p \leq 2$, is isomorphic to a closed subgroup of U_∞ . In particular, ℓ_1 is (isomorphic to) a closed subgroup of U_∞ . ■*

These facts were noted by Megrelishvili in [13].

Thus we have the following in view of Corollary 2.18.

COROLLARY 3.3. *Every abelian Polish group is isomorphic to a factor group of a closed abelian subgroup of U_∞ . ■*

In the remaining part of this section we consider the topological group $L^0(X, U(1))$. Here X is an arbitrary uncountable standard Borel space with a non-atomic Borel measure. The group $L^0(X, U(1))$ consists of all (equivalence classes of) measurable functions from X into the circle rotation group $U(1)$, and the topology of $L^0(X, U(1))$ is that of convergence in measure. The topological group $L^0(X, U(1))$ is the unitary group of the abelian von Neumann algebra $L^\infty(X)$, equipped with the ultraweak topology. This can also be considered as the strong operator topology with regard to the standard representation of $L^\infty(X)$ by multiplication operators in $L^2(X)$.

The following is another well known theorem for which it is hard to find a standard reference. However, arguments that are sufficient to establish the theorem can be found in many sources, e.g., [3] and [26]. (The result as stated appears, for instance, in the paper [7], but it had certainly been known to experts for a long time before that.)

THEOREM 3.4. *Let G be a separable metrizable abelian group. Then G is isomorphic to a topological subgroup of U_∞ iff G is isomorphic to a topological subgroup of $L^0(X, U(1))$.*

Proof. The remarks above give an embedding of $L^0(X, U(1))$ into U_∞ . Now suppose G is an abelian topological subgroup of U_∞ . We can extend G to a maximal abelian von Neumann algebra W . Then W is isomorphic to $L^\infty(X)$, and the ultraweak topology on the unitary group of the latter (the topology generated by the von Neumann algebra predual) coincides with the strong operator topology induced from U_∞ . The result follows. ■

Thus we have the following immediate corollary.

COROLLARY 3.5. *Every abelian Polish group G is isomorphic to a topological factor group of a closed subgroup of $L^0(X, U(1))$.*

Proof. Since ℓ_1 is a closed subgroup of U_∞ by Proposition 3.2, ℓ_1 is a closed subgroup of $L^0(X, U(1))$ by Theorem 3.4. The statement now follows from Corollary 2.18. ■

In fact, because we are dealing with abelian groups here, a stronger statement ensues.

COROLLARY 3.6. *Every abelian Polish group G is isomorphic to a closed subgroup of a topological factor group of $L^0(X, U(1))$.*

Proof. Let F be a closed subgroup of ℓ_1 , given by Corollary 2.18, with the property that G embeds into ℓ_1/F as a closed subgroup. Embed ℓ_1 into $L^0(X, U(1))$ as a closed subgroup using Theorem 3.4. Then G is isomorphic to a closed subgroup of $L^0(X, U(1))/F$. ■

The topological group $L^0(X, U(1))$ seems to play an important role in the theory of “large” topological groups. Among other known properties of $L^0(X, U(1))$ are the following two.

THEOREM 3.7. *$L^0(X, U(1))$ is a monothetic topological group.* ■

(See e.g. [6], or a simple proof from [16].)

THEOREM 3.8 ([6]; also Furstenberg and Weiss, unpublished). *The topological group $L^0(X, U(1))$ is extremely amenable, that is, every continuous action of $L^0(X, U(1))$ on a compact space has a fixed point.* ■

Embeddability into monothetic groups is closed under taking factor groups [16]. It is also evident that a topological factor group of an extremely amenable group is extremely amenable. Thus combining the previous three results, we not only obtain a different proof of the main result of [16], but also strengthen it as follows.

THEOREM 3.9. *Every separable metrizable abelian topological group embeds as a topological subgroup into a monothetic extremely amenable metrizable topological group. This group is a topological factor group of $L^0(X, U(1))$.* ■

It was previously known that every topological group embeds into an extremely amenable group [18], but an abelian version of the result appears here for the first time.

4. Borel actions. If a Polish group G is a topological factor group of a Polish group H , then any Borel G -space X is trivially a Borel H -space. Moreover, the orbit equivalence relations E_G^X and E_H^X are the same.

The following can be found e.g. in [2, Theorem 2.3.5].

THEOREM 4.1 (Mackey). *If G is a closed subgroup of a Polish group H , then any Borel G -space X can be extended to a Borel H -space Y such that every G -orbit in X is contained in exactly one H -orbit in Y . Moreover, it follows that $E_G^X \leq_B E_H^Y$.* ■

Combining these results, we obtain:

COROLLARY 4.2. *If G is a topological factor group of a closed subgroup of H , then for any Borel G -space X there is a Borel H -space Y such that $E_G^X \leq_B E_H^Y$. ■*

The same conclusion holds if G is a closed subgroup of a topological factor group of H . But for an abelian Polish group H these two conditions are in fact equivalent.

Our results from previous sections thus imply Borel reducibility results for orbit equivalence relations induced by actions of the groups mentioned. Let us first summarize the universality results we have essentially proved.

THEOREM 4.3. *Every abelian Polish group is the topological factor group of a closed subgroup of any of the following groups:*

- (i) *the additive group of the Banach space ℓ_1 ;*
- (ii) *the additive group of the Banach space $C([0, 1])$;*
- (iii) *$L^0(X, U(1))$, where X is any uncountable standard Borel space with a non-atomic Borel measure;*
- (iv) *the full unitary group U_∞ . ■*

Clause (ii) follows from the well known fact that the Banach space $C([0, 1])$ is a universal separable Banach space.

One could also add two other groups to the list. One is the projectively universal abelian Polish group $\hat{A}(\mathcal{N})$, where \mathcal{N} is the Baire space of infinite sequences of natural numbers. The proof of its universality follows from Proposition 2.16 modulo known properties of the Baire space (equipped with the standard ultrametric). See [11, 22].

The other group is the injectively universal abelian Polish group recently constructed by Shkarin [23]. Both these groups are not quite as familiar as the universal groups we consider here.

Our main Borel reducibility result is now immediate.

THEOREM 4.4. *Let G be an abelian Polish group and let X be a Borel G -space. Then there is a Borel U_∞ -space Y such that $E_G^X \leq_B E_{U_\infty}^Y$.*

Orbit equivalence relations induced by abelian Polish group actions are a rich source of examples in the descriptive set theory of equivalence relations. To name a few important equivalence relations that have been studied intensively: the shift equivalence relations on $\mathbb{R}^{\mathbb{N}}$ by classical Banach spaces ℓ_p ($p \geq 1$) or c_0 , the equivalence relations on $\mathcal{P}(\mathbb{N})$ (the power set of \mathbb{N}) by the natural actions of Polishable Borel ideals. There has been some hope that an example of an equivalence relation not Borel reducible to any U_∞ -orbit

equivalence relation would be among these examples; now we know there can be none.

Of course Theorem 4.3 also implies that any orbit equivalence relation induced by an abelian Polish group action must be Borel reducible to one induced by an action of either ℓ_1 , $C([0, 1])$ or $L^0(X, U(1))$. Furthermore we can identify some universal equivalence relations among those induced by abelian Polish group actions. These are summarized in the following theorem.

THEOREM 4.5. *Let G be either ℓ_1 , $C([0, 1])$ or $L^0(X, U(1))$. Let $\mathcal{F}(G)$ be the space of all closed subsets of G with the Effros Borel structure. Let G act on $\mathcal{F}(G)$ by multiplication and denote the orbit equivalence relation by E_G . Then E_G is universal among all orbit equivalence relations induced by abelian Polish group actions, i.e., for any abelian Polish group H and Borel H -space X , $E_H^X \leq_B E_G$.*

Proof. By Theorem 3.5.3 of [2], it suffices to note that $G \times \mathbb{Z}$ is isomorphic to a closed subgroup of G . ■

The following problems seem to be open: Is Theorem 4.5 true for ℓ_p ($p > 1$), especially ℓ_2 , and c_0 ?

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