FUNDAMENTA
MATHEMATICAE
179 (2003)

The gap between I_3 and the wholeness axiom

by

Paul Corazza (Fairfield, IA)

Abstract. $\exists\kappa\, \mathrm{I}_3(\kappa)$ is the assertion that there is an elementary embedding $i: V_\lambda \to V_\lambda$ with critical point below λ, and with λ a limit. The Wholeness Axiom, or WA, asserts that there is a nontrivial elementary embedding $j : V \to V$; WA is formulated in the language $\{\in, \mathbf{j}\}$ and has as axioms an Elementarity schema, which asserts that $\mathbf{j}$ is elementary; a Critical Point axiom, which asserts that there is a least ordinal moved by $\mathbf{j}$; and includes every instance of the Separation schema for $\mathbf{j}$-formulas. Because no instance of Replacement for $\mathbf{j}$-formulas is included in WA, Kunen's inconsistency argument is not applicable. It is known that an I_3 embedding $i : V_\lambda \to V_\lambda$ induces a transitive model $\langle V_\lambda, \in, i\rangle$ of ZFC + WA. We study here the gap in consistency strength between I_3 and WA. We formulate a sequence of axioms $\langle \mathrm{I}_4^n : n \in \omega\rangle$ each of which asserts the existence of a transitive model of ZFC + WA having strong closure properties. We show that I_3 represents the "limit" of the axioms I_4^n in a sense that is made precise.

1. Introduction. $\mathrm{I}_3(\kappa)$ is the statement that there is an elementary embedding $i : V_\lambda \to V_\lambda$ having critical point κ, where λ is a limit ordinal. An I_3 *embedding* is an embedding $i : V_\lambda \to V_\lambda$ that witnesses $\mathrm{I}_3(\kappa)$ for some κ. The axiom $\exists\kappa\ \mathrm{I}_3(\kappa)$ arose in the work of Kunen [Ku] who observed that his proof in KM-set theory that there is no elementary embedding $V \to V$ also showed that there could be no embedding $V_{\lambda+2} \to V_{\lambda+2}$; he noted that his proof did not rule out the possible existence of embeddings $V_\lambda \to V_\lambda$ (λ a limit) or $V_{\lambda+1} \to V_{\lambda+1}$.

The Wholeness Axiom, or WA, is an axiom schema asserting the existence of a nontrivial elementary embedding $j : V \to V$, formulated in the extended language $\{\in, \mathbf{j}\}$. WA is comprised of an Elementarity schema, which asserts that $\mathbf{j}$ is an elementary embedding; a Nontriviality axiom, which asserts that for some x, $\mathbf{j}(x) \neq x$; and all instances of the Separation schema for instances of Separation in which the symbol $\mathbf{j}$ occurs—we call the latter schema $\mathrm{Separation}_{\mathbf{j}}$. The Wholeness Axiom was developed in an effort to formulate a schema asserting the existence of a $j : V \to V$ that is

2000 *Mathematics Subject Classification*: Primary 03E55; Secondary 03E65.

Key words and phrases: large cardinal, I_3, wholeness axiom, elementary embedding, WA, huge cardinal, extender.

not provably inconsistent with ZFC via Kunen's methods, but that retains "most" of the strength of Reinhardt's original (inconsistent) version of the axiom. WA is successful in this regard because, on the one hand, it is known to imply the existence of most large cardinals; yet, on the other hand, it is not known to be inconsistent since no instance of Replacement for **j** formulas is assumed. (Such an instance is needed in order to arrive at Kunen's inconsistency, because Replacement is what allows one to prove the critical sequence $\kappa, j(\kappa), j^2(\kappa), \ldots$ forms a set, and therefore has a supremum. Here, as usual, j^n denotes the nth iterate of j under composition.) The Wholeness Axiom has been studied in [Co1]–[Co4] and [Ha]. It is known to have consistency strength strictly between that of a cardinal that is super-n-huge for every n and that of the axiom $\exists\kappa\ \mathrm{I}_3(\kappa)$.

The purpose of this paper is to see "how big" the gap is between WA and I_3. All the well-known large cardinal notions that are weaker than I_3 are also known to be weaker than WA, so we would expect the gap to be "small" in a certain sense. Our approach to the problem is based on the simple observation that if $i : V_\lambda \to V_\lambda$ is an I_3 embedding, then $\langle V_\lambda, \in, i\rangle$ is a transitive model of $\mathrm{ZFC} + \mathrm{WA}$ ([Co3]). Our perspective here is that V_λ is a highly closed transitive domain for a model of $\mathrm{ZFC} + \mathrm{WA}$. This perspective suggests a way to measure the gap between I_3 and WA: Formulate a sequence of axioms asserting the existence of increasingly closed transitive models of WA so that in the "limit", we arrive at I_3.

In the third section of this paper, we carry out this plan. We define, for each $m \in \omega$, an axiom I_4^m asserting the existence of a certain type of transitive model of $\mathrm{ZFC} + \mathrm{WA}$, and then we proceed to describe the sense in which I_3 represents the limit of these new axioms. Before proceeding with the main results, we establish some preliminaries in the next section.

In concluding this introduction, we gratefully acknowledge the efforts of the referee, which led to clarifications in a number of proofs.

2. Preliminaries. We begin with several definitions and propositions, which form a proper subset of the material developed in [Co1]. The second half of this section provides a quick review of extenders.

When working with the theory $\mathrm{ZFC} + \mathrm{WA}$, we will always be working in the language $\mathcal{L} = \{\in, \mathbf{j}\}$, where $\mathbf{j}$ is a unary function symbol. **j**-*formulas* are $\mathcal{L}$-formulas that have an occurrence of the symbol $\mathbf{j}$. Like ordinary $\in$-formulas, $\mathcal{L}$-formulas can be classified into complexity classes; the class that will concern us here is the Σ_0 $\mathcal{L}$-formulas which consist of the $\mathcal{L}$-formulas in which every quantifier is bound.

We will need to refer to certain subtheories of $\mathrm{ZFC} + \mathrm{WA}$. For this purpose, it is helpful to list a number of axioms that pertain to the language $\{\in, \mathbf{j}\}$ (see [Co1] for a complete list):

- **Elementarity**. Each of the following $\mathbf{j}$-sentences is an axiom, where $\phi(x_1, \ldots, x_m)$ is an $\in$-formula:

$$\forall x_1, \ldots, x_m\ (\phi(x_1, \ldots, x_m) \Leftrightarrow \phi(\mathbf{j}(x_1), \ldots, \mathbf{j}(x_m))).$$

- **Critical Point**: There is a least ordinal moved by $\mathbf{j}$.
- **BTEE**: Elementarity + Critical Point.
- **Separation$_{\mathbf{j}}$**: All instances of Separation for $\mathbf{j}$-formulas.
- Σ_0**-Separation**: All Σ_0 instances of Separation for $\mathbf{j}$-formulas.
- **Cofinal Axiom**: $\forall\alpha\ \exists n \in \omega\ (\kappa_n$ exists and $\alpha \le \kappa_n)$.
- **WA**: Elementarity + Critical Point + Separation$_{\mathbf{j}}$.
- **WA$_0$**: Elementarity + Critical Point + Σ_0-Separation$_{\mathbf{j}}$.

Whenever Critical Point is included as an axiom in one of our subtheories of ZFC + WA, the critical point of $\mathbf{j}$ is denoted κ; κ is understood to be a constant added to the language by definitional extension. The reader will notice that we refer to 'κ' in our formulation of the Cofinal Axiom; we adopt the convention that the Cofinal Axiom is not included in any of our subtheories unless Critical Point is also included. Moreover, in the Cofinal Axiom, we define $\kappa_0 = \kappa$ and, for $n \ge 1$, 'κ_n' stands for $\mathbf{j}^n(\kappa)$, the nth iterate of $\mathbf{j}$ applied to κ. To formulate this notion precisely, one defines the relation $y = \mathbf{j}^n(x)$ as a three-place predicate with an appropriate formula $\Phi(n, x, y)$; then the statement "κ_n exists" (as in the statement of the Cofinal Axiom) is simply an abbreviation for "$\exists y\ \Phi(n, \kappa, y)$". The formula $\Phi(n, x, y)$ is discussed in detail in [Co1].

The three theories that will concern us in this paper are:

(1) ZFC + BTEE,
(2) ZFC + WA$_0$,
(3) ZFC + WA.

To specify the strengths of these theories, and for later work, let us first recall the definition of n-huge cardinals and some variants (see [Ka]): For $1 \le n < \omega$, κ is *n-huge* if there exists an inner model M and an elementary embedding $j : V \to M$ such that $\mathrm{crit}(j) = \kappa$ and M is closed under $j^n(\kappa)$-sequences; $j(\kappa)$ is called the *target* of j and j is called an *n-huge embedding.* κ is *super-n-huge* if, for every $\lambda > \kappa$, there is an n-huge embedding j such that $\mathrm{crit}(j) = \kappa$ and $j(\kappa) > \lambda$. An equivalent form of the definition of n-huge in terms of ultrafilters is well-known (see [Ka]). We state this equivalent form in a somewhat nonstandard way, in terms of *n-huge indexes*, for use in later sections.

Definition 2.1. Suppose $1 \leq n < \omega$, κ is an uncountable cardinal, and $\lambda > \kappa$. Then λ is an *n-huge index for* κ if there is a κ-complete normal ultrafilter U over $P(\lambda)$ and cardinals $\kappa = \lambda_0 < \lambda_1 < \ldots < \lambda_n = \lambda$ so that for each $i < n$, $\{x \in P(\lambda) : \mathrm{ot}(x \cap \lambda_{i+1}) = \lambda_i\} \in U$.

Now, by the usual proof, κ is n-huge if and only if there is an n-huge index for κ. In that case, the witnessing ultrafilter is called an *n-huge ultrafilter* (over $P(\lambda)$).

Proposition 2.2. (1) *The consistency strength of* ZFC + BTEE *lies between that of an ineffable cardinal and that of* $0^{\#}$.

(2) *The consistency strength of* ZFC + WA *lies between that of a cardinal that is n-huge for every n and the existence of an* I_3 *embedding.*

To give the consistency strength of $\mathrm{ZFC} + \mathrm{WA}_0$ we need to observe that it is possible in a model of $\mathrm{ZFC} + \mathrm{WA}_0$ that κ_n fails to exist for some nonstandard integer n. This peculiar problem (which does not arise in the theory ZFC + WA) is mollified considerably by part (2) of the following:

Proposition 2.3. (1) *The theory* ZFC + WA *proves*

$$V_{\kappa_0} \prec V_{\kappa_1} \prec \ldots \prec V_{\kappa_n} \prec \ldots \prec V.$$

(2) *The theory* $\mathrm{ZFC} + \mathrm{WA}_0$ *proves that if* $\mathbf{A}$ *is the class of* $n \in \omega$ *for which* κ_n *exists*, *then* $\{\kappa_n : n \in \mathbf{A}\}$ *is cofinal in* ON, *and the* V_{κ_n} *for which* $n \in \mathbf{A}$ *form an elementary chain whose union is* V.

As a corollary, we have:

Proposition 2.4. *Cofinal Axiom is derivable from each of the theories* ZFC + WA *and* $\mathrm{ZFC} + \mathrm{WA}_0$.

Finally, we remark that different kinds of models of the language $\{\in, \mathbf{j}\}$—in particular, of the three main theories mentioned above—are possible, depending on one's assumptions about the surrounding universe. In this paper (as in [Co1]), all models will live in a ZFC universe $\langle V, \in\rangle$, fixed once and for all, and in particular, if $\langle M, E, i\rangle$ is a model of $\mathrm{ZFC}_{\mathbf{j}}$, we assume i is definable in V. Using the terminology of [Co1], we call such models *sharp-like*. An important consequence of this assumption is the following:

Metatheorem 2.5. *There is no proper class sharp-like transitive model of* $\mathrm{ZFC} + \mathrm{WA}_0$; *that is*, *if* $\langle M, \in, i\rangle \models \mathrm{ZFC} + \mathrm{WA}_0$ *and* M *is sharp-like and transitive*, *then* M *is a set.*

Proof. If $\langle M, \in, i\rangle \models \mathrm{ZFC} + \mathrm{WA}_0$, then in particular the Cofinal Axiom holds in M. If M is a sharp-like transitive model containing all the ordinals, $(\text{Cofinal Axiom})^M$ would imply that there is a cofinal ω-sequence in the real class ON of ordinals, violating Replacement. Therefore, if M is transitive, it must only be a set. ∎

Metatheorem 2.5 does not rule out the possibility that $\langle V, \in, j\rangle$, where j is an elementary embedding, is a model of $\mathrm{ZFC}+\mathrm{WA}$ or $\mathrm{ZFC}+\mathrm{WA}_0$; by the theorem, though, such a model would not be sharp-like.

We begin the second half of this section with a review of extenders. Our treatment follows [Ka, Section 26]. As in [Ka], we start with a discussion of extenders derived from an embedding, though here we restrict attention to embeddings $i : M \to M$, and to just the $(\kappa, i(\kappa))$-extenders derived from i. After fixing notation and stating some of the known results, we discuss the more general M-(κ, λ)-extenders, and conclude with some lemmas that will be needed in the next section.

For the next few paragraphs, we fix an elementary embedding $i : M \to M$ with critical point κ, where M is a transitive set model ([1]) of ZFC. Let $\lambda = i(\kappa)$. For $a \in [\lambda]^{<\omega}$, define E_a from i by

$$E_a = \{X \subseteq P(\kappa^{|a|}) \cap M : a \in i(X)\}.$$

Then $\langle M, \in, E_a\rangle \models$ "E_a is a κ-complete ultrafilter over $[\kappa]^{|a|}$". For $f, g \in M$ with $f : [\kappa]^{|a|} \to M$ and $g : [\kappa]^{|a|} \to M$, we write $f \sim_a g$ iff $\{b \in [\kappa]^{|a|} : f(b) = g(b)\} \in E_a$. We denote the equivalence class containing f by $[f]_{E_a}$. (Recall that Scott's trick is used in this step.) Now the *ultrapower of M by E_a* is denoted $\mathrm{Ult}(M, E_a)$ and consists of all $[f]_{E_a}$ for which $f : [\kappa]^{|a|} \to M$ and $f \in M$. It has a membership relation $\in_a$ defined by $[f]_{E_a} \in_a [g]_{E_a}$ iff $\{b \in [\kappa]^{|a|} : f(b) \in g(b)\} \in E_a$. This relation is well-defined. $(\mathrm{Ult}(M, E_a), \in_a)$ is well-founded; we identify it with its transitive collapse $(M_a, \in)$. The canonical embedding $j_a : M \to M_a$ is given by $j_a(x) = [c_x^a]_{E_a}$, where $c_x^a : \kappa^{|a|} \to M$ is the constant function with value x. Moreover, $k_a : M_a \to M$ is defined by $k_a([f]_{E_a}) = i(f)(a)$; it is a well-defined elementary embedding and $i = k_a \circ j_a$.

The (κ, λ)-*extender derived from i* is now defined to be

$$E = \langle E_a : a \in [\lambda]^{<\omega}\rangle.$$

E gives rise to a directed system of models as follows: Suppose $a \subseteq b$ and that both are elements of $[\lambda]^{<\omega}$ with $|b| = n$, $|a| = m$. Define $\pi_{ba} : [\kappa]^n \to [\kappa]^m$ by

$$\pi_{ba}(\{\xi_1, \ldots, \xi_n\}) = \{\xi_{i_1}, \ldots, \xi_{i_m}\}, \tag{2.1}$$

where we adopt the convention that elements $\xi_1, \ldots, \xi_n$ occur in increasing order and that if $b = \{\alpha_1 < \ldots < \alpha_n\}$, then $a = \{\alpha_{i_1}, \ldots, \alpha_{i_m}\}$. Now for such a, b, define $i_{ab} : M_a \to M_b$ by

$$i_{ab}([f]_{E_a}) = [f \circ \pi_{ba}]_{E_b}. \tag{2.2}$$

Then i_{ab} is a well-defined elementary embedding, and

([1]) In [Ka], extenders are developed relative to M that are *proper classes*, but it is pointed out in [MS] that the same treatment goes through for transitive set models of ZFC.

(a) $i_{ab} \circ j_a = j_b$,
(b) $k_b \circ i_{ab} = k_a$,
(c) $\langle M_a; i_{ab} : a \subseteq b, a, b \in [\lambda]^{<\omega}\rangle$ is a directed system.

We let $\langle M_E, \in_E\rangle$ denote the direct limit of this directed system. In order to fix notation, we review the elements of this construction. Let $B = \{(a, [f]_{E_a}) : a \in [\lambda]^{<\omega}$ and $[f]_{E_a} \in M_a\}$ (the disjoint union of the classes $\{a\} \times M_a$). Define $\sim$ on B by

$$(a, [f]_{E_a}) \sim (b, [g]_{E_b}) \quad \text{iff} \quad \exists c \supseteq a \cup b\ (i_{ac}([f]_{E_a}) = i_{bc}([g]_{E_b})).$$

The equivalence class containing $(a, [f]_{E_a})$ will be denoted by $[a, [f]_{E_a}]$. An easy computation shows that

$$(2.3) \quad (a, [f]_{E_a}) \sim (b, [g]_{E_b}) \quad \text{iff} \quad \exists c \supseteq a \cup b\ (\{d \in [\kappa]^{|c|} : f(\pi_{ca}(d)) = g(\pi_{cb}(d))\} \in E_c).$$

The membership relation $\in_E$ is defined similarly by

$$(a, [f]_{E_a}) \in_E (b, [g]_{E_b}) \quad \text{iff} \quad \exists c \supseteq a \cup b\ (i_{ac}([f]_{E_a}) \in_c i_{bc}([g]_{E_b})).$$

We have the following criterion for membership, analogous to (2.3):

$$(2.4) \quad (a, [f]_{E_a}) \in_E (b, [g]_{E_b}) \quad \text{iff} \quad \exists c \supseteq a \cup b\ (\{d \in [\kappa]^{|c|} : f(\pi_{ca}(d)) \in g(\pi_{cb}(d))\} \in E_c).$$

The class M_E is defined by

$$M_E = \{[a, [f]_{E_a}] : a \in [\lambda]^{<\omega},\ f : [\kappa]^{|a|} \to M, \text{ and } f \in M\}.$$

By an abuse of notation, M_E is also denoted $\mathrm{Ult}(M, E)$, the *extender ultrapower of M by E*. $(M_E, \in_E)$ is well-founded and is therefore identified with its transitive collapse; both notations—M_E and $\mathrm{Ult}(M, E)$—are used to denote the transitive collapse.

The embeddings $k_{aE} : M_a \to M_E$, $j_E : M \to M_E$, and $k_E : M_E \to M$ are defined by:

$$\begin{aligned} k_{aE}([f]_{E_a}) &= [a, [f]_{E_a}], \\ j_E(x) &= k_{aE}(j_a(x)) \qquad \text{for some (any) } a \in [\lambda]^{<\omega}, \\ k_E([a, [f]_{E_a}]) &= i(f)(a). \end{aligned}$$

These functions are well-defined elementary embeddings satisfying the following relations:

(A) $k_E \circ j_E = i$,
(B) $k_{aE} \circ j_a = j_E$,
(C) $k_E \circ k_{aE} = k_a$.

An easy computation using (B) yields

$$(2.5) \qquad j_E(x) = [a, [c^a_x]_{E_a}] \quad \text{for some (any) } a \in [\lambda]^{<\omega}.$$

We have the following standard lemma:

LEMMA 2.6 [Ka, Lemma 26.1(b),(c)]. *Suppose* $i : M \to M$ *is an elementary embedding with critical point* κ. *Let* $\lambda = i(\kappa)$. *Let* E *be the* (κ, λ)-*extender derived from* i. *Let* $j_E : M \to M_E$ *denote the canonical embedding. Then*:

(1) $\lambda = j_E(\kappa)$.
(2) *If* $(\lambda = |V_\lambda|)^M$, *then* $V_\lambda^M = V_\lambda^{M_E}$.

We turn to a discussion of the abstract notion of an extender:

DEFINITION 2.7 (extenders, [Ka, p. 354]). Suppose M is a transitive set model of ZFC, κ is a cardinal in M, $\lambda > \kappa$, , and $E = \langle E_a : a \in [\lambda]^{<\omega} \rangle$. Then E is an M-(κ, λ)-*extender* (or simply a (κ, λ)-*extender* whenever $M = V$) if:

(1) For each $a \in [\lambda]^{<\omega}$, $\langle M, \in, E_a \rangle \models$ "E_a is a κ-complete ultrafilter over $[\kappa]^{|a|}$", and:

  (a) For at least one such a, $\langle M, \in, E_a \rangle \models$ "E_a is not κ^+-complete".

  (b) For each $\xi \in \kappa$, there is such an a satisfying $\{s \in [\kappa]^{|a|} : \xi \in s\} \in E_a$.

(2) (Coherence) Suppose $a \subseteq b$ are both in $[\lambda]^{<\omega}$. Then

$$X \in E_a \quad \text{iff} \quad \{s : \pi_{ba}(s) \in X\} \in E_b,$$

where π_{ba} is defined as in (2.1).

(3) (Well-foundedness) Whenever $a_m \in [\lambda]^{<\omega}$ and $X_m \in E_{a_m}$ for $m \in \omega$, there is $d : \bigcup_m a_m \to \kappa$ such that $d'' a_m \in X_m$ for every such m.

(4) (Normality) Whenever $a \in [\lambda]^{<\omega}$, $f : [\kappa]^{|a|} \to M$ is an element of M, and $\{s \in [\kappa]^{|a|} : f(s) \in \max(s)\} \in E_a$, there is a $b \in [\lambda]^{<\omega}$ with $a \subseteq b$ such that

$$\{s \in [\kappa]^{|b|} : f(\pi_{ba}(s)) \in s\} \in E_b,$$

where π_{ba} is defined as in (2.1).

With this definition, one builds a model M_E in much the same way as described earlier: For each $a \in [\lambda]^{<\omega}$, let $\mathrm{Ult}(M, E_a)$ be the ultrapower of M by E_a, again using only functions that lie in M. Since $\mathrm{Ult}(M, E_a)$ is well-founded, we again identify it with its transitive collapse M_a. We again have the canonical map $j_a : M \to M_a$, defined as before. We again obtain a directed system $\langle M_a; i_{ab} : a \subseteq b$ both in $[\lambda]^{<\omega} \rangle$, where the maps i_{ab} are defined as before in (2.2). The direct limit M_E is obtained exactly as before, and its membership relation $\in_E$ and the maps k_{aE}, j_E have the same definitions and properties as before (see (A) and (B) above). By an abuse of notation, M_E is sometimes referred to as $\mathrm{Ult}(M, E)$, though M_E is not itself an ultrapower. As before (but for different reasons), $(M_E, \in_E)$ is well-founded, and so we identify it with its transitive collapse. Again, both M_E and $\mathrm{Ult}(M,E)$ may be used to denote this transitive collapse.

One difference in the constructions is that we no longer define the maps k_a, $a \in [\lambda]^{<\omega}$, since in the present abstract context, we do not have an ambient embedding $i : M \to M$; therefore, properties (c) and (C) above are no longer relevant here. Also, the analogue to Lemma 2.6 is somewhat weaker:

LEMMA 2.8 [Ka, Lemma 26.2(b)]. *Suppose M is a transitive set model of* ZFC, *κ is a cardinal in M, $\lambda > \kappa$, $E = \langle E_a : a \in [\lambda]^{<\omega}\rangle$ is an M-(κ,λ)-extender, and M_E is the direct limit of the corresponding ultrapowers. Then* $\mathrm{crit}(j_E) = \kappa$ *and* $j_E(\kappa) \geq \lambda$.

In Lemma 2.8, it is possible that $j_E(\kappa) > \lambda$. Recall that κ is *superstrong* if there is a nontrivial elementary embedding $j : V \to M$ with critical point κ such that $V_{j(\kappa)} \subseteq M$.

PROPOSITION 2.9 [Ka, Exercise 26.7(c)]. *κ is superstrong iff there is a (κ,λ)-extender E for some $\lambda > \kappa$ such that $j_E(\kappa) = \lambda$ and $V_{j_E(\kappa)} \subseteq M_E$, where j_E is the canonical embedding.*

Actually, in [Ka], the condition "$j_E(\kappa)=\lambda$" in Proposition 2.9 is dropped, since it is not strictly necessary; however, if E is derived from a superstrong embedding $j : V \to M$ with critical point κ, it follows from [Ka, Lemma 26.1(c)] that $j_E(\kappa) = j(\kappa)$.

Because Lemma 2.8 is so weak, in order to verify that a given (κ,λ)-extender E satisfies $j_E(\kappa) = \lambda$ and $V_\lambda \subseteq M_E$ (whence κ is superstrong), the easiest approach (one that we will use) is to obtain E from another embedding, and then make use of Lemma 2.6, or some variation thereof.

The following proposition shows the connection between the two types of extenders we have discussed:

PROPOSITION 2.10 [Ka, Exercise 26.3(a)]. *Suppose $i : M \to M$ is an elementary embedding with critical point κ, where M is a transitive set model of* ZFC. *Let E be the $(\kappa, i(\kappa))$-extender derived from i. Then E is an M-$(\kappa, i(\kappa))$-extender.*

We introduce a definition pertaining to superstrong cardinals:

DEFINITION 2.11. Suppose κ is superstrong and $\lambda > \kappa$. Then λ is a *target* for κ if there is an elementary embedding $j : V \to M$ with critical point κ such that $V_{j(\kappa)} \subseteq M$ and $\lambda = j(\kappa)$. Equivalently, λ is a target for κ if there is a (κ,λ)-extender E such that if $j_E : V \to M_E$ is the canonical embedding, then $j_E(\kappa) = \lambda$ and $V_\lambda \subseteq M_E$.

Our last technical lemma of this section recalls the well-known fact that, in the definition of Normality (Definition 2.7(4)), one may restrict attention to functions f whose range lies in a sufficiently large rank. To state this in the form of a lemma, we introduce two variants of the Normality condition: Normality$_V$ and Normality$_{V_\gamma}$, where $\gamma \geq \kappa$; Normality$_V$ is what one gets when $M = V$ in the definition given above.

- **Normality$_V$**: Whenever $a \in [\lambda]^{<\omega}$, $f : [\kappa]^{|a|} \to V$, and $\{s \in [\kappa]^{|a|} : f(s) \in \max(s)\} \in E_a$, there is a $b \in [\lambda]^{<\omega}$ with $a \subseteq b$ such that, with π_{ba} as in (2.1),
$$\{s \in [\kappa]^{|b|} : f(\pi_{ba}(s)) \in s\} \in E_b.$$
Suppose $\gamma \geq \kappa$.
- **Normality$_{V_\gamma}$**: Whenever $a \in [\lambda]^{<\omega}$, $f : [\kappa]^{|a|} \to V_\gamma$, and $\{s \in [\kappa]^{|a|} : f(s) \in \max(s)\} \in E_a$, there is a $b \in [\lambda]^{<\omega}$ with $a \subseteq b$ such that, with π_{ba} as in (2.1),
$$\{s \in [\kappa]^{|b|} : f(\pi_{ba}(s)) \in s\} \in E_b.$$

LEMMA 2.12. *Suppose $\gamma \geq \kappa$. Then the following are equivalent*:

(1) *Normality$_V$.*
(2) *Normality$_{V_\gamma}$.*

Proof. (1)⇒(2) is obvious. For (2)⇒(1), let $a \in [\lambda]^{<\omega}$, $f : [\kappa]^{|a|} \to V$, and assume $A \in E_a$ where
$$A = \{s \in [\kappa]^{|a|} : f(s) \in \max(s)\}.$$
We show that

(2.6) there is a $b \in [\lambda]^{<\omega}$ with $a \subseteq b$ such that
$$\{s \in [\kappa]^{|b|} : f(\pi_{ba}(s)) \in s\} \in E_b.$$

Notice that for all $s \in A$, $\mathrm{rank}(f(s)) < \gamma$. Define $g : [\kappa]^{|a|} \to V_\gamma$ by
$$g(s) = \begin{cases} f(s) & \text{if } s \in A, \\ s & \text{if } s \notin A. \end{cases}$$
Clearly $f{\restriction}A = g{\restriction}A$. By Normality$_{V_\gamma}$, there is a $b \in [\lambda]^{<\omega}$ with $a \subseteq b$ such that $B \in E_b$, where
$$B = \{s \in [\kappa]^{|b|} : g(\pi_{ba}(s)) \in s\}.$$
By Coherence, we also have $C \in E_b$, where
$$C = \{s \in [\kappa]^{|b|} : \pi_{ba}(s) \in A\}.$$
Then, letting $D = B \cap C$, we have $D \in E_b$ as well. We show $D \subseteq \{s \in [\kappa]^{|b|} : f(\pi_{ba}(s)) \in s\}$, and this will establish (2.6) and complete the proof. Let $s \in D$. Then $\pi_{ba}(s) \in A$ and $g(\pi_{ba}(s)) \in s$; since f and g agree on A, it follows that $f(\pi_{ba}(s)) \in s$, as required. ■

3. Main results. In this section, we formulate a sequence $\langle \mathrm{I}_4^n : n \in \omega\rangle$ of axioms, each asserting the existence of a transitive model of ZFC + WA having closure properties that increase in strength as n increases; we show that, in a sense to be made precise, I_3 represents the limit of this sequence of axioms. To motivate the formulation of these axioms, we begin by considering several kinds of closure, typically imposed on transitive set models in studies of this kind. We start by showing that even countable closure of transitive models of WA is inconsistent, and then turn to three other familiar notions of closure, which we show are equivalent in this context. Motivated by this observation, we use these latter notions of closure to formulate our ω-sequence of axioms I_4^n. We compare the strengths of these axioms with known large cardinals and finally demonstrate the sense in which the axioms I_4^n "converge" to I_3. We begin with an observation that was made in Section 2:

PROPOSITION 3.1. *No transitive model $\langle M, \in, i\rangle$ of* $\mathrm{ZFC} + \mathrm{WA}_0$ *satisfies* ${}^{\omega}M \subseteq M$.

Proof. Suppose $\mathcal{M} = \langle M, \in, i\rangle \models \mathrm{ZFC} + \mathrm{WA}_0$ and ${}^{\omega}M \subseteq M$. By hypothesis, $f : \omega \to M : n \mapsto i^n(\kappa)$ is a set in M. Therefore, the Cofinal Axiom fails. This contradicts Proposition 2.4. ∎

THEOREM 3.2. *Suppose $\langle M, \in, i\rangle$ is a transitive model of* ZFC + WA, *and* $\mathrm{crit}(i) = \kappa$. *Let $U_M = \{X \in P(\kappa) \cap M : \kappa \in i(X)\}$. Then the following are equivalent*:

(1) *$P(\kappa) \subseteq M$.*
(2) *U_M is a κ-complete nonprincipal ultrafilter on κ (in V).*
(3) *$V_{\kappa+1} \subseteq M$.*
(4) *Whenever $f : \kappa \to X$ and $X \subseteq V_\kappa^M$, we have $f \in M$.*

REMARK. We use the following elementary facts without special mention (see [Je, p. 21]):

(a) There is a definable bijection $\Gamma : \mathrm{ON} \times \mathrm{ON} \to \mathrm{ON}$ having the property that for any ordinal α for which

$$\alpha = \omega^{\alpha} \quad \text{(ordinal exponentiation)}, \tag{3.1}$$

we have $\Gamma''\alpha \times \alpha = \alpha$.

(b) Every cardinal satisfies (3.1).

(c) The property (3.1) is absolute for transitive models of ZFC.

Proof of (1)⇒(2). Assuming (1), we first show that κ is a regular uncountable cardinal. Suppose $\alpha < \kappa$ and $f : \alpha \to \kappa$ (f is a potential witness to "κ is not a cardinal" or "κ is singular"). We do not know yet that κ is a cardinal; however, as κ is a cardinal in M, we see that "$\kappa = \omega^{\kappa}$" (under ordinal exponentiation) holds in M and therefore in V. Thus, f can be coded as a subset of κ using Γ, and hence must already lie in M. But in M, κ is a regular cardinal, so f must be bounded in M, and hence in V by absoluteness. Thus, κ is a regular cardinal in V. Finally, since $i(\omega) = \omega$, κ must be uncountable.

By (1), $U_M = \{X \subseteq \kappa : \kappa \in i(X)\}$; also, U_M is a nonprincipal M-κ-complete ultrafilter. To prove that it is κ-complete, suppose that $S = \{X_\alpha : \alpha < \gamma\} \subseteq U_M$ (where $\gamma < \kappa$). We can code S as the disjoint union of its elements: Let $\overline{S} = \{(\alpha, x) \in \kappa \times \kappa : \alpha < \gamma \text{ and } x \in X_\alpha\}$. $\overline{S}$ in turn can be coded as a subset of κ using Γ. Thus, $S \in M$, and by M-κ-completeness, $\bigcap S \in U_M$, as required. ∎

Proof of (2)⇒(3). To begin, observe that $P(\kappa) \subseteq M$: If $X \in P(\kappa) \setminus M$, then certainly $\kappa \setminus X \notin M$, and U_M would fail to be an ultrafilter in V, violating (2). Next, we prove that (3) holds. We show by induction that for all $\alpha \leq \kappa+1$, $V_\alpha \in M$. Assuming $V_\alpha \in M$, the inaccessiblity of κ implies that there is a $\gamma \leq \kappa$ and a bijection $f : \gamma \to V_\alpha$ with $f \in M$. Let $X \subseteq V_\alpha$. There is, in V, a set $A \subseteq \gamma$ such that $A = f^{-1}(X)$. Because $P(\kappa) \subseteq M$, $A \in M$. Thus $X = f''A \in M$. This shows that $V_{\alpha+1} \subseteq M$, whence $V_{\alpha+1} \in M$. The limit case is easy, since unions are absolute. This completes the induction, and we have $V^M_{\kappa+1} = V_{\kappa+1}$, as required. ∎

Proof of (3)⇒(4). Given $f : \kappa \to X \subseteq V^M_\kappa$, notice $f \in V_{\kappa+1}$. The result follows by (3). ∎

Proof of (4)⇒(1). Let $X \subseteq \kappa$ and let $f : \kappa \to X$ be a surjection. Since $X \subseteq V^M_\kappa$, by (4), $f \in M$. But then $X = f''\kappa \in M$. It follows that $P(\kappa) \subseteq M$. ∎

The theorem can be proven for transitive models of ZFC + BTEE—the full strength of WA (or even WA_0) is unnecessary. (Note that in the proof of (1)⇒(2), we do not need to know whether or not U_M is a set in M; to establish that it is (apparently) requires an application of $\mathrm{Separation}_{\mathbf{j}}$, but set-hood of U_M is not needed for the proof.)

The theorem leads us to the definition of the following sequence $\{\mathrm{I}^n_4 : n \in \omega\}$ of axioms:

$\mathrm{I}^n_4(\kappa)$: there exist sets M, i such that M is transitive, $\langle M, \in, i\rangle \models \mathrm{ZFC} + \mathrm{WA}$, $\mathrm{crit}(i) = \kappa$, and $V_{i^n(\kappa)+1} \subseteq M$.

In this context, we interpret $i^0(\kappa)$ as κ.

Clearly, $\mathrm{I}_3(\kappa)$ implies $\forall n \in \omega\ \mathrm{I}^n_4(\kappa)$. Also notice that $\mathrm{I}^0_4(\kappa)$ is already slightly stronger than WA since it asserts the existence of a transitive model of WA.

Before proving the large cardinal consequences of each of these axioms, we first verify that for each n, the axiom $\mathrm{I}^n_4(\kappa)$ is strong enough to ensure that $i^n(\kappa)$ is a measurable cardinal:

PROPOSITION 3.3. *Assume* $\mathrm{I}_4^n(\kappa)$ *is true, with witness* $\langle M, \in, i\rangle$. *Then* $i^n(\kappa)$ *is a measurable cardinal.*

Proof. The case $n = 0$ follows from the fact that (3)⇒(2) in Theorem 3.2. For $n > 0$, let $\langle M, \in, i\rangle$ satisfy the properties of $\mathrm{I}_4^n(\kappa)$. As in the proof of (1)⇒(2) in Theorem 3.2, note that any function $f : \alpha \to i^n(\kappa)$ that is a potential witness to $i^n(\kappa)$ being either a noncardinal or a singular cardinal must already lie in M by coding. Thus $i^n(\kappa)$ is, in V, a regular cardinal. Let $U = \{X \subseteq \kappa : \kappa \in i(X)\}$, defined in M. In M, U is a κ-complete nonprincipal ultrafilter on κ. By elementarity, in M, $i^n(U)$ is an $i^n(\kappa)$-complete nonprincipal ultrafilter on $i^n(\kappa)$. Since $i^n(U) \subseteq P(i^n(\kappa)) \subseteq M$, one can use a coding argument as in the proof of (1)⇒(2) of Theorem 3.2 to conclude that $i^n(U)$ is an $i^n(\kappa)$-complete nonprincipal ultrafilter on $i^n(\kappa)$ in V. The result follows. ∎

PROPOSITION 3.4. $\mathrm{I}_4^0(\kappa)$ *implies that* κ *is measurable*, *and that the set* $\{\alpha < \kappa : \alpha$ *is measurable*$\}$ *has normal measure* 1.

Proof. Proposition 3.3 shows κ is measurable. Let $\langle M, \in, i\rangle$ be a witness to $\mathrm{I}_4^0(\kappa)$. Since $U = \{X \subseteq \kappa : \kappa \in i(X)\} \in M$, and since for all $\alpha < \kappa$, "α is measurable" is absolute for M, the second clause of the proposition follows, where U is the required normal measure. ∎

THEOREM 3.5. $\mathrm{I}_4^1(\kappa)$ *implies that* κ *is superstrong*, *and that the set* $\{\alpha < \kappa : \alpha$ *is superstrong*$\}$ *has normal measure* 1.

Proof. Let $\langle M, \in, i\rangle$ be a model given by $\mathrm{I}_4^1(\kappa)$. Let $\lambda = i(\kappa)$. By Proposition 2.10, $E = \langle E_a : a \in [\lambda]^{<\omega}\rangle$ is an M-(κ, λ)-extender, where $E_a = \{X \in P([\kappa]^{|a|}) \cap M : a \in i(X)\}$. We wish to prove:

(A) E is a (κ, λ)-extender (in V),
(B) the canonical embedding $j_E : V \to \mathrm{Ult}(V, E)$ satisfies $j_E(\kappa) = \lambda$,
(C) $V_\lambda \subseteq \mathrm{Ult}(V, E)$.

The first part of the theorem will then follow by Proposition 2.9.

To prove (A), we verify that conditions (1)–(4) of Definition 2.7 hold true, with M in each case being replaced by V. Several of the conditions follow immediately from absoluteness between M and V; in particular, conditions (2) and (3) can be verified in this way. Also, for each E_a that is a κ-complete ultrafilter in V, conditions (1a) and (1b) also hold by absoluteness. What remains is to verify the following:

(i) Each E_a, $a \in [i(\kappa)]^{<\omega}$, is a κ-complete ultrafilter (in V).
(ii) $\mathrm{Normality}_V$.

Part (i) follows immediately from our assumption that $V_{i(\kappa)+1} \subseteq M$. For (ii), Lemma 2.12 tells us that it suffices to demonstrate $\mathrm{Normality}_{V_\lambda}$. But again, $\mathrm{Normality}_{V_\lambda}$ follows immediately from the fact that $V_{i(\kappa)+1} \subseteq M$.

To prove (B) and (C), we begin with a number of easy observations. For the next several paragraphs, until indicated otherwise, we let $\mathrm{Ult}(V,E)$, $\mathrm{Ult}(M,E)$, and $\mathrm{Ult}^M(V,E)$ denote the "raw" direct limits, prior to applying the Mostowski collapsing function. (To eliminate possible confusion here, the notation $\mathrm{Ult}^M(V,E)$ signifies the extender ultrapower construction carried out entirely in M; in this context, we understand V to be the "extension" of the predicate $x=x$, and so $V^M = M$.) We explicitly refer to the relevant collapsing functions as follows: $p_E : \mathrm{Ult}(V,E) \to V_E$, $p_{M,E} : \mathrm{Ult}(M,E) \to M_E$, and $p_E^M : \mathrm{Ult}^M(V,E) \to V_E^M$, respectively. We denote the corresponding ultrapower membership relations by $\in_E, \in_{M,E}$ and $\in_E^M$, respectively. Our strategy is to show that initial segments of all three of these extender ultrapowers are isomorphic and that, in fact, their transitive collapse is in every case equal to V_λ. As we will show, this will establish (B) and (C).

We isolate the relevant "initial segments" as follows. Let

$$\begin{aligned} A &= \{[a,[f]_{E_a}] : a \in [\lambda]^{<\omega} \text{ and } f : [\kappa]^{|a|} \to V_\kappa\} \subseteq \mathrm{Ult}(V,E), \\ A_M &= \{[a,[f]_{E_a}] : a \in [\lambda]^{<\omega} \text{ and } f : [\kappa]^{|a|} \to V_\kappa,\ f \in M\} \subseteq \mathrm{Ult}(M,E), \\ A^M &= (\{[a,[f]_{E_a}] : a \in [\lambda]^{<\omega} \text{ and } f : [\kappa]^{|a|} \to V_\kappa\})^M \subseteq \mathrm{Ult}^M(V,E). \end{aligned}$$

Before proving that A, A_M and A^M are equal, we first mention that they are not equal merely by definition. For instance, to verify that $A_M = A^M$, we wish to verify the fact that for all $a \in [\lambda]^{<\omega}$ and $f : [\kappa]^{|a|} \to V_\kappa$ in M, $[a,[f]_{E_a}] = [a,[f]_{E_a}]^M$.

Because Scott's trick is used in the formation of equivalence classes of the form $[f]_{E_a}$ in forming the ultrapowers by E_a, all functions $g : [\kappa]^{|a|} \to N$ (where $N = V$ or $N = M$, depending on which model we are building) that belong in the class $[f]_{E_a}$, including $f : [\kappa]^{|a|} \to V_\kappa$ itself, must have the least possible rank, namely κ. Thus, as a set, $[f]_{E_a} \subseteq V_{\kappa+1}$. Since the a's lie in $[\lambda]^{<\omega}$, each pair $(a,[f]_{E_a})$ has rank below λ, and the class $[a,[f]_{E_a}]$ belongs to $V_{\lambda+1}$. Since $V_{\lambda+1} \subseteq M$, it follows that for each $a \in [\lambda]^{<\omega}$ and each $f : [\kappa]^{|a|} \to V_\kappa$, $[a,[f]_{E_a}]$ is the same set as $([a,[f]_{E_a}])^M$. Moreover, it is clear that the sets A, A_M and A^M are equal. It also follows that each of the structures $\mathcal{A} = \langle A, \in_E\rangle$, $\mathcal{A}_M = \langle A_M, \in_{M,E}\rangle$, and $\mathcal{A}^M = \langle A^M, \in_E^M\rangle$ is well-founded in V. This observation is obvious for $\mathcal{A}$ and $\mathcal{A}_M$ since $\mathrm{Ult}(V,E)$ and $\mathrm{Ult}(M,E)$ are themselves well-founded (see the discussion in Section 2 or in [Ka, Chapter 26]). It is perhaps less obvious in the case of $\mathcal{A}^M$ because M is not even countably closed, as we showed in Lemma 3.1. However, if $F : \omega \to A^M$ witnesses ill-foundedness, with $F(n) = [a_n,[f_n]_{E_{a_n}}]$, then

$$\exists\{c_n : n \geq 1\}\ \forall n\ \{d : f_n(\pi_{c_n a_n}(d)) \in f_{n-1}(\pi_{c_{n-1}a_{n-1}}(d))\} \in E_{c_n}. \tag{3.2}$$

In fact, (3.2) is equivalent to ill-foundedness not only of $\mathcal{A}^M$, but of $\mathcal{A}_M$ as well. This contradicts the fact that $\mathcal{A}_M$ is well-founded.

Next we observe that, as $\in$-structures, $\mathcal{A}$, $\mathcal{A}_M$, and $\mathcal{A}^M$ are isomorphic. The proof in each case is easy. We show how to verify $\mathcal{A}_M \cong \mathcal{A}^M$ and leave the other cases to the reader. We must show that for all $a, b \in [\lambda]^{<\omega}$ and all $f : [\kappa]^{|a|} \to V_\kappa$ and $g : [\kappa]^{|b|} \to V_\kappa$ in M,

$$[a, [f]_{E_a}] \in_{M,E} [b, [g]_{E_b}] \quad \text{iff} \quad ([a, [f]_{E_a}] \in_E^M [b, [g]_{E_b}])^M. \tag{3.3}$$

Using the membership criterion (2.4), the display (3.3) is equivalent to

$$\begin{aligned} &\text{for some } c \supseteq a \cup b,\ \{d : f(\pi_{ca}(d)) \in g(\pi_{cb}(d))\} \in E_c \\ &\qquad\text{iff} \quad (\text{for some } c \supseteq a \cup b,\ \{d : f(\pi_{ca}(d)) \in g(\pi_{cb}(d))\} \in E_c)^M. \end{aligned}$$

Since the statement "for some $c \supseteq a \cup b$, $\{d : f(\pi_{ca}(d)) \in g(\pi_{cb}(d))\} \in E_c$" is absolute for M (since $V_{\lambda+1} \subseteq M$), the displayed equivalence is true.

The fact that $\mathcal{A}, \mathcal{A}_M$, and $\mathcal{A}^M$ are isomorphic leads to the conclusion that their transitive collapses are all equal. We show that these in turn are equal to V_λ. We do this by showing that the transitive collapse of $\mathcal{A}_M$ is $V_\lambda^{\mathrm{Ult}(M,E)}$; this will suffice because, by Lemma 2.6(2), $V_\lambda^{\mathrm{Ult}(M,E)} = V_\lambda$ (note that $\lambda = |V_\lambda|$ because λ is inaccessible).

Recalling that $p_{M,E} : \mathrm{Ult}(M, E) \to M_E$ denotes the collapsing function, we now show that $p''_{M,E}\mathcal{A}_M = V_\lambda^{\mathrm{Ult}(M,E)}$. Recall $\lambda = i(\kappa) = (p_{M,E} \circ i_E)(\kappa)$, where $i_E : M \to \mathrm{Ult}(M, E)$ is the canonical embedding and $i_E(\kappa) = [a, [c^a_\kappa]]$ for some (any) $a \in [\lambda]^{<\omega}$. Thus, $V_\lambda^{\mathrm{Ult}(M,E)} = p_{M,E}([a, [c^a_{V_\kappa}])$ for some (any) $a \in [\lambda]^{<\omega}$.

Suppose $x \in V_\lambda^{\mathrm{Ult}(M,E)}$. Let b, g be such that $x = p_{M,E}([b, [g]_{E_b}])$. Since $x \in V_\lambda^{\mathrm{Ult}(M,E)}$, it follows that $[b, [g]_{E_b}] \in_{M,E} [a, [c^a_{V_\kappa}]_{E_a}]$. Thus, for some $c \supseteq a \cup b$,

$$D = \{d \in [\kappa]^{|c|} : g(\pi_{cb}(d)) \in V_\kappa\} \in E_c. \tag{3.4}$$

Let $B = \{e \in [\kappa]^{|b|} : g(e) \in V_\kappa\}$. By Coherence, $B \in E_b$ iff $D' \in E_c$ where $D' = \{d \in [\kappa]^{|c|} : \pi_{cb}(d) \in B\}$. But this follows easily since $D \subseteq D'$. We have shown that g has values in V_κ on the set $B \in E_b$. Define g' by $g' \restriction B = g \restriction B$ and $g'(e) = 0$ if $e \notin B$. Clearly, $(b, [g']_{E_b}) \sim (b, [g]_{E_b})$, and so we can write $x = p_{M,E}([b, [g']_{E_b}])$ where now the representative function g' has range in V_κ. We have shown that $V_\lambda^{\mathrm{Ult}(M,E)} \subseteq p''_{M,E}A_M$. For the converse, suppose $[b, [g]_{E_b}] \in A_M$ with $g : [\kappa]^{|b|} \to V_\kappa$. From our work above, it suffices to prove that for some $a \in [\lambda]^{<\omega}$ and $c \supseteq a \cup b$, equation (3.4) holds. Let $a \in [\lambda]^{<\omega}$ and $c \supseteq a \cup b$ be chosen arbitrarily. Defining B as above by $B = \{e \in [\kappa]^{|b|} : g(e) \in V_\kappa\}$, it is clear that $B \in E_b$. By Coherence, we must have $D' \in E_c$, where $D' = \{d \in [\kappa]^{|c|} : \pi_{cb}(d) \in B\}$. As is easily checked, $D' \subseteq D$, whence $D \in E_c$, as required.

We have therefore shown that V_λ is a subset of each of the (transitive collapses of the) extender ultrapower models, namely V_E, M_E, and V_E^M.

Next we wish to show that the canonical embeddings all take on the same value at κ. For the rest of the proof of the theorem, we return to the point of view that each of $\mathrm{Ult}(V,E)$, $\mathrm{Ult}(M,E)$ and $\mathrm{Ult}^M(V,E)$ is identified with its transitive collapse. As a matter of notation, we denote the canonical embeddings as follows:

$$j_E : V \to \mathrm{Ult}(V,E), \quad i_E : M \to \mathrm{Ult}(M,E), \quad j_E^M : M \to \mathrm{Ult}^M(V,E).$$

By Lemma 2.6, $i_E(\kappa) = \lambda$. We compute $j_E(\kappa)$: Suppose $[b,[g]_{E_b}] \in_E [a,[c_\kappa^a]]$. For some $c \supseteq a \cup b$, $\{d : g(\pi_{cb}(d)) \in \kappa\} \in E_c$. Reasoning as above, it follows that on a set in E_b, g takes values in V_κ, and therefore the (transitive collapse of) $[b,[g]_{E_b}]$ lies in V_λ. Therefore $j_E(\kappa) \le \lambda$. But by Lemma 2.8, $j_E(\kappa) \ge \lambda$, and the result follows. Exactly the same reasoning shows that $j_E^M(\kappa) = \lambda$.

We have proven (B) and (C). To prove that the set $\{\alpha < \kappa : \alpha \text{ is superstrong}\}$ has normal measure 1, we first observe that $M \models$ "κ is superstrong": Because $V_{\lambda+1} \subseteq M$, $M \models$ "E is a (κ,λ)-extender". As we observed above, $j_E^M(\kappa) = \lambda$ and $V_\lambda \subseteq \mathrm{Ult}^M(V,E)$. It follows from Proposition 2.9 that κ is superstrong in M.

The fact that κ is superstrong in M is enough to ensure that those α below κ that are superstrong in M form a normal measure 1 set; however, to ensure that such α are also superstrong in V, we need to make fuller use of elementarity. Recalling Definition 2.11, we observe that

$$M \models \text{“}\kappa \text{ is superstrong and has an inaccessible target } < i(\lambda)\text{”}.$$

(In fact, λ is such a target in M.)

It follows therefore that if $U = \{X \subseteq \kappa : \kappa \in i(X)\}$ and $X = \{\alpha < \kappa : (\alpha \text{ is superstrong and has an inaccessible target } < \lambda)^M\}$, then $X \in U$. The following Claim will complete the proof of the theorem; note that the required normal measure (mentioned in the statement of the theorem) is U.

Claim. *For each α in X, α is superstrong.*

Proof of Claim. Let $\alpha \in X$ and let $\gamma < \lambda$ be, in M, an inaccessible target for α. Moreover, let F be, in M, an (α,γ)-extender for which the canonical embedding j_F^M (defined in M) maps α to γ and for which $V_\gamma^M = V_\gamma$ is a subset of the resulting extender ultrapower $\mathrm{Ult}^M(V,F)$. By considering sets analogous to A, A_M, and A^M as we did earlier, one can perform a similar analysis of the models $\mathrm{Ult}(V,F)$, $\mathrm{Ult}(M,F)$ and $\mathrm{Ult}^M(V,F)$ to conclude that V_γ is a subset of all of them and that the canonical embedding in each case maps α to γ. It follows that α is superstrong, and the Claim is proved. ∎

Theorem 3.6. *For each $n \ge 1$, $\mathrm{I}_4^{n+1}(\kappa)$ implies that κ is n-huge and that the set $\{\alpha < \kappa : \alpha$ is n-huge$\}$ has normal measure* 1.

Proof. Let $\langle M, \in, i\rangle$ witness $\mathrm{I}_4^{n+1}(\kappa)$. By Proposition 3.3, $i^{n+1}(\kappa)$ is inaccessible. As usual, define an n-huge ultrafilter $U \subseteq P(P(i^n(\kappa)))$ by putting

$X \in U$ if and only if $X \subseteq P(i^n(\kappa))$ and $i''(i^n(\kappa)) \in i(X)$. The definition makes sense since $i''(i^n(\kappa)) \in M$. (Note that $i''(i^n(\kappa))$ lies in a proper initial segment of $i^{n+1}(\kappa)$, and so must lie in $V_{i^{n+1}(\kappa)}$.) Verification that U is an n-huge normal measure involves verifying that U is closed under diagonal intersections, is fine, and contains all collections of the form $\{x \in P(i^n(\kappa)) : \mathrm{ot}(x \cap i^{m+1}(\kappa)) = i^m(\kappa)\}$ for $m < n$; all these arguments go through because the relevant sets lie in $V_{i^{n+1}(\kappa)+1}$.

To prove the second part of the theorem, first note that as $U \in V_{i^n(\kappa)} \subseteq M$ (and since $i^n(\kappa)$ is inaccessible), we have $M \models$ "κ is n-huge". Let $\lambda = i^n(\kappa)$. Recalling Definition 2.1, we also have

$$M \models \text{“}\kappa \text{ is } n\text{-huge and has an } n\text{-huge index below } i(\lambda).\text{”}$$

(In particular, λ itself is an n-huge index for κ in M.) It follows therefore that if $D = \{X \subseteq \kappa : \kappa \in i(X)\}$, then $X \in D$, where

$$X = \{\alpha < \kappa : M \models \text{“}\alpha \text{ is } n\text{-huge and has an } n\text{-huge index below } \lambda\text{”}\}.$$

Since D is a normal measure, the proof of the second part of the theorem will be complete after we have demonstrated the following claim:

Claim. *For all $\alpha \in X$, α is n-huge* (*in V*).

Proof of Claim. Given $\alpha \in X$, let γ_α be, in M, an n-huge index for α such that $\gamma_\alpha < \lambda$. It follows that there is a $U_\alpha \in M$ such that

$$M \models \text{“}U_\alpha \text{ is an } n\text{-huge ultrafilter on } P(\gamma_\alpha).\text{”}$$

Since $U_\alpha \in V_{i^n(\kappa)} \subseteq M$ and $i^n(\kappa)$ is inaccessible, U_α is an n-huge ultrafilter on $P(\gamma_\alpha)$ in V. It follows that α is n-huge in V. ∎

We do not believe that $\forall n\ \mathrm{I}_4^n(\kappa)$ implies $\mathrm{I}_3(\kappa)$, but we do not have a proof. We can show, however, that if we have embeddings $i_n : M_n \to M_n$, $n \in \omega$, each witnessing $\mathrm{I}_4^n(\kappa)$ and collectively exhibiting enough "coherence", then we can obtain $\mathrm{I}_3(\kappa)$. We will need the following definition:

Definition 3.7 (coherence of I_4 embeddings). Suppose that $I = \{i_n : M_n \to M_n \mid n \in \omega\}$ is an enumeration of elementary embeddings, all having critical point κ. Then I is called $\mathrm{I}_4(\kappa)$-*coherent* if for each n, $\langle M_n, \in, i_n\rangle$ witnesses $\mathrm{I}_4^n(\kappa)$, and for each $m < n$,

$$i_m \restriction V_{\kappa_{(m)}+1} = i_n \restriction V_{\kappa_{(m)}+1}, \tag{3.5}$$

where $\kappa_{(m)} = i_m^m(\kappa)$ for each $m \geq 1$ and $\kappa_{(0)} = \kappa$.

In the theorem below, we show that an I_3 embedding can be obtained from an $\mathrm{I}_4(\kappa)$-coherent set of embeddings by taking an appropriate "limit."

Theorem 3.8. *The following are equivalent*:

(1) $\mathrm{I}_3(\kappa)$.

(2) *There is an* $\mathrm{I}_4(\kappa)$-*coherent set of embeddings.*

Proof. For (1)⇒(2), let $M_n = V_\lambda$ and $i_n = i$ for each n, where $i : V_\lambda \to V_\lambda$ is an I_3 embedding. For (2)⇒(1), let $\{i_n : M_n \to M_n \mid n \in \omega\}$ be an $\mathrm{I}_4(\kappa)$-coherent set of embeddings, and as in the definition of coherence, let $\kappa_{(m)} = i_m^m(\kappa)$ for each $m \geq 1$, and $\kappa_{(0)} = \kappa$. Let $\lambda = \sup\{\kappa_{(m)} : m \in \omega\}$. Then $V_\lambda = \bigcup_{m\in\omega} V_{\kappa_{(m)}}$.

Claim. $V_{\kappa_{(0)}} \prec V_{\kappa_{(1)}} \prec \ldots \prec V_{\kappa_{(n)}} \prec \ldots \prec V_\lambda$ *forms an elementary chain.*

Proof of Claim. It suffices to show that $V_{\kappa_{(n)}} \prec V_{\kappa_{(n+1)}}$ for all n. Clearly,

$$V_\kappa \prec V_{\kappa_{(1)}}. \tag{3.6}$$

We wish to apply the embedding i_n^n to (3.6). First, we observe that by (3.5),

(a) for all $n \geq 1$, $i_n(i_1(\kappa)) = i_{n+1}(i_1(\kappa))$; so by induction on n, $i_n^n(i_1(\kappa)) = i_{n+1}^n(i_1(\kappa))$,

(b) for all $n \in \omega$, $i_1(\kappa) = i_{n+1}(\kappa)$.

Applying (a) and (b), we obtain

$$i_n^n(\kappa_{(1)}) = i_n^n(i_1(\kappa)) = i_{n+1}^n(i_1(\kappa)) = i_{n+1}^n(i_{n+1}(\kappa)) = \kappa_{(n+1)}.$$

Now, applying i_n^n to (3.6) yields $V_{\kappa_{(n)}} \prec V_{\kappa_{(n+1)}}$, as required. ■

Continuation of proof of Theorem. We define $i : V_\lambda \to V_\lambda$ by $i(x) = i_m(x)$ for some (any) m for which $x \in V_{\kappa_{(m)}}$. That i is well-defined follows from (3.5). To prove that i is elementary, we use the Tarski–Vaught criterion: For a given $\phi(x, y)$ and set $c \in V_\lambda$, we show that if for some $b \in V_\lambda$, $V_\lambda \models \phi[b, i(c)]$, then there is $a \in V_\lambda$ such that $V_\lambda \models \phi[i(a), i(c)]$. Let n be such that $b, c \in V_{\kappa_{(n)}}$. By the definition of i, $i(c) = i_{n+1}(c)$. By the claim, $V_{\kappa_{(n+1)}} \prec V_\lambda$. Hence $V_{\kappa_{(n+1)}} \models \phi[b, i_{n+1}(c)]$. Using (3.5), it is easy to verify that $i_{n+1}{\restriction}V_{\kappa_{(n)}} : V_{\kappa_{(n)}} \to V_{\kappa_{(n+1)}}$ is elementary. Thus, there is $a \in V_{\kappa_{(n)}}$ such that $V_{\kappa_{(n+1)}} \models \phi[i_{n+1}(a), i_{n+1}(c)]$. It follows that $V_\lambda \models \phi[i_{n+1}(a), i_{n+1}(c)]$ and $V_\lambda \models \phi[i(a), i(c)]$. Thus $i : V_\lambda \to V_\lambda$ is elementary and the result follows. ■

4. Open questions. The axioms I_4^n clearly appear to grow in strength as n increases, but we are unable to prove this:

Question 1. Show that if $m < n$, then $\mathrm{ZFC} + \mathrm{I}_4^n(\kappa) \vdash \mathrm{Con}(\mathrm{ZFC} + \mathrm{I}_4^m(\kappa))$.

In a similar vein:

Question 2. Is there a model of $\mathrm{ZFC} + (\forall n \in \omega\ \mathrm{I}_4^n(\kappa)) + \neg\mathrm{I}_3(\kappa)$?

References

[Co1] P. Corazza, *The spectrum of elementary embeddings $j : V \to V$*, Ann. Pure Appl. Logic, submitted.

[Co2] P. Corazza, *Forcing over the language* $\{\in, \mathbf{j}\}$, Trans. Amer. Math. Soc., submitted.

[Co3] —, *Consistency of* $V =$ HOD *with the wholeness axiom*, Arch. Math. Logic 39 (2000), 219–226.

[Co4] —, *The wholeness axiom and Laver sequences*, Ann. Pure Appl. Logic 105 (2000), 157–260.

[Ha] J. D. Hamkins, *The wholeness axioms and* $V =$ HOD, Arch. Math. Logic 40 (2001), 1–8.

[Je] T. Jech, *Set Theory*, Academic Press, New York, 1978.

[Ka] A. Kanamori, *The Higher Infinite*, Springer, New York, 1994.

[Ku] K. Kunen, *Elementary embeddings and infinitary combinatorics*, J. Symbolic Logic 36 (1971), 407–413.

[MS] D. A. Martin and J. Steel, *A proof of projective determinacy*, J. Amer. Math. Soc. 2 (1989), 71–125.

310 East Washington Ave.
Fairfield, IA 52556, U.S.A.
E-mail: pcorazza@kdsi.net

Received 8 October 2002;
in revised form 15 January 2003