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Shadowing in actions of some Abelian groups

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Abstract. We study shadowing properties of continuous actions of the groups $\mathbb{Z}^p$ and $\mathbb{Z}^p \times \mathbb{R}^p$. Necessary and sufficient conditions are given under which a linear action of $\mathbb{Z}^p$ on $\mathbb{C}^m$ has a Lipschitz shadowing property.

1. Introduction. One of the main fields of the classical theory of dynamical systems (i.e., of actions of the groups $\mathbb{Z}$ and $\mathbb{R}$) is the theory of structural stability.

This theory has influenced, in particular, the theory of shadowing of approximate trajectories (pseudotrajectories) in dynamical systems. At present, shadowing theory is well developed (see, for example, the monographs [6, 7]).

In parallell to the classical theory of dynamical systems, global qualitative properties of actions of groups more general than $\mathbb{Z}$ and $\mathbb{R}$ have been studied (let us mention structural stability and ergodicity of Anosov actions [1, 8] and rigidity properties of hyperbolic actions [2, 3]).

In this paper, we study shadowing properties of the groups $\mathbb{Z}^p$ and $\mathbb{Z}^p \times \mathbb{R}^q$. We reduce the shadowing problem for a continuous action $\Phi(n, \cdot)$, where $n \in \mathbb{Z}^p$, to well known shadowing and expansivity properties of a single homeomorphism $\Phi(\nu, \cdot)$ (Theorem 1). It is shown that a similar result holds for actions of some infinite-dimensional groups (Theorem 1$'$).

We give necessary and sufficient conditions under which a linear action of $\mathbb{Z}^p$ on $\mathbb{C}^m$ has a shadowing property (Theorem 2).

Finally, we study shadowing properties of the group $\mathbb{Z}^p \times \mathbb{R}^q$ (Theorem 3).

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2. Actions of $\mathbb{Z}^p$. Let (M,ϱ) be a metric space and let $H(M)$ be the set of homeomorphisms of M. Let $\mathcal{G}$ be an Abelian group with operation $+$. A continuous action

$$\Phi : \mathcal{G} \times M \to M \tag{1}$$

is defined by the following conditions:

(i) $\Phi(n,\cdot) \in H(M)$ for $n \in \mathcal{G}$;
(ii) $\Phi(0,x) = x$ for $x \in M$;
(iii) $\Phi(n+m,\cdot) = \Phi(n,\Phi(m,\cdot))$ for $n,m \in \mathcal{G}$.

We begin with actions of the group $\mathcal{G} = \mathbb{Z}^p$.

Let us introduce some notation. Denote by $N(a,A)$ the a-neighborhood of a set $A \subset M$.

Let $I = \{1,\dots,p\}$. Fix $n = (n_1,\dots,n_p) \in \mathbb{Z}^p$, $i \in I$, and $k \in \mathbb{Z}$. We denote by $n(i,k)$ the element $n' \in \mathbb{Z}^p$ such that $n'_j = n_j$ for $j \in I$, $j \neq i$, and $n'_i = n_i + k$. According to this notation, $n(i,0) = n$ for any i.

The definition of Φ implies that the homeomorphism

$$f_{i,k}(n) = \Phi(n(i,k),\cdot) \circ \Phi^{-1}(n,\cdot) \tag{2}$$

does not depend on $n \in \mathbb{Z}^p$. We denote it by $f_{i,k}$.

Fix a positive number d. We say that a set $\xi = \{x_n \in M : n \in \mathbb{Z}^p\}$ is a *d-pseudotrajectory* of Φ if

$$\varrho(x_{n(i,\pm 1)}, f_{i,\pm 1}(x_n)) < d \quad \text{for any } n \in \mathbb{Z}^p \text{ and } i \in I. \tag{3}$$

This definition is a natural generalization of the definition of a pseudotrajectory of a homeomorphism $h \in H(M)$ (see [7]). Let us recall that a sequence $\xi = \{x_n \in M : n \in \mathbb{Z}\}$ is called a *d-pseudotrajectory* of $h \in H(M)$ if

$$\varrho(x_{n+1}, h(x_n)) < d \quad \text{for any } n \in \mathbb{Z}. \tag{4}$$

Assume that h^{-1} is uniformly continuous on M. In this case, for any $d' > 0$ there exists $d \in (0,d')$ such that inequalities (4) imply

$$\varrho(x_{n-1}, h^{-1}(x_n)) < d',$$

so that any d-pseudotrajectory of h is a d'-pseudotrajectory of the action $\Psi : \mathbb{Z} \times M \to M$, where $\Psi(n,x) = h^n(x)$.

The following two properties of continuous dynamical systems are well known (see, for example, [7]).

We say that a homeomorphism $h \in H(M)$ has the *shadowing property* on a set $V \subset M$ if given $\varepsilon > 0$ there exists $d > 0$ such that for any d-pseudotrajectory $\{x_n \in V : n \in \mathbb{Z}\}$ of h there is a point $x \in M$ such that

$$\varrho(h^n(x), x_n) < \varepsilon \quad \text{for } n \in \mathbb{Z}. \tag{5}$$

We say that a homeomorphism $h \in H(M)$ is *expansive* on a set $U \subset M$ if there exists a constant $b > 0$ (*expansivity constant*) such that if for two points x, y, we have

$$h^n(x), h^n(y) \in U, \qquad \varrho(h^n(x), h^n(y)) < b, \qquad \text{for all } n \in \mathbb{Z},$$

then $x = y$.

Consider two sets $V, U \subset M$. We say that a homeomorphism $h \in H(M)$ is *topologically Anosov* with respect to the pair (V, U) if the following conditions are satisfied:

(TA1) there exists $\delta > 0$ such that $N(\delta, V) \subset U$;
(TA2) h has the shadowing property on V;
(TA3) h is expansive on U.

In the case $V = U = M$, the definition above coincides with the standard definition of a topologically Anosov homeomorphism [7].

Let us formulate a theorem giving sufficient conditions under which action (1) has a property of shadowing its pseudotrajectories.

Recall that a family $\mathcal{F}$ of mappings of M is called *equicontinuous* if given $\varepsilon > 0$ there exists $\delta > 0$ such that if $x, y \in M$ and $\varrho(x, y) < \delta$, then $\varrho(f(x), f(y)) < \varepsilon$ for any $f \in \mathcal{F}$.

THEOREM 1. *Assume that there exist $V, U \subset M$ and $\nu \in \mathbb{Z}^p$ such that the homeomorphism $f = \Phi(\nu, \cdot)$ is topologically Anosov with respect to the pair (V, U). Assume, in addition, that the family $\{f_{i,\pm 1} : i \in I\}$ is equicontinuous. Then for any $\varepsilon > 0$ there exists $d > 0$ with the following property: if $\{x_n \in V : n \in \mathbb{Z}^p\}$ is a d-pseudotrajectory of Φ then there exists a unique point x such that*

$$\varrho(\Phi(n, x), x_n) < \varepsilon \quad \textit{for all } n \in \mathbb{Z}^p. \tag{6}$$

REMARK 1. Obviously, the assumptions of Theorem 1 are satisfied if M is a closed smooth manifold and $V \subset M$ is a hyperbolic set of a diffeomorphism $f = \Phi(\nu, \cdot)$. It is well known that in this case there exists a neighborhood U of the compact set V such that f is topologically Anosov with respect to the pair (V, U) (see [7]). Since M is compact, the family $\{f_{i,\pm 1} : i \in I\}$ is obviously equicontinuous.

In addition, in this case the dependence of ε on d is Lipschitz, i.e., there exist positive constants L and d_0 such that if $\{x_n \in V : n \in \mathbb{Z}\}$ is a d-pseudotrajectory of f with $d \le d_0$, then there is a point x such that (5) holds with $h = f$ and $\varepsilon = Ld$. The proof of Theorem 1 below shows that if $f = \Phi(\nu, \cdot)$ has such a Lipschitz shadowing property (instead of the usual one), then there exist positive constants L and d_0' such that if $\{x_n \in V : n \in \mathbb{Z}^p\}$ is a d-pseudotrajectory of Φ with $d \le d_0'$, then there is a point x such that (6) holds with $\varepsilon = Ld$.

A particular case of the above-mentioned situation is the so-called *Anosov action* of $\mathbb{Z}^p$ on M (see [1]), i.e., an action of $\mathbb{Z}^p$ by diffeomorphisms such that some diffeomorphism $f = \Phi(\nu, \cdot)$ is Anosov (this means that the manifold M is a hyperbolic set of f).

Proof of Theorem 1. Let $N(\delta, V) \subset U$ and let b be an expansivity constant of f on U. Fix $\varepsilon > 0$. Decreasing it if necessary, we may assume that

$$4\varepsilon < \min(4\delta, b) \tag{7}$$

and that $\varrho(x, y) < \varepsilon$ implies

$$2\varrho(f_{i,\pm1}(x), f_{i,\pm1}(y)) < b \quad \text{for any } i \in I. \tag{8}$$

For the fixed ε, we find $d' > 0$ such that any d'-pseudotrajectory of f in V is ε-shadowed by a trajectory of f (i.e., analogs of inequalities (5) hold).

Since the family $\{f_{i,\pm1} : i \in I\}$ is equicontinuous, there exists $d > 0$ (depending only on $\sum_{i \in I} |\nu_i|$) such that if $\{x_n\}$ is a d-pseudotrajectory of (1), then

$$\varrho(f(x_n), x_{n+\nu}) < d' \quad \text{for any } n \in \mathbb{Z}^p. \tag{9}$$

We assume, in addition, that

$$4d < b. \tag{10}$$

We claim that this d has the desired property. Fix $\mu \in \mathbb{Z}^p$ and consider the sequence

$$y_k = x_{\mu + k\nu}, \quad k \in \mathbb{Z}.$$

It follows from (9) that $\{y_k\}$ is a d'-pseudotrajectory of f (lying in V). Hence, there exists a point $z(\mu)$ such that

$$\varrho(f^k(z(\mu)), y_k) < \varepsilon, \quad k \in \mathbb{Z}. \tag{11}$$

Since $y_k \in V$, inequalities (11) and (7) imply that

$$f^k(z(\mu)) \in U, \quad k \in \mathbb{Z}. \tag{12}$$

Let $z'(\mu)$ be another point for which (11) holds. In this case (12) holds for $z'(\mu)$ as well. It follows from (7) that then

$$\varrho(f^k(z(\mu)), f^k(z'(\mu))) < 2\varepsilon < b, \quad k \in \mathbb{Z},$$

and the expansivity of f on U implies that $z(\mu) = z'(\mu)$. Hence, the point $z(\mu)$ with property (11) is unique.

Now we fix $i \in I$. Let μ' be any of the points $\mu(i, \pm 1)$ and let χ be the corresponding homeomorphism $f_{i,1}$ or $f_{i,-1}$. Consider the sequence

$$y'_k = x_{\mu' + k\nu}, \quad k \in \mathbb{Z},$$

and apply the same reasoning as above to find the point z' such that

$$\varrho(f^k(z'), y'_k) < \varepsilon, \quad k \in \mathbb{Z}. \tag{13}$$

We claim that

$$z' = \chi(z(\mu)). \tag{14}$$

By the choice of ε, it follows from (11) that

$$2\varrho(\chi(f^k(z(\mu))), \chi(y_k)) < b, \qquad k \in \mathbb{Z}. \tag{15}$$

Since $\{x_n\}$ is a d-pseudotrajectory of (1),

$$\varrho(\chi(y_k), y_k') < d. \tag{16}$$

Combining (13)–(16) and taking into account the equality $\chi(f^k(z(\mu))) = f^k(\chi(z(\mu)))$, we see that

$$\begin{aligned}\varrho(f^k(z'), f^k(\chi(z(\mu)))) &\le \varrho(f^k(z'), y_k') + \varrho(y_k', \chi(y_k)) + \varrho(\chi(y_k), \chi(f^k(z(\mu))))\\ &< \varepsilon + d + b/2 < b\end{aligned}$$

(see (7) and (10)). Since (13) implies $f^k(z') \in U$, it follows from the estimate above and from (12) that equality (14) holds.

Now (11) implies that

$$\varrho(z(\mu), x_\mu) < \varepsilon \quad \text{for any } \mu \in \mathbb{Z}^p. \tag{17}$$

Our reasoning above shows that for any pair μ and $\mu' = \mu(i, \pm 1)$, $i \in I$, the points $z(\mu)$ and $z(\mu')$ satisfy

$$z(\mu(i, \pm 1)) = f_{i,\pm 1}(z(\mu)).$$

It follows immediately that $z(\mu) = \Phi(\mu, z(0))$. Now (17) implies that

$$\varrho(\Phi(\mu, z(0)), x_\mu) < \varepsilon \quad \text{for any } \mu \in \mathbb{Z}^p.$$

This differs from the desired inequality (6) only in notation.

To establish the uniqueness of a point x satisfying (6), note that such an x must satisfy

$$\varrho(f^k(x), y_k) < \varepsilon, \qquad k \in \mathbb{Z},$$

where $y_k = x_{k\nu}$, and $z(0)$ is the unique point having this property.

The theorem is proved.

A statement similar to Theorem 1 can be proved for actions of some infinite-dimensional groups. Let $\mathcal{G}$ be the subgroup of $\mathbb{Z}^\infty$ defined by the following condition: $n = \{n_i : i \in \mathbb{Z}\} \in \mathcal{G}$ if and only if

$$\sum_{i \in \mathbb{Z}} |n_i| < \infty$$

(see, for example, [4]). For $n \in \mathcal{G}$ and $i, k \in \mathbb{Z}$, we define $n(i, k) \in \mathcal{G}$ and homeomorphisms (2) in the same way as above. The definition of a d-pseudotrajectory of (1) is similar to that for an action of $\mathbb{Z}^p$ (with $\mathbb{Z}^p$ and I replaced by $\mathcal{G}$ and $\mathbb{Z}$, respectively).

For any $n, n' \in \mathcal{G}$, there exists a finite sequence

$$n_0 = n, n_1, \ldots, n_l = n'$$

with the following property: for any $j \in \{1, \ldots, l-1\}$ there exists $i \in \mathbb{Z}$ such that either $n_{j+1} = n_j(i, 1)$ or $n_{j+1} = n_j(i, -1)$.

Now it is easy to see that the reasoning in the proof of Theorem 1 yields the following statement.

THEOREM 1′. *Assume that there exist* $V, U \subset M$ *and* $\nu \in \mathcal{G}$ *such that the homeomorphism* $f = \Phi(\nu, \cdot)$ *is topologically Anosov with respect to the pair* (V, U). *Assume, in addition, that the family* $\{f_{i,\pm 1} : i \in \mathbb{Z}\}$ *is equicontinuous. Then for any* $\varepsilon > 0$ *there exists* $d > 0$ *with the following property: if* $\{x_n \in V : n \in \mathcal{G}\}$ *is a d-pseudotrajectory of* (1), *then there exists a unique point* x *such that*

$$\varrho(\Phi(n, x), x_n) < \varepsilon \quad \text{for all } n \in \mathcal{G}.$$

3. Linear actions of $\mathbb{Z}^p$. Consider a linear action of $\mathbb{Z}^p$ on $\mathbb{C}^m$. In this case, we fix p nonsingular $m \times m$ matrices $A_1, \ldots, A_p$. Assuming that they pairwise commute, we get the action

$$\Phi : \mathbb{Z}^p \times \mathbb{C}^m \to \mathbb{C}^m \tag{18}$$

defined by

$$\Phi(n, x) = A_1^{n_1} \ldots A_p^{n_p} x \tag{19}$$

for $n = (n_1, \ldots, n_p) \in \mathbb{Z}^p$ and $x \in \mathbb{C}^m$.

It is known [5] that for any family of pairwise commuting matrices A_i there exists a unitary matrix U such that each matrix $T_i = U^{-1} A_i U$ is upper triangular. Obviously, the change of variables $x = Uy$ preserves any shadowing and expansivity properties. Hence, we may assume that the matrices A_i are upper triangular.

Denote by λ_{ij} the jth diagonal element (i.e., the (j, j) entry) of A_i.

THEOREM 2. *Under the above conditions, the following statements are equivalent*:

(1) *action* (18) *has the Lipschitz shadowing property, i.e., there exists a constant* $L > 0$ *such that for any d-pseudotrajectory* $\{x_n : n \in \mathbb{Z}^p\}$ *of* Φ *there is a point* x *satisfying*

$$|\Phi(n, x) - x_n| \le Ld, \qquad n \in \mathbb{Z}^p, \tag{20}$$

where $|\cdot|$ *is the standard norm of* $\mathbb{C}^m$;

(2) *for any* $j \in \{1, \ldots, m\}$ *there exists* $i \in \{1, \ldots, p\}$ *such that* $|\lambda_{ij}| \ne 1$;

(3) *there is no vector* $v \ne 0$ *such that*

$$A_i v = \mu_i v, \quad i = 1, \ldots, p, \quad \text{where} \quad |\mu_i| = 1. \tag{21}$$

Proof. Denote by $(n1)$, $(n2)$, and $(n3)$ the negations of (1), (2), and (3), respectively. We prove the implications

$$(n1) \Rightarrow (n2) \Rightarrow (n3) \Rightarrow (n1).$$

First we prove $(n1) \Rightarrow (n2)$.

It follows from Theorem 1.3.2 of [7] that if a matrix A is hyperbolic (i.e., its eigenvalues λ_i satisfy $|\lambda_i| \neq 1$), then the homeomorphism $f(x) = Ax$ of $\mathbb{C}^m$ has the Lipschitz shadowing property described in statement (1). Obviously, this homeomorphism is expansive on $\mathbb{C}^m$. Thus, it follows from Theorem 1 and Remark 1 that to establish (1) it is enough to show that there exists $n = (n_1, \dots, n_p) \in \mathbb{Z}^p$ such that the matrix

$$A = A_1^{n_1} \dots A_p^{n_p} \tag{22}$$

is hyperbolic.

For contradiction, assume that condition (2) is satisfied while any matrix (22) has an eigenvalue λ with $|\lambda| = 1$. Since the matrices A_i are upper triangular, the set of eigenvalues of (22) is

$$\{\lambda_{1j}^{n_1} \dots \lambda_{pj}^{n_p} : j = 1, \dots, m\}.$$

By our assumption, for any $n = (n_1, \dots, n_p)$ there is j such that

$$|\lambda_{1j}^{n_1} \dots \lambda_{pj}^{n_p}| = 1. \tag{23}$$

To proceed, we need the following auxiliary statement.

LEMMA 1. *Assume that numbers μ_{ij}, where $i = 1, \dots, p$ and $j = 1, \dots, m$, satisfy the following condition: for any $n_1, \dots, n_p \in \mathbb{Z}$ there exists j such that*

$$n_1\mu_{1j} + \dots + n_p\mu_{pj} = 0. \tag{24}$$

Then there exists j such that

$$\mu_{ij} = 0, \qquad i = 1, \dots, p. \tag{25}$$

Proof. We apply induction on p. If $p = 1$, our condition implies that for $n_1 = 1$ there exists j such that $n_1\mu_{1j} = 0$. Thus, $\mu_{1j} = 0$, as required.

Now assume that our statement holds for $p - 1$. Fix $n_2, \dots, n_p \in \mathbb{Z}$ and define $u_j = n_2\mu_{2j} + \dots + n_p\mu_{pj}$. By our assumption, for any $n_1 \in \mathbb{Z}$ there exists j such that $n_1\mu_{1j} + u_j = 0$, hence

$$P := \prod_{j=1}^{m} (n_1\mu_{1j} + u_j) = 0. \tag{26}$$

Since P is a polynomial in n_1, and it equals zero for any n_1, its coefficients are zero. The leading coefficient (of n_1^m) equals

$$\mu_{11}\mu_{12} \cdots \mu_{1m},$$

hence at least one of the μ_{1j} is zero.

Let $J \subset \{1, \dots, m\}$ be the set of all j such that $\mu_{1j} = 0$. Let k be the number of elements of J. The coefficient of n_1^{m-k} in P equals

$$\prod_{j \in J} u_j \prod_{j \notin J} \mu_{1j} = 0.$$

Since the second product is nonzero, we see that

$$\prod_{j\in J} u_j = 0.$$

Thus, there exists $j \in J$ such that $u_j = 0$. Our reasoning shows that for any $n_2, \dots, n_p$ there exists j' such that

$$u_{j'} = n_2\mu_{2j'} + \ldots + n_p\mu_{pj'} = 0.$$

This means that the numbers $\mu_{2j}, \dots, \mu_{pj}$, $j \in J$, satisfy the assumption of our lemma. By the induction assumption, there exists j such that

$$\mu_{2j} = \ldots = \mu_{pj} = 0.$$

Since $j \in J$, we also have $\mu_{1j} = 0$, which completes the induction step. Our lemma is proved.

Setting $\mu_{ij} = \log|\lambda_{ij}|$, we reduce condition (23) to (24). By Lemma 1, there exists j such that $|\lambda_{ij}| = 1$ for $i = 1, \dots, p$. The contradiction obtained proves the implication $(n1)\Rightarrow(n2)$.

Before proving $(n2)\Rightarrow(n3)$, we establish an auxiliary statement.

LEMMA 2. *Let $A'_1, \dots, A'_p$ be pairwise commuting linear operators on $\mathbb{C}^m$ such that*

$$\ker(a_1A'_1 + \ldots + a_pA'_p) \neq \{0\} \tag{27}$$

for any real numbers $a_1, \dots, a_p$. Then

$$\bigcap_{i=1}^{p} \ker A'_i \neq \{0\}. \tag{28}$$

Proof. In the proof, we often use the following simple statement. Let A and B be commuting linear operators and let $\ker B = Y$. Then $A(Y) \subset Y$. Indeed, if $y \in Y$, then $By = 0$, so $ABy = 0$ and $B(Ay) = 0$. Thus, $Ay \in Y$.

We prove the lemma by induction on p. The case $p = 1$ is trivial. Let $p = 2$. Define

$$C_k = A'_1 + kA'_2 \quad \text{and} \quad X_k = \ker C_k$$

for nonnegative integer k. Obviously, C_i and C_j commute for any i and j. Our statement above implies that

$$C_i(X_j) \subset X_j. \tag{29}$$

Let $\mathrm{Lin}(Y_1, \dots, Y_r)$ be the linear hull of the linear subspaces $Y_1, \dots, Y_r$.

We claim that there exists n such that

$$X_{n+1} \cap \mathrm{Lin}(X_0, \dots, X_n) \neq \{0\}. \tag{30}$$

Indeed, if $X_{n+1} \cap \mathrm{Lin}(X_0, \dots, X_n) = \{0\}$ for any n, then

$$\dim \mathrm{Lin}(X_0, \dots, X_n) = \sum_{i=0}^{n} \dim X_i \geq n + 1,$$

which is impossible if $n \geq m$.

Let n be minimal satisfying (30). Consider $x \neq 0$ such that

$$x \in X_{n+1} \cap \mathrm{Lin}(X_0, \ldots, X_n). \tag{31}$$

Represent x in the form $x_0 + \ldots + x_n$, where $x_i \in X_i$. Note that

$$C_{n+1}x = C_{n+1}x_0 + \ldots + C_{n+1}x_n = 0. \tag{32}$$

It follows from (29) that

$$y_i := C_{n+1}x_i \in X_i.$$

Relation (32) implies that

$$y_0 + \ldots + y_n = 0.$$

If we assume that $y_i \neq 0$ for some i, and consider the maximal i with this property, then

$$y_i = -(y_0 + \ldots + y_{i-1}) \in \mathrm{Lin}(X_0, \ldots, X_{i-1}),$$

contradicting the choice of n and the inequality $i - 1 < n$. Thus,

$$y_i = 0 \quad \text{for } 0 \leq i \leq n. \tag{33}$$

Since $x \neq 0$, there exists $k \in \{1, \ldots, n\}$ such that $x_k \neq 0$. The equalities

$$y_k = C_{n+1}x_k = 0 \quad \text{and} \quad C_k x_k = 0$$

(recall that $x_k \in \ker C_k$) written in the form

$$(A'_1 + (n+1)A'_2)x_k = 0 \quad \text{and} \quad (A'_1 + kA'_2)x_k = 0$$

imply that $x_k \in \ker A'_1 \cap \ker A'_2$. Thus, our lemma is proved for $p = 2$.

Now assume that our statement holds for p. Define

$$B_{i,k} = A'_1 + kA'_i \quad \text{and} \quad X_{i,k} = \ker B_{i,k}.$$

Since the operators A'_i pairwise commute, so do $B_{i,k}$ and A'_i. Condition (27) implies that any linear combination of $B_{i,k}$ and A'_i has a nonempty kernel.

By the induction assumption applied to $A'_{p+1}, B_{2,k}, \ldots, B_{p,k}$ with any k,

$$Y_k := \ker A'_{p+1} \cap X_{2,k} \cap \ldots \cap X_{p,k} \neq \{0\}.$$

The same reasoning as above shows that there exists n such that

$$Y_{n+1} \cap \mathrm{Lin}(Y_0, \ldots, Y_n) \neq \{0\}.$$

Consider the minimal n with this property. Take $y_i \in Y_i$ such that

$$y_{n+1} = y_0 + \ldots + y_n \tag{34}$$

and $y_{n+1} \neq 0$. Applying the operator $B_{2,n+1}$ to (34) and taking into account that $y_{n+1} \in Y_{n+1} \subset X_{2,n+1}$, we see that

$$0 = B_{2,n+1}y_0 + \ldots + B_{2,n+1}y_n. \tag{35}$$

Since $B_{2,n+1}y_l \in Y_l$, the reasoning applied to establish (33) shows that $B_{2,n+1}y_l = 0$ for any $l \in \{0, \ldots, n\}$.

Consider $y_l \neq 0$. We claim that

$$y_l \in \ker A'_1 \cap \ker A'_2 \cap \ldots \cap \ker A'_{p+1}. \tag{36}$$

Since $y_l \in Y_l$, we have $y_l \in \ker A'_{p+1}$. The relations $B_{2,n+1} y_l = 0$ and $y_l \in Y_i \subset X_{2,l}$ imply that

$$(A'_1 + (n+1)A'_2)y_l = 0 \quad \text{and} \quad (A'_1 + lA'_2)y_l = 0.$$

It follows that $A'_1 y_l = 0$ and $A'_2 y_l = 0$. Since $y_l \in X_{i,l} = \ker B_{i,l}$ for any $l \in \{3, \ldots, p\}$, we see, in addition, that

$$(A'_1 + lA'_i)y_l = 0.$$

Thus, $A'_i y_l = 0$ for these l, relation (36) holds, and the lemma is proved.

This lemma implies an important property of our pairwise commuting triangular matrices $A_1, \ldots, A_n$ generating action (18). Obviously, the desired implication $(n2) \Rightarrow (n3)$ follows from this property.

COROLLARY. *For any $j \in \{1, \ldots, m\}$ there exists a vector $v \neq 0$ such that*

$$A_i v = \lambda_{ij} v, \quad i = 1, \ldots, p. \tag{37}$$

Proof. Fix j and consider the matrices

$$A'_i = A_i - \lambda_{ij} E_m,$$

where E_m is the identity $m \times m$ matrix. These matrices are triangular and pairwise commute. Their jth diagonal elements are zero.

Hence, condition (27) of Lemma 2 is satisfied. By Lemma 2, there exists a vector $v \neq 0$ such that $A'_i v = 0$ for $i = 1, \ldots, p$. Obviously, v satisfies (37).

Now let us prove the implication $(n3) \Rightarrow (n1)$. Fix a vector v with $|v| = 1$ satisfying (21). We claim that action (18) does not have the shadowing property. Let us construct a pseudotrajectory as follows. Fix a positive number d and a sequence $\{c_l : l \in \mathbb{Z}\}$ of integers with the following properties: $|c_{l+1} - c_l| = 1$ for any l, the sequence $|c_l|$ is unbounded, and the limits

$$\lim_{|l| \to \infty} c_l$$

do not exist. Define $\Lambda = \mu_1$ and set $a_l = dc_l \Lambda^l v$ and

$$x_n = A_2^{n_2} \ldots A_p^{n_p} a_{n_1} \quad \text{for } n = (n_1, \ldots, n_p).$$

The sequence $\{x_n\}$ is a $2d$-pseudotrajectory of (18). Indeed,

$$x_{(n_1,\ldots,n_p)(i,\pm 1)} = A_i^{\pm 1} x_{(n_1,\ldots,n_p)}$$

for $i \geq 2$. Since v is an eigenvector of $A_2, \ldots, A_p$ with eigenvalues $|\mu_i| = 1$, it follows from

$$|A_1^{\pm 1} x_{(n_1,\ldots,n_p)} - x_{(n_1 \pm 1, n_2, \ldots, n_p)}| = |A_2^{n_2} \ldots A_p^{n_p} (A_1^{\pm 1} a_{n_1} - a_{n_1 \pm 1})|$$

and

$$A_1^{\pm 1} a_{n_1} - a_{n_1 \pm 1} = \varepsilon d v, \quad \text{where} \quad |\varepsilon| = 1,$$

that

$$|A_1^{\pm 1} x_{(n_1,\dots,n_p)} - x_{(n_1 \pm 1, n_2, \dots, n_p)}| = |\varepsilon d v| = d.$$

To complete the proof, we claim that

$$\sup_n |\Phi(n,y) - x_n| = \infty$$

for any $y \in \mathbb{C}^m$. To see this, it is enough to show that

$$\sup_{(n_1,0,\dots,0)} |\Phi((n_1,0,\dots,0),y) - x_{(n_1,0,\dots,0)}| = \infty \tag{38}$$

for any $y \in \mathbb{C}^m$. Fix a basis $e_1,\dots,e_m$ in $\mathbb{C}^m$ as follows:

$$e_1 = v, \qquad A_1 e_i = \Lambda e_i + e_{i-1} \quad \text{for } 2 \le i \le k,$$

and

$$A_1 = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$$

in this basis, where B and C are $k \times k$ and $(m-k) \times (m-k)$ matrices, respectively.

If $a^{(1)}$ is the first coordinate of a vector $a \in \mathbb{C}^m$ in the chosen basis, then

$$x^{(1)}_{(l,0,\dots,0)} = d c_l \Lambda^l.$$

For $y = (y_1,\dots,y_m) \in \mathbb{C}^m$, write $y' = (y_1,\dots,y_k)$.

The matrix B has the form $\Lambda E_k + J$, where $J^i = 0$ for $i \ge k$. Hence, for any $y \in \mathbb{C}^m$,

$$(A_1^l y)^{(1)} = (B^l y')^{(1)} = \left(\sum_{i=0}^{k-1} \frac{l!}{(l-i)!i!} \Lambda^{l-i} J^i y' \right)^{(1)} = \Lambda^l P(l),$$

where $P(l)$ is a polynomial in l of degree not exceeding $k-1$ (determined by the fixed vector y).

If (38) does not hold for some $y \in \mathbb{C}^m$, then the expression

$$|(A_1^l y)^{(1)} - x^{(1)}_{(l,0,\dots,0)}| = |d c_l - P(l)|$$

is bounded in l. This contradicts the choice of the sequence c_l since either $P(l)$ is constant (while c_l is unbounded) or $|P(l)| \to \infty$ as $|l| \to \infty$ (while c_l does not have limits as $|l| \to \infty$).

The proof is complete.

4. Actions of the group $\mathbb{Z}^p \times \mathbb{R}^q$. Now we pass to continuous actions of the group $\mathcal{G} = \mathbb{Z}^p \times \mathbb{R}^q$. As above, (M, ϱ) is a metric space and $H(M)$ is the set of homeomorphisms of M.

We represent $n \in \mathbb{Z}^p \times \mathbb{R}^q$ in the form $n = (n^D, n^C)$, where $n^D = (n_1, \dots, n_p)$ and $n^C = (n_{p+1}, \dots, n_{p+q})$. Let $I_1 = \{1, \dots, p\}$, $I_2 = \{p+1, \dots, p+q\}$, and $I = I_1 \cup I_2$. For any set $m = \{m_i\}$, where $i \in I_1$, $i \in I_2$, or $i \in I$, we define $|m| = \sum_i |m_i|$.

Fix $n \in \mathbb{Z}^p \times \mathbb{R}^q$, $i \in I$, and $k \in \mathbb{Z}$ (if $i \in I_1$) or $k \in \mathbb{R}$ (if $i \in I_2$). As above, we denote by $n(i,k)$ the element $n' \in \mathbb{Z}^p \times \mathbb{R}^q$ such that $n'_j = n_j$ for $j \in I$, $j \neq i$, and $n'_i = n_i + k$.

Consider the homeomorphisms

$$f_{i,\pm 1} = \Phi(n(i, \pm 1), \cdot) \circ \Phi^{-1}(n, \cdot), \quad i \in I_1,$$
$$g_{i-p}(t, \cdot) = \Phi(n(i,t), \cdot) \circ \Phi^{-1}(n, \cdot), \quad i \in I_2.$$

We use different notation for dependence on "time" to emphasize the difference between the "discrete-time" generators $f_{i,\pm 1}$, $i \in I_1$, and "continuous-time" generators $g_{i-p}(t, \cdot)$, $i \in I_2$.

Fix a positive number d. We say that a set $\xi = \{x_n \in M : n \in \mathbb{Z}^p \times \mathbb{R}^q\}$ is a *d-pseudotrajectory* of Φ if

$$\varrho(x_{n(i,\pm 1)}, f_{i,\pm 1}(x_n)) < d, \quad i \in I_1, \tag{39}$$
$$\varrho(x_{n(i,t)}, g_{i-p}(t, x_n)) < d, \quad |t| \le 1, \, i \in I_2, \tag{40}$$

for any $n \in \mathbb{Z}^p \times \mathbb{R}^q$.

In the case of a flow (i.e., for $I_1 = \emptyset$ and $I_2 = \{1\}$), this definition corresponds to the standard definition of a pseudotrajectory (see Remark 2 below and the books [6, 7] for the details and discussion).

Let us formulate two properties which we need to give conditions under which Φ has a shadowing property.

Denote by $\mathcal{R}$ the set of orientation preserving homeomorphisms $\alpha : \mathbb{R}^q \to \mathbb{R}^q$ such that $\alpha(0) = 0$. Let $\alpha = (\alpha_1, \dots, \alpha_q)$.

Fix $\nu = \{\nu_1, \dots, \nu_{p+q}\} \in \mathbb{Z}^p \times \mathbb{R}^q$ and consider the corresponding homeomorphism $f = \Phi(\nu, \cdot)$. Fix, in addition, a homeomorphism $\alpha \in \mathcal{R}$ and consider the mapping

$$\Psi(k, \alpha) = f_1^{k\nu_1} \circ \dots \circ f_p^{k\nu_p} \circ g_1(\alpha_1(k\nu^C), \cdot) \circ \dots \circ g_q(\alpha_q(k\nu^C), \cdot),$$

where $k \in \mathbb{Z}$.

To simplify the presentation, we formulate the properties (and the main result of this section) for the phase space M (instead of its subsets); possible generalizations (similar to properties defined in Sec. 2) are trivial.

We say that the homeomorphism $f = \Phi(\nu, \cdot)$ has the *flow-type shadowing property* (FTS property) if given $\varepsilon > 0$ there exists $d > 0$ such that for any d-pseudotrajectory $\{y_k : k \in \mathbb{Z}\}$ of f there is a homeomorphism $\alpha \in \mathcal{R}$ and a point $z \in M$ such that

$$\varrho(\Psi(k, \alpha)(z), y_k) < \varepsilon \quad \text{for } k \in \mathbb{Z}. \tag{41}$$

We say that the homeomorphism $f = \Phi(\nu, \cdot)$ is *flow-type expansive* (has the FTE property) if given $\Delta > 0$ there exists $\delta > 0$ with the following property: if $\alpha, \alpha' \in \mathcal{R}$ and

$$\varrho(\Psi(k,\alpha)(z), \Psi(k',\alpha')(z')) < \delta, \qquad k \in \mathbb{Z}, \tag{42}$$

for some points z and z', then there exist numbers $\tau_1, \ldots, \tau_q$ such that

$$|\tau_1| + \ldots + |\tau_q| < \Delta, \tag{43}$$

$$z' = g_1(\tau_1, \cdot) \circ \ldots \circ g_q(\tau_q, \cdot)(z). \tag{44}$$

Remark 2. The introduced properties are natural generalizations of the corresponding properties for flows.

Let ϕ be the flow of an autonomous system of differential equations on a smooth manifold. Assume that the system has a hyperbolic set Λ containing no rest points. It is known (see [6]) that the flow ϕ has the following analogs of the FTS and FTE properties in a neighborhood U of Λ.

We say that $\{y_t : t \in \mathbb{R}\}$ is a $(d,1)$-*pseudotrajectory* of ϕ if

$$\varrho(y_{t+\tau}, \phi(\tau, y_t)) < d \quad \text{for any } t \in \mathbb{R} \text{ and } |\tau| \le 1.$$

Given $\varepsilon > 0$ there exists $d > 0$ such that if $\{y_t \in U : t \in \mathbb{R}\}$ is a $(d,1)$-pseudotrajectory of ϕ, then there is a homeomorphism $\alpha \in \mathcal{R}$ (here $q = 1$) and a point $z \in M$ such that

$$\varrho(\phi(\alpha(t), z), y_t) < \varepsilon \quad \text{for } t \in \mathbb{R}. \tag{45}$$

Given $\Delta > 0$ there exists $\delta > 0$ with the following property: if $\alpha, \alpha' \in \mathcal{R}$,

$$\phi(\alpha(t), z), \phi(\alpha'(t), z') \in U,$$
$$\varrho(\phi(\alpha(t), z), \phi(\alpha'(t), z')) < \delta, \qquad t \in \mathbb{R},$$

for some points z and z', then there exists a number τ such that

$$|\tau| < \Delta, \tag{46}$$

$$z' = \phi(\tau, z). \tag{47}$$

In addition, there exist numbers $L, d_0, \delta_0 > 0$ such that if $d \le d_0$, then (45) holds with ε replaced by Ld, and if $\delta \le \delta_0$, then (46) holds with Δ replaced by $L\delta$.

The same reasoning as in the proof of Theorem 1 (with obvious modifications) establishes the following statement.

Theorem 3. *Assume that there exists $\nu \in \mathbb{Z}^p \times \mathbb{R}^q$ such that the homeomorphism $f = \Phi(\nu, \cdot)$ has the FTS and FTE properties. Assume, in addition, that the family*

$$\mathcal{F} = \{f_{i,\pm 1} : i \in I_1\} \cup \{g_{i-p}(t, \cdot) : |t| \le 1,\ i \in I_2\}$$

is equicontinuous. Then for any $\varepsilon > 0$ there exists $d > 0$ with the following property: if $\{x_n \in V : n \in \mathbb{Z}^p \times \mathbb{R}^q\}$ is a d-pseudotrajectory of Φ, then there *exists a point* x *and a mapping* $\tau : \mathbb{Z}^p \times \mathbb{R}^q \to \mathbb{Z}^p \times \mathbb{R}^q$ *such that*

$$\varrho(\Phi(\tau(n), x), x_n) < \varepsilon, \quad n \in \mathbb{Z}^p \times \mathbb{R}^q, \tag{48}$$

$$(\tau(n))^D = n^D, \quad |(\tau(n))^C - n^C| \leq \varepsilon(q + |n|). \tag{49}$$

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