FUNDAMENTA
MATHEMATICAE
228 (2015)

On sets with rank one in simple homogeneous structures

by

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Abstract. We study definable sets D of SU-rank 1 in $\mathcal{M}^{\mathrm{eq}}$, where $\mathcal{M}$ is a countable homogeneous and simple structure in a language with finite relational vocabulary. Each such D can be seen as a 'canonically embedded structure', which inherits all relations on D which are definable in $\mathcal{M}^{\mathrm{eq}}$, and has no other definable relations. Our results imply that if no relation symbol of the language of $\mathcal{M}$ has arity higher than 2, then there is a close relationship between triviality of dependence and $\mathcal{D}$ being a reduct of a binary random structure. Somewhat more precisely: (a) if for every $n \geq 2$, every n-type $p(x_1, \dots, x_n)$ which is realized in D is determined by its sub-2-types $q(x_i, x_j) \subseteq p$, then the algebraic closure restricted to D is trivial; (b) if $\mathcal{M}$ has trivial dependence, then $\mathcal{D}$ is a reduct of a binary random structure.

1. Introduction. We call a countable first-order structure $\mathcal{M}$ *homogeneous* if it has a finite relational vocabulary (also called signature) and every isomorphism between finite substructures of $\mathcal{M}$ can be extended to an automorphism of $\mathcal{M}$. (The terminology *ultrahomogeneous* is used in some texts.) For surveys about homogeneous structures and connections with other areas, see [25] and the first chapter of [3]. It is possible to construct 2^ω countable homogeneous structures, even for a vocabulary with only a binary relation symbol, as shown by Henson [13]. But it is also known that in several cases, such as partial orders, undirected graphs, directed graphs or stable structures with finite relational vocabulary, all countable homogeneous structures in each class can be classified in a more or less explicit way [3, 10, 11, 15, 20, 21, 22, 23, 27, 28]. Ideas from stability theory and the study of homogeneous structures have been used to obtain a good understanding of structures that are ω-categorical and ω-stable (which need not be homogeneous) [4] and, more generally, of smoothly approximable structures [5, 16].

2010 *Mathematics Subject Classification*: 03C50, 03C45, 03C15, 03C30.
Key words and phrases: model theory, homogeneous structure, simple theory, pregeometry, rank, reduct, random structure.

DOI: 10.4064/fm228-3-2

Simplicity [2, 30] is a notion that is more general than stability. The structures that are stable, countable and homogeneous are well understood, by the work of Lachlan and others; see for example the survey [21]. However, little appears to be known about countable homogeneous structures that are simple, even for a *binary vocabulary*, i.e. a *finite* relational vocabulary where every relation symbol has arity at most 2. Besides the present work, [19] and the dissertation of Aranda López [1] have results in this direction. A *binary structure* is one with binary vocabulary.

We say that a structure $\mathcal{M}$ is a *reduct* of a structure $\mathcal{M}'$ (possibly with another vocabulary) if they have the same universe and for every positive integer n and every relation $R \subseteq M^n$, if R is definable in $\mathcal{M}$ without parameters, then R is definable in $\mathcal{M}'$ without parameters. For any structure $\mathcal{M}$, $\mathcal{M}^{\mathrm{eq}}$ denotes the extension of $\mathcal{M}$ by *imaginary elements* [14, 29]. Note that understanding what kind of structures can be defined in $\mathcal{M}^{\mathrm{eq}}$ is roughly the same as understanding which structures can be interpreted in $\mathcal{M}$.

We address the following problems: *Suppose that $\mathcal{M}$ has finite relational vocabulary, is homogeneous and simple, $E \subset M$ is finite, $D \subseteq M^{\mathrm{eq}}$ is E-definable, only finitely many 1-types over E are realized in D, and for every $d \in D$ the SU-rank of the type of d over E is 1.*

(A) What are the possible behaviours of the algebraic closure restricted to D if elements of E may be used as constants?

(B) Let $\mathcal{D}$ be the structure with universe D which for every n and E-definable $R \subseteq D^n$ has a relation symbol which is interpreted as R (and the vocabulary of $\mathcal{D}$ has no other symbols). We call $\mathcal{D}$ a *canonically embedded structure over E*. Note that the vocabulary of $\mathcal{D}$ is relational but *not* finite. Now we ask: is $\mathcal{D}$ necessarily a reduct of a homogeneous structure with finite relational vocabulary?

Macpherson [24] has shown that no infinite vector space over a finite field can be interpreted in a homogeneous structure over a finite relational language, which implies that, in (A), the pregeometry of D induced by the algebraic closure cannot be isomorphic to the pregeometry induced by linear span in a vector space over a finite field. If we assume, in addition to the assumptions made above (before (A)), that $\mathcal{M}$ is one-based, then it follows from [24] and work of De Piro and Kim [6, Corollary 3.23] that algebraic closure restricted to D is *trivial*, i.e. if $d \in D$, $B \subseteq D$ and $d \in \mathrm{acl}(B \cup E)$, then there is $b \in B$ such that $d \in \mathrm{acl}(\{b\} \cup E)$. But what if we do not assume that $\mathcal{M}$ is one-based?

Let $\mathcal{D}_0$ be the reduct of $\mathcal{D}$ to the relation symbols with arity at most 2. If the answer to the question in (B) is 'yes' in the strong sense that $\mathcal{D}$ is a reduct of $\mathcal{D}_0$ and $\mathcal{D}_0$ is homogeneous, then Remark 3.9 below implies that algebraic closure and dependence restricted to D are trivial. If $\mathcal{M}$ is supersimple with finite SU-rank, and the assumptions about $\mathcal{D}$ and $\mathcal{D}_0$ hold not only for this particular $\mathcal{D}$, but for all $\mathcal{D}$, then it follows from [12, Corollary 4.7], [6, Corollary 3.23] and some additional straightforward arguments that *the theory of* $\mathcal{M}$ has *trivial dependence* (Definition 3.5 below).

In the other direction, if $\mathcal{M}$ is binary, has trivial dependence and moreover $\mathrm{acl}(\{d\} \cup E) \cap D = \{d\}$ for all $d \in D$ (so D is a *geometry*), then, by Theorem 5.1, $\mathcal{D}$ is a reduct of a binary homogeneous structure; in fact $\mathcal{D}$ is a reduct of a *binary random structure* in the sense of Section 2.3.

Thus we establish that, at least for binary $\mathcal{M}$, the problems (A) and (B) are closely related, although we do not know whether our partial conclusions to (A) and (B) in the binary case are equivalent. Neither do we solve any one of problems (A) or (B). So in particular, the problem whether algebraic closure restricted to D (and using constants from E) can be nontrivial for some binary, homogeneous and simple $\mathcal{M}$ remains open. Nevertheless, Theorem 5.1 is used in [19] where a subclass of the countable, binary, homogeneous, simple and one-based structures is classified in a fairly concrete way; namely, as the class of such structures which have *height* 1 in the sense of [7], roughly meaning that the structure is "coordinatized" by a definable set of SU-rank 1.

This article is organized as follows. In Section 2 we recall definitions and results about homogeneous structures and simple structures, in particular the independence theorem and consequences of ω-categoricity and simplicity together, especially with regard to imaginary elements. We also explain what is meant by a binary random structure.

In Section 3 we prove results implying that if $\mathcal{M}$ and D are as assumed before (A) above and $\mathcal{M}$ is binary, then algebraic closure and dependence restricted to D are trivial. In Section 5 we prove the next main result, Theorem 5.1, saying that if $\mathcal{M}$ and D are as assumed before (A), $\mathcal{M}$ is binary and its theory has trivial dependence, then $\mathcal{D}$ is a reduct of a binary random structure. In order to prove Theorem 5.1 we use a more technical result, Theorem 4.6, which is proved in Section 4, where most of the technical (and simplicity-theoretic) work is done. The proofs assume a working knowledge in stability/simplicity theory, as can be found in [2, 30].

2. Preliminaries

2.1. General notation and terminology. A vocabulary (or signature) is called *relational* if it only contains relation symbols. For a finite relational vocabulary the maximal k such that some relation symbol has arity k is called its *maximal arity.* If V is a *finite* vocabulary and the maximal arity is 2 then we call V *binary* (although it may contain unary relation symbols), and in this case a V-structure is called a *binary structure.*

We denote (first-order) structures by $\mathcal{A}, \mathcal{B}, \ldots, \mathcal{M}, \mathcal{N}, \ldots$ and their respective universes by $A, B, \ldots, M, N, \ldots$. By the cardinality of a structure we mean the cardinality of its universe. To emphasize the cardinality of a finite structure we sometimes call a structure with cardinality $k < \omega$ a *k-structure*, or *k-substructure* if it is seen as a substructure of some other structure. Finite sequences (tuples) of elements of some structure (or set in general) will be denoted $\bar{a}, \bar{b}, \ldots$, while $a, b, \ldots$ usually denote elements from the universe of some structure.

The notation $\bar{a} \in A$ means that every element in the sequence $\bar{a}$ belongs to A. Sometimes we write $\bar{a} \in A^n$ to show that the length of $\bar{a}$, denoted $|\bar{a}|$, is n and all elements of $\bar{a}$ belong to A. By $\mathrm{rng}(\bar{a})$, the *range of* $\bar{a}$, we denote the set of elements that occur in $\bar{a}$. In order to compress notation, we sometimes, in particular together with type notation and the symbol '$\mathop{\smile\!\!\!\!\!\!\mid}$' (for independence), write 'AB' instead of '$A \cup B$', or '$\bar{a}$' instead of '$\mathrm{rng}(\bar{a})$'.

Suppose that $\mathcal{M}$ is a structure, $A \subseteq M$ and $\bar{a} \in M$. Then $\mathrm{acl}_{\mathcal{M}}(A)$, $\mathrm{dcl}_{\mathcal{M}}(A)$ and $\mathrm{tp}_{\mathcal{M}}(\bar{a}/A)$ denote the *algebraic closure* of A with respect to $\mathcal{M}$, the *definable closure* of A with respect to $\mathcal{M}$ and the *complete type of $\bar{a}$ over A* with respect to $\mathcal{M}$, respectively (see for example [14] for definitions). By $S_n^{\mathcal{M}}(A)$ we denote the set of all complete n-types over A with respect to $\mathcal{M}$. We abbreviate $\mathrm{tp}_{\mathcal{M}}(\bar{a}/\emptyset)$ to $\mathrm{tp}_{\mathcal{M}}(\bar{a})$. The notation $\mathrm{acl}_{\mathcal{M}}(\bar{a})$ is an abbreviation of $\mathrm{acl}_{\mathcal{M}}(\mathrm{rng}(\bar{a}))$, and similarly for 'dcl'.

We say that $\mathcal{M}$ is *ω-categorical*, respectively *simple*, if $\mathrm{Th}(\mathcal{M})$ has that property, where $\mathrm{Th}(\mathcal{M})$ is the complete theory of $\mathcal{M}$ (see [14] and [2, 30] for definitions). Let $A \subseteq M$ and $R \subseteq M^k$. We say that R is *A-definable* (with respect to $\mathcal{M}$) if there is a formula $\varphi(\bar{x}, \bar{y})$ (without parameters) and $\bar{a} \in A$ such that $R = \{\bar{b} \in M^k : \mathcal{M} \models \varphi(\bar{b}, \bar{a})\}$. In this case we also denote R by $\varphi(\mathcal{M}, \bar{a})$. Similarly, for a type $p(\bar{x})$ (possibly with parameters) we let $p(\mathcal{M})$ be the set of all tuples of elements in $\mathcal{M}$ that realize p, and $\mathcal{M} \models p(\bar{a})$ means that $\bar{a}$ realizes p in $\mathcal{M}$. *Definable without parameters* means the same as $\emptyset$-definable.

Definition 2.1. (i) If $\mathcal{M}$ is a structure with relational vocabulary and $A \subseteq M$, then $\mathcal{M}{\restriction}A$ denotes the substructure of $\mathcal{M}$ with universe A.

(ii) If $\mathcal{M}$ is a V-structure and $V' \subseteq V$, then $\mathcal{M}{\restriction}V'$ denotes the *reduct of $\mathcal{M}$ to the vocabulary V'*.

Note that if $\mathcal{M}$ is a V-structure and $V' \subseteq V$, then $\mathcal{M}{\restriction}V'$ is a reduct of $\mathcal{M}$ in the sense defined in Section 1.

2.2. Homogeneity, Fraïssé limits and ω-categoricity

Definition 2.2. (i) Let V be a relational vocabulary and $\mathcal{M}$ be a V-structure. We call $\mathcal{M}$ *homogeneous* if its universe is countable and for all finite substructures $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{M}$, every isomorphism from $\mathcal{A}$ to $\mathcal{B}$ can be extended to an automorphism of $\mathcal{M}$.

(ii) A structure $\mathcal{M}$ (for any vocabulary) is called *ω-homogeneous* if whenever $0 < n < \omega$, $a_1, \ldots, a_n, a_{n+1}, b_1, \ldots, b_n \in M$ and $\mathrm{tp}(a_1, \ldots, a_n) = \mathrm{tp}(b_1, \ldots, b_n)$, there is $b_{n+1} \in M$ such that $\mathrm{tp}(a_1, \ldots, a_{n+1}) = \mathrm{tp}(b_1, \ldots, b_{n+1})$.

DEFINITION 2.3. Let V be a finite relational vocabulary and let $\mathbf{K}$ be a class of finite V-structures which is *closed under isomorphism*, that is, if $\mathcal{A} \in \mathbf{K}$ and $\mathcal{B} \cong \mathcal{A}$, then $\mathcal{B} \in \mathbf{K}$.

(i) $\mathbf{K}$ has the *hereditary property*, abbreviated HP, if $\mathcal{A} \subseteq \mathcal{B} \in \mathbf{K}$ implies that $\mathcal{A} \in \mathbf{K}$.
(ii) $\mathbf{K}$ has the *amalgamation property*, abbreviated AP, if the following holds: if $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbf{K}$ and $f_{\mathcal{B}} : \mathcal{A} \to \mathcal{B}$ and $f_{\mathcal{C}} : \mathcal{A} \to \mathcal{C}$ are embeddings then there are $\mathcal{D} \in \mathbf{K}$ and embeddings $g_{\mathcal{B}} : \mathcal{B} \to \mathcal{D}$ and $g_{\mathcal{C}} : \mathcal{C} \to \mathcal{D}$ such that $g_{\mathcal{B}} \circ f_{\mathcal{B}} = g_{\mathcal{C}} \circ f_{\mathcal{C}}$.
(iii) If $\mathcal{M}$ is a V-structure, then $\mathbf{Age}(\mathcal{M})$ is the class of all V-structures that are isomorphic with some finite substructure of $\mathcal{M}$.

We allow structures with empty universe if the vocabulary is relational (as generally assumed in this article), so if $\mathbf{K}$ has the hereditary property then the structure with empty universe belongs to $\mathbf{K}$. It follows that if the vocabulary is relational then the *joint embedding property* [14] is a consequence of the amalgamation property, which is the reason why we need not bother about the former in the present context. The following result of Fraïssé ([9], [14, Theorems 7.1.2 and 7.1.7]) relates homogeneous structures to finite structures, and shows how the former can be constructed from the latter.

FACT 2.4. *Let V be a finite relational vocabulary.*

(i) *Suppose that $\mathbf{K}$ is a class of finite V-structures which is closed under isomorphism and has HP and AP. Then there is a unique, up to isomorphism, countable V-structure $\mathcal{M}$ such that $\mathcal{M}$ is homogeneous and $\mathbf{Age}(\mathcal{M}) = \mathbf{K}$.*
(ii) *If $\mathcal{M}$ is a homogeneous V-structure, then $\mathbf{Age}(\mathcal{M})$ has HP and AP.*

DEFINITION 2.5. Suppose that V is a finite relational vocabulary and that $\mathbf{K}$ is a class of finite V-structures which is closed under isomorphism and has HP and AP. The unique (up to isomorphism) countable structure $\mathcal{M}$ such that $\mathbf{Age}(\mathcal{M}) = \mathbf{K}$ is called the *Fraïssé limit* of $\mathbf{K}$.

Part (i) in the next fact is Corollary 7.4.2 in [14] (for example). Part (ii) follows from the well known characterization of ω-categorical structures by Engeler, Ryll-Nardzewski and Svenonius [14, Theorem 7.3.1], which will frequently be used without further reference. Part (iii) follows from a straightforward back-and-forth argument.

Fact 2.6. *Let V be a relational vocabulary and $\mathcal{M}$ an infinite countable V-structure.*

(i) *If V is finite then $\mathcal{M}$ is homogeneous if and only if $\mathcal{M}$ is ω-categorical and has elimination of quantifiers.*
(ii) *If $\mathcal{M}$ is ω-categorical, then $\mathcal{M}$ is ω-saturated and ω-homogeneous.*
(iii) *If $\mathcal{M}$ is ω-homogeneous, then the following holds: if $0 < n < \omega$, $a_1, \dots, a_n, b_1, \dots, b_n \in M$ and $\mathrm{tp}_{\mathcal{M}}(a_1, \dots, a_n) = \mathrm{tp}_{\mathcal{M}}(b_1, \dots, b_n)$, then there is an automorphism f of $\mathcal{M}$ such that $f(a_i) = b_i$ for all i.*

2.3. Binary random structures. Let V be a binary vocabulary (and therefore finite).

Definition 2.7. A class $\mathbf{K}$ of finite V-structures is called 1-*adequate* if it has HP and the following property with respect to 1-structures:

- If $\mathcal{A}, \mathcal{B} \in \mathbf{K}$ are 1-structures, then there is $\mathcal{C} \in \mathbf{K}$ such that $\mathcal{A} \subseteq \mathcal{C}$ and $\mathcal{B} \subseteq \mathcal{C}$.

Construction of a binary random structure. Let $\mathbf{P}_2$ be a 1-adequate class of V-structures such that $\mathbf{P}_2$ contains a 2-structure. We think of $\mathbf{P}_2$ as containing the isomorphism types of "permitted" 1-(sub)structures and 2-(sub)structures. Then let $\mathbf{RP}_2$ be the class of all finite V-structures $\mathcal{A}$ such that for $k = 1, 2$ every k-substructure of $\mathcal{A}$ is isomorphic to some member of $\mathbf{P}_2$. Obviously, $\mathbf{RP}_2$ has HP, because the 1-adequateness of $\mathbf{P}_2$ implies that $\mathbf{P}_2$ has HP. The 1-adequateness of $\mathbf{P}_2$ implies that any two 1-structures of $\mathbf{RP}_2$ can be embedded into a 2-structure of $\mathbf{P}_2$. From this it easily follows that $\mathbf{RP}_2$ has AP. Let $\mathcal{F}$ be the Fraïssé limit of $\mathbf{RP}_2$. We call $\mathcal{F}$ the *random structure over* $\mathbf{P}_2$, or more generally a *binary random structure*. This is motivated by the remark below. But first we show that the well known "random graph" (or "Rado graph") is a binary random structure in this sense.

Example 2.8 (The random graph). Let $V = \{R\}$, where R is a binary relation symbol, and let $\mathbf{P}_2$ be the following class (in fact a set) of V-structures, where (A, B) denotes the $\{R\}$-structure with universe A and where R is intepreted as $B \subseteq A^2$:

$$\mathbf{P}_2 = \big\{(\emptyset, \emptyset), (\{1\}, \emptyset), (\{1, 2\}, \emptyset), (\{1, 2\}, \{(1, 2), (2, 1)\})\big\}.$$

If $\mathbf{RP}_2$ is as in the construction above, then $\mathbf{RP}_2$ is the class of all finite undirected graphs (without loops), which has HP and AP, and the Fraïssé limit of it is (in a model-theoretic context) often called the *random graph.*

Remark 2.9 (Random structures and zero-one laws). Let $\mathbf{P}_2$ and $\mathbf{RP}_2$ be as in the construction of a binary random structure above. Then $\mathbf{RP}_2$ is a *parametric class* in the sense of [8, Definition 4.2.1] or [26, Section 2]. Hence, by [8, Theorem 4.2.3], $\mathbf{RP}_2$ has a (labelled) 0-1 law (with the uniform probability measure). This is proved by showing that all extension axioms that are compatible with $\mathbf{RP}_2$ hold with probability approaching 1 as the (finite) cardinality of members of $\mathbf{RP}_2$ approaches infinity; see [8, statement (5), p. 76]. (Alternatively, one can use the terminology of [18] and show that $\mathbf{RP}_2$ "admits k-substitutions" for every positive integer k, and then apply [18, Theorem 3.15].) It follows that if $T_{\mathbf{RP}_2}$ is the set of all V-sentences φ with asymptotic probability 1 (in $\mathbf{RP}_2$), then all extension axioms that are compatible with $\mathbf{RP}_2$ belong to $T_{\mathbf{RP}_2}$. Let $\mathcal{F}$ be the Fraïssé limit of $\mathbf{RP}_2$. Then $\mathcal{F}$ satisfies every extension axiom which is compatible with $\mathbf{RP}_2$ (since if $\mathcal{A} \subseteq \mathcal{F}$ and $\mathcal{A} \subseteq \mathcal{B} \in \mathbf{RP}_2$, then there is an embedding of $\mathcal{B}$ into $\mathcal{F}$ which is the identity on $\mathcal{A}$). By a standard back-and-forth argument, it follows that if $\mathcal{M}$ is a countable model of $T_{\mathbf{RP}_2}$, then $\mathcal{M} \cong \mathcal{F}$ and hence $\mathcal{F} \models T_{\mathbf{RP}_2}$.

The construction of a binary random structure can of course be generalized to any finite relational (not necessarity binary) vocabulary.

2.4. Simple ω-categorical structures, imaginary elements and rank. We will work with concepts from stability/simplicty theory, including imaginary elements. That is, we work in the structure $\mathcal{M}^{\mathrm{eq}}$ obtained from a structure $\mathcal{M}$ by adding "imaginary" elements, in the way explained in [14, 29], for example. In the case of ω-categorical simple theories, some notions and results become easier than in the general case. For example, every ω-categorical simple theory has elimination of hyperimaginaries, so we need not consider "hyperimaginary elements" or the "bounded closure"; it suffices to consider imaginary elements and algebraic closure, so we need not go beyond $\mathcal{M}^{\mathrm{eq}}$. The results about ω-categorical simple structures that will be used, often without explicit reference, are stated below, with proofs or at least indications of how they follow from well known results in stability/simplicity theory or model theory in general.

Let $\mathcal{M}$ be a V-structure. Although we assume familiarity with $\mathcal{M}^{\mathrm{eq}}$, the universe of which is denoted M^{eq}, we recall part of its construction (as in [14, 29] for instance), since the distinction between different "sorts" of elements of $\mathcal{M}^{\mathrm{eq}}$ matters in the present work. For every $0 < n < \omega$ and every equivalence relation E on M^n which is $\emptyset$-definable in $\mathcal{M}$, V^{eq} (the vocabulary of $\mathcal{M}^{\mathrm{eq}}$) contains a unary relation symbol P_E and an $(n+1)$-ary relation symbol F_E (which do not belong to V). Here P_E is interpreted as the set of E-equivalence classes and, for all $\bar{a} \in (M^{\mathrm{eq}})^n$ and each $c \in M^{\mathrm{eq}}$, $\mathcal{M}^{\mathrm{eq}} \models F_E(\bar{a}, c)$ if and only if $\bar{a} \in M^n$, c is an E-equivalence class and $\bar{a}$ belongs to c. (So the interpretation of F_E is the graph of a function from M^n to the set of all E-equivalence classes.) *The notation $F(\bar{a}, c)$ means that $\mathcal{M}^{\mathrm{eq}} \models F_E(\bar{a}, c)$ for some n and some $\emptyset$-definable equivalence relation E on M^n.*

A *sort* of $\mathcal{M}^{\mathrm{eq}}$ is, by definition, a set of the form $S_E = \{a \in M^{\mathrm{eq}} : \mathcal{M}^{\mathrm{eq}} \models P_E(a)\}$ for some E as above. If $A \subseteq M^{\mathrm{eq}}$ and there are only finitely many E such that $A \cap S_E \neq \emptyset$, then we say that *only finitely many sorts are represented in* A. Note that '$=$', the identity relation, is an $\emptyset$-definable equivalence relation on M and every $=$-class is a singleton. Therefore M can (and will) be identified with the sort $S_=$, which we call the *real sort*. Hence every element of M^{eq} belongs to S_E for some E. If $\mathcal{N} \equiv \mathcal{M}^{\mathrm{eq}}$ then every $a \in N$ such that $\mathcal{N} \models P_=(a)$ is called a *real element* and every $a \in N$ such that $\mathcal{N} \models P_E$ for some E is called an *imaginary element* (so real elements are special cases of imaginary elements). However, the set

$$\{\neg P_E(x) : E \text{ is a } \emptyset\text{-definable equivalence relation on } M^n \text{ for some } n\}$$

is consistent with $\mathrm{Th}(\mathcal{M}^{\mathrm{eq}})$ (by compactness), so some model of $\mathrm{Th}(\mathcal{M}^{\mathrm{eq}})$ will contain elements which are neither real nor imaginary. This also shows that $\mathcal{M}^{\mathrm{eq}}$ is *not* ω-saturated even if $\mathcal{M}$ is (which is the case if $\mathcal{M}$ is ω-categorical). However, if $\mathcal{M}$ is ω-categorical and $A \subseteq M^{\mathrm{eq}}$ is finite, then every type $p \in S_n^{\mathcal{M}^{\mathrm{eq}}}(A)$ which is realized by an n-tuple of imaginary elements in some elementary extension of $\mathcal{M}^{\mathrm{eq}}$ is already realized in $\mathcal{M}^{\mathrm{eq}}$, as stated in Fact 2.14 below.

The first result below follows from [14, Theorem 4.3.3] or from [29, Lemma III.6.4].

Fact 2.10. *For all $\bar{a}, \bar{b} \in M$, $\mathrm{tp}_{\mathcal{M}}(\bar{a}) = \mathrm{tp}_{\mathcal{M}}(\bar{b})$ if and only if $\mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(\bar{a}) = \mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(\bar{b})$.*

Fact 2.11. *Suppose that $\mathcal{M}$ is ω-categorical, let $A \subseteq M^{\mathrm{eq}}$ and suppose that only finitely many sorts are represented in A.*

(i) *For every $n < \omega$ and finite $B \subseteq \mathcal{M}^{\mathrm{eq}}$, only finitely many types from $S_n^{\mathcal{M}^{\mathrm{eq}}}(\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(B))$ are realized by n-tuples in A^n.*
(ii) *For every finite $B \subseteq \mathcal{M}^{\mathrm{eq}}$, $\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(B) \cap A$ is finite.*

Proof. Let $B' \subseteq M$ be finite and such that $B \subseteq \mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(B')$. By ω-categoricity, there are, up to equivalence in $\mathcal{M}$, only finitely many formulas in free variables $x_1, \ldots, x_n$ with parameters from B', so part (i) is a consequence of [29, Lemma 6.4 of Chapter III] (or use [14, Theorem 4.3.3]). Part (ii) follows from (i). ∎

Definition 2.12. Suppose that $A \subseteq M^{\mathrm{eq}}$ is finite. We say that a structure $\mathcal{N}$ is *canonically embedded in $\mathcal{M}^{\mathrm{eq}}$ over A* if N is an A-definable subset of M^{eq}, and for every $0 < n < \omega$, and every relation $R \subseteq N^n$ which is A-definable in $\mathcal{M}^{\mathrm{eq}}$, there is a relation symbol in the vocabulary of $\mathcal{N}$ which is interpreted as R, and the vocabulary of $\mathcal{N}$ contains no other relation symbols (and no constant or function symbols).

The following is immediate from the definition:

FACT 2.13. *If $A \subseteq M^{\mathrm{eq}}$ is finite and $\mathcal{N}$ is canonically embedded in $\mathcal{M}^{\mathrm{eq}}$ over A, then for all $\bar{a}, \bar{b} \in N$ and all $C \subseteq N$, $\mathrm{acl}_{\mathcal{N}}(C) = \mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(CA) \cap N$ and $\mathrm{tp}_{\mathcal{N}}(\bar{a}/C) = \mathrm{tp}_{\mathcal{N}}(\bar{b}/C)$ if and only if $\mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(\bar{a}/CA) = \mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(\bar{b}/CA)$.*

FACT 2.14. *Suppose that $\mathcal{M}$ is ω-categorical.*

(i) *If $\mathcal{N}$ is canonically embedded in $\mathcal{M}^{\mathrm{eq}}$ over a finite $A \subseteq M^{\mathrm{eq}}$ and only finitely many sorts are represented in N, then $\mathcal{N}$ is ω-categorical and therefore ω-saturated.*
(ii) *If $A \subseteq M^{\mathrm{eq}}$ is finite and $\bar{a} \in M^{\mathrm{eq}}$, then $\mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(\bar{a}/\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(A))$ is isolated.*
(iii) *If $A \subseteq \mathcal{M}^{\mathrm{eq}}$ is finite, $n < \omega$ and $p \in S_n^{\mathcal{M}^{\mathrm{eq}}}(\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(A))$ is realized in some elementary extension of $\mathcal{M}^{\mathrm{eq}}$ by an n-tuple of imaginary elements, then p is realized in $\mathcal{M}^{\mathrm{eq}}$.*
(iv) *If $\mathcal{M}$ is countable, then $\mathcal{M}^{\mathrm{eq}}$ is ω-homogeneous.*

Proof. (i) If $\mathcal{M}$ is ω-categorical, then, by the characterization of its complete theory by Engeler, Ryll-Nardzewski and Svenonius (the characterization by isolated types), Fact 2.13 and, for example, [29, Lemma 6.4 of Chapter III] (or [7, Fact 1.1]), it follows that $\mathcal{N}$ is ω-categorical (and hence ω-saturated).

(ii) For ω-categorical $\mathcal{M}$, finite $A \subseteq M^{\mathrm{eq}}$ and $\bar{a} \in M^{\mathrm{eq}}$, it follows from Fact 2.13 and part (i) that $\mathrm{tp}(\bar{a}/A)$ is isolated. From the assumption that $\mathrm{tp}(\bar{a}/\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(A))$ is not isolated it is straightforward to derive a contradiction to Fact 2.11.

Parts (iii) and (iv) follow from (ii). ∎

Besides the above stated consequences of ω-categoricity, the proofs in Sections 3 and 4 use the so called *independence theorem* for simple theories [2, 30]. Every ω-categorical simple theory has elimination of hyperimaginaries and, with respect to it, ‘Lascar strong types’ are equivalent with strong types ([2, Theorem 18.14], [30, Lemma 6.1.11]), from which it follows that any two finite tuples $\bar{a}, \bar{b} \in M^{\mathrm{eq}}$ have the same Lascar strong type over a finite set $A \subseteq M^{\mathrm{eq}}$ if and only if they have the same type over $\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(A)$. Therefore the independence theorem implies the following, which is the version of it that we will use:

FACT 2.15 (The independence theorem for simple ω-categorical structures and finite sets). *Let $\mathcal{M}$ be a simple and ω-categorical structure and let $A, B, C \subseteq \mathcal{M}^{\mathrm{eq}}$ be finite. Suppose that $B \underset{A}{\downarrow} C$, $n < \omega$, $\bar{b}, \bar{c} \in (M^{\mathrm{eq}})^n$, $\bar{b} \underset{A}{\downarrow} B$, $\bar{c} \underset{A}{\downarrow} C$ and*

$$\mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(\bar{b}/\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(A)) = \mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(\bar{c}/\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(A)).$$

Then there is $\bar{d} \in (M^{\mathrm{eq}})^n$ *such that*

$$\mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(\bar{d}/B \cup \mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(A)) = \mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(\bar{b}/B \cup \mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(A)),$$
$$\mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(\bar{d}/C \cup \mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(A)) = \mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(\bar{c}/C \cup \mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(A))$$

and $\bar{d}$ *is independent of* $B \cup C$ *over* A.

By induction one easily gets the following, which is sometimes more practical:

Corollary 2.16. *Let* $\mathcal{M}$ *be a simple and* ω*-categorical structure,* $2 \leq k < \omega$ *and let* $A, B_1, \ldots, B_k \subseteq \mathcal{M}^{\mathrm{eq}}$ *be finite. Suppose that* $\{B_1, \ldots, B_k\}$ *is independent over* A*,* $n < \omega$*,* $\bar{b}_1, \ldots, \bar{b}_k \in (M^{\mathrm{eq}})^n$ *and, for all* $i, j \in \{1, \ldots, k\}$*,* $\bar{b}_i \mathop{\smile\!\!\!\!\!\!|}_A B_i$ *and*

$$\mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(\bar{b}_i/\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(A)) = \mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(\bar{b}_j/\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(A)).$$

Then there is $\bar{b} \in (M^{\mathrm{eq}})^n$ *such that, for all* $i = 1, \ldots, k$*,*

$$\mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(\bar{b}/B_i \cup \mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(A)) = \mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(\bar{b}_i/B_i \cup \mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(A))$$

and $\bar{b}$ *is independent of* $B_1 \cup \cdots \cup B_k$ *over* A.

Suppose that T is a simple theory. For every type p (possibly over a set of parameters) with respect to T, there is a notion of SU-*rank* of p, denoted $\mathrm{SU}(p)$; it is defined in [2, 30] for instance. We abbreviate $\mathrm{SU}(\mathrm{tp}_{\mathcal{M}}(\bar{a}/A))$ to $\mathrm{SU}(\bar{a}/A)$. For any type p, $\mathrm{SU}(p)$ is either ordinal valued or undefined (or alternatively given the value ∞).

3. Sets of rank one in simple homogeneous structures. In this section we derive consequences for sets with rank one in simple homogeneous structures with the n-dimensional amalgamation property for strong types (defined below), where n is the maximal arity of the vocabulary. A consequence of the independence theorem is that all simple structures have the 2-dimensional amalgamation property for strong types. We will use the notation $\mathcal{P}(S)$ for the powerset of the set S, and let $\mathcal{P}^-(S) = \mathcal{P}(S) \setminus \{S\}$. Every $n < \omega$ is identified with the set $\{0, \ldots, n-1\}$, and hence the notation $\mathcal{P}(n)$ makes sense. For a type p, $\mathrm{dom}(p)$ denotes the set of all parameters that occur in formulas in p. We now consider the 'strong n-dimensional amalgamation property for Lascar strong types', studied by Kolesnikov in [17] (Definition 4.5). However, we only need it for real elements, and in the present context 'Lascar strong type' is the same as 'type over an algebraically closed set'.

Definition 3.1. Let T be an ω-categorical and simple theory, and let $n < \omega$.

(i) A set of types $\{p_s(\bar{x}) : s \in \mathcal{P}^-(n)\}$ (with respect to $\mathcal{M}^{\mathrm{eq}}$ for some $\mathcal{M} \models T$) is called an *n-independent system of strong types over A*

(where $A \subseteq \mathcal{M}^{\mathrm{eq}}$) if it satisfies the following properties:

- $\mathrm{dom}(p_\emptyset) = A$;
- p_t is a nondividing extension of p_s for all $s, t \in \mathcal{P}^-(n)$ such that $s \subseteq t$;
- $\mathrm{dom}(p_s) \underset{\mathrm{dom}(p_{s\cap t})}{\perp} \mathrm{dom}(p_t)$ for all $s, t \in \mathcal{P}^-(n)$;
- p_s and p_t extend the same type over $\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(\mathrm{dom}(p_{s\cap t}))$ for all $s, t \in \mathcal{P}^-(n)$.

(ii) We say that T (and any $\mathcal{N} \models T$) has the *n-dimensional amalgamation property for strong types* if for every $\mathcal{M} \models T$ and every n-independent system of strong types $\{p_s(\bar{x}) : s \in \mathcal{P}^-(n)\}$ over some set $A \subseteq M^{\mathrm{eq}}$, there is a type p^* such that p^* extends p_s for each $s \in \mathcal{P}^-(n)$ and p^* does not divide over $\bigcup_{s\in\mathcal{P}^-(n)} \mathrm{dom}(p_s)$.

REMARK 3.2. By the independence theorem (in the general case when the sets of parameters of the given types may be infinite [2, 30]), every ω-categorical and simple theory has the 2-dimensional amalgamation property for strong types.

THEOREM 3.3. *Suppose that $\mathcal{M}$ has a finite relational vocabulary with maximal arity ρ. Also assume that $\mathcal{M}$ is countable, homogeneous and simple, and has the ρ-dimensional amalgamation property for strong types. Let $D, E \subseteq M$ where E is finite, D is E-definable and* $\mathrm{SU}(a/E) = 1$ *for every $a \in D$. If $a \in D$, $B \subseteq D$ and $a \in \mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(BE)$, then $a \in \mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(B'E)$ for some $B' \subseteq B$ with $|B'| < \rho$.*

Proof. Assume that $a \in D$, $B \subseteq D$ and $a \in \mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(BE)$. Without loss of generality we may assume that B is finite. By induction on $|B|$ we prove that there is $B' \subseteq B$ such that $|B'| < \rho$ and $a \in \mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(B'E)$. The base case is when $|B| < \rho$ and we are automatically done.

So suppose that $|B| \geq \rho$. If B is not independent over E then there is $b \in B$ such that $b \underset{E}{\not\perp} (B \setminus \{b\})$, and as $\mathrm{SU}(b/E) = 1$ (by assumption) we get $b \in \mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}((B \setminus \{b\}) \cup E)$. Hence $B \subseteq \mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(B'E)$ where $B' = B \setminus \{b\}$ is a proper subset of B, so by the induction hypothesis we are done.

Therefore suppose now, in addition, that B is independent over E. If $a \in \mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(B'E)$ for some proper subset $B' \subset B$, then we are done by the induction hypothesis. So assume moreover that $a \notin \mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(B'E)$ for every proper subset $B' \subset B$.

Let $n = |B|$, so $n \geq \rho$, and enumerate B as $B = \{b_0, \ldots, b_{n-1}\}$. For each $S \in \mathcal{P}^-(\rho)$, let

$$B_S = \mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(\{b_t : t \in S\} \cup \{b_\rho, \ldots, b_{n-1}\} \cup E).$$

From the assumptions that B is independent over E and $a \notin \mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(B'E)$ for every proper subset $B' \subset B$ it follows that the types $\mathrm{tp}(a/B_S)$ form a ρ-independent system of strong types over $\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(E \cup \{b_\rho, \ldots, b_{n-1}\})$. As $\mathrm{Th}(\mathcal{M})$ has the ρ-dimensional amalgamation property for strong types (and by Fact 2.14), there is $a' \in D$ such that for every $i \in \{0, \ldots, \rho-1\}$ and $S_i = \{0, \ldots, \rho-1\} \setminus \{i\}$ we have

$$\mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(a'/B_{S_i}) = \mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(a/B_{S_i}) \quad \text{and} \quad a' \notin \mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(BE). \tag{3.1}$$

Claim. *The bijection $f : \mathcal{M}{\restriction}BEa' \to \mathcal{M}{\restriction}BEa$ defined by $f(b) = b$ for all $b \in BE$ and $f(a') = a$ is an isomorphism.*

Proof of the Claim. Let R be a relation symbol of the vocabulary of $\mathcal{M}$, so the arity of R is at most ρ. It suffices to show that if $\bar{a} \in BEa'$ then $\mathcal{M} \models R(\bar{a})$ if and only if $\mathcal{M} \models R(f(\bar{a}))$. But this is immediate from (3.1) and the definition of f. ∎

Since $\mathcal{M}$ is homogeneous and B and E are finite, there is an automorphism g of $\mathcal{M}$ which extends f from the Claim. Then $g(a') = a$ and g fixes BE pointwise. But since $a \in \mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(BE)$ and, by (3.1), $a' \notin \mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(BE)$, this contradicts g being an automorphism. ∎

By using the previous theorem and Remark 3.2 we get the following:

Corollary 3.4. *Suppose that $\mathcal{M}$ is a countable, binary, homogeneous and simple structure. Let $D, E \subseteq M$ where E is finite, D is E-definable and* $\mathrm{SU}(a/E) = 1$ *for every $a \in D$. If $a \in D$, $B \subseteq D$ and $a \in \mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(BE)$, then $a \in \mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(\{b\} \cup E)$ for some $b \in B$.*

Definition 3.5. Let T be a simple theory.

(i) Suppose that $\mathcal{M} \models T$ and $E \subseteq M$. We say that $D \subseteq M^{\mathrm{eq}}$ has *n-degenerate dependence over E* if for all $A, B, C \subseteq D$ such that $A \underset{CE}{\not\perp} B$ there is $B_0 \subseteq B$ such that $|B_0| \leq n$ and $A \underset{CE}{\not\perp} B_0$.

(ii) We say that T has *trivial dependence* if whenever $\mathcal{M} \models T$, $A, B, C_1, C_2 \subseteq M^{\mathrm{eq}}$ and $A \underset{B}{\not\perp} C_1C_2$, then $A \underset{B}{\not\perp} C_i$ for $i = 1$ or $i = 2$. A simple structure $\mathcal{M}$ has *trivial dependence* if its complete theory $\mathrm{Th}(\mathcal{M})$ does.

Theorem 3.6. *Suppose that $\mathcal{M}$ has a finite relational vocabulary with maximal arity ρ. Also assume that $\mathcal{M}$ is countable, homogeneous and simple, and has the ρ-dimensional amalgamation property for strong types. Let $D, E \subseteq M$ where E is finite, D is E-definable and* $\mathrm{SU}(a/E) = 1$ *for every $a \in D$. Then D has $(\rho - 1)$-degenerate dependence over E.*

Proof. This is essentially an application of Theorem 3.3, basic properties of SU-rank and the Lascar (in)equalities (see for example [30, Section 5.1, in particular Theorem 5.1.6]).

Suppose that $B, C \subseteq D$, $\bar{a} = (a_1, \ldots, a_n) \in D^n$ and

$$\bar{a} \underset{CE}{\not\perp} B. \tag{3.2}$$

If $\bar{a}$ is not independent over CE, then (since $\mathrm{SU}(d/E) = 1$ for all $d \in D$) there is a proper subsequence $\bar{a}'$ of $\bar{a}$ such that $\mathrm{rng}(\bar{a}) \subseteq \mathrm{acl}(\bar{a}'CE)$ and hence $\bar{a}' \underset{CE}{\not\perp} B$. If, in addition, $B' \subsetneq B$ and $\bar{a}' \underset{CE}{\not\perp} B'$, then $\bar{a} \underset{CE}{\not\perp} B'$. Therefore we may assume that

$$\bar{a} \text{ is independent over } CE. \tag{3.3}$$

Moreover, we may assume that

$$\mathrm{SU}(a_i/CE) = 1 \quad \text{for every } i. \tag{3.4}$$

For otherwise, $a_i \in \mathrm{acl}(CE)$ for some i, which implies that $\bar{a}$ is not independent over CE, contradicting (3.3).

Now (3.3), (3.4) and the Lascar equalities (for finite ranks) give

$$\mathrm{SU}(\bar{a}/CE) = n. \tag{3.5}$$

Then (3.2) and (3.5) (together with [30, Lemma 5.1.4] for example) give

$$\mathrm{SU}(\bar{a}/BCE) < n,$$

so $\bar{a}$ is not independent over BCE and hence there is i such that

$$a_i \underset{BCE}{\not\perp} (\{a_1, \ldots, a_n\} \setminus \{a_i\}),$$

whence (by monotonicity of dependence)

$$a_i \underset{CE}{\not\perp} B \cup (\{a_1, \ldots, a_n\} \setminus \{a_i\}),$$

and by (3.4),

$$a_i \in \mathrm{acl}(BCE \cup (\{a_1, \ldots, a_n\} \setminus \{a_i\})).$$

By Theorem 3.3, there is $X \subseteq BC \cup (\{a_1, \ldots, a_n\} \setminus \{a_i\})$ such that $|X| < \rho$ and $a_i \in \mathrm{acl}(XE)$. Then

$$a_i \in \mathrm{acl}((X \cap B) \cup CE \cup (\{a_1, \ldots, a_n\} \setminus \{a_i\})).$$

Let $\bar{a}'$ be the proper subsequence of $\bar{a}$ in which a_i is removed. Then, by (3.5),

$$\mathrm{SU}(\bar{a}/(X \cap B) \cup CE) = \mathrm{SU}(\bar{a}'/(X \cap B) \cup CE) < n = \mathrm{SU}(\bar{a}/CE),$$

and hence $\bar{a} \underset{CE}{\not\perp} (X \cap B)$ where $|X \cap B| < \rho$. ∎

REMARK 3.7. Suppose that $\mathcal{M}$ is homogeneous and simple and that $E \subseteq M$ is finite. Let $\mathcal{M}_E$ be the expansion of $\mathcal{M}$ with a unary relation symbol P_e for each $e \in E$, and interpret P_e as $\{e\}$. It is straightforward to verify that any isomorphism between finite substructures of $\mathcal{M}_E$ can be extended to an automorphism of $\mathcal{M}_E$, so it is homogeneous. Moreover, since the notion of simplicity only depends on which relations are definable with parameters, and exactly the same relations are definable with parameters in $\mathcal{M}_E$ as in $\mathcal{M}$, it follows that $\mathcal{M}_E$ is simple (see for example [2, Remark 2.26]). For the same reason, if $\mathcal{M}$ has trivial dependence, then so does $\mathcal{M}_E$.

Corollary 3.8. *Suppose that $\mathcal{M}$ is countable, homogeneous, simple with a finite relational vocabulary with maximal arity ρ, and with the ρ-dimensional amalgamation property for strong types. Let $D \subseteq M^{\mathrm{eq}}$ be E-definable for finite $E \subseteq M$, suppose that only finitely many sorts are represented in D and that $\mathrm{SU}(d/E) = 1$ for all $d \in D$. Moreover, suppose that if $n < \omega$, $a_1, \ldots, a_n, b_1, \ldots, b_n \in D$ and*

$$\mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(a_{i_1}, \ldots, a_{i_\rho}/E) = \mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(b_{i_1}, \ldots, b_{i_\rho}/E) \quad \textit{for all } i_1, \ldots, i_\rho \in \{1, \ldots, n\},$$

then $\mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(a_1, \ldots, a_n/E) = \mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}(b_1, \ldots, b_n/E)$. Then D has $(\rho-1)$-degenerate dependence over E.

Proof. Let $\mathcal{M}_E$ be the expansion of $\mathcal{M}$ by a unary relation symbol P_e for every $e \in E$ where P_e is interpreted as $\{e\}$. By Remark 3.7, $\mathcal{M}_E$ is homogeneous and simple. Moreover, D is $\emptyset$-definable in $(\mathcal{M}_E)^{\mathrm{eq}}$, so it is the universe of a canonically embedded structure $\mathcal{D}$ in $(\mathcal{M}_E)^{\mathrm{eq}}$ over $\emptyset$. By 2.14, $\mathcal{D}$ is ω-categorical and hence ω-homogeneous. As $\mathcal{M}$, and thus $\mathcal{D}$, is countable it follows that if $0 < n < \omega$, $a_1, \ldots, a_n, b_1, \ldots, b_n \in D$ and $\mathrm{tp}_{\mathcal{D}}(a_1, \ldots, a_n) = \mathrm{tp}_{\mathcal{D}}(b_1, \ldots, b_n)$, then there is an automorphism f of $\mathcal{D}$ such that $f(a_i) = b_i$ for all i. By assumption and Fact 2.10, if $n < \omega$, $a_1, \ldots, a_n, b_1, \ldots, b_n \in D$ and $\mathrm{tp}_{\mathcal{D}}(a_{i_1}, \ldots, a_{i_\rho}) = \mathrm{tp}_{\mathcal{D}}(b_{i_1}, \ldots, b_{i_\rho})$ for all $i_1, \ldots, i_\rho \in \{1, \ldots, n\}$, then

$$\mathrm{tp}_{\mathcal{D}}(a_1, \ldots, a_n) = \mathrm{tp}_{\mathcal{D}}(b_1, \ldots, b_n).$$

Hence the reduct $\mathcal{D}_0$ of $\mathcal{D}$ to the relation symbols of arity at most ρ is homogeneous (and $\mathcal{D}$ is a reduct of $\mathcal{D}_0$). Clearly, D is an $\emptyset$-definable subset in $\mathcal{D}$ and by Fact 2.13 we have $\mathrm{SU}(d/\emptyset) = 1$ for all $d \in D$ when 'SU' is computed in $\mathcal{D}$, as well as in $\mathcal{D}_0$ (since $\mathcal{D}$ is a reduct of $\mathcal{D}_0$). Hence, Theorem 3.6 applied to $\mathcal{M} = \mathcal{D}_0$ implies that D has $(\rho-1)$-degenerate dependence when we consider D as a $\emptyset$-definable set within $\mathcal{D}_0$, and hence when D is considered as a $\emptyset$-definable set within $\mathcal{D}$. From Fact 2.13 it follows that D has $(\rho-1)$-degenerate dependence when we consider D as a $\emptyset$-definable set within $(\mathcal{M}_E)^{\mathrm{eq}}$. It follows that D has $(\rho-1)$-definable dependence over E when we consider D as an E-definable set within $\mathcal{M}^{\mathrm{eq}}$. ∎

Remark 3.9. As mentioned earlier, every ω-categorical simple theory has the 2-dimensional amalgamation property for strong types. So if $\mathcal{M}$ in Corollary 3.8 is binary, that is, if $\rho \leq 2$, then the assumption that $\mathcal{M}$ has the ρ-dimensional amalgamation property can be removed and the conclusion still holds.

4. Technical implications of trivial dependence in binary homogeneous structures. In this section we define the notion of 'acl-complete set' and prove Theorem 4.6, and its corollary, which shows, roughly speaking, that on any $\emptyset$-definable acl-complete subset of $\mathcal{M}^{\mathrm{eq}}$ with rank 1 where $\mathcal{M}$ is binary, homogeneous and simple with trivial dependence, the "type structure" is determined by the 2-types.

ASSUMPTION 4.1. Throughout this section, including Theorem 4.6 and Corollary 4.7, we assume that

(i) $\mathcal{M}$ is countable, binary, homogeneous, simple, with trivial dependence, and
(ii) $D \subseteq \mathcal{M}^{\mathrm{eq}}$ is $\emptyset$-definable, only finitely many sorts are represented in D and $\mathrm{SU}(d) = 1$ for every $d \in D$.

NOTATION 4.2. In the rest of the article, '$\mathrm{tp}_{\mathcal{M}^{\mathrm{eq}}}$', '$\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}$' and '$\mathrm{dcl}_{\mathcal{M}^{\mathrm{eq}}}$' are abbreviated to 'tp', 'acl' and 'dcl', respectively. (But when types etc. are used with respect to other structures, we indicate this with a subscript.)

Recall the notation '$F(\bar{a}, b)$' explained at the beginning of Section 2.4.

LEMMA 4.3. *Suppose that $a_1, \ldots, a_n, b_1, \ldots, b_n \in M^{\mathrm{eq}}$. Then the following are equivalent:*

(1) $\mathrm{tp}(a_1, \ldots, a_n) = \mathrm{tp}(b_1, \ldots, b_n)$.
(2) *There are finite sequences $\bar{a}_1, \ldots, \bar{a}_n, \bar{b}_1, \ldots, \bar{b}_n \in M$ and an isomorphism $f : \mathcal{M}{\restriction}\bar{a}_1 \ldots \bar{a}_n \to \mathcal{M}{\restriction}\bar{b}_1 \ldots \bar{b}_n$ such that $F(\bar{a}_i, a_i)$, $F(\bar{b}_i, b_i)$ and $f(\bar{a}_i) = \bar{b}_i$ for all $i = 1, \ldots, n$.*

Proof. If $\mathrm{tp}(a_1, \ldots, a_n) = \mathrm{tp}(b_1, \ldots, b_n)$, then since $\mathcal{M}^{\mathrm{eq}}$ is ω-homogeneous and countable, there is an automorphism f of $\mathcal{M}^{\mathrm{eq}}$ such that $f(a_i) = b_i$ for all i. Let $\bar{a}_i \in M$ be such that $F(\bar{a}_i, a_i)$ for each i, and let $\bar{b}_i = f(\bar{a}_i)$. Then the restriction of f to $\mathrm{rng}(\bar{a}_1) \cup \cdots \cup \mathrm{rng}(\bar{a}_n)$ is an isomorphism from $\mathcal{M}{\restriction}\bar{a}_1 \ldots \bar{a}_n$ to $\mathcal{M}{\restriction}\bar{b}_1 \ldots \bar{b}_n$.

For the other direction, note that $F(\bar{a}_i, a_i)$ implies that $a_i \in \mathrm{dcl}(\bar{a}_i)$, and similarly for $\bar{b}_i$ and b_i. So if (2) holds then, as $\mathcal{M}$ is homogeneous, $\mathrm{tp}(\bar{a}_1, \ldots, \bar{a}_n) = \mathrm{tp}(\bar{b}_1, \ldots, \bar{b}_n)$, and hence $\mathrm{tp}(a_1, \ldots, a_n) = \mathrm{tp}(b_1, \ldots, b_n)$. ■

DEFINITION 4.4. We call D *acl-complete* if for all $a \in D$ and $\bar{a}, \bar{a}' \in M$, if $F(\bar{a}, a)$ and $F(\bar{a}', a)$, then $\mathrm{tp}(\bar{a}/\mathrm{acl}(a)) = \mathrm{tp}(\bar{a}'/\mathrm{acl}(a))$.

LEMMA 4.5. *There is $D' \subseteq M^{\mathrm{eq}}$ such that D' satisfies Assumption* 4.1(ii), *D' is acl-complete and*

(1) *for every $d \in D$ there is $d' \in D'$ (not necessarily unique) such that $d \in \mathrm{dcl}(d')$ and $d' \in \mathrm{acl}(d)$, and*
(2) *for every $d' \in D'$ there is $d \in D$ such that $d \in \mathrm{dcl}(d')$ and $d' \in \mathrm{acl}(d)$.*

Proof. Let p be any one of the finitely many complete 1-types over $\emptyset$ which are realized in D, and let the equivalence relation E_p on M^n (for some n) define the sort of the elements which realize p. By Fact 2.14, the following equivalence relation on M^n is $\emptyset$-definable:

$$E'_p(\bar{x},\bar{y}) \Leftrightarrow \exists z\big(p(z) \wedge F_{E_p}(\bar{x},z) \wedge F_{E_p}(\bar{y},z) \wedge \mathrm{tp}(\bar{x}/\mathrm{acl}(z)) = \mathrm{tp}(\bar{y}/\mathrm{acl}(z))\big).$$

Moreover, by the same fact, every E_p-class is a union of finitely many E'_p-classes. By replacing, for every complete 1-type p over $\emptyset$ that is realized in D, the elements realizing p with the elements of $\mathcal{M}^{\mathrm{eq}}$ which correspond to E'_p-classes, we get D'. This set has the properties stated in the lemma. ∎

Recall Assumption 4.1 and Notation 4.2. The following theorem is the main result of this section.

THEOREM 4.6. *Suppose that D is acl-complete, $1 < n < \omega$, $a_i, b_i \in D$ for $i = 1,\dots,n$, $\{a_1,\dots,a_n\}$ is independent over $\emptyset$, $\{b_1,\dots,b_n\}$ is independent over $\emptyset$ and $\mathrm{tp}(a_i,a_j) = \mathrm{tp}(b_i,b_j)$ for all $i,j = 1,\dots,n$. Then $\mathrm{tp}(a_1,\dots,a_n) = \mathrm{tp}(b_1,\dots,b_n)$.*

COROLLARY 4.7. *Suppose that D is acl-complete. Let $n < \omega$, $\bar{a}_i, \bar{b}_i \in D$, $i = 1,\dots,n$, and suppose that $\mathrm{SU}(\bar{a}_i) = \mathrm{SU}(\bar{b}_i) = 1$, $\mathrm{acl}(\bar{a}_i) \cap D = \mathrm{rng}(\bar{a}_i)$ and $\mathrm{acl}(\bar{b}_i) \cap D = \mathrm{rng}(\bar{b}_i)$ for all i. Furthermore, asssume that $\{\bar{a}_1,\dots,\bar{a}_n\}$ is independent over $\emptyset$, $\{\bar{b}_1,\dots,\bar{b}_n\}$ is independent over $\emptyset$ and $\mathrm{tp}(\bar{a}_i,\bar{a}_j) = \mathrm{tp}(\bar{b}_i,\bar{b}_j)$ for all i and j. Then there is a permutation $\bar{b}'_i$ of $\bar{b}_i$, for each i, such that $\mathrm{tp}(\bar{a}_1,\dots,\bar{a}_n) = \mathrm{tp}(\bar{b}'_1,\dots,\bar{b}'_n)$.*

Proof. Suppose that $\bar{a}_i = (a_{i,1},\dots,a_{i,k_i})$, $\bar{b}_i = (b_{i,1},\dots,b_{i,k_i})$, $i = 1,\dots,n$, satisfy the assumptions of the theorem. In particular we have $\mathrm{tp}(a_{i,1},a_{j,1}) = \mathrm{tp}(b_{i,1},b_{j,1})$ for all i and j, and both $\{a_{1,1},\dots,a_{n,1}\}$ and $\{b_{1,1},\dots,b_{n,1}\}$ are independent over $\emptyset$. By Theorem 4.6,

$$\mathrm{tp}(a_{1,1},\dots,a_{n,1}) = \mathrm{tp}(b_{1,1},\dots,b_{n,1}).$$

By ω-homogeneity of $\mathcal{M}^{\mathrm{eq}}$ (and Fact 2.6) there is an automorphism f of $\mathcal{M}^{\mathrm{eq}}$ such that $f(a_{i,1}) = b_{i,1}$ for all $i = 1,\dots,n$. From $\mathrm{SU}(\bar{a}_i) = 1$ it follows that $\bar{a}_i \in \mathrm{acl}(a_{i,1})$ for every i, and for the same reason $\bar{b}_i \in \mathrm{acl}(b_{i,1})$ for every i. Hence $f(\mathrm{rng}(\bar{a}_i)) = \mathrm{rng}(\bar{b}_i)$ for all i, and consequently there is a permutation $\bar{b}'_i$ of $\bar{b}_i$ for each i such that $\mathrm{tp}(\bar{a}_1,\dots,\bar{a}_n) = \mathrm{tp}(\bar{b}'_1,\dots,\bar{b}'_n)$. ∎

4.1. Proof of Theorem 4.6. Let $D \subseteq M^{\mathrm{eq}}$ and $a_i, b_i \in D$, $i = 1,\dots,n$, satisfy the assumptions of the theorem. We prove that $\mathrm{tp}(a_1,\dots,a_n) = \mathrm{tp}(b_1,\dots,b_n)$ by induction on $n = 2,3,4,\dots$. The case $n = 2$ is trivial, so we assume that $n > 2$ and, by the induction hypothesis, that

$$\mathrm{tp}(a_1,\dots a_{n-1}) = \mathrm{tp}(b_1,\dots,b_{n-1}). \tag{4.1}$$

Suppose that we can find $\bar{a}_i, \bar{b}_i \in M$, $i = 1, \ldots, n$, such that $F(\bar{a}_i, a_i)$, $F(\bar{b}_i, b_i)$ for all i and

$$\mathrm{tp}(\bar{a}_1, \ldots, \bar{a}_n) = \mathrm{tp}(\bar{b}_1, \ldots, \bar{b}_n).$$

Then Lemma 4.3 implies that

$$\mathrm{tp}(a_1, \ldots, a_n) = \mathrm{tp}(b_1, \ldots, b_n),$$

which is what we want to prove. Our aim is to find $\bar{a}_1, \ldots, \bar{a}_n, \bar{b}_1, \ldots, \bar{b}_n$ as above.

We now prove three technical lemmas. Then a short argument which combines these lemmas proves the theorem.

LEMMA 4.8. *There are $\bar{a}_i \in M$ for $i = 1, \ldots, n$ such that $F(\bar{a}_i, a_i)$ for every i and $\{\bar{a}_1, \ldots, \bar{a}_n\}$ is independent over $\emptyset$.*

Proof. By induction we prove that for each $k = 1, \ldots, n$ there are $\bar{a}'_1, \ldots, \bar{a}'_k \in M$ such that $F(\bar{a}'_i, a_i)$ for every i, and $\{\bar{a}'_1, \ldots, \bar{a}'_k\}$ is independent over $\emptyset$. The case $k = 1$ is trivial, so we assume that $0 < k < n$ and that we have found $\bar{a}_1, \ldots, \bar{a}_k \in M$ such that $F(\bar{a}_i, a_i)$ for every i and $\{\bar{a}_1, \ldots, \bar{a}_k\}$ is independent over $\emptyset$.

Choose any $a^*_{k+1} \in D$ such that $\mathrm{tp}(a^*_{k+1}/a_1, \ldots, a_k, \bar{a}_1, \ldots, \bar{a}_k)$ is a nondividing extension of $\mathrm{tp}(a_{k+1}/a_1, \ldots, a_k)$, so in particular $\mathrm{tp}(a^*_{k+1}, a_1, \ldots, a_k) = \mathrm{tp}(a_{k+1}, a_1, \ldots, a_k)$ and

$$a^*_{k+1} \mathop{\smile\!\!\!\!\!\!|}\limits_{a_1, \ldots, a_k} \bar{a}_1, \ldots, \bar{a}_k,$$

and since (by assumption) $\{a_1, \ldots, a_{k+1}\}$ is independent over $\emptyset$, we see that $a^*_{k+1} \mathop{\smile\!\!\!\!\!\!|}\, a_1, \ldots, a_k$. By transitivity of dividing,

$$a^*_{k+1} \mathop{\smile\!\!\!\!\!\!|}\, a_1, \ldots, a_k, \bar{a}_1, \ldots, \bar{a}_k,$$

so by monotonicity

$$a^*_{k+1} \mathop{\smile\!\!\!\!\!\!|}\, \bar{a}_1, \ldots, \bar{a}_k. \tag{4.2}$$

As $\mathrm{tp}(a^*_{k+1}, a_1, \ldots, a_k) = \mathrm{tp}(a_{k+1}, a_1, \ldots, a_k)$ and $\mathcal{M}^{\mathrm{eq}}$ is ω-homogeneous (and countable) there is an automorphism f of $\mathcal{M}^{\mathrm{eq}}$ which maps $(a^*_{k+1}, a_1, \ldots, a_k)$ to $(a_{k+1}, a_1, \ldots, a_k)$. Let $f(\bar{a}_i) = \bar{a}'_i$ for $i = 1, \ldots, k$. Then $F(\bar{a}'_i, a_i)$ for $i = 1, \ldots, k$ and

$$\mathrm{tp}(a^*_{k+1}, a_1, \ldots, a_k, \bar{a}_1, \ldots, \bar{a}_k) = \mathrm{tp}(a_{k+1}, a_1, \ldots, a_k, \bar{a}'_1, \ldots, \bar{a}'_k),$$

so in view of (4.2),

$$a_{k+1} \mathop{\smile\!\!\!\!\!\!|}\, \bar{a}'_1, \ldots, \bar{a}'_k, \tag{4.3}$$

and as $\{\bar{a}_1, \ldots, \bar{a}_k\}$ is independent over $\emptyset$ (by induction hypothesis),

$$\{\bar{a}'_1, \ldots, \bar{a}'_k\} \text{ is independent over } \emptyset. \tag{4.4}$$

Choose any $\bar{a}_{k+1} \in M$ such that $F(\bar{a}_{k+1}, a_{k+1})$. There exist $\bar{a}^*_1, \ldots, \bar{a}^*_k$ in M^{eq} such that $\mathrm{tp}(\bar{a}^*_1, \ldots, \bar{a}^*_k/a_{k+1}, \bar{a}_{k+1})$ is a nondividing extension of

$\mathrm{tp}(\bar{a}'_1, \ldots, \bar{a}'_k / a_{k+1})$. Then

$$\bar{a}^*_1, \ldots, \bar{a}^*_k \underset{a_{k+1}}{\downarrow} \bar{a}_{k+1}$$

and $\mathrm{tp}(\bar{a}^*_1, \ldots, \bar{a}^*_k, a_{k+1}) = \mathrm{tp}(\bar{a}'_1, \ldots, \bar{a}'_k, a_{k+1})$, so $\bar{a}^*_1, \ldots, \bar{a}^*_k \downarrow a_{k+1}$. By transitivity, $\bar{a}^*_1, \ldots, \bar{a}^*_k \downarrow a_{k+1}, \bar{a}_{k+1}$, and by monotonicity,

$$\bar{a}^*_1, \ldots, \bar{a}^*_k \downarrow \bar{a}_{k+1}. \tag{4.5}$$

By the ω-homogeneity of $\mathcal{M}^{\mathrm{eq}}$ there is an automorphism g of $\mathcal{M}^{\mathrm{eq}}$ that maps $(\bar{a}^*_1, \ldots, \bar{a}^*_k, a_{k+1})$ to $(\bar{a}'_1, \ldots, \bar{a}'_k, a_{k+1})$. Let $g(\bar{a}_{k+1}) = \bar{a}'_{k+1}$. Then

$$\mathrm{tp}(\bar{a}^*_1, \ldots, \bar{a}^*_k, \bar{a}_{k+1}, a_{k+1}) = \mathrm{tp}(\bar{a}'_1, \ldots, \bar{a}'_k, \bar{a}'_{k+1}, a_{k+1}),$$

so $F(\bar{a}'_{k+1}, a_{k+1})$ and, by (4.5),

$$\bar{a}'_1, \ldots, \bar{a}'_k \downarrow \bar{a}'_{k+1}.$$

From (4.4) it follows that $\{\bar{a}'_1, \ldots, \bar{a}'_{k+1}\}$ is independent over $\emptyset$. ∎

Lemma 4.9. *Let $I \subseteq \{1, \ldots, n-1\}$. Suppose that $\bar{c}_i \in M$ for $i = 1, \ldots, n-1$ and $\bar{d}_j \in M$ for $j \in I$ are such that $F(\bar{c}_i, b_i)$ for every $1 \le i \le n-1$ and $F(\bar{d}_j, b_n)$ for every $j \in I$. Then there are $\bar{c}'_i \in M$ for $i = 1, \ldots, n-1$ and $\bar{d}'_j \in M$ for $j \in I$ such that $F(\bar{c}'_i, b_i)$ for every $1 \le i \le n-1$, $F(\bar{d}'_j, b_n)$ for every $j \in I$, $\mathrm{tp}(\bar{c}'_1, \ldots, \bar{c}'_{n-1}) = \mathrm{tp}(\bar{c}_1, \ldots, \bar{c}_{n-1})$, $\mathrm{tp}(\bar{c}'_j, \bar{d}'_j) = \mathrm{tp}(\bar{c}_j, \bar{d}_j)$ for all $j \in I$ and $b_n \notin \mathrm{acl}(\bar{c}'_1, \ldots, \bar{c}'_{n-1})$.*

Proof. Suppose on the contrary that $b_n \in \mathrm{acl}(\bar{c}'_1, \ldots, \bar{c}'_{n-1})$ for all $\bar{c}'_1, \ldots, \bar{c}'_{n-1} \in M$ such that

$$\begin{aligned}
&F(\bar{c}'_i, b_i) \text{ for every } i = 1, \ldots, n-1,\\
&\mathrm{tp}(\bar{c}'_1, \ldots, \bar{c}'_{n-1}) = \mathrm{tp}(\bar{c}_1, \ldots, \bar{c}_{n-1}),\\
&\text{for every } i \in I \text{ there is } \bar{d}'_i \in M \text{ such that } F(\bar{d}'_i, b_n) \text{ and}\\
&\mathrm{tp}(\bar{c}'_i, \bar{d}'_i) = \mathrm{tp}(\bar{c}_i, \bar{d}_i).
\end{aligned} \tag{4.6}$$

Note that by the ω-categoricity of $\mathcal{M}$ the condition

"$b_n \in \mathrm{acl}(\bar{c}'_1, \ldots, \bar{c}'_{n-1})$ for all $\bar{c}'_1, \ldots, \bar{c}'_{n-1} \in M$ such that (4.6) holds"

can be expressed by a formula $\varphi(x_1, \ldots, x_n)$ such that $\mathcal{M}^{\mathrm{eq}} \models \varphi(b_1, \ldots, b_n)$. By assumption, $\{b_1, \ldots, b_n\}$ is independent over $\emptyset$, so $b_n \notin \mathrm{acl}(b_1, \ldots, b_{n-1})$ and hence there are distinct $b_{n,i}$, for all $i < \omega$, such that

$$\mathrm{tp}(b_1, \ldots, b_{n-1}, b_{n,i}) = \mathrm{tp}(b_1, \ldots, b_{n-1}, b_n) \quad \text{for all } i < \omega.$$

Then $\mathcal{M}^{\mathrm{eq}} \models \varphi(b_1, \ldots, b_{n-1}, b_{n,i})$ for all $i < \omega$. Since (4.6) is satisfied if we let $\bar{c}'_i = \bar{c}_i$ for $i = 1, \ldots, n-1$ and $\bar{d}'_i = \bar{d}_i$ for $i \in I$, it follows that $b_{n,i} \in \mathrm{acl}(\bar{c}_1, \ldots, \bar{c}_{n-1})$ for all $i < \omega$. This contradicts the ω-categoricity of $\mathcal{M}$ (via Fact 2.11) because $\mathrm{tp}(b_{n,i}) = \mathrm{tp}(b_{n,j})$ for all i and j. ∎

By Lemma 4.8, let $\bar{a}_i \in M$ for $i = 1, \ldots, n$ be such that $F(\bar{a}_i, a_i)$ for every i and

$$\{\bar{a}_1, \ldots, \bar{a}_n\} \text{ is independent over } \emptyset. \tag{4.7}$$

LEMMA 4.10. *Let I be a proper subset of $\{1, \ldots, n-1\}$. Suppose that $\bar{b}_i \in M$ for $i = 1, \ldots, n-1$ and $\bar{b}_{n,j} \in M$ for $j \in I$ are such that*

$$\begin{aligned} &F(\bar{b}_i, b_i) \text{ for all } i = 1, \ldots, n-1, F(\bar{b}_{n,j}, b_n) \text{ for all } j \in I, \\ &\mathrm{tp}(\bar{b}_1, \ldots, \bar{b}_{n-1}) = \mathrm{tp}(\bar{a}_1, \ldots, \bar{a}_{n-1}), \\ &\mathrm{tp}(\bar{b}_j, \bar{b}_{n,j}) = \mathrm{tp}(\bar{a}_j, \bar{a}_n) \quad \text{for all } j \in I, \\ &b_n \notin \mathrm{acl}(\bar{b}_1, \ldots, \bar{b}_{n-1}). \end{aligned} \tag{4.8}$$

Let $j \in \{1, \ldots, n-1\} \setminus I$ and $J = I \cup \{j\}$. Then there are $\bar{b}'_i \in M$ for $i = 1, \ldots, n-1$ and $\bar{b}'_{n,j} \in M$ for $j \in J$ such that (4.8) *holds if $\bar{b}$ is replaced with $\bar{b}'$ and I with J.*

Proof. Suppose that (4.8) holds. Note that the second line of it together with (4.7) implies that

$$\{\bar{b}_1, \ldots, \bar{b}_{n-1}\} \text{ is independent over } \emptyset. \tag{4.9}$$

Without loss of generality we assume that $I = \{1, \ldots, k\}$ where $k < n-1$. The case $k = 0$ is interpreted as meaning that $I = \emptyset$. By assumption (of Theorem 4.6), $\mathrm{tp}(b_{k+1}, b_n) = \mathrm{tp}(a_{k+1}, a_n)$, so there are $\bar{b}^*_{k+1}, \bar{b}_{n,k+1} \in M$ such that $F(\bar{b}^*_{k+1}, b_{k+1})$, $F(\bar{b}_{n,k+1}, b_n)$ and

$$\mathrm{tp}(\bar{b}^*_{k+1}, \bar{b}_{n,k+1}, b_{k+1}, b_n) = \mathrm{tp}(\bar{a}_{k+1}, \bar{a}_n, a_{k+1}, a_n).$$

Since $\mathrm{tp}(\bar{b}_{n,k+1}/\bar{b}^*_{k+1}, b_n)$ has a nondividing extension to

$$\bar{b}^*_{k+1}, b_n, \bar{b}_1, \ldots, \bar{b}_k, \bar{b}_{k+2}, \ldots, \bar{b}_{n-1}$$

we may without loss of generality assume that $\bar{b}_{n,k+1}$ realizes such a nondividing extension and hence

$$\bar{b}_{n,k+1} \mathop{\smile\!\!\!\!\!\!\;|}_{b_n, \bar{b}^*_{k+1}} \bar{b}_1, \ldots, \bar{b}_k, \bar{b}_{k+2}, \ldots, \bar{b}_{n-1}. \tag{4.10}$$

From $\mathrm{tp}(\bar{b}^*_{k+1}, \bar{b}_{n,k+1}) = \mathrm{tp}(\bar{a}_{k+1}, \bar{a}_n)$ and (4.7) we get $\bar{b}^*_{k+1} \mathop{\smile\!\!\!\!\!\!\;|} \bar{b}_{n,k+1}$, which since $b_{k+1} \in \mathrm{dcl}(\bar{b}^*_{k+1})$ implies that $\bar{b}^*_{k+1}, b_{k+1} \mathop{\smile\!\!\!\!\!\!\;|} \bar{b}_{n,k+1}$ and hence

$$\bar{b}^*_{k+1} \mathop{\smile\!\!\!\!\!\!\;|}_{b_{k+1}} \bar{b}_{n,k+1}. \tag{4.11}$$

Since $b_{k+1} \in \mathrm{dcl}(\bar{b}_{k+1})$ it follows from (4.9) that

$$\bar{b}_{k+1} \mathop{\smile\!\!\!\!\!\!\;|}_{b_{k+1}} \bar{b}_1, \ldots, \bar{b}_k, \bar{b}_{k+2}, \ldots, \bar{b}_{n-1}. \tag{4.12}$$

As $F(\bar{b}_{k+1}, b_{k+1})$ and $F(\bar{b}^*_{k+1}, b_{k+1})$, the assumption that D is acl-complete implies that

$$\operatorname{tp}(\bar{b}^*_{k+1}/\operatorname{acl}(b_{k+1})) = \operatorname{tp}(\bar{b}_{k+1}/\operatorname{acl}(b_{k+1})). \tag{4.13}$$

We have already concluded that $\bar{b}_{n,k+1} \downarrow \bar{b}^*_{k+1}$ and since $b_n \in \operatorname{dcl}(\bar{b}_{n,k+1})$ we get

$$\bar{b}_{n,k+1} \underset{b_n}{\downarrow} \bar{b}^*_{k+1},$$

which together with (4.10) and transitivity gives

$$\bar{b}_{n,k+1} \underset{b_n}{\downarrow} \bar{b}^*_{k+1}, \bar{b}_1, \ldots, \bar{b}_k, \bar{b}_{k+2}, \ldots, \bar{b}_{n-1}. \tag{4.14}$$

Now we claim that

$$\bar{b}_{n,k+1} \underset{b_{k+1}}{\downarrow} \bar{b}_1, \ldots, \bar{b}_k, \bar{b}_{k+2}, \ldots, \bar{b}_{n-1}. \tag{4.15}$$

Suppose on the contrary that (4.15) is false. Then

$$\bar{b}_{n,k+1} \not\downarrow b_{k+1}, \bar{b}_1, \ldots, \bar{b}_k, \bar{b}_{k+2}, \ldots, \bar{b}_{n-1}.$$

Since $\bar{b}_{n,k+1} \downarrow \bar{b}^*_{k+1}$ (as we have seen above) and $b_{k+1} \in \operatorname{dcl}(\bar{b}^*_{k+1})$ we get $\bar{b}_{n,k+1} \downarrow b_{k+1}$. By the triviality of dependence we must have $\bar{b}_{n,k+1} \not\downarrow \bar{b}_i$ for some $i \neq k+1$, so

$$\bar{b}_{n,k+1}, b_n \not\downarrow \bar{b}_i.$$

Since $\mathrm{SU}(b_n) = 1$ it follows from the last line of (4.8) that $b_n \downarrow \bar{b}_i$. From (4.14) we get $\bar{b}_{n,k+1} \underset{b_n}{\downarrow} \bar{b}_i$, so by transitivity $\bar{b}_{n,k+1}, b_n \downarrow \bar{b}_i$, which contradicts what we got above. Hence (4.15) is proved.

By the independence theorem (Fact 2.15) applied over $\operatorname{acl}(b_{k+1})$ together with (4.11), (4.12), (4.13) and (4.15), there is $\bar{b}'_{k+1}$ such that

$$\operatorname{tp}(\bar{b}'_{k+1}, \bar{b}_{n,k+1}) = \operatorname{tp}(\bar{b}^*_{k+1}, \bar{b}_{n,k+1}) = \operatorname{tp}(\bar{a}_{k+1}, \bar{a}_n),$$
$$\operatorname{tp}(\bar{b}_1, \ldots, \bar{b}_k, \bar{b}'_{k+1}, \bar{b}_{k+2}, \ldots, \bar{b}_{n-1}) = \operatorname{tp}(\bar{b}_1, \ldots, \bar{b}_{n-1}) = \operatorname{tp}(\bar{a}_1, \ldots, \bar{a}_{n-1}).$$

By Lemma 4.9 with $I = \{1, \ldots, k\}$, $\bar{c}_i = \bar{b}_i$ for $i \in \{1, \ldots, n-1\} \setminus \{k+1\}$, $\bar{c}_{k+1} = \bar{b}'_{k+1}$ and $\bar{d}_i = \bar{b}_{n,i}$ for $i \in I$, we find $\bar{b}'_i$ for $i \in \{1, \ldots, n-1\}$ and $\bar{b}'_{n,j}$ for $j \in J = I \cup \{k+1\}$ such that (4.8) holds with $\bar{b}'$ and J in place of $\bar{b}$ and J, respectively. ■

Now we are ready to complete the proof of Theorem 4.6. By induction on $k = 1, \ldots, n-1$ and applying Lemma 4.10 with $I = \{1, \ldots, k\}$ for $k < n-1$, we find $\bar{b}_1, \ldots, \bar{b}_{n-1} \in M$ and $\bar{b}_{n,1}, \ldots, \bar{b}_{n,n-1} \in M$ such that (4.8) holds with $I = \{1, \ldots, n-1\}$. With the use of (4.7) it follows that $\bar{b}_{n,i} \downarrow \bar{b}_i$ for all $i = 1, \ldots, n-1$ and since $b_n \in \operatorname{dcl}(\bar{b}_{n,i})$ we get

$$\bar{b}_{n,i} \underset{b_n}{\downarrow} \bar{b}_i \quad \text{for all } i = 1, \ldots, n-1. \tag{4.16}$$

Since D is acl-complete we have

$$(4.17)\qquad \mathrm{tp}(\bar{b}_{n,i}/\mathrm{acl}(b_n)) = \mathrm{tp}(\bar{b}_{n,j}/\mathrm{acl}(b_n)) \quad \text{for all } i, j = 1, \ldots, n-1.$$

Moreover, we claim that

$$(4.18)\qquad \{\bar{b}_1, \ldots, \bar{b}_{n-1}\} \text{ is independent over } \{b_n\}.$$

Suppose on the contrary that (4.18) is false. By triviality of dependence, $\bar{b}_i \not\mathop{\smile}\limits^{|}_{b_n} \bar{b}_j$ for some $i \neq j$, and hence $\bar{b}_i \not\mathop{\smile}\limits^{|} b_n\bar{b}_j$. By triviality of dependence again, $\bar{b}_i \not\mathop{\smile}\limits^{|} b_n$ or $\bar{b}_i \not\mathop{\smile}\limits^{|} \bar{b}_j$. But $\bar{b}_i \not\mathop{\smile}\limits^{|} b_n$ implies that $b_n \in \mathrm{acl}(\bar{b}_i)$ (since $\mathrm{SU}(b_n) = 1$), which contradicts the choice of $\bar{b}_1, \ldots, \bar{b}_{n-1}$. And $\bar{b}_i \not\mathop{\smile}\limits^{|} \bar{b}_j$ also contradicts the choice of $\bar{b}_1, \ldots, \bar{b}_{n-1}$ since $\mathrm{tp}(\bar{b}_i, \bar{b}_j) = \mathrm{tp}(\bar{a}_i, \bar{a}_j)$ where $\bar{a}_i \mathop{\smile}\limits^{|} \bar{a}_j$. Hence (4.18) is proved.

The independence theorem (Corollary 2.16) together with (4.16)–(4.18) implies that there is $\bar{b}_n \in M$ such that $F(\bar{b}_n, b_n)$ and $\mathrm{tp}(\bar{b}_n, \bar{b}_i) = \mathrm{tp}(\bar{b}_{n,i}, \bar{b}_i) = \mathrm{tp}(\bar{a}_n, \bar{a}_i)$ for all $i = 1, \ldots, n-1$. Moreover, by the choice of $\bar{b}_1, \ldots, \bar{b}_{n-1}$, $\mathrm{tp}(\bar{b}_1, \ldots, \bar{b}_{n-1}) = \mathrm{tp}(\bar{a}_1, \ldots, \bar{a}_{n-1})$. As the language is binary, there is an isomorphism f from $\mathcal{M}{\restriction}\bar{a}_1 \ldots \bar{a}_n$ to $\mathcal{M}{\restriction}\bar{b}_1 \ldots \bar{b}_n$ such that $f(\bar{a}_i) = \bar{b}_i$ for each i, so by Lemma 4.3, $\mathrm{tp}(a_1, \ldots, a_n) = \mathrm{tp}(b_1, \ldots, b_n)$, and the proof of Theorem 4.6 is finished.

5. Trivial dependence implies that any canonically embedded geometry is a reduct of a binary random structure. Throughout this section we use the conventions of Notation 4.2.

THEOREM 5.1. *Let $\mathcal{M}$ be countable, binary, homogeneous and simple with trivial dependence. Suppose that $G \subseteq M^{\mathrm{eq}}$ is A-definable where $A \subseteq M$ is finite, only finitely many sorts are represented in G, $\mathrm{SU}(a/A) = 1$ and $\mathrm{acl}(\{a\} \cup A) \cap G = \{a\}$ for every $a \in G$. Let $\mathcal{G}$ denote the canonically embedded structure in $\mathcal{M}^{\mathrm{eq}}$ over A with universe G. Then $\mathcal{G}$ is a reduct of a binary random structure.*

5.1. Proof of Theorem 5.1. Let $\mathcal{M}$, $G \subseteq M^{\mathrm{eq}}$ and $A \subseteq M$ be as assumed in the theorem. By Remark 3.7, we may without loss of generality assume that $A = \emptyset$, implying that G is $\emptyset$-definable in $\mathcal{M}^{\mathrm{eq}}$ and that $\mathcal{G}$ is a canonically embedded structure in $\mathcal{M}^{\mathrm{eq}}$ over $\emptyset$. By Lemma 4.5 applied to G, there is $D \subseteq M^{\mathrm{eq}}$ with rank 1 such that D is $\emptyset$-definable, acl-complete and

$$(5.1)\qquad \begin{array}{l} \text{for every } a \in G \text{ there is } d \in D \text{ with } a \in \mathrm{dcl}(d) \text{ and } d \in \mathrm{acl}(a), \\ \text{for every } d \in D \text{ there is } a \in G \text{ with } a \in \mathrm{dcl}(d) \text{ and } d \in \mathrm{acl}(a). \end{array}$$

REMARK 5.2. Observe that the independence theorem implies the following: *Suppose $n < \omega$, $\{a_1, \ldots, a_n\} \subseteq D$ is independent over $\emptyset$, $b_1, \ldots, b_n \in D$ and $b_i \mathop{\smile}\limits^{|} a_i$ for all $i = 1, \ldots, n$ and* $\mathrm{tp}(b_i/\mathrm{acl}(\emptyset)) = \mathrm{tp}(b_j/\mathrm{acl}(\emptyset))$ *for all*

i and j. Then there is $b \in D$ such that $\mathrm{tp}(b/\mathrm{acl}(\emptyset)) = \mathrm{tp}(b_i/\mathrm{acl}(\emptyset))$ *and* $\mathrm{tp}(b, a_i) = \mathrm{tp}(b_i, a_i)$ *for all $i = 1, \ldots, n$, and $b \mathop{\smile\hspace{-0.9em}\mid\hspace{0.9em}} \{a_1, \ldots, a_n\}$.*

Let $p_1, \ldots, p_r$ be all complete 1-types over $\mathrm{acl}(\emptyset)$ which are realized in D, and let $p_{r+1}, \ldots, p_s$ be all complete 2-types over $\emptyset$ which are realized in D and, for each $r < i \leq s$, have the property that if $p_i(a, b)$, then $a \neq b$ and $\{a, b\}$ is independent. For each $i = 1, \ldots, s$, let R_i be a relation symbol with arity 1 if $i \leq r$ and otherwise with arity 2. Let $V = \{R_1, \ldots, R_s\}$ and let $\mathcal{D}$ denote the V-structure with universe D such that for every $\bar{a} \in D$, $\mathcal{D} \models R_i(\bar{a})$ if and only if $\mathcal{M}^{\mathrm{eq}} \models p_i(\bar{a})$.

Now define $\mathbf{K}$ to be the class of all *finite* V-structures $\mathcal{N}$ such that there is an embedding $f : \mathcal{N} \to \mathcal{D}$ such that $f(N)$ is an independent set. Let $\mathbf{P}_2$ be the class of all $\mathcal{N} \in \mathbf{K}$ such that $|N| \leq 2$. Recall the definition of $\mathbf{RP}_2$ in Section 2.3.

LEMMA 5.3. $\mathbf{K} = \mathbf{RP}_2$, *where* $\mathbf{RP}_2$ *has the hereditary property and the amalgamation property.*

Proof. Clearly, $\mathbf{P}_2$ is a 1-adequate class, so (as observed in Section 2.3) $\mathbf{RP}_2$ has the hereditary property and the amalgamation property. Evidently $\mathbf{K} \subseteq \mathbf{RP}_2$, so it remains to prove that $\mathbf{RP}_2 \subseteq \mathbf{K}$. For this it suffices to show that if $\mathcal{N} \subset \mathcal{N}' \in \mathbf{RP}_2$, $N' = N \cup \{a\}$ and $f : \mathcal{N} \to \mathcal{D}$ is an embedding such that $f(N)$ is independent, then there is an embedding $f' : \mathcal{N}' \to \mathcal{D}$ which extends f and $f'(N')$ is independent. But this follows immediately from Remark 5.2 together with the definitions of the structures involved. ■

By Lemma 5.3, $\mathbf{K} = \mathbf{RP}_2$ has the hereditary property and the amalgamation property, so let $\mathcal{F}$ be the Fraïssé limit of $\mathbf{K}$. Hence $\mathcal{F}$ is homogeneous and a binary random structure. Since $\mathcal{F}$ is the Fraïssé limit of $\mathbf{K}$, it follows that if $\mathcal{N} \subseteq \mathcal{N}' \in \mathbf{K}$ and $f : \mathcal{N} \to \mathcal{F}$ is an embedding, then there is an embedding $f' : \mathcal{N}' \to \mathcal{F}$ which extends f. By using this together with the definition of $\mathbf{K}$ and Remark 5.2 it is straightforward to prove, by a back-and-forth argument, that there is $D' \subseteq D$ such that

(a) D' is independent,
(b) $\mathcal{F} \cong \mathcal{D} \restriction D'$, and
(c) for every $d \in D$ there is $d' \in D'$ with $\mathrm{acl}(d) = \mathrm{acl}(d')$.

Let $a \in G$. By (5.1), $a \in \mathrm{dcl}(d)$ for some $d \in D$. By (c), there is $d' \in D'$ such that $\mathrm{acl}(d) = \mathrm{acl}(d')$ and hence $a \in \mathrm{acl}(d')$. For a contradiction suppose that there is $a' \in G$ such that $a' \neq a$, $\mathrm{tp}(a') = \mathrm{tp}(a)$ and $a' \in \mathrm{acl}(d')$. By (5.1) and (c) there is $d'' \in D' \cap \mathrm{acl}(a')$. As $a' \in \mathrm{acl}(d')$ this implies that $d'' \in \mathrm{acl}(d')$, which by the independence of D' gives $d'' = d'$. Then $a \in \mathrm{acl}(d') = \mathrm{acl}(d'') \subseteq \mathrm{acl}(a')$, which contradicts the assumptions about G. Thus we conclude that

(d) every $a \in G$ belongs to $\mathrm{dcl}(d')$ for some $d' \in D'$.

From the assumptions about G, D and from (a) it follows that for every $a \in G$, $\mathrm{acl}(a) \cap D'$ contains a unique element, which we denote $g(a)$. It also follows from the assumptions about G, D and from (a) that $g : G \to D'$ is bijective, and, because of (d), that

(e) for every $a \in G$, we have $a \in \mathrm{dcl}(g(a))$.

Observe that we are not assuming, and we have not proved, that D' or g are definable (over any set).

Now we define a V-structure $\mathcal{G}'$ with universe G as follows. For each $R_i \in V$ and every $\bar{a} \in G$, let

$$\mathcal{G}' \models R_i(\bar{a}) \quad \text{if and only if} \quad \mathcal{D}{\restriction}D' \models R_i(g(\bar{a})).$$

Since g is bijective it is clear that $\mathcal{G}' \cong \mathcal{D}{\restriction}D'$ and by (b) we get $\mathcal{G}' \cong \mathcal{F}$ so $\mathcal{G}'$ is a binary random structure. From the definition of $\mathcal{G}'$ (through the definitions of $\mathcal{D}$, D' and $\mathcal{F}$) it follows that for every $a \in G$ there is R_i, $1 \le i \le r$, such that $\mathcal{G} \models R_i(a)$, and for all distinct $a, b \in G$ there is R_i, $r < i \le s$, such that $\mathcal{G} \models R_i(a, b)$.

Lemma 5.4. *If $n < \omega$, $a_1, \ldots, a_n, b_1, \ldots, b_n \in G$ and $\mathrm{tp}_{\mathcal{G}'}(a_1, \ldots, a_n) = \mathrm{tp}_{\mathcal{G}'}(b_1, \ldots, b_n)$, then $\mathrm{tp}_{\mathcal{G}}(a_1, \ldots, a_n) = \mathrm{tp}_{\mathcal{G}}(b_1, \ldots, b_n)$.*

Proof. Suppose that $a_1, \ldots, a_n, b_1, \ldots, b_n \in G$ and $\mathrm{tp}_{\mathcal{G}'}(a_1, \ldots, a_n) = \mathrm{tp}_{\mathcal{G}'}(b_1, \ldots, b_n)$. Since $\mathcal{G}$ is a canonically embedded structure in $\mathcal{M}^{\mathrm{eq}}$, it follows that $\mathrm{tp}_{\mathcal{G}}(\bar{a}) = \mathrm{tp}_{\mathcal{G}}(\bar{b})$ if and only if $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{b})$, for all finite tuples $\bar{a}, \bar{b} \in G$. So it suffices to prove that $\mathrm{tp}(a_1, \ldots, a_n) = \mathrm{tp}(b_1, \ldots, b_n)$. We may assume that all $a_1, \ldots, a_n$ are distinct and that all $b_1, \ldots, b_n$ are distinct.

The assumptions and the definitions of $\mathcal{G}$, $\mathcal{D}$ and D' imply $\mathrm{tp}(g(a_i), g(a_j)) = \mathrm{tp}(g(b_i), g(b_j))$ for all i and j. Since $g : G \to D'$ is bijective and D' is independent it follows from Theorem 4.6 that

$$\mathrm{tp}(g(a_1), \ldots, g(a_n)) = \mathrm{tp}(g(b_1), \ldots, g(b_n)).$$

By (e) we have $a_i \in \mathrm{dcl}(g(a_i))$ and $b_i \in \mathrm{dcl}(g(b_i))$ for each i, and therefore $\mathrm{tp}(a_1, \ldots, a_n) = \mathrm{tp}(b_1, \ldots, b_n)$. ∎

To prove that $\mathcal{G}$ is a reduct of $\mathcal{G}'$ it suffices to show that for every $1 < n < \omega$ and every complete n-type p over $\emptyset$ of $\mathcal{G}$ there is a V-formula $\varphi_p(\bar{x})$ such that for all n-tuples $\bar{a} \in G$, $\mathcal{G} \models p(\bar{a})$ if and only if $\mathcal{G}' \models \varphi_p(\bar{a})$. As $\mathcal{G}'$ has elimination of quantifiers it has only finitely many complete n-types over $\emptyset$, say $q_1, \ldots, q_m$. Let q_i be isolated by $\varphi_i(\bar{x})$. By Lemma 5.4, for each i either

- for all $\bar{a} \in G$, if $\mathcal{G}' \models \varphi_i(\bar{a})$, then $\mathcal{G} \models p(\bar{a})$, or
- for all $\bar{a} \in G$, if $\mathcal{G}' \models \varphi_i(\bar{a})$, then $\mathcal{G} \not\models p(\bar{a})$.

Let I be the set of all i for which the first case holds. If $\varphi_p(\bar{x}) = \bigvee_{i \in I} \varphi_i(\bar{x})$ then, for all n-tuples $\bar{a} \in G$, $\mathcal{G} \models p(\bar{a})$ if and only if $\mathcal{G}' \models \varphi_p(\bar{a})$. This concludes the proof of Theorem 5.1.

Remark 5.5. The conclusion of Theorem 5.1 is that

(f) $\mathcal{G}$ is a reduct of a binary random structure.

A stronger conclusion, essentially saying that $\mathcal{G}$ is a binary random structure, would be:

(g) If $\mathcal{G}_0$ is the reduct of $\mathcal{G}$ to all relation symbols with arity at most 2, then $\mathcal{G}$ is a reduct of $\mathcal{G}_0$, and $\mathcal{G}_0$ is a binary random structure.

What extra assumptions do we need in order to get the conclusion (g)? It is straightforward to verify the following implications, where we use notation from the above proof:

D' is $\emptyset$-definable in $\mathcal{M}^{\mathrm{eq}}$

$$\begin{aligned}
&\Leftrightarrow \text{(the graph of) } g \text{ is } \emptyset\text{-definable} \\
&\Rightarrow g(a) \in \mathrm{dcl}(a) \text{ for every } a \in G \\
&\Rightarrow \text{for all } 0 < n < \omega \text{ and all } a_1, \ldots, a_n, b_1, \ldots, b_n \in G, \\
&\qquad \mathrm{tp}(a_1, \ldots, a_n) = \mathrm{tp}(b_1, \ldots, b_n) \text{ if and only if} \\
&\qquad \mathrm{tp}(g(a_1), \ldots, g(a_n)) = \mathrm{tp}(g(b_1), \ldots, g(b_n)).
\end{aligned}$$

Hence the condition that D' is definable over $\emptyset$ in $\mathcal{M}^{\mathrm{eq}}$, as well as the equivalent condition, guarantees that the conclusion of the proof of Theorem 5.1 is (g). The next example shows that (g) does not in general follow from the assumptions of Theorem 5.1.

Example 5.6. This example, due to the anonymous referee, shows that there are $\mathcal{M}$ and $G \subseteq M^{\mathrm{eq}}$ which satisfy the assumptions of Theorem 5.1, but for which (g) fails if we let $\mathcal{G}$ be the canonically embedded structure (in $\mathcal{M}^{\mathrm{eq}}$) with universe G. Consequently, for such $\mathcal{M}$ and G, *no* D' as in the proof of Theorem 5.1 is $\emptyset$-definable.

Let $\mathcal{F}$ be the random graph, that is, $\mathcal{F}$ is the Fraïssé limit of the class of all finite undirected loopless graphs. We now construct a new graph $\mathcal{M}$ (viewed as a first-order structure) as follows. The universe of $\mathcal{M}$ is $M = F \times \{0, 1\}$ (where F is the universe of $\mathcal{F}$). If $a, b \in F$ are adjacent then (a, i) and (b, i) are adjacent in $\mathcal{M}$ for $i = 0, 1$. If $a, b \in F$ are nonadjacent (so in particular if $a = b$) then (a, i) and $(b, 1 - i)$ are adjacent in $\mathcal{M}$ for $i = 0, 1$. There are no other adjacencies in $\mathcal{M}$.

Now we define

$$\text{for } (a, i), (b, j) \in M, \quad E((a, i), (b, j)) \text{ if and only if } a = b.$$

Clearly E is an equivalence relation such that each one of its classes has cardinality 2. Moreover, it is straightforward to see that $E(x,y)$ is $\emptyset$-definable by the formula

$$x = y \vee (x \neq y \wedge \neg\exists z(z \sim_{\mathcal{M}} x \wedge z \sim_{\mathcal{M}} y)),$$

where '$\sim_{\mathcal{M}}$' denotes adjacency in $\mathcal{M}$. For every $u \in M$ let u' denote the unique $v \neq u$ such that $E(u,v)$ holds (or in other words, for $u = (a,i) \in M$, $u' = (a, 1-i)$). Note that for all $u \in M$, $(u')' = u$ and $u \sim_{\mathcal{M}} u'$. Let

$$M_0 = \{(a,0) : a \in F\} \quad \text{and} \quad M_1 = \{(a,1) : a \in F\}$$

and note that the set M is the disjoint union of M_0 and M_1 and that $\mathcal{M}{\restriction}M_i$ is a copy of the random graph for $i = 0, 1$. The following is a straightforward consequence of the definition of $\mathcal{M}$:

Claim 1. *For all distinct $u, v \in M$,*

$$u \sim_{\mathcal{M}} v \ \Leftrightarrow\ u' \sim_{\mathcal{M}} v' \ \Leftrightarrow\ u \not\sim_{\mathcal{M}} v' \ \Leftrightarrow\ u' \not\sim_{\mathcal{M}} v.$$

We now prove that $\mathcal{M}$ is homogeneous. The above claim tells us that if $n < \omega$, $u_1, \ldots, u_n, v_1, \ldots, v_n \in M$ and $f(u_i) = v_i$ for $i = 1, \ldots, n$ is a partial isomorphism, then f can be extended to a partial isomorphism which maps u'_i to v'_i for all $i = 1, \ldots, n$. So to prove that $\mathcal{M}$ is homogeneous it suffices (by the symmetry of M_0 and M_1) to prove the following:

Claim 2. *Let $u_1, \ldots u_n, v_1, \ldots, v_n \in M_0$ and suppose that the map $f(u_i) = v_i$ and $f(u'_i) = v'_i$ for $i = 1, \ldots, n$ is a partial isomorphism. Then for every $u_{n+1} \in M_0$ there is $v_{n+1} \in M_0$ such that f can be extended to a partial isomorphism g such that $g(u_{n+1}) = v_{n+1}$ and $g(u'_{n+1}) = g(v'_{n+1})$.*

We do not give the details of the proof of this claim but just note that the argument is straightforward and uses the fact that $\mathcal{F}$ is the random graph, the construction of $\mathcal{M}$ and the first claim.

By representing 0 and 1 with two distinct elements of $a_0, a_1 \in F$ it is straightforward to verify that $\mathcal{M}$ is interpretable in $\mathcal{F}$ with the parameters a_0 and a_1. It follows (from [2, Remarks 2.26 and 2.27]) that $\mathcal{M}$ is simple. The natural way of interpreting $\mathcal{M}$ in $\mathcal{F}$ (with the parameters a_0, a_1) is by letting $F^- = F \setminus \{a_0, a_1\}$, so $\mathcal{F}{\restriction}F^- \cong \mathcal{F}$, and then identifying the universe of M with $F^- \times \{a_0, a_1\}$. Then $\mathrm{SU}(u/\{a_0, a_1\}) = 1$ for every $u \in M$ (where SU-rank is with respect to $\mathcal{F}$) and it follows that $\mathcal{M}$ is supersimple with SU-rank 1. Moreover, since $\mathcal{F}$ has trivial dependence it follows that the same is true for $\mathcal{M}$. Because if there where subsets of N^{eq} for some $\mathcal{N} \equiv \mathcal{M}$ that witnessed nontrivial dependence, then, by supersimplicity, we may assume that they are finite, so by ω-categoricity of $\mathcal{M}$ we may assume that they are subsets of $\mathcal{M}^{\mathrm{eq}}$, and finally the same sets with a_0 and a_1 added would witness nontrivial dependence in $\mathcal{F}$, a contradiction. Hence $\mathcal{M}$ satisfies the assumptions of Theorem 5.1.

For every $u \in M$ let $[u]$ be its equivalence class with respect to E. Let

$$G = \{[u] : u \in M\}.$$

Then $G \subseteq M^{\mathrm{eq}}$ and G satisfies the assumptions of Theorem 5.1. Let $\mathcal{G}$ be the canonically embedded structure with universe G and let $\mathcal{G}_0$ be the reduct of $\mathcal{G}$ to the relation symbols of arity at most 2. It remains prove that $\mathcal{G}$ is *not* a reduct of $\mathcal{G}_0$.

First we show the following:

Claim 3. *For all distinct $u_1, u_2 \in G$ and all distinct $v_1, v_2 \in G$, we have $\mathrm{tp}_{\mathcal{G}}(u_1, u_2) = \mathrm{tp}_{\mathcal{G}}(v_1, v_2)$.*

Let $g_1, g_2, h_1, h_2 \in G$ be such that $g_1 \neq g_2$ and $h_1 \neq h_2$. Then there are $u_1, u_2, v_1, v_2 \in M_0$ such that $g_i = \{u_i, u_i'\}$ and $h_i = \{v_i, v_i'\}$ for $i = 1, 2$.

We consider four cases: (1) $u_1 \sim_{\mathcal{M}} u_2$ and $v_1 \sim_{\mathcal{M}} v_2$, (2) $u_1 \not\sim_{\mathcal{M}} u_2$ and $v_1 \not\sim_{\mathcal{M}} v_2$, (3) $u_1 \sim_{\mathcal{M}} u_2$ and $v_1 \not\sim_{\mathcal{M}} v_2$, and (4) $u_1 \not\sim_{\mathcal{M}} u_2$ and $v_1 \sim_{\mathcal{M}} v_2$. In the first two cases the map given by $u_i \mapsto v_i$ and $u_i' \mapsto v_i'$ for $i = 1, 2$ is a partial isomorphism, so by the homogeneity of $\mathcal{M}$ it extends to an automorphism of $\mathcal{M}$ and hence we get $\mathrm{tp}_{\mathcal{M}}(u_1, u_2, u_1', u_2') = \mathrm{tp}_{\mathcal{M}}(v_1, v_2, v_1', v_2')$, which in turn gives $\mathrm{tp}_{\mathcal{G}}(g_1, g_2) = \mathrm{tp}_{\mathcal{G}}(h_1, h_2)$ (since $g_i \in \mathrm{dcl}_{\mathcal{M}^{\mathrm{eq}}}(u_i)$ and similarly for h_i). In the third and fourth cases the map given by $u_1 \mapsto v_1$, $u_1' \mapsto v_1'$, $u_2 \mapsto v_2'$ and $u_2' \mapsto v_2$ is a partial isomorphism so we get $\mathrm{tp}_{\mathcal{M}}(u_1, u_2, u_1', u_2') = \mathrm{tp}_{\mathcal{M}}(v_1, v_2', v_1', v_2)$ and hence $\mathrm{tp}_{\mathcal{G}}(g_1, g_2) = \mathrm{tp}_{\mathcal{G}}(h_1, h_2)$ (as $h_2 \in \mathrm{dcl}_{\mathcal{M}^{\mathrm{eq}}}(v_2')$). This concludes the proof of Claim 3.

Observe that Claim 3 implies that every isomorphism between finite substructures of $\mathcal{G}_0$ can be extended to an automorphism of $\mathcal{G}_0$, so $\mathcal{G}_0$ is a binary homogeneous structure (with finite vocabulary). This and Claim 3 easily imply the following:

Claim 4. *For every $n < \omega$, all distinct $g_1, \ldots, g_n \in G$ and all distinct $h_1, \ldots, h_n \in G$, we have $\mathrm{tp}_{\mathcal{G}_0}(g_1, \ldots, g_n) = \mathrm{tp}_{\mathcal{G}_0}(h_1, \ldots, h_n)$.*

To prove that $\mathcal{G}$ is not a reduct of $\mathcal{G}_0$ it now suffices to show that there are distinct $g_1, g_2, g_3 \in G$ and distinct $h_1, h_2, h_3 \in G$ such that $\mathrm{tp}_{\mathcal{G}}(g_1, g_2, g_3) \neq \mathrm{tp}_{\mathcal{G}}(h_1, h_2, h_3)$. Since $\mathcal{M}$ restricted to M_0 is a copy of the random graph it follows that there are distinct $u_1, u_2, u_3 \in M_0$ and distinct $v_1, v_2, v_3 \in M_0$ such that

$$u_1 \sim_{\mathcal{M}} u_2, \quad u_1 \sim_{\mathcal{M}} u_3, \quad u_2 \not\sim_{\mathcal{M}} u_3 \quad \text{and} \quad v_1, v_2, v_3 \text{ form a 3-cycle.}$$

By Claim 1 we see that

$$\mathcal{M}{\restriction}\{u_1, u_2, u_3, u_1', u_2', u_3'\} \not\cong \mathcal{M}{\restriction}\{v_1, v_2, v_3, v_1', v_2', v_3'\}.$$

Let $g_i = [u_i]$ and $h_i = [v_i]$ for $i = 1, 2, 3$. Then g_1, g_2, g_3 are distinct and the same holds for h_1, h_2, h_3. Moreover,

$$\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(g_1, g_2, g_3) \cap M = \{u_1, u_2, u_3, u_1', u_2', u_3'\},$$
$$\mathrm{acl}_{\mathcal{M}^{\mathrm{eq}}}(h_1, h_2, h_3) \cap M = \{v_1, v_2, v_3, v_1', v_2', v_3'\}.$$

It follows that $\mathrm{tp}_{\mathcal{G}}(g_1, g_2, g_3) \neq \mathrm{tp}_{\mathcal{G}}(h_1, h_2, h_3)$, and this finishes the proof that this example has the claimed properties.

We know from Theorem 5.1 that $\mathcal{G}$ is a reduct of a binary random structure. In this example we can explicitly describe such a binary random structure. We can simply expand $\mathcal{M}$ with a unary relation symbol interpreted as M_0. Call this expansion $\mathcal{M}^*$. Let $\mathcal{G}^*$ be the canonically embedded structure of $(\mathcal{M}^*)^{\mathrm{eq}}$ with universe G. Let $\mathcal{G}_0^*$ be the reduct of $\mathcal{G}^*$ to the relation symbols of arity at most 2. One can now prove that $\mathcal{G}_0^*$ is a binary random structure and that $\mathcal{G}$ is a reduct of $\mathcal{G}_0^*$.

Acknowledgements. We thank the anonymous referee for supplying Example 5.6 and for careful reading of the article.

References

[1] A. Aranda López, *Omega-categorical simple theories*, Ph.D. thesis, Univ. of Leeds, 2014.

[2] E. Casanovas, *Simple Theories and Hyperimaginaries*, Lecture Notes in Logic 39, Assoc. Symbolic Logic and Cambridge Univ. Press, 2011.

[3] G. L. Cherlin, *The classification of countable homogeneous directed graphs and countable homogeneous n-tournaments*, Mem. Amer. Math. Soc. 131 (1998), no. 621.

[4] G. Cherlin, L. Harrington and A. H. Lachlan, *$\aleph_0$-categorical, $\aleph_0$-stable structures*, Ann. Pure Appl. Logic 28 (1985), 103–135.

[5] G. Cherlin and E. Hrushovski, *Finite Structures with Few Types*, Ann. Math. Stud. 152, Princeton Univ. Press, 2003.

[6] T. de Piro and B. Kim, *The geometry of 1-based minimal types*, Trans. Amer. Math. Soc. 355 (2003), 4241–4263.

[7] M. Djordjević, *Finite satisfiability and $\aleph_0$-categorical structures with trivial dependence*, J. Symbolic Logic 71 (2006), 810–830.

[8] H.-D. Ebbinghaus and J. Flum, *Finite Model Theory*, 2nd ed., Springer, 1999.

[9] R. Fraïssé, *Sur l'extension aux relations de quelques propriétés des ordres*, Ann. Sci. École Norm. Sup. 71 (1954), 363–388.

[10] A. Gardiner, *Homogeneous graphs*, J. Combin. Theory Ser. B 20 (1976), 94–102.

[11] Ya. Yu. Gol'fand and M. H. Klin, *On k-homogeneous graphs*, in: Algorithmic Studies in Combinatorics, Nauka, Moscow, 1978, 76–85 (in Russian).

[12] B. Hart, B. Kim and A. Pillay, *Coordinatisation and canonical bases in simple theories*, J. Symbolic Logic 65 (2000), 293–309.

[13] C. W. Henson, *Countable homogeneous relational structures and $\aleph_0$-categorical theories*, J. Symbolic Logic 37 (1972), 494–500.

[14] W. Hodges, *Model Theory*, Cambridge Univ. Press, 1993.

[15] T. Jenkinson, J. K. Truss and D. Seidel, *Countable homogeneous multipartite graphs*, Eur. J. Combin. 33 (2012), 82–109.

[16] W. M. Kantor, M. W. Liebeck and H. D. Macpherson, *$\aleph_0$-categorical structures smoothly approximated by finite substructures*, Proc. London Math. Soc. 59 (1989), 439–463.

[17] A. S. Kolesnikov, *n-Simple theories*, Ann. Pure Appl. Logic 131 (2005), 227–261.

[18] V. Koponen, *Asymptotic probabilities of extension properties and random l-colourable structures*, Ann. Pure Appl. Logic 163 (2012), 391–438.

[19] V. Koponen, *Homogeneous 1-based structures and interpretability in random structures*, arXiv:1403.3757 (2014).

[20] A. H. Lachlan, *Countable homogeneous tournaments*, Trans. Amer. Math. Soc. 284 (1984), 431–461.

[21] A. H. Lachlan, *Stable finitely homogeneous structures: a survey*, in: B. T. Hart et al. (eds.), Algebraic Model Theory, Kluwer, 1997, 145–159.

[22] A. H. Lachlan and A. Tripp, *Finite homogeneous 3-graphs*, Math. Logic Quart. 41 (1995), 287–306.

[23] A. H. Lachlan and R. E. Woodrow, *Countable ultrahomogenous undirected graphs*, Trans. Amer. Math. Soc. 262 (1980), 51–94.

[24] D. Macpherson, *Interpreting groups in ω-categorical structures*, J. Symbolic Logic 56 (1991), 1317–1324.

[25] D. Macpherson, *A survey of homogeneous structures*, Discrete Math. 311 (2011), 1599–1634.

[26] W. Oberschelp, *Asymptotic 0-1 laws in combinatorics*, in: D. Jungnickel (ed.), Combinatorial Theory, Lecture Notes in Math. 969, Springer, 1982, 276–292.

[27] J. H. Schmerl, *Countable homogeneous partially ordered sets*, Algebra Universalis 9 (1979), 317–321.

[28] J. Sheehan, *Smoothly embeddable subgraphs*, J. London Math. Soc. 9 (1974), 212–218.

[29] S. Shelah, *Classification Theory*, North-Holland, 1990.

[30] F. O. Wagner, *Simple Theories*, Kluwer, 2000.

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Received 12 March 2014;
in revised form 24 June 2014