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Non-additivity of the fixed point property for tree-like continua

by

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Dedicated to the memory of Howard Cook

Abstract. We investigate the fixed point property for tree-like continua that are unions of tree-like continua. We obtain a positive result if finitely many tree-like continua with the fixed point property have dendrites for pairwise intersections. Using Bellamy's seminal example, we define (i) a countable wedge $\hat{X}$ of tree-like continua, each having the fpp, and $\hat{X}$ admitting a fixed-point-free homeomorphism, and (ii) two tree-like continua H and K such that H, K, and $H \cap K$ have the fixed point property, but $H \cup K$ admits a fixed-point-free homeomorphism. In an appendix we verify some of the properties of Bellamy's continuum.

1. Introduction and definitions. A continuous function between topological spaces will be called a *map* or *mapping*. A topological space X has the *fixed point property* (fpp) if each self-mapping on X has a fixed point. The additivity of the fpp, or lack thereof, has been a topic of interest in topological fixed point theory for more than fifty years. Specifically, if X, Y, and $X \cap Y$ are topological spaces with the fpp, must $X \cup Y$ have the fpp?

It is easy to prove that additivity of the fpp holds for the wedge of two spaces, that is, when $X \cap Y$ is degenerate. K. Borsuk's theory of absolute retracts from the 1940's gives a positive result if each of X, Y, and $X \cap Y$ is an absolute retract. M. Shtan'ko [27] showed in 1964 that if X and Y are 1-dimensional continua with the fpp and $X \cap Y$ is a dendrite, then $X \cup Y$ has the fpp. We give a slight generalization of Shtan'ko's result in §2.

In 1982, E. Duda and J. Kell III [7] showed that if $X \cup Y$ is an atriodic, hereditarily unicoherent continuum, then $X \cup Y$ has zero semispan if $X \cap Y$ is connected and each of X and Y has zero semispan. Since zero semispan is equivalent to zero span for continua (see [6]), it follows that, in the class of atriodic, hereditarily unicoherent continua, additivity of the fpp holds for

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continua that have zero span. As a corollary, additivity of the fpp also holds for arc-like (or chainable) continua whose union is in this class.

In the three years from 1967 to 1969, three examples were provided showing that, in general, the fpp is not additive:

(1) (W. Lopez [14], 1967) An example where X is a 17-dimensional polyhedron with the fpp, Y is a disk, $X \cap Y$ is an arc, but $X \cup Y$ does not have the fpp. (See [3, Th. 17] for discussion.)
(2) (A. L. Yandl [28], 1968) An example where each of X and Y is a uniquely arcwise connected continuum with the fpp, $X \cap Y$ is a chainable continuum, and $X \cup Y$ is a 1-dimensional, planar, arcwise connected continuum that does not have the fpp.
(3) (R. H. Bing [3], 1969) An example where X is a 1-dimensional arcwise connected continuum with the fpp, Y is a disk, $X \cap Y$ is an arc, but $X \cup Y$ does not have the fpp.

Bing's example is not planar. In 2005, C. L. Hagopian and J. R. Prajs [13] constructed a planar example that has the same properties as Bing's example. In [17, 18], R. Mańka investigated the additivity of the fpp for 1-dimensional continua when both X and Y are uniquely arcwise connected. He constructed examples where $X \cap Y$ is also uniquely arcwise connected, yet additivity of the fpp fails. In [18], $X \cap Y$ is a contractible harmonic brush, but still additivity of the fpp fails.

In all of these examples, neither X nor Y is tree-like. In light of this, we investigate the additivity of the fpp for tree-like continua where the union is also tree-like. We show, in §5, that the fpp is not additive in this setting either. Additionally, we show that countable wedges of tree-like continua with the fpp may not have the fpp.

A *continuum* is a non-degenerate, compact, connected metric space. A continuum is *indecomposable* if it is not the union of two proper subcontinua. Let x be a point of a continuum X. The *x-composant* of X is the union of all proper subcontinua of X that contain x. If X is indecomposable, then X is the union of uncountably many dense disjoint composants. A continuum X is *hereditarily unicoherent* if each pair of its intersecting subcontinua has a connected intersection. A *dendroid* is an arcwise connected, hereditarily unicoherent continuum. A *dendrite* is a locally connected dendroid; or equivalently, it is locally connected and contains no simple closed curve. For a set S in a continuum X, $\overline{S}$ will denote the closure of S in X.

Given $\epsilon > 0$, a mapping $f\colon X \to Y$ is an *ϵ-mapping* if for each $y \in Y$, $\mathrm{diam}(f^{-1}(y)) < \epsilon$. A continuum X is *arc-like* if for each $\epsilon > 0$, there exists an ϵ-mapping from X onto $[0,1]$. A continuum X is *tree-like* if for each $\epsilon > 0$, there exists a tree T and an ϵ-mapping from X onto T. A non-constant mapping $f\colon X \to Y$ is *atomic* if for each subcontinuum K of X such that

$f(K)$ is non-degenerate, we have $f^{-1}(f(K)) = K$. It was shown in [8] that each atomic map is monotone. The following lemma is well-known and easy to prove. We will use it later in the paper to establish the indecomposablity of certain subcontinua of our examples.

LEMMA 1. *Suppose that $f\colon X \to Y$ is a surjective atomic map of continua. Then X is indecomposable if and only if Y is indecomposable.*

2. A positive fixed-point result. We begin this section with a definition, and two lemmas that will be used in the proof of Theorem 1.

Let $a \neq b$ be points in a uniquely arcwise connected continuum D. Let $[a, b]$ denote the unique arc from a to b and let $f\colon [a, b] \to D$ be a mapping. We say that $[a, b]$ is *stretched over itself by f* if a lies in the unique arc $[f(a), b]$ and b lies in the unique arc $[a, f(b)]$. Note that if $[a, b]$ is stretched over itself, then $[a, b] \subset [f(a), f(b)]$.

LEMMA 2. *Suppose that $[a, b]$ is an arc in a uniquely arcwise connected continuum D and $f\colon [a, b] \to D$ is a mapping that stretches $[a, b]$ over itself. Then f has a fixed point.*

Proof. First we note that for $x \in (a, b)$, either f stretches $[a, x]$ over itself or f stretches $[x, b]$ over itself. To see this, let β be the unique arc in D from $f(x)$ to $[a, b]$, and simply note that β meets exactly one of $[a, x]$ or $(x, b]$.

Suppose that f is fixed-point-free and $\epsilon > 0$ is such that $d(x, f(x)) \geq \epsilon$ for all $x \in X$. Let $\delta > 0$ be such that if α is a subarc of $[a, b]$ with $\operatorname{diam} \alpha < \delta$, then $\operatorname{diam} f(\alpha) < \epsilon$. Partition $[a, b]$ by points $a = a_0 < a_1 < a_2 < \cdots < a_n = b$ such that for each $0 \leq i < n$, $\operatorname{diam}[a_i, a_{i+1}] < \delta$. It follows from the observation in the first paragraph of the proof that there exists $0 \leq j < n$ such that f stretches $[a_j, a_{j+1}]$ over itself. So, $[a_j, a_{j+1}] \subset [f(a_j), f(a_{j+1})]$. But since a_j and $f(a_j)$ are in $[f(a_j), f(a_{j+1})] \subset f([a_j, a_{j+1}])$, it follows that $\operatorname{diam} f([a_j, a_{j+1}]) \geq \epsilon$, a contradiction. ∎

LEMMA 3. *Suppose H and K are continua, $H \cap K$ is a dendrite, and neither H nor K contains a simple closed curve. Then $H \cup K$ contains no simple closed curve.*

Proof. Assume $H \cup K$ contains a simple closed curve S. By hypothesis, S must intersect both $H - K$ and $K - H$. So, let $x \in S \cap (H \cap K)$ and let $a \in S \cap (H - K)$. There are two arcs A and B with common endpoint set $\{a, x\}$ such that $A \cup B = S$. Let $[a, c]$ be the unique arc in A such that $[a, c] \cap (H \cap K) = \{c\}$, and let $[a, b]$ be the unique arc in B such that $[a, b] \cap (H \cap K) = \{b\}$. Note that $[a, c] \cup [a, b] \subset H$. Since $H \cap K$ is a dendrite, there is a unique arc $[b, c]$ in $H \cap K$. We find that $[a, c] \cup [b, c] \cup [a, b]$ is a simple closed curve lying in H, which is a contradiction. ∎

The following theorem is a generalization, to higher-dimensional continua, of M. Shtan'ko's theorem mentioned in §1.

Theorem 1. *If H and K are continua with the fpp, $H \cap K$ is a dendrite, and neither H nor K contains a simple closed curve, then $H \cup K$ has the fpp.*

Proof. Suppose there exists a fixed-point-free map f of $H \cup K$ into itself. Let $E = H \cap K$ and note that $E \cap f(E) \neq \emptyset$. By Lemma 3, $H \cup K$ contains no simple closed curve. Hence, $D = E \cup f(E)$ is a dendrite. Let r be the retraction of D onto E that takes each $x \in D - E$ to $y \in E$ such that the half-open arc $[x, y)$ in D misses E. Since E has the fixed point property, $rf|_E$ has a fixed point p. Note that p is the only fixed point of $rf|_E$, for otherwise, an arc in E would be stretched over itself by f, producing a fixed point in E by Lemma 2, and thus contradicting that f is fixed-point-free.

Either $f(p) \in H - K$ or $f(p) \in K - H$. Assume without loss of generality that $f(p) \in H - K$. Let $k\colon K \to E$ be an extension of the retraction $r|_{D\cap K}\colon D \cap K \to E$. Define $h\colon H \to H$ by

$$h(x) = \begin{cases} f(x) & \text{if } f(x) \in H, \\ kf(x) & \text{if } f(x) \in K. \end{cases}$$

We claim that h is fixed-point-free. If $f(x) \in H$, then $h(x) = f(x) \neq x$. Suppose $f(x) \in K$ and $x = h(x)$. Then $x = kf(x)$ and so $x \in E$. Since $f(E) \subset D$, we have $f(x) \in D \cap K$. So, $x = kf(x) = rf(x)$. Since p is the only fixed point of $rf|_E$, it follows that $x = p$. But then $f(x) \in H - K$, contradicting $f(x) \in K$.

But H has the fpp, so $H \cup K$ admits no fixed-point-free mapping. ∎

We now state a few lemmas that will be used to establish corollaries that extend Theorem 1 from two to finitely many continua.

Lemma 4. *Suppose $H_1, \ldots, H_n$ are continua, $H_i \cap H_j$ is a dendrite for each $1 \leq i < j \leq n$, and no H_i contains a simple closed curve. Then $\bigcap_{i=1}^n H_i$ is a dendrite if and only if $\bigcap_{i\in J} H_i$ is a dendrite for each non-degenerate $J \subset \{1, \ldots, n\}$.*

Proof. $\Leftarrow$: Obvious.

$\Rightarrow$: First we note that by Lemma 3, no simple closed curve is contained in the union of two H_i's. Suppose that $\bigcap_{i\in J} H_i$ is disconnected for some $J \subset \{1, \ldots, n\}$. By hypothesis, J contains more than two elements. Assume, without loss of generality, that $1, 2 \in J$. Let a and b be points in different components of $\bigcap_{i\in J} H_i$. Since $H_1 \cap H_2$ is a dendrite, there is an arc $A \subset H_1 \cap H_2$ with endpoints a and b. For some $j \in J$, H_j does not contain some point p in A. Since $H_1 \cap H_j$ is a dendrite, there is an arc $B \subset H_1 \cap H_j$ with endpoints a and b. Since $p \notin B$, $A \cup B$ contains a simple closed curve. But $A \cup B \subset H_1 \cup H_j$, contradicting our initial observation (or Lemma 3). Hence, $\bigcap_{i\in J} H_i$ is connected. Since $\bigcap_{i\in J} H_i \subset H_1 \cap H_2$, it follows that $\bigcap_{i\in J} H_i$ is a dendrite. ∎

LEMMA 5. *Suppose $D_1, \ldots, D_n$ are dendrites, $D_i \cap D_j$ is a dendrite for each $1 \le i < j \le n$, and $\bigcap_{i=1}^n D_i$ is a dendrite. Then $\bigcup_{i=1}^n D_i$ is a dendrite.*

Proof. That $\bigcup_{i=1}^n D_i$ is connected and locally connected follows from the hypothesis. So, we only need to see that $\bigcup_{i=1}^n D_i$ contains no simple closed curve. We use induction on n.

For $n = 1$, the result is trivial. Assume the statement holds for some $n-1 \ge 1$. Note that by Lemma 4, the dendrites $D_1, \ldots, D_{n-1}$ satisfy the hypothesis. So, $\bigcup_{i=1}^{n-1} D_i$ is a dendrite. Set-theoretically, $(\bigcup_{i=1}^{n-1} D_i) \cap D_n = \bigcup_{i=1}^{n-1} (D_i \cap D_n)$. We next observe that the dendrites $D_i \cap D_n$, for $1 \le i \le n-1$, satisfy the inductive hypothesis. For each i, j, $(D_i \cap D_n) \cap (D_j \cap D_n) = D_i \cap D_j \cap D_n$ is a dendrite by Lemma 4. Also, $\bigcap_{i=1}^{n-1} (D_i \cap D_n) = \bigcap_{i=1}^n D_i$ is a dendrite. So, by inductive assumption, $\bigcup_{i=1}^{n-1}(D_i \cap D_n) = (\bigcup_{i=1}^{n-1} D_i) \cap D_n$ is a dendrite, and by Lemma 3, $(\bigcup_{i=1}^{n-1} D_i) \cup D_n = \bigcup_{i=1}^n D_i$ contains no simple closed curve. Therefore, $\bigcup_{i=1}^n D_i$ is a dendrite. ∎

LEMMA 6. *Suppose $H_1, \ldots, H_n$ are continua, $H_i \cap H_j$ is a dendrite for each $1 \le i < j \le n$, and $\bigcap_{i=1}^n H_i$ is a dendrite. Then*

(1) *each H_i is tree-like if and only if $\bigcup_{i=1}^n H_i$ is tree-like, and*
(2) *no H_i contains a simple closed curve if and only if $\bigcup_{i=1}^n H_i$ contains no simple closed curve.*

Proof. (1) $\Leftarrow$: Obvious.

$\Rightarrow$: H. Cook [4, Theorem 2] showed that the union of two tree-like continua that intersect in a connected set is tree-like. So, for $n = 2$ the result follows from Cook's theorem. Assume the result holds for some $n-1 \ge 2$. Note that by Lemma 4, $H_1, \ldots, H_{n-1}$ satisfy the hypothesis. So, by inductive assumption, $\bigcup_{i=1}^{n-1} H_i$ is tree-like. Also, $(\bigcup_{i=1}^{n-1} H_i) \cap H_n = \bigcup_{i=1}^{n-1} (H_i \cap H_n)$ is a dendrite by Lemma 5. So, again by Cook's theorem, $(\bigcup_{i=1}^{n-1} H_i) \cup H_n = \bigcup_{i=1}^n H_i$ is tree-like.

(2) The proof, using Lemma 3, is similar to the proof of (1). ∎

COROLLARY 1. *Suppose $H_1, \ldots, H_n$ are continua with the fpp, $H_i \cap H_j$ is a dendrite for all $1 \le i < j \le n$, $\bigcap_{i=1}^n H_i$ is a dendrite, and no H_i contains a simple closed curve. Then $\bigcup_{i=1}^n H_i$ has the fpp.*

Proof. We use induction to establish the corollary. For $n = 1$, the result is obvious. For $n = 2$, it follows from Theorem 1. Assume it holds for some $n-1 \ge 2$. By Lemma 4, the continua $H_1, \ldots, H_{n-1}$ also satisfy the conditions of the hypothesis. So, $\bigcup_{i=1}^{n-1} H_i$ is a continuum with the fpp by inductive assumption, and contains no simple closed curve by Lemma 6(2).

As we have previously seen, $(\bigcup_{i=1}^{n-1} H_i) \cap H_n = \bigcup_{i=1}^{n-1}(H_i \cap H_n)$ is a dendrite. It follows from Theorem 1 that $\bigcup_{i=1}^{n} H_i$ has the fpp. ∎

Corollary 2. *If $H_1, \ldots, H_n$ are tree-like continua with the fpp, $H_i \cap H_j$ is a dendrite for each $1 \leq i < j \leq n$, and $\bigcap_{i=1}^{n} H_i$ is a dendrite, then $\bigcup_{i=1}^{n} H_i$ is a tree-like continuum with the fpp.*

Proof. This follows immediately from Corollary 1 and Lemma 6(1). ∎

Corollary 3. *If $H_1, \ldots, H_n$ are tree-like continua with the fpp, and C is a dendrite such that $H_i \cap H_j = C$ for all $1 \leq i < j \leq n$, then $\bigcup_{i=1}^{n} H_i$ is a tree-like continuum with the fpp.*

The collection of tree-like continua in Corollary 3 is a clump of tree-like continua as defined by Cook in [5]. Example 1 in §5 below shows that Corollary 3 cannot be extended to countably many tree-like continua, even when the intersection is degenerate.

The authors wish to thank R. Mańka for calling our attention to M. Shtan'ko's theorem [27], and J. R. Prajs for pointing out that Shtan'ko's theorem can be generalized to higher-dimensional continua. We also thank the referee for suggesting the corollaries above which generalized the corollaries that we first had in mind.

We end this section with two related questions.

Question 1. *Suppose H and K are tree-like continua with the fpp and $H \cap K$ is a dendroid. Does $H \cup K$ have the fpp?*

Question 2. *Does additivity of the fpp hold for arc-like* (*chainable*) *continua? That is, if H, K, and $H \cap K$ are chainable continua, does $H \cup K$ have the fpp?*

Question 2 has an affirmative answer if $H \cup K$ is embeddable in the plane.

3. Definitions and constructions. A *matchbox manifold* X is a separable metric space in which each point p of X has an open neighborhood that is homeomorphic to the product of a 0-dimensional set and the real numbers $\mathbb{R}$. So, in particular, if C is a compact 0-dimensional set and $(0,1)$ is the open segment in $\mathbb{R}$ between 0 and 1, then $C \times (0,1)$ is a matchbox manifold. We will call a space that is homeomorphic to $C \times (0,1)$ an *open matchbox* and a space that is homeomorphic to $C \times [0,1]$ a *closed* (or *compact*) *matchbox*. The sets $C \times \{0\}$ and $C \times \{1\}$ will be called, respectively, the *left* and *right boundaries* of both an open matchbox and a closed matchbox.

For homeomorphic topological spaces X and Y, we write $X \overset{T}{\approx} Y$. Let $M \overset{T}{\approx} C \times [0,1]$ be a closed matchbox. Identify M with $C \times [0,1] \times \{0\}$ in

$C \times [0,1] \times \mathbb{R}$. Let $K = \bigcup_{c \in C} K_c$ be a compact subset of $C \times [0,1] \times \mathbb{R}$, where for each $c \in C$, K_c is a continuum such that $K_c \subset \{c\} \times [0,1] \times \mathbb{R}$, $K_c \cap (\{c\} \times \{0\} \times \mathbb{R}) = \{(c,0,0)\}$, and $K_c \cap (\{c\} \times \{1\} \times \mathbb{R}) = \{(c,1,0)\}$. We refer to K as an *mbox replacement of the matchbox* M. We let $\pi \colon K \to M$ be the natural projection onto M. Note that M and K have the same left and right boundaries, namely $C \times \{0\} \times \{0\}$ and $C \times \{1\} \times \{0\}$.

Suppose $f \colon C \to D$ is a homeomorphism between compact 0-dimensional sets. Let λ be a positive real number. Let $f_\lambda \colon C \times [0,1] \to D \times [0,\infty)$ be the embedding given by $f_\lambda(c,t) = (f(c), \lambda t)$. We call f_λ a λ-*rearrangement* of the matchbox $C \times [0,1]$. Note that f_λ stretches or shrinks $C \times [0,1]$ as λ is greater or less than one. Any map, similarly defined (conjugate to such a map), on a compact matchbox will be referred to analogously. In our examples, we will typically have a homeomorphism, say f, already defined on an entire matchbox $C \times [0,1]$. When we introduce the notation f_λ relative to this map, we mean $f_\lambda = (f|_{C \times \{0\}})_\lambda$, according to our definition above.

Let $M \overset{T}{\approx} C \times [0,1]$ be a closed matchbox and $N = f_\lambda(M)$, where $f_\lambda \colon M \to N$ is a λ-rearrangement of M. Let μ be a positive real number and let K be an mbox replacement of M. Let $f_{\lambda\mu} \colon K \to N \times [0,\infty)$ be the embedding given by $f_{\lambda\mu}(c,t,s) = (f(c), \lambda t, \mu s)$. So, $f_{\lambda\mu}$ is a homeomorphism onto its image. Thus, $f_{\lambda\mu}(K)$ is an mbox replacement of N. We call $f_{\lambda\mu}$ a $\lambda\mu$-*rearrangement* of the mbox K. If $\lambda = \mu$, we write f_λ for $f_{\lambda\lambda}$. Note that a rearrangement of a matchbox either stretches or shrinks only in the second coordinate, while a rearrangement of an mbox may stretch or shrink in both the second and third coordinates.

The map $\pi \circ f_{\lambda\mu}$ will be called a λ-*flattening map*. Note that the image of a flattening map is a closed matchbox.

In our examples, we will be interested in defining an mbox replacement in which exactly one arc is replaced by a continuum that is homeomorphic to one of S, $-S$, or S^2 defined below. Each is a variation of a topologist's sine curve. Our building block S is a topologist's sine curve with an "extended" limit bar, which is a subset of $[0,1] \times [0,1]$. Let

$$S = \big([0, \tfrac{1}{2}] \times \{0\}\big) \cup \big(\{\tfrac{1}{2}\} \times [0,1]\big) \cup \big\{\big(x, \tfrac{1}{2}\big(1 + \sin \tfrac{3\pi}{4x-2}\big)\big) \;\big|\; \tfrac{1}{2} < x \le 1\big\}.$$

Note that $\pi_1 \colon S \to [0,1]$ is an atomic map that is one-to-one for $x \neq \frac{1}{2}$. Also, $\pi_1^{-1}\big(\{\frac{1}{2}\}\big)$ is the limit bar of the topologist's sine curve. The points of the sine curve that have second coordinate 1 are the points with first coordinates $t_k = \frac{2k+2}{4k+1}$ for $k \ge 1$. We let T_k be the points of S whose first coordinates are greater than or equal to t_k.

Let $-S$ denote the reflection of S through the line $x = \frac{1}{2}$, and let

$$S^2 = \big\{(x,y) \in -S \;\big|\; 0 \le x \le \tfrac{1}{2}\big\} \cup \big\{(x,y) \in S \;\big|\; \tfrac{1}{2} \le x \le 1\big\}.$$

Note that there is no surjective map between any two of S, $-S$, and S^2 that fixes $(0,0)$.

For $M \overset{T}{\approx} C \times [0,1]$ a closed matchbox and $p \in C$, we define an mbox replacement of M so that K_p is S (or $-S$, or S^2) and K_c is an arc for $c \in C - \{p\}$. We call such K an S_p-*mbox* (or $-S_p$-*mbox*, or S^2_p-*mbox*). If the point of C is unimportant or unspecified, we will simply use the terms S-*mbox*, $-S$-*mbox*, and S^2-*mbox*. There are numerous ways in which this "replacement" construction can be done. One such way is indicated below for an S_p-mbox.

So, assume K_p is S and has the arc $\{p\} \times [0,1] \times \{0\}$ in $\{p\} \times [0,1] \times [0,1]$ replaced. We wish to replace all arcs of the form $\{c\} \times [0,1] \times \{0\}$, for $c \in C - \{p\}$, with arcs K_c having endpoints $(c,0,0)$ and $(c,1,0)$, and so that $\bigcup_{c \in C} K_c$ is compact. For $c \in C$, let d_c denote the distance of c to p. Assume, without loss of generality, that $d_c < 1$ for all $c \in C$. For $c \neq p$ and $\frac{1}{k+1} \leq d_c < \frac{1}{k}$, let

$$\begin{aligned} K_c = {} & \big(\{c\} \times \big[0, \tfrac{1}{2}\big] \times \{0\}\big) \cup \big(\{c\} \times \big\{\tfrac{1}{2}\big\} \times [0,1]\big) \\ & \cup \big(\{c\} \times \big[\tfrac{1}{2}, \tfrac{2k+2}{4k+1}\big] \times \{1\}\big) \cup T_k. \end{aligned}$$

Let $S_p = \bigcup_{c \in C} K_c$. Note that the left and right boundaries of S_p are the same as the left and right boundaries of M. Also, S_p has the remaining desired properties.

Analogous constructions give $-S$-mbox or S^2-mbox replacements for a given matchbox.

4. Attaching sequences of arcs to Bellamy's second tree-like continuum. In 1979, D. Bellamy [1] answered, in the negative, a question that ten years earlier R. H. Bing had called one of the most interesting questions in geometric topology. Namely, "Does each tree-like continuuum have the fixed point property?" Using a modified 6-adic Knaster continuum, Bellamy constructed an indecomposable tree-like continuum admitting a fixed-point-free map. Bellamy applied to his example a technique used by J. B. Fugate and L. Mohler [10] to get an indecomposable tree-like arc-continuum admitting a fixed-point-free homeomorphism. We refer to this example as *Bellamy's second example*.

In 1980, L. G. Oversteegen and J. T. Rogers, Jr. [25] gave an inverse limit description of a tree-like arc-continuum admitting an induced fixed-point-free map. To obtain a simple description of the bonding maps, they used rather complicated factor spaces. In 1982, the same authors [26] defined two more examples using inverse limits. These examples have similar properties to their first example, but additionally the fixed-point-free map is a homeomorphism and the factor spaces are trees. The nature of the symmetry in the factor spaces and the folding of the bonding maps is reminiscent of Bellamy's example.

In 1993, L. Fearnley and D. G. Wright [9] gave a geometric description of a tree-like continuum, consisting of a chainable continuum and a Cantor fan, without the fixed point property.

In a number of papers between 1992 and 2000, P. Minc [20–24] provided tree-like examples that answered other important questions in continuum fixed point theory; for example, in [21], he constructed a periodic-point-free homeomorphism on a tree-like continuum. For each positive integer j, Minc altered an n-adic Knaster continuum, where

$$n = 2(4^1 - 1) \cdots (4^j - 1),$$

by replacing an arc containing the endpoint with a fan over a 0-dimensional set. The resulting indecomposable tree-like continuum B_j admits a map with no periodic points of period less than or equal to j. The continuum B_1 is quite similar to Bellamy's original example, a modification of the 6-adic Knaster continuum. Applying the Fugate–Mohler technique, Minc defined tree-like arc-continua $\tilde{B}_j$ that admit homeomorphisms with analogous properties. Minc used the sequences $\{B_j\}$ and $\{\tilde{B}_j\}$ to construct his examples.

In 2012, C. L. Hagopian, M. M. Marsh, and J. R. Prajs [12] used Bellamy's second example to construct an indecomposable tree-like continuum that admits a composant-preserving fixed-point-free homeomorphism.

Although defined in various ways, all of these examples either are continua or contain subcontinua that are fundamentally similar to either Bellamy's first or second example.

We will use Bellamy's second example in the construction of our examples in §5.

Let B be Bellamy's indecomposable tree-like continuum in [1] that contains a fan and admits a fixed-point-free mapping g. In Bellamy's paper, B is $\hat{D}$ and g is $\hat{F}$. Some properties of B that we will use are the following:

- The endpoint set of the fan is invariant under g (see Appendix).
- The vertex of the fan is pulled by g out of the fan.
- There exist endpoints p and q such that $g(p) = q$ and $g(q) = p$ (see Appendix).

Let $\hat{B}$ be Bellamy's second indecomposable tree-like continuum obtained by applying the Fugate–Mohler technique, that is, $\hat{B} = \varprojlim\{B, g\}$. Some properties of $\hat{B}$ that we will use are the following:

- The shift map $\sigma\colon \hat{B} \to \hat{B}$ given by $\sigma(x_1, x_2, \ldots) = (g(x_1), x_1, x_2, \ldots)$ is a fixed-point-free homeomorphism on $\hat{B}$ that interchanges composants that have endpoints.
- Each proper subcontinuum of $\hat{B}$ is an arc.

- If $x \in \hat{B}$ is not an endpoint, then x has an open matchbox neighborhood.
- If C_a and C_b are composants of $\hat{B}$ with endpoints a and b respectively, where $\sigma(a) = b$, then, relative to arc length, the distance of b to $\sigma(x)$ is twice the distance of a to x for a point $x \in C_a$.
- The points $\hat{p} = (p, q, p, \dots)$ and $\hat{q} = (q, p, q, \dots)$ are endpoints of $\hat{B}$ with $\sigma(\hat{p}) = \hat{q}$ and $\sigma(\hat{q}) = \hat{p}$.

It may be helpful to refer to Figure 1 for all constructions that follow. While the section of the continuum pictured in Figure 1 appears to be planar, there is no such assumption. The figure is a simplified schematic representation.

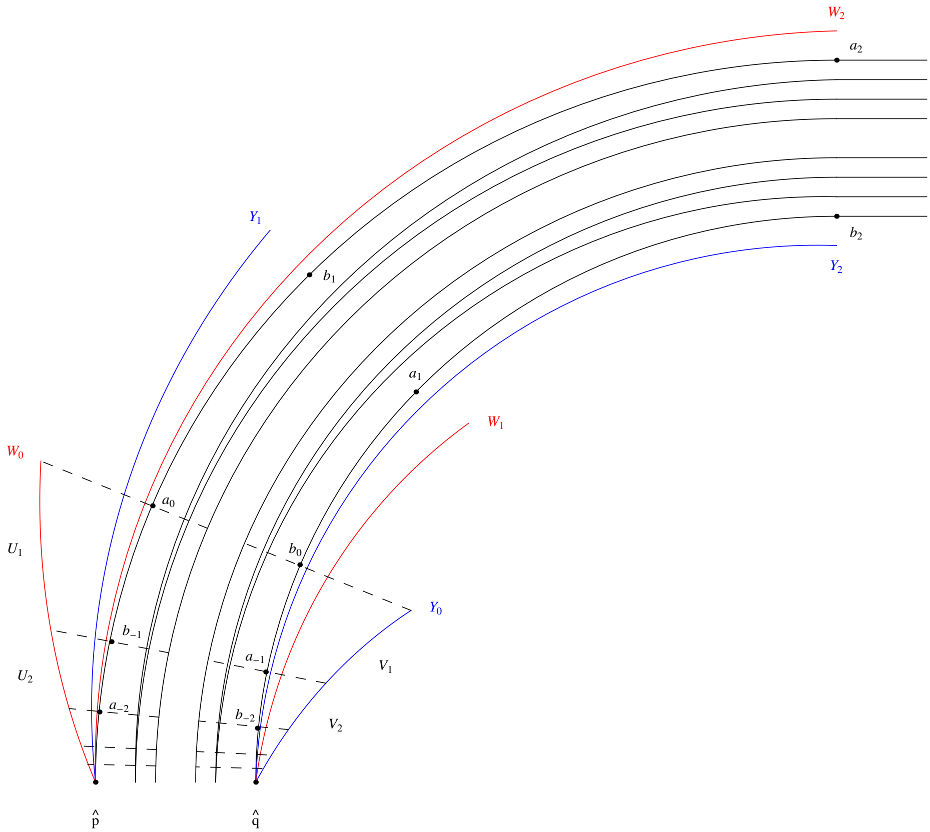

Fig. 1

Let C_1 and C_2 be the composants of $\hat{B}$ containing $\hat{p}$ and $\hat{q}$ respectively. Let a_0 and b_0 be points of C_1 and C_2, respectively, whose arc-distances to $\hat{p}$ and $\hat{q}$ are assumed, without loss of generality, to be one. Let $a_n = \sigma^n(a_0)$ and $b_n = \sigma^n(b_0)$ for $n \in \mathbb{Z}$ ($\sigma^0 = \mathrm{id}$). Note that $a_n \in C_1$ for even n, and $a_n \in C_2$

for odd n. Also, $b_n \in C_2$ for even n, and $b_n \in C_1$ for odd n. Furthermore, by our assumption, for $n < -1$ and even, $d(a_n, \hat{p}) = \frac{1}{2} d(a_{n+1}, \hat{q})$. Analogous statements hold for odd n, and for the sequence of b_n's.

Let $P = C_1 \times [0,1] \cup C_2 \times [0,1]$. Define $\alpha\colon P \to P$ by $\alpha(x,t) = \bigl(\sigma(x), \frac{1}{2}t\bigr)$. Let W_0, Y_0 be the subsets of P given by $W_0 = \{((1-t)\hat{p} + ta_0, t) \mid 0 \le t \le 1\}$ and $Y_0 = \{((1-t)\hat{q} + tb_0, t) \mid 0 \le t \le 1\}$. By our assumption, we may think of W_0 as the diagonal of $[\hat{p}, a_0] \times [0,1] \subset C_1 \times [0,1]$ and Y_0 as the diagonal of $[\hat{q}, b_0] \times [0,1] \subset C_2 \times [0,1]$. Note that α is one-to-one, and that $\alpha(\hat{p}, 0) = (\hat{q}, 0)$ and $\alpha(\hat{q}, 0) = (\hat{p}, 0)$. For $n \ge 1$, let $W_n = \alpha^n(W_0)$ and $Y_n = \alpha^n(Y_0)$. Observe that for even n, W_n is an arc in $C_1 \times [0, 1/2^n]$ with endpoints $(\hat{p}, 0)$ and $(a_n, 1/2^n)$, and for odd n, W_n is an arc in $C_2 \times [0, 1/2^n]$ with endpoints $(\hat{q}, 0)$ and $(a_n, 1/2^n)$. There are analogous statements for the $\{Y_n\}$ sequence. Let F_1 be the union of the W_n's for all even n, and F_2 be the union of the W_n's for all odd n. Let G_1 be the union of the Y_n's for all odd n, and G_2 be the union of the Y_n's for all even n. We see that $\overline{F}_i - F_i = C_i$ and $\overline{G}_i - G_i = C_i$ in the space P for $i \in \{1, 2\}$.

Let $Z = \hat{B} \cup F_1 \cup F_2$ and let $\hat{Z} = Z \cup G_1 \cup G_2$. We will be using modifications of both Z and $\hat{Z}$ in our examples. It is easy to see that α maps F_1 homeomorphically onto F_2, and maps F_2 homeomorphically onto $F_1 - (W_0 - \{\hat{p}\})$. Also, $\alpha \cup \sigma\colon Z \to Z$ is an extension of $\sigma\colon \hat{B} \to \hat{B}$. So, $\alpha \cup \sigma$ is a fixed-point-free embedding of Z into Z. For notational convenience, we will hereafter let σ denote the extended map $\alpha \cup \sigma$ on Z.

Remark 1. We point out that σ and Z could be modified to have a surjective homeomorphism by adding two null sequences of arcs to Z, one sequence at $\hat{p}$ and one at $\hat{q}$, and extending σ to these sequences in the obvious way; namely, σ maps each arc attached to $\hat{p}$ to the arc twice as long that is attached to $\hat{q}$, and vice versa. Again, there are analogous statements for the continuum $\hat{Z}$ and the extended map σ.

The continua Z and $\hat{Z}$ and the embedding σ of Z into Z (or $\hat{Z}$ into $\hat{Z}$) will be basic items in the construction of our examples. Note that Z is tree-like since Z is hereditarily unicoherent and each indecomposable subcontinuum of Z is tree-like [4, Th. 1]. Similarly, $\hat{Z}$ is tree-like.

We will be replacing matchboxes in Z and $\hat{Z}$ with S-mboxes, $-S$-mboxes, and S^2-mboxes. So, it should be helpful to refer to Figure 1 and think of the mbox replacements as being inserted vertically above the matchboxes in Z and in $\hat{Z}$.

5. Examples

Example 1. *There exists a countable wedge of tree-like continua X_n such that each X_n has the fpp, but $X = \bigcup X_n$ is a tree-like continuum that does not have the fpp. Furthermore, all but one of the X_n's are arcs.*

Proof. We begin with the continuum Z and the fixed-point-free embedding $\sigma\colon Z \to Z$. So, for construction of this example, we are using $\hat{B}$ with the attached W_n arcs shown in Figure 1.

Let U_1 be an open set in Z such that U_1 is an open matchbox, U_1 contains a segment of W_n for all even n, and $U_1 \cap C_1$ is the segment from b_{-1} to a_0. Let B_{-1} and A_0 be the compact 0-dimensional sets containing b_{-1} and a_0 respectively, and so that $\overline{U}_1 = U_1 \cup B_{-1} \cup A_0$. We think of B_{-1} as the 0-level left boundary of $\overline{U}_1$, and A_0 as the 1-level right boundary of $\overline{U}_1$. So, $\overline{U}_1$ is a closed matchbox. Note that $\sigma^{-1}(U_1)$ is an open matchbox such that $\sigma^{-1}(U_1) \cap C_2 = (b_{-2}, a_{-1})$. Let $B_{-2} = \sigma^{-1}(B_{-1})$ and $A_{-1} = \sigma^{-1}(A_0)$. We see that $\sigma^{-1}(\overline{U}_1) = \sigma^{-1}(U_1) \cup B_{-2} \cup A_{-1}$, which is a closed matchbox with left boundary B_{-2} and right boundary A_{-1}. Now let K_1 be an $S_{b_{-1}}$-mbox replacement of $\overline{U}_1 - W_0$. Since σ^{-1} maps $B_{-1} - W_0$ homeomorphically onto B_{-2}, $\sigma^{-1}_{.5}$ is a $\frac{1}{2}$-rearrangement of the closed matchbox $\overline{U}_1 - W_0$. So, $K_2 = \sigma^{-1}_{.5}(K_1)$ is an $S_{b_{-2}}$-mbox replacement of $\sigma^{-1}(\overline{U}_1 - W_0)$.

Let U_2 be the open set in Z whose right boundary is the left boundary of U_1 and such that $U_2 \cap C_1 = (a_{-2}, b_{-1})$. Clearly, U_2 is an open matchbox and $V_1 = \sigma(U_2)$ is an open matchbox whose left boundary is the right boundary of $\sigma^{-1}(U_1)$. Note that $V_1 \cap C_2 = (a_{-1}, b_0)$. Repeat for $\overline{V}_1$ the mbox replacement we applied to $\overline{U}_1 - W_0$, getting an $S_{a_{-1}}$-mbox replacement of $\overline{V}_1$. Call this mbox L_1. Let $L_2 = \sigma^{-1}_{.5}(L_1)$ and note that L_2 is an $S_{a_{-2}}$-mbox replacement of $\overline{U}_2$. As above, let B_0 denote the right boundary of L_1, and let A_{-2} denote the left boundary of L_2. Observe that the left boundary of L_1 is A_{-1} and the right boundary of L_2 is B_{-1}. We see that σ_2 is a 2-rearrangement of the mbox $K_2 \cup L_2$ onto the mbox $K_1 \cup L_1$.

Continue this backward construction toward $\hat{p}$ and $\hat{q}$ inductively from L_n and K_n, getting L_{n+1} and K_{n+1} so that σ_2 is a 2-rearrangement of $L_{n+1} \cup K_{n+1}$ onto $L_n \cup K_n$. Also note that the backward sequences of topologist's sine curves that are introduced into C_1 and C_2 are null sequences since they are generated by $\frac{1}{2}$-mbox rearrangements. Clearly, they converge to $\hat{p}$ and $\hat{q}$.

Remark 2. We could have used S^2-mboxes, rather than S-mboxes, in this construction, and in Example 3 we will do just that.

Let X be the continuum that results after replacing the sequence $\{\overline{U}_n\}$ of matchboxes with the sequence $K_1, L_2, K_3, L_4, \ldots$ of S-mboxes and the sequence $\{\overline{V}_n\}$ of matchboxes with the sequence $L_1, K_2, L_3, K_4, \ldots$ of S-mboxes, and otherwise leaving Z unaltered. The W_n's are altered arcs in K_i and L_i for all $i \geq 1$. We keep the labels W_n for these altered arcs in the continuum X. Let $\tilde{B} = X - \bigcup_{n \geq 0}(W_n - \{\hat{p}, \hat{q}\})$. So, $\tilde{B}$ is Bellamy's second continuum with two backward null sequences of copies of S, one sequence converging to $\hat{p}$ and the other converging to $\hat{q}$. The map from X

onto Z that collapses the limit bars of S to points and otherwise is one-to-one is an atomic map. Since the fibers of this atomic map are arcs, it follows from [15, (6.14), p. 18] that X is tree-like. Also, the restriction of this map to $\tilde{B}$ has image $\hat{B}$. Since $\hat{B}$ is indecomposable, Lemma 1 shows that $\tilde{B}$ is indecomposable.

Define the fixed-point-free map $f\colon X \to X$ as follows. Let $f = \sigma$ on $X - \bigcup(K_n \cup L_n) = Z - \bigcup(\overline{U}_n \cup \overline{V}_n)$. Let $f = \pi \circ \sigma_2$ on $K_1 \cup L_1$. That is, f is the 2-flattening map on $K_1 \cup L_1$. Finally, let $f = \sigma_2$ on $K_n \cup L_n$ for each $n \geq 2$. That is, f is the 2-rearrangement of the mboxes K_n and L_n, defined according to the behavior of σ on the left boundaries of these sets.

Clearly, f is continuous and fixed-point-free by construction. Also, f is one-to-one on $X - (K_1 \cup L_1)$. The continuum X is the wedge promised in the statement of this example. To see this, we define certain subcontinua of X.

Let $X_0 = (X - \bigcup\{W_n \mid n \text{ is odd}\}) \cup \{\hat{q}\}$. For $i \geq 1$, let $X_i = W_{2i-1}$. So, $X = \bigcup_{i \geq 0} X_i$ is a wedge of tree-like continua with common intersection $\{\hat{q}\}$. For $i \neq 0$, X_i is an arc. Thus, all that remains is to see that X_0 has the fpp.

Let $F = \bigcup\{W_n \mid n \text{ is even}\}$, let $\tilde{C}_1$ be the composant of $\tilde{B}$ that contains $\hat{p}$, and let $\tilde{C}_2$ be the composant of $\tilde{B}$ that contains $\hat{q}$.

Suppose $\ell\colon X_0 \to X_0$ is a fixed-point-free mapping. Suppose $\ell(\hat{p}) \in F$. Then $\ell(\hat{p}) \in W_n$ for some even n. It follows that ℓ has a fixed point in W_n, which is a contradiction. So, $\ell(\hat{p}) \in \tilde{B}$. Since F is an arc component of X_0, $\ell(F)$ is a subset of the arc component of $\tilde{B}$ that contains $\ell(\hat{p})$. Thus, $\ell(F) \subset \tilde{B}$. Since $\tilde{B}$ is a subset of the closure of F, it follows that $\ell(\tilde{B}) \subset \tilde{B}$. Since λ-dendroids have the fixed point property (see [16]), and each proper subcontinuum of $\tilde{B}$ is a tree-like, hereditarily decomposable continuum (a λ-dendroid), it follows that $\ell(\tilde{B})$ cannot be a proper subset of $\tilde{B}$. So, $\ell(\tilde{B}) = \tilde{B}$.

Since $\tilde{B}$ is atriodic and tree-like, $\ell|_{\tilde{B}}$ is weakly confluent (see [11]). Pick a non-degenerate subcontinuum C of $\tilde{C}_1$ that contains $\hat{p}$. Since ℓ is weakly confluent, some subcontinuum of $\tilde{B}$ must be mapped by ℓ onto C. Since C is non-locally connected in each ϵ-neighborhood of $\hat{p}$, it follows that either $\ell(\hat{p}) = \hat{p}$ or $\ell(\hat{q}) = \hat{p}$. Since ℓ is fixed-point-free, we have $\ell(\hat{q}) = \hat{p}$, and similarly $\ell(\hat{p}) = \hat{q}$. Again, since F is an arc component of X_0, $\ell(F) = \{\hat{q}\}$. Thus, $\ell(\overline{F}) = \ell(X_0) = \{\hat{q}\}$, a contradiction. Hence, ℓ has a fixed point. ∎

There is a 1-dimensional non-tree-like example, similar to Example 1 above, in [19, Example 4], where a double Warsaw circle plays the role of the modified Bellamy continuum.

Example 2. *There exists a countable wedge of tree-like continua $\hat{X}_n$ such that each $\hat{X}_n$ has the fpp, but $\hat{X} = \bigcup \hat{X}_n$ is a tree-like continuum that admits a fixed-point-free homeomorphism. Furthermore, all but one of the $\hat{X}_n$'s are arcs.*

Proof. As pointed out in Example 1, the fixed-point-free map $f\colon X \to X$ is not surjective and is not a homeomorphism. We will modify X and f to get $\hat{X}$ in this example. The basic idea is to add null sequences of arcs to $\hat{p}$ and $\hat{q}$ (see Remark 1) to make α (from §4) surjective, which in turn will make f (from Example 1) surjective. Thereafter, we apply the Fugate–Mohler technique to the pair (X, f), getting the desired example.

Recalling the construction of Z (see end of §4), in $C_1 \times [0,1]$ we add a null sequence $\{W_{-n}\}$ of arcs, for even positive n, to $\hat{p}$, and in $C_2 \times [0,1]$ we add a null sequence $\{W_{-n}\}$ of arcs, for odd positive n, to $\hat{q}$, so that $W_i \cap W_j \subset \{\hat{p}, \hat{q}\}$ for all $i, j \in \mathbb{Z}$. Also, we extend α, and thus f, so that for all $n \in \mathbb{Z}$, f maps W_n homeomorphically onto W_{n+1}. One way to do this is to define in P, for $n \geq 1$, the sets $W_{-2n} = \{((1-t)\hat{p} + ta_{-2n}, t/2^n) \mid 0 \leq t \leq 1\}$ and $W_{-2n+1} = \{((1-t)\hat{q} + ta_{-2n+1}, t/2^{n-1}) \mid 0 \leq t \leq 1\}$.

So, assume that X and f from Example 1 have been modified by the addition of the arcs W_{-n} and the extension of f described. Throughout the remainder of Example 2, the new X and new f will still be referred to as X and f. Note that $f\colon X \to X$ is now surjective.

However, f is still not a homeomorphism, but is not one-to-one only on the limit bars of K_1 and L_1. By adding two null sequences of copies of S into $\tilde{C}_1$ and $\tilde{C}_2$, respectively, so that $f^n(K_1)$ and $f^n(L_1)$, for $n \geq 1$, contain copies of S alternating between $\tilde{C}_2$ and $\tilde{C}_1$, and between $\tilde{C}_1$ and $\tilde{C}_2$, respectively, we can modify f to be a homeomorphism.

This is accomplished by simply applying the Fugate–Mohler technique to the pair (X, f). That is, let $\hat{X} = \varprojlim\{X, f\}$ and let $\hat{f}$ be either the left or the right shift map on the inverse limit space. The map $\hat{f}\colon \hat{X} \to \hat{X}$ is a homeomorphism. That $\hat{X}$ has the desired properties is straightforward to verify. In this regard, it may be helpful to note the following.

Let D_1 be the limit bar in $K_1 \cap \tilde{C}_1 \overset{T}{\approx} S$ and let E_1 be the limit bar in $L_1 \cap \tilde{C}_2 \overset{T}{\approx} S$. Let $f(D_1) = \{v\}$, where $v \in \tilde{C}_2$, and let $f(E_1) = \{u\}$, where $u \in \tilde{C}_1$. Note that since the bonding map f is one-to-one except on $D_1 \cup E_1$, the only points of $\hat{X}$ that are not uniquely determined by their first coordinates are of the form $(f^{n-1}(u), \ldots, f(u), u, t, f^{-1}(t), \ldots)$ for $t \in D_1$, and $(f^{n-1}(v), \ldots, f(v), v, t, f^{-1}(t), \ldots)$ for $t \in E_1$. So, for given $n \geq 1$, these points form new limit bars in the inverse limits $\tilde{C}_1 \overset{f}{\leftarrow} \tilde{C}_2 \overset{f}{\leftarrow} \tilde{C}_1 \overset{f}{\leftarrow} \cdots$ and $\tilde{C}_2 \overset{f}{\leftarrow} \tilde{C}_1 \overset{f}{\leftarrow} \tilde{C}_2 \overset{f}{\leftarrow} \cdots$, respectively. The shift map $\hat{f}$ takes each of these limit bars to the next, alternating between the two inverse limits which are composants of $\hat{X}$. ■

Example 3. *There exist tree-like continuua H and K such that H, K, and $H \cap K$ have the fpp, but $H \cup K$ is a tree-like continuum that does not have the fpp.*

Proof. To build our third example, we begin with the continuum $\hat{Z}$ and the fixed-point-free embedding $\sigma\colon \hat{Z} \to \hat{Z}$ (see §4 and Figure 1). Recall that $\hat{Z}$ only differs from Z by having an additional sequence of arcs $\{Y_n\}$ that are exact copies of the W_n's in Z, but if W_n is attached to $\hat{p}$, then Y_n is attached to $\hat{q}$, and vice versa. We apply the construction, as done in Example 1, to the continuum $\hat{Z}$, but we use S^2-mbox replacements rather than the S-mbox replacements we used in Example 1, getting a continuum X' that has "backward" null sequences of copies of S^2 placed into the arcs $[\hat{p}, a_0]$ and $[\hat{q}, b_0]$. This produces an indecomposable subcontinuum $\tilde{B}$ of X' that is a modified Bellamy continuum. Also, the sequences $\{W_n\}$ and $\{Y_n\}$ of arcs are modified to limit to the modified composants $\tilde{C}_1$ and $\tilde{C}_2$ of $\tilde{B}$, whose endpoints are $\hat{p}$ and $\hat{q}$ respectively. Let $f'\colon X' \to X'$ denote the fixed-point-free embedding, defined analogously to Example 1, from X' onto $X' - (W_0 \cup Y_0 - \{\hat{p}, \hat{q}\})$.

Let D_1 be the limit bar in $\tilde{C}_1 \cap K_1 \overset{T}{\approx} S^2$, and let E_1 be the limit bar in $\tilde{C}_2 \cap L_1 \overset{T}{\approx} S^2$. For $n \geq 2$, let $D_n = {f'}^{-1}(D_{n-1}) \subset K_n$ and $E_n = {f'}^{-1}(E_{n-1}) \subset L_n$. So, the D_n's and E_n's are limit bars in copies of S^2 that alternate between $\tilde{C}_1$ and $\tilde{C}_2$, each one half the length of its predecessor.

To complete the construction we will replace a null sequence of arcs in each W_i with copies of S and a null sequence of arcs in each Y_i with copies of $-S$. The resulting continuum will be our example. We begin by describing the process for the $\{W_i\}$ sequence.

For even $i \geq 0$, let $W_i(1) \subset W_i$ be a sequence of arcs converging to D_1, and for odd $i \geq 1$, let $W_i(1) \subset W_i$ be a sequence of arcs converging to E_1. For all $i \geq 1$ and $n \geq 2$, let $W_i(n) = {f'}^{-1}(W_i(n-1))$. Note that for fixed $i \geq 0$, $\{W_i(n)\}_{n\geq 1}$ is a null sequence of arcs in W_i.

We note that

$$M_1 = D_1 \cup \Big(\bigcup_{\text{even}\, i} W_i(1)\Big) \cup E_1 \cup \Big(\bigcup_{\text{odd}\, i} W_i(1)\Big)$$

is a closed matchbox. Let N_1 be the mbox replacement of M_1, where D_1 and E_1 remain unchanged, and each $W_i(1)$ is replaced with a copy of S, which we call $\tilde{W}_i(1)$, so that the limit bar in $\tilde{W}_{i+1}(1)$ has length equal to one half of the length of the limit bar in $\tilde{W}_i(1)$. We observe that for even i, $\{\tilde{W}_i(1)\}$ converges to D_1, and for odd i, $\{\tilde{W}_i(1)\}$ converges to E_1. So, indeed, N_1 is an mbox.

We note that ${f'}_{.5}^{-1}$ is a $\frac{1}{2}$-rearrangement of $M_1 - W_0$ onto its image, which we denote by M_2. As discussed in §3, it follows that $N_2 = {f'}_{.5}^{-1}(N_1 - \tilde{W}_0(1))$ is an mbox replacement of ${f'}_{.5}^{-1}(M_1 - W_0)$. For $i \geq 0$, let $\tilde{W}_i(2)$ denote the copy of S that replaced the arc $M_2 \cap W_i$. Note that for even i, $\{\tilde{W}_i(2)\}$

converges to E_2, and for odd i, $\{\tilde{W}_i(2)\}$ converges to D_2. Also, we see that each $\tilde{W}_i(2)$ is one half the size of $\tilde{W}_{i+1}(1)$.

We continue this process, getting a sequence $\{N_n\}$ of mboxes replacing $\{M_n\}$. Furthermore, for each $n \geq 1$, we have copies $\tilde{W}_i(n)$, for $i \geq 0$, of S in N_n such that:

(1) for n fixed and even,

 (i) for odd i, $\{\tilde{W}_i(n)\}$ converges to D_n,

 (ii) for even i, $\{\tilde{W}_i(n)\}$ converges to E_n; and

(2) for n fixed and odd,

 (i) for even i, $\{\tilde{W}_i(n)\}$ converges to D_n,

 (ii) for odd i, $\{\tilde{W}_i(n)\}$ converges to E_n.

Since $D_1, E_2, D_3, E_4, \ldots$ is a backward null sequence of limit bars in $\tilde{C}_1$, and $E_1, D_2, E_3, D_4, \ldots$ is a backward null sequence of limit bars in $\tilde{C}_2$, we see that for $n \geq 1$ and fixed, for even i, $\{\tilde{W}_i(n)\}$ converges to a limit bar in $\tilde{C}_1$, and for odd i, $\{\tilde{W}_i(n)\}$ converges to a limit bar in $\tilde{C}_2$.

For $i \geq 0$, let $\tilde{W}_i$ be the continuum obtained by replacing the arcs $W_i(n)$ in W_i with the copies $\tilde{W}_i(n)$ of S. Let $\tilde{F}_1 = \bigcup \tilde{W}_i$ for even i, and $\tilde{F}_2 = \bigcup \tilde{W}_i$ for odd i.

Now, we do a similar construction on the sequence $\{Y_i\}$, but we replace arcs converging to limit bars in $\tilde{C}_1$ and $\tilde{C}_2$ with copies of $-S$. We get mboxes Q_n and new continua $\tilde{Y}_i$ replacing the arcs Y_i and having analogous subcontinua $\{\tilde{Y}_i(n)\}$, for $n \geq 1$, that are copies of $-S$. Let $\tilde{G}_1 = \bigcup \tilde{Y}_i$ for even i, and $\tilde{G}_2 = \bigcup \tilde{Y}_i$ for odd i.

Let

$$\tilde{X} = \tilde{B} \cup \tilde{F}_1 \cup \tilde{F}_2 \cup \tilde{G}_1 \cup \tilde{G}_2.$$

Define the fixed-point-free map $\tilde{f}\colon \tilde{X} \to \tilde{X}$ as follows. Let $\tilde{f} = f'$ on $\tilde{B}$. Let $\tilde{f} = \pi \circ f_2'$ on $N_1 \cup Q_1$. That is, $\tilde{f}$ is the 2-flattening map on the mboxes N_1 and Q_1. And let $\tilde{f} = f_2'$ on $N_n \cup Q_n$ for each $n \geq 2$. By construction, $\tilde{f}$ is continuous and fixed-point-free. Also $\tilde{f}$ is one-to-one on $\tilde{X} - (N_1 \cup Q_1)$, and only $(\tilde{W}_0 - \{\hat{p}\}) \cup (\tilde{Y}_0 - \{\hat{q}\})$ is not in the image of $\tilde{f}$.

The continuum $\tilde{X}$ is tree-like for reasons analogous to those that implied X is tree-like.

To see that $\tilde{X}$ is a union of two subcontinua H and K such that H, K, and $H \cap K$ have the fpp, we define the following subcontinua.

Let $H = \tilde{B} \cup \tilde{F}_1 \cup \tilde{G}_1$ and $K = \tilde{B} \cup \tilde{F}_2 \cup \tilde{G}_1 \cup \tilde{G}_2$. Note that $H \cap K = \tilde{B} \cup \tilde{G}_1$ and $\tilde{X} = H \cup K$. The proofs that H, K, and $H \cap K$ have the fpp are similar to the proof that X_0 has the fpp in Example 1. In reconstructing the basic idea of that proof, one should recall that there is no surjective map between any two of S, $-S$, and S^2 that fixes an endpoint. ∎

EXAMPLE 4. *There exist tree-like continuua H and K such that H, K, and $H \cap K$ have the fpp, but $H \cup K$ is a tree-like continuum that admits a fixed-point-free homeomorphism.*

Proof. We modify Example 3 to get this example in a manner analogous to how we modified Example 1 to produce Example 2. ∎

Appendix: On Bellamy's example. Let B be Bellamy's indecomposable tree-like continuum that contains a fan and admits a fixed-point-free map (see [1], [2]). In [1], B is $\hat{D}$. Otherwise, we use notation and terminology that was introduced in [1]. The reader will need to be familiar with Bellamy's definitions and notation. In fact, it would be helpful to review Bellamy's paper and have it at hand while reading this Appendix.

We provide proofs for several properties of Bellamy's example that are not immediately evident from his paper. The properties primarily involve the behavior of the points in the set J, which is the endpoint set of the fan in B, under the fixed-point-free map $\hat{F}$. We will show that $\hat{F}(J) = J$, $\hat{F}^{-1}(J) = J$, $\hat{F}|_J$ is not one-to-one, and J contains periodic points of period two and of period three under $\hat{F}$. The first observation of the existence of periodic points of period two and three in J was made by Howard Cook after a presentation by the second author of Bellamy's example in the Cook–Ingram faculty/student seminar at the University of Houston, circa 1977–1978.

We start with a few number-theoretic lemmas that will be used later. The first two are easily established.

LEMMA A1.

(i) *If n is an even non-negative integer, then* $2^n \equiv 1 \pmod 3$.
(ii) *If n is an odd positive integer, then* $2^n \equiv 2 \pmod 3$.

LEMMA A2.

(i) *If n is an even non-negative integer, then* $2^{3n} \equiv 1 \pmod 9$.
(ii) *If n is an odd positive integer, then* $2^{3n} \equiv -1 \pmod 9$.

LEMMA A3. *For $x, n \in \mathbb{N}$, $x + 2^n k \equiv 0 \pmod 3$ for some unique k in $\{0, 1, 2\}$.*

Proof. We consider three cases.

(i) Suppose $x \equiv 0 \pmod 3$. Then $x + 2^n k \equiv 0 + 2^n k \pmod 3$. Note that the unique solution, for $k \in \{0, 1, 2\}$, of $2^n k \equiv 0 \pmod 3$ is $k = 0$.

(ii) Suppose $x \equiv 1 \pmod 3$. Then $x + 2^n k \equiv 1 + 2^n k \pmod 3$. By Lemma A1, the unique solution of $1 + 2^n k \equiv 0 \pmod 3$ is $k = 1$ for odd n, and is $k = 2$ for even n.

(iii) If $x \equiv 2 \pmod 3$, the proof is similar to case (ii). ∎

Hereafter, we identify a point e^{it} on S^1 with the number t (mod 2π). So, exponentiation is replaced with multiplication, and multiplication with addition. Since $e^{i2k\pi} = 1$ and $e^{i(2k+1)\pi} = -1$ for all integers k, we identify values $t = 2k\pi$ and $t = (2k+1)\pi$ with 1 and -1, respectively.

At this point, we begin using definitions and notation from Bellamy's paper. The set M, of real parts of the second non-real coordinates of points of E (defined below), is an important set in Bellamy's construction. We let M' be a set of circle points whose real parts constitute M. We list the points of M' for reference:

$$M' = \Big\{\tfrac{\pi}{9}, \tfrac{4\pi}{9}, \tfrac{7\pi}{9}, \tfrac{10\pi}{9}, \tfrac{13\pi}{9}, \tfrac{16\pi}{9}, \tfrac{\pi}{18}, \tfrac{7\pi}{18}, \tfrac{13\pi}{18}, \tfrac{19\pi}{18}, \tfrac{25\pi}{18}, \tfrac{31\pi}{18}, \tfrac{\pi}{12}, \tfrac{5\pi}{12}, \tfrac{3\pi}{4}, \tfrac{13\pi}{12}, \tfrac{17\pi}{12}, \tfrac{7\pi}{4}, \tfrac{\pi}{36}, \tfrac{5\pi}{36}, \tfrac{13\pi}{36}, \tfrac{17\pi}{36}, \tfrac{25\pi}{36}, \tfrac{29\pi}{36}, \tfrac{37\pi}{36}, \tfrac{41\pi}{36}, \tfrac{49\pi}{36}, \tfrac{53\pi}{36}, \tfrac{61\pi}{36}, \tfrac{65\pi}{36}\Big\}.$$

Recall that Bellamy begins his construction with the 6-adic solenoid $\varSigma$. The unit circle S^1 in the complex plane is, of course, a group under complex multiplication. So, $\varSigma$ is a topological group under coordinatewise multiplication with identity element $e = (1, 1, 1, \ldots)$. The squaring map $z \mapsto z^2$ on S^1 is a homomorphism, and induces a squaring map f on $\varSigma$ that is also a homomorphism. By [1, Lemma 1], $\ker f = \{e\}$. Thus, f is one-to-one; in fact, f is a homeomorphism. The point e is the only fixed point of f, and the composant of $\varSigma$ that contains e is stretched away from e by f. In fact, for each point x of this composant, $f(x)$ is twice as far from e as x, relative to arc length in the composant.

The set H is the set of points of $\varSigma$ whose first coordinates are 1, and $E = H - \{e\}$. A compactification of E is given so that the remainder J is a totally disconnected compact set. The points of J become the endpoints of the fan as the construction continues. The endpoint set J is constructed in the product of two Cantor sets, namely $K = H \times \prod_{i=0}^{\infty} M$. For a point $z \in E \subset \varSigma$, $\phi(z)$ is the real part of the second non-real coordinate of z. So, ϕ is a mapping from E into M. The squaring map f on $\varSigma$ restricted to E is called s, the map $g\colon E \to K$ is defined by $g(x) = (x, \langle \phi s^n(x)\rangle_{n=0}^{\infty})$, and $\overline{s}\colon K \to K$ is defined by $\overline{s}(x, \langle t_k\rangle_{k=0}^{\infty}) = (s(x), \langle t_{k+1}\rangle_{k=0}^{\infty})$. The map $\hat{s} = \overline{s}|_{\overline{g(E)}}$ is an extension of the squaring map $s = f|_E$. Identifying E with $g(E)$, we see that $J = \overline{g(E)} - E$. So, J is the remainder in the compactification of E. As mentioned above, the points of J are the eventual endpoints of the fan in B.

An arc containing e is removed from $\varSigma$ to get $\varSigma_0$, and $\hat{\varSigma}$ is obtained by replacing the removed arc in $\varSigma_0$ with a suspension over the set J. The squaring map f induces a "squaring map" $\hat{f}\colon \hat{\varSigma} \to \hat{\varSigma}$ that agrees with f on $\varSigma_0$. Also, $\hat{f}|_J = \hat{s}|_J = \overline{s}|_J$. There are several steps remaining in the construction of B, but the set J is maintained throughout the construction, and the final fixed-point-free map $\hat{F}\colon B \to B$ interchanges the points of J in the same manner as $\hat{f}$ (or $\hat{s}$, or $\overline{s}$).

The behavior of the fixed-point-free map on J. We claim that $\hat{f}(J) = J$ and $\hat{f}^{-1}(J) = J$. The inclusion $\hat{f}(J) \subset J$ follows from the definition of $\hat{f}$ and $\overline{s}$.

Since the composant C of Σ that contains e is mapped to itself under f, it follows from the one-to-one-ness of f that no other composant of Σ is mapped by f into C. Let $\hat{C}$ be the composant of $\hat{\Sigma}$ that contains the set J. From Bellamy's construction and the observation above, it follows that $\hat{f}(\hat{C}) = \hat{C}$, and no other composant of $\hat{\Sigma}$ is mapped by $\hat{f}$ into $\hat{C}$. We also know from the construction that each point x of $\hat{C} - J$ is mapped by $\hat{f}$ twice as far from J as x. So, if $x \in \hat{C} - J$, then $\hat{f}(x) \notin J$. Thus $\hat{f}^{-1}(J) \subset J$.

Since $f\colon \Sigma \to \Sigma$ is surjective and $f(C) = C$, we get $f(\Sigma - C) = \Sigma - C$. Since $\hat{f}$ agrees with f on Σ_0, it follows that $\hat{f}(\hat{\Sigma} - \hat{C}) = \hat{\Sigma} - \hat{C}$. Hence,

$$\hat{f}(\hat{\Sigma}) = \hat{f}(\overline{\hat{\Sigma} - \hat{C}}) \subset \overline{\hat{f}(\hat{\Sigma} - \hat{C})} = \overline{\hat{\Sigma} - \hat{C}} = \hat{\Sigma}.$$

So, $\hat{f}\colon \hat{\Sigma} \to \hat{\Sigma}$ is surjective. Since $J \subset \hat{\Sigma}$ and $\hat{f}^{-1}(J) \subset J$, it follows that $J \subset \hat{f}(J)$. Together with $\hat{f}(J) \subset J$ from above, we have $\hat{f}(J) = J$ and $\hat{f}^{-1}(J) = J$, establishing our claims.

For t_1 a positive integer, we define a sequence $\{t_n\}$ recursively by letting $t_n = \frac{1}{3}(t_{n-1} + 2^{n+1}k_n)$, where k_n is chosen from the set $\{0, 1, 2\}$ so that t_n is an integer. It follows from Lemma A3 that $t_1, t_2, t_3, \ldots$ is well-defined and uniquely determined from the choice of t_1. We will refer to a sequence defined in this way as the *sequence generated by* t_1.

Define $x_n = \frac{1}{2^{n+1}} t_n \pi$ for $n \geq 1$. We call $\{x_n\}$ the *point of* Σ *derived from* $\{t_n\}$. Note that for $n \geq 2$,

$$\begin{aligned} 6x_n = \frac{6}{2^{n+1}} t_n \pi &= \frac{3}{2^n} t_n \pi \\ &= \frac{3}{2^n} \frac{1}{3}(t_{n-1} + 2^{n+1}k)\pi \qquad \text{for some } k \in \{0, 1, 2\} \\ &= \frac{1}{2^n} t_{n-1}\pi + 2k\pi \equiv \frac{1}{2^n} t_{n-1}\pi \pmod{2\pi}. \end{aligned}$$

That is, $6x_n \equiv x_{n-1} \pmod{2\pi}$.

It follows that the point x in $\prod_{n \geq 1} S^1$ whose nth coordinate is x_n is a point of the 6-adic solenoid Σ.

A period two point of J

Lemma A4. *Let $u_1 = 1$ and consider the integer sequence $\{u_n\}_{n \geq 1}$ generated by $u_1 = 1$. For $n \geq 1$,*

$$u_n \equiv \begin{cases} 3 \pmod 8 & \textit{for even } n, \\ 1 \pmod 8 & \textit{for odd } n. \end{cases}$$

Proof. Note that $u_1 = 1$ and $u_2 = \frac{1}{3}(1 + 2^3(1)) = \frac{9}{3} = 3$.

CASE (i). Assume the lemma is true for some $2n \geq 2$. So, $u_{2n} \equiv 3 \pmod 8$. Now, $u_{2n+1} = \frac{1}{3}(u_{2n} + 2^{2n+2}k) = \frac{1}{3}(u_{2n} - 3 + 2^{2n+2}k + 3)$. So, $u_{2n+1} - 1 = \frac{1}{3}(8\ell + 2^{2n+2}k) = 8\frac{1}{3}(\ell + 2^{2n-1}k)$. Since $u_{2n+1} - 1$ is an integer, $\frac{1}{3}(\ell + 2^{2n-1}k)$ is also an integer. It follows that $u_{2n+1} \equiv 1 \pmod 8$.

CASE (ii). Assume the lemma is true for some $2n+1 \geq 3$. So, $u_{2n+1} \equiv 1 \pmod 8$. Now, $u_{2n+2} = \frac{1}{3}(u_{2n+1} + 2^{2n+3}k) = \frac{1}{3}(u_{2n+1} - 1 + 2^{2n+3}k + 1 + 2^3 - 2^3) = \frac{1}{3}(8\ell + (2^{2n}k - 1)2^3 + 9)$. So, $u_{2n+2} - 3 = \frac{8}{3}(\ell + 2^{2n}k - 1)$. Since $u_{2n+2} - 3$ is an integer, $\frac{1}{3}(\ell + 2^{2n}k - 1)$ is also an integer. It follows that $u_{2n+2} \equiv 3 \pmod 8$. ∎

LEMMA A5. *Let $v_1 = 3$ and consider the integer sequence $\{v_n\}_{n\geq 1}$ generated by $v_1 = 3$. For $n \geq 1$,*

$$v_n \equiv \begin{cases} 1 \pmod 8 & \text{for even } n, \\ 3 \pmod 8 & \text{for odd } n. \end{cases}$$

Proof. Note that $v_2 = \frac{1}{3}(3 + 2^3 \cdot 0) = 1$, and hence $v_{n+1} = u_n$ for all $n \geq 1$. The result follows from Lemma A4. ∎

LEMMA A6. *Let $\{x_n\}_{n\geq 1}$ be the point of Σ derived from $\{u_n\}_{n\geq 1}$, and let $\{y_n\}_{n\geq 1}$ be the point of Σ derived from $\{v_n\}_{n\geq 1}$.*

(1) *For $n \geq 1$,*

$$2^{n-1}x_n \equiv \begin{cases} \frac{\pi}{4} \pmod{2\pi} & \text{for odd } n, \\ \frac{3\pi}{4} \pmod{2\pi} & \text{for even } n. \end{cases}$$

(2) *For $n \geq 1$,*

$$2^{n-1}y_n \equiv \begin{cases} \frac{\pi}{4} \pmod{2\pi} & \text{for even } n, \\ \frac{3\pi}{4} \pmod{2\pi} & \text{for odd } n. \end{cases}$$

Proof. (1) Note that $2^{n-1}x_n = 2^{n-1}\frac{1}{2^{n+1}}u_n\pi = \frac{1}{4}u_n\pi$. If n is odd, by Lemma A4,

$$\tfrac{1}{4}u_n\pi = \tfrac{1}{4}((u_n - 1)\pi + \pi) = \tfrac{1}{4}(8\ell\pi + \pi) = 2\ell\pi + \tfrac{\pi}{4}.$$

So, $2^{n-1}x_n \equiv \frac{\pi}{4} \pmod{2\pi}$.

If n is even, by Lemma A4,

$$\tfrac{1}{4}u_n\pi = \tfrac{1}{4}((u_n - 3)\pi + 3\pi) = \tfrac{1}{4}(8\ell\pi + 3\pi) = 2\ell\pi + \tfrac{3\pi}{4}.$$

So, $2^{n-1}x_n \equiv \frac{3\pi}{4} \pmod{2\pi}$.

(2) follows analogously from Lemma A5. ∎

For $n \geq 1$, let $p_1 = \big(1, -1, \frac{3\pi}{2}, x_1, x_2, \ldots\big)$, $p_2 = \big(1, 1, -1, \frac{3\pi}{2}, x_1, x_2, \ldots\big)$, and in general, $p_n = \big(1, 1, \ldots, 1, -1, \frac{3\pi}{2}, x_1, x_2, x_3, \ldots\big)$, where each of the first n coordinates of p_n is 1, the $(n+1)$st coordinate is -1, the $(n+2)$nd coordinate is $\frac{3\pi}{2}$, and the x sequence begins in the $(n+3)$rd coordinate of p_n. Note that $p_n \in \Sigma$ for each n, and more specifically, $p_n \in E$. Clearly, $\{p_n\}$ converges to the point $e = (1, 1, 1, \ldots)$.

Analogously define a sequence $\{q_n\}$ using the y sequence beginning in the $(n+3)$rd coordinate of q_n. That is, $q_1 = \big(1, -1, \frac{\pi}{2}, y_1, y_2, \ldots\big)$, $q_2 = \big(1, 1, -1, \frac{\pi}{2}, y_1, y_2, \ldots\big)$, etc. Each q_n is in E, and $\{q_n\}$ converges to the point $e = (1, 1, 1, \ldots)$.

Lemma A7. *For each $n \geq 1$,*

$$\langle \phi(p_n), \phi(s(p_n)), \phi(s^2(p_n)), \ldots \rangle = \big\langle \tfrac{\sqrt{2}}{2}, -\tfrac{\sqrt{2}}{2}, \tfrac{\sqrt{2}}{2}, \ldots \big\rangle,$$
$$\big\langle \phi(q_n), \phi(s(q_n)), \phi(s^2(q_n)), \ldots \rangle = \langle -\tfrac{\sqrt{2}}{2}, \tfrac{\sqrt{2}}{2}, -\tfrac{\sqrt{2}}{2}, \ldots \big\rangle.$$

Proof. Recalling the definition of ϕ, note that the second non-real coordinate of p_n is $x_1 = \frac{\pi}{4}$. So, $\phi(p_n) = \frac{\sqrt{2}}{2}$. Since s is the squaring map on E, through our identification, s "multiplies by 2" in each coordinate of p_n. Since $2 \cdot \frac{3\pi}{2} = 3\pi \equiv -1$ and $(-1)^2 = 1$, the second non-real coordinate of $s(p_n)$ is $2x_2$. Similarly, the second non-real coordinate of $s^k(p_n)$ is $2^k x_{k+1}$. By Lemma A6(1), $2x_2 \equiv \frac{3\pi}{4} \pmod{2\pi}$; so $\phi(s(p_n)) = -\frac{\sqrt{2}}{2}$. In general, $2^k x_{k+1}$ is equivalent $\pmod{2\pi}$ to either $\frac{\pi}{4}$ or $\frac{3\pi}{4}$ as k is even or odd. It follows that $\phi(s^2(p_n)) = \frac{\sqrt{2}}{2}$, $\phi(s^3(p_n)) = -\frac{\sqrt{2}}{2}$, and the values of $\phi(s^k(p_n))$ continue to alternate between $\frac{\sqrt{2}}{2}$ and $-\frac{\sqrt{2}}{2}$.

Similarly, the second non-real coordinate of q_n is $y_1 = \frac{3\pi}{4}$. So, $\phi(q_n) = -\frac{\sqrt{2}}{2}$, and by Lemma A6(2) the values of $\phi(s^k(q_n))$ alternate between $-\frac{\sqrt{2}}{2}$ and $\frac{\sqrt{2}}{2}$. ∎

Let $p = \big(e, \langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \ldots \rangle\big)$ and $q = \big(e, \langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, \ldots \rangle\big)$.

Theorem A1. *The points p and q form a period two orbit in J. That is, p and q are in J, $\overline{s}(p) = q$, and $\overline{s}(q) = p$.*

Proof. By Lemma A7 and the definition of g, we see that $g(p_n) = \big(p_n, \langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \ldots \rangle\big)$ for each $n \geq 1$. It follows that $\{g(p_n)\}$ converges to $p = \big(e, \langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \ldots \rangle\big)$.

Similarly, $g(q_n) = \big(q_n, \langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, \ldots \rangle\big)$ for each $n \geq 1$, and $\{g(q_n)\}$ converges to $q = \big(e, \langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, \ldots \rangle\big)$.

By construction, we see that $p, q \in \overline{g(E)} - E = J \subset K$. Recalling the definition of $\overline{s}\colon K \to K$, we conclude that $\overline{s}(p) = q$ and $\overline{s}(q) = p$. ∎

So, p is a point of period 2 under $\overline{s}$, and hence a point of period 2 under $\hat{F}$.

To complete this section, we wish to show that no point of $\hat{\Sigma}$ other than q maps to p, and vice versa. That is, $\hat{f}^{-1}(p) = \{q\}$ and $\hat{f}^{-1}(q) = \{p\}$.

From the section concerning the behavior of $\hat{f}$ on J, we know that if $\hat{f}(x) = \overline{s}(x) = p$ for some $x \in \hat{\Sigma}$, then $x \in J$. Let $x = (e, \langle x_1, x_2, \ldots \rangle)$ and suppose that $\overline{s}(x) = p$. We have

$$\overline{s}(x) = (e, \langle x_2, x_3, \ldots \rangle) = \big(e, \big\langle \tfrac{\sqrt{2}}{2}, -\tfrac{\sqrt{2}}{2}, \tfrac{\sqrt{2}}{2}, \ldots \big\rangle\big).$$

So, $x = \big(e, \big\langle x_1, \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, \ldots \big\rangle\big)$. Since $x \in J$, there exists a sequence $\{w_n\}$ of points in $\hat{E} = \overline{g(E)}$ that converges to x. For $n \geq 1$, let $w_n = (z_n, \langle \phi(z_n), \phi(s(z_n)), \ldots \rangle)$. Since M is finite, the sequence $\{\phi(s(z_n))\}$ must eventually be the constant sequence $\big\{\frac{\sqrt{2}}{2}\big\}$. So, for large n, the second non-real coordinate of $s(z_n)$ is either $\frac{\pi}{4}$ or $\frac{7\pi}{4}$. Hence, the third non-real coordinate of z_n is $\frac{9\pi}{8}$, $\frac{\pi}{8}$, $\frac{7\pi}{8}$, or $\frac{15\pi}{8}$. Thus, the second non-real coordinate of z_n must be $6 \cdot \frac{9\pi}{8}$, $6 \cdot \frac{\pi}{8}$, $6 \cdot \frac{7\pi}{8}$, or $6 \cdot \frac{15\pi}{8}$. But each of these circle points has real part $-\frac{\sqrt{2}}{2}$. So, $\phi(z_n) = -\frac{\sqrt{2}}{2}$ for large n, and thus $x_1 = -\frac{\sqrt{2}}{2}$. So, $x = q$.

The proof that $\hat{f}^{-1}(q) = \{p\}$ is similar.

A period three point of J

LEMMA A8. *Let $n \in \mathbb{N}$.*

(i) *If $n \equiv 0 \pmod 3$, then $2^n \cdot \frac{2\pi}{9} \equiv \frac{2\pi}{9} \pmod{2\pi}$ for even n, and $2^n \cdot \frac{2\pi}{9} \equiv \frac{16\pi}{9} \pmod{2\pi}$ for odd n.*

(ii) *If $n \equiv 1 \pmod 3$, then $2^n \cdot \frac{2\pi}{9} \equiv \frac{4\pi}{9} \pmod{2\pi}$ for even n, and $2^n \cdot \frac{2\pi}{9} \equiv \frac{14\pi}{9} \pmod{2\pi}$ for odd n.*

(iii) *If $n \equiv 2 \pmod 3$, then $2^n \cdot \frac{2\pi}{9} \equiv \frac{8\pi}{9} \pmod{2\pi}$ for even n, and $2^n \cdot \frac{2\pi}{9} \equiv \frac{10\pi}{9} \pmod{2\pi}$ for odd n.*

Proof. (i) Suppose first that $n \equiv 0 \pmod 3$ and n is even. Then by Lemma A2(i), $2^n - 1 = 9k$ for some integer k. So,

$$\frac{2^n - 1}{9} = k,$$
$$\frac{2^n 2\pi}{9} - \frac{2\pi}{9} = 2\pi k.$$

That is, $2^n \cdot \frac{2\pi}{9} \equiv \frac{2\pi}{9} \pmod{2\pi}$.

Suppose $n \equiv 0 \pmod 3$ and n is odd. Then by Lemma A2(ii), $2^n + 1 = 9k$ for some integer k. So,

$$2^n - 8 = 9(k-1),$$
$$\frac{2^n - 8}{9} = k - 1,$$
$$\frac{2^n 2\pi}{9} - \frac{16\pi}{9} = 2\pi(k-1).$$

That is, $2^n \cdot \frac{2\pi}{9} \equiv \frac{16\pi}{9} \pmod{2\pi}$.

The proofs of cases (ii) and (iii) are similar. ∎

Note that in each case of Lemma A8, the real part of $2^n \cdot \frac{2\pi}{9}$ is the same whether n is even or odd.

For $n \geq 1$, define the sequences

$$\left\{a_n = \frac{\pi}{2^{n-2}3^{n+1}}\right\}, \quad \left\{b_n = \frac{\pi}{2^{n-3}3^{n+1}}\right\}, \quad \left\{c_n = \frac{\pi}{2^{n-4}3^{n+1}}\right\}.$$

It is easy to check that these sequences are points of Σ.

Define three sequences of points in E as follows. For $n \geq 1$, let $\alpha_n = \big(1, 1, \ldots, 1, \frac{4\pi}{3}, a_1, a_2, \ldots\big)$, where the $\{a_i\}$ sequence starts in the $(n+2)$nd coordinate, which is the second non-real coordinate of α_n. For $n \geq 1$, define β_n and γ_n analogously using the sequences $\{b_i\}$ and $\{c_i\}$. That is, $\beta_n = \big(1, 1, \ldots, 1, \frac{2\pi}{3}, b_1, b_2, \ldots\big)$ and $\gamma_n = \big(1, 1, \ldots, 1, \frac{4\pi}{3}, c_1, c_2, \ldots\big)$.

Let t_2, t_4, and t_8 be, respectively, the real parts of $\frac{2\pi}{9}$, $\frac{4\pi}{9}$, and $\frac{8\pi}{9}$. Applying Lemma A8, it is easy to see that for each $n \geq 1$,

$$\begin{aligned}
\langle \phi(\alpha_n), \phi(s(\alpha_n)), \phi(s^2(\alpha_n)), \phi(s^3(\alpha_n)), \ldots \rangle &= \langle t_2, t_4, t_8, t_2, \ldots \rangle, \\
\langle \phi(\beta_n), \phi(s(\beta_n)), \phi(s^2(\beta_n)), \phi(s^3(\beta_n)), \ldots \rangle &= \langle t_4, t_8, t_2, t_4, \ldots \rangle, \\
\langle \phi(\gamma_n), \phi(s(\gamma_n)), \phi(s^2(\gamma_n)), \phi(s^3(\gamma_n)), \ldots \rangle &= \langle t_8, t_2, t_4, t_8, \ldots \rangle.
\end{aligned}$$

So, $\{g(\alpha_n)\}$ converges to the point $\alpha = (e, \langle t_2, t_4, t_8, t_2, \ldots \rangle)$, $\{g(\beta_n)\}$ converges to the point $\beta = (e, \langle t_4, t_8, t_2, t_4, \ldots \rangle)$, and $\{g(\gamma_n)\}$ converges to the point $\gamma = (e, \langle t_8, t_2, t_4, t_8, \ldots \rangle)$. Hence, we have the following theorem.

THEOREM A2. *The points α, β, and γ form a period three orbit in J. That is, α, β, and γ are in J, $\overline{s}(\alpha) = \beta$, $\overline{s}(\beta) = \gamma$, and $\overline{s}(\gamma) = \alpha$.*

Unlike the period two orbit, other points of J can be mapped by $\overline{s}$, and thus by $\hat{F}$, into the period three orbit. To see this, we note that $\overline{s}^{-1}(\alpha) \neq \{\gamma\}$. Let t_1 be the real part of $\frac{\pi}{9}$, which is not equal to any of t_2, t_4, or t_8. We can show, in a similar manner that we showed p, q, α, β, and γ are in J, that the point $y = (e, \langle t_1, t_2, t_4, t_8, t_2, t_4, \ldots \rangle)$ is in J. Also, $y \neq \gamma$. By definition of $\overline{s}$ we see that $\overline{s}(y) = \alpha$. So, $\overline{s}$ is not one-to-one on J, and therefore $\hat{F}|_J$ is not one-to-one.

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