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On the Conley index in Hilbert spaces in the absence of uniqueness

by

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Abstract. Consider the ordinary differential equation

$$\dot{x} = Lx + K(x) \tag{1}$$

on an infinite-dimensional Hilbert space E, where L is a bounded linear operator on E which is assumed to be *strongly indefinite* and $K : E \to E$ is a completely continuous but *not necessarily locally Lipschitzian* map. Given any isolating neighborhood N relative to equation (1) we define a Conley-type index of N. This index is based on Galerkin approximation of equation (1) by finite-dimensional ODEs and extends to the non-Lipschitzian case the $\mathcal{LS}$-Conley index theory introduced in [9]. This extended $\mathcal{LS}$-Conley index allows applications to strongly indefinite variational problems $\nabla\Phi(x) = 0$ where $\Phi : E \to \mathbb{R}$ is merely a C^1-function.

1. Introduction. Let E be an infinite-dimensional Banach space. Consider the ordinary differential equation

$$\dot{x} = f(x) := Lx + K(x) \tag{1}$$

where L is a bounded linear operator on E which is assumed to be *strongly indefinite* and $K : E \to E$ is a completely continuous map.

Equations of type (1) are interesting mainly because their equilibria, i.e. the solutions of the equation

$$Lx + K(x) = 0 \tag{2}$$

often arise as periodic solutions of Hamiltonian systems or solutions of strongly indefinite elliptic systems.

If the map K is locally Lipschitzian then (1) generates a local flow π_f on E and one may try to use methods of Conley index theory to obtain existence and multiplicity results for equilibria of this equation. However, the two known direct infinite-dimensional versions of the Conley index theory

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are not suitable for this purpose. The infinite-dimensional Conley index theory developed by the second author in [15], though applicable to wide classes of parabolic and even hyperbolic equations, cannot be applied to equation (1), the reason being that bounded subsets of E are not necessarily π_f-admissible in the sense of [16]. The second infinite-dimensional Conley index theory defined in [3] does apply in such cases but gives only trivial index $\overline{0}$.

Recently a very elementary extension of the Conley index to Hilbert spaces E was proposed in [9], based on finite-dimensional approximations of (1). The so called $\mathcal{LS}$-Conley index $h(\pi_f, N)$ of an isolating neighborhood relative to the flow π_f is defined as the homotopy type of a certain *spectrum*.

Although much simpler than the theories developed in [15] and [3], the $\mathcal{LS}$-Conley index is applicable and can be nontrivial for the flow π_f. In fact, some applications to Hamiltonian systems have been given in [9]–[11].

On the other hand, in many situations occurring in applications to Hamiltonian or strongly indefinite elliptic systems the right hand side of (1) has the form

$$Lx + K(x) = \nabla \varPhi$$

where $\varPhi : E \to \mathbb{R}$ is only a C^1-function. In such cases the Cauchy problem for equation (1) does not necessarily have unique solutions, so (1) does not generate a flow and the index of [9] cannot be used.

In view of such applications we present in this paper an extension of the index theory from [9] to the case of equations of type (1) with a merely *continuous* right hand side. For every isolating neighborhood N relative to f we define an index $h(f, N)$ and show that all properties of the $\mathcal{LS}$-Conley index theory proved in [9] hold in this more general setting. In addition, we show that the index depends only on the isolated invariant set in question and not on the choice of its isolating neighborhood.

It should be remarked that whenever the $\mathcal{LS}$-Conley index theory from [9] is applicable, then so is the theory presented here and the two indices are the same. This follows from Remark 4.18 below.

The reader interested in a concrete application of the results of this article to strongly indefinite elliptic systems is referred to the paper [12], in which, among other things, we give simple proofs of some results obtained earlier by Angenent and Van der Vorst [1] using a version of Floer homology.

This paper is organized as follows.

In Section 2 we explain some notation and introduce a few basic definitions.

In Section 3 we present an extension of the classical Conley theory to finite-dimensional ordinary differential equations with a merely continuous right-hand side and prove various properties of this index.

In Section 4 the extended $\mathcal{LS}$-Conley index is defined and its basic properties are stated.

Section 5 contains the proofs of the results from Section 4.

Finally, in Section 6 we show that all the results of the previous sections carry over to the G-equivariant case. This more refined *G-equivariant $\mathcal{LS}$-Conley index* is particularly useful for problems with symmetries.

2. Notation and preliminaries. In this paper we use standard notation, denoting, in particular, the set of real numbers by $\mathbb{R}$, the set of integers by $\mathbb{Z}$ and the set of positive integers by $\mathbb{N}$. Given sets A, B, C with $A \subset B$ and a function $f : B \to C$ we denote by $f|_A$ the restriction of f to A. Given a topological space X and a subset S of X we denote respectively by $\mathrm{Int}_X S$, $\mathrm{Cl}_X S$ and $\partial_X S$ the *interior*, *closure* and *boundary* of the set S relative to the topology of X.

Throughout this paper, unless otherwise specified, $(E, \|\cdot\|)$ is an arbitrary Banach space, U an open subset of E and N a closed bounded subset of E with $N \subset U$. By $\mathcal{C}$ we denote the set of all continuous maps from U to E.

For every function $f : U \to E$ and $S \subset U$ we set

$$|f|_S = \sup_{x\in S} \|f(x)\|.$$

Note that $|f|_S \in [0, \infty]$.

Given an arbitrary function $x : \mathbb{R} \to E$ we denote by $\omega(x)$ (resp. $\alpha(x)$) the set of all $a \in E$ for which there is a sequence $(t_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ such that $t_n \to \infty$ (resp. $t_n \to -\infty$) and $x(t_n) \to a$ as $n \to \infty$. It is clear that $\omega(x) = \bigcap_{t\in[0,\infty[} \mathrm{Cl}_E\, x([t, \infty[)$ and $\alpha(x) = \bigcap_{t\in]-\infty,0]} \mathrm{Cl}_E\, x(]-\infty, t])$ so these sets are closed in E.

Let $f \in \mathcal{C}$. By a *solution of* f we mean a differentiable function $x : \mathbb{R} \to E$ mapping $\mathbb{R}$ into U and such that

$$\dot{x}(t) = f(x(t)) \quad \text{for all } t \in \mathbb{R}.$$

(Note that, in this paper, a solution is what is usually termed a *complete* or *full* solution.) Given $S \subset U$ we denote by $\mathrm{Sol}(f, S)$ the set of all solutions x of f with $x(\mathbb{R}) \subset S$.

Given $f \in \mathcal{C}$ and $S \subset U$ we say that (f, S) is *gradient-like with respect to* φ if φ is a continuous function from S to $\mathbb{R}$ such that:

1. whenever $x \in \mathrm{Sol}(f, S)$ then the function $\varphi \circ x$ is nonincreasing;
2. whenever $x \in \mathrm{Sol}(f, S)$ and the function $\varphi \circ x$ is constant, then the function x is constant.

Note that if E is a Hilbert space and $f = \nabla F$, where $F : U \to \mathbb{R}$ is a C^1-function, then (f, U) is gradient-like with respect to $\varphi := -F$.

Let us now make the following basic

Definition 2.1. Given $f \in \mathcal{C}$ and $S \subset U$ we define $\mathrm{Inv}(f, S)$ to be the set of all points $a \in S$ for which there is an $x \in \mathrm{Sol}(f, S)$ such that $x(0) = a$. The set S is called an *invariant set relative to* f if $\mathrm{Inv}(f, S) = S$.

A set M is called an *isolating neighborhood relative to* f if M is bounded and closed in E and $\mathrm{Inv}(f,M) \subset \mathrm{Int}_E M$. A set S is called an *isolated invariant set relative to* f if there is a set M, bounded and closed in E and such that $S = \mathrm{Inv}(f,M)$ and $\mathrm{Inv}(f,M) \subset \mathrm{Int}_E M$. In this case we say that M is an *isolating neighborhood of* S *relative to* f.

REMARK 2.2. If E is finite-dimensional and $f \in \mathcal{C}$ is locally Lipschitzian, then for S and M compact the notions defined in Definition 2.1 coincide with the corresponding concepts of the classical Conley index theory for the (local) flow induced by the map f (see [5]). If f is merely continuous then, due to the lack of uniqueness of solutions, the map f, in general, does not define a flow on U. The situation is then somewhat similar to the (local) semiflow case, where, in general, there is no uniqueness (or even existence) of backward-time solutions. In fact, Definition 2.1 is inspired by the Conley index theory for semiflows (see [15]).

3. The finite-dimensional case. Throughout this section let E be *finite-dimensional*. We assume that the reader is familiar with the classical Conley index theory for local flows on a locally compact metric space, as expounded in the monographs [5] or [16].

We will now present an extension of the Conley index theory to the case of ordinary differential equations on U with a merely *continuous* right hand side, analogous to the extension, made by Leray and Schauder, of Brouwer mapping degree to infinite dimensions.

Several results of this section are known and were obtained by P. Baiti in his master thesis [2]. However, our proofs are simpler and we also establish additional important results (e.g. Proposition 3.8). For the (more special) case of the cohomological Conley index, the results of this section also follow from very general results of Mrozek in [14] for multivalued-differential equations. Again, the approach taken here is much simpler.

We begin with the following well known result, which follows from the Arzelà–Ascoli Theorem, and is a special case of Kamke's Theorem for finite-dimensional ordinary differential equations:

PROPOSITION 3.1. *For every* $n \in \mathbb{N}$ *let* $f_n \in \mathcal{C}$ *and* $x_n \in \mathrm{Sol}(f_n, N)$*. If* $f \in \mathcal{C}$ *and* $|f_n - f|_N \to 0$ *as* $n \to \infty$ *then there is a subsequence* $(x_{n_k})_{k\in\mathbb{N}}$ *of* $(x_n)_{n\in\mathbb{N}}$ *and an* $x \in \mathrm{Sol}(f,N)$ *such that* $x_{n_k} \to x$ *as* $k \to \infty$*, uniformly on compact subsets of* $\mathbb{R}$*.* ∎

REMARK 3.2. Note that Proposition 3.1 is also a special case of Proposition 4.3, stated and proved below.

Proposition 3.3. *Suppose $f \in \mathcal{C}$ and N is an isolating neighborhood relative to f. Then there is an $\varepsilon > 0$ such that whenever $g \in \mathcal{C}$ and $|g - f|_N < \varepsilon$ then N is an isolating neighborhood relative to g.*

We denote by $\varepsilon(f, N)$ the *supremum* of all such numbers $\varepsilon > 0$.

Proof. If the proposition is not true then there is a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{C}$ with $|f_n - f|_N \to 0$ as $n \to \infty$, and there is a sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n \in \mathrm{Sol}(f_n, N)$ and $x_n(0) \in \partial_E N$ for every $n \in \mathbb{N}$. Proposition 3.1 implies that there is a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ and an $x \in \mathrm{Sol}(f, N)$ such that $x_{n_k} \to x$ as $k \to \infty$, uniformly on compact subsets of $\mathbb{R}$. In particular, $x(0) \in \partial_E N$ so $\mathrm{Inv}(f, N) \not\subset \mathrm{Int}_E N$, a contradiction. ∎

Remark 3.4. Note that, for $f \in \mathcal{C}$, the fact that N is closed and Proposition 3.1 with $f_n \equiv f$ imply that $\mathrm{Inv}(f, N)$ is closed, i.e. compact.

Note that if $g \in \mathcal{C}$ is *locally Lipschitzian*, N is an isolating neighborhood relative to g and π_g is the local flow generated by g then the Conley index $h(g, S) := h(\pi_g, S)$ of the set $S := \mathrm{Inv}(g, N)$ is well defined (see [5] or [16]). Since the isolating neighborhood N uniquely determines the isolated invariant set S we also write $h(g, N)$ instead of $h(g, S)$. This will not lead to confusion.

We can now extend the concept of the Conley index to ODEs with nonunique solutions:

Definition 3.5 (cf. [2]). Suppose $f \in \mathcal{C}$ and N is an isolating neighborhood relative to f. Set

$$h(f, N) := h(g, N)$$

where $g \in \mathcal{C}$ is an arbitrary *locally Lipschitzian map* with $|g - f|_N < \varepsilon(f, N)$. We call $h(f, N)$ the *Conley index of the isolating neighborhood N relative to f*. If $H = (H_q)_{q\in\mathbb{Z}}$ is an arbitrary homology or cohomology theory, then we call the graded group $H(h(f, N)) = (H_q(h(f, N)))_{q\in\mathbb{Z}}$ the *(co)homological Conley index of the isolating neighborhood N relative to f.*

The following result shows that this definition is independent of the choice of g:

Proposition 3.6. *If $g, g' \in \mathcal{C}$ are locally Lipschitzian maps with $|g - f|_N < \varepsilon(f, N)$ and $|g' - f|_N < \varepsilon(f, N)$ then*

$$h(g, N) = h(g', N).$$

Proof. For $\theta \in [0, 1]$ set $g_\theta := (1 - \theta)g + \theta g'$. It follows that g_θ is locally Lipschitzian and $|g_\theta - f|_N < \varepsilon(f, N)$ for all $\theta \in [0, 1]$. By the continuation invariance property of the classical Conley index we see that $h(g_\theta, N)$ is independent of $\theta \in [0, 1]$. This proves the proposition. ∎

Remark 3.7. The index just defined is also independent of the open set U with $N \subset U$. In fact, if U_1, U_2 are open subsets of E with $N \subset U_1 \cap U_2$ and $f_1 : U_1 \to E$, $f_2 : U_2 \to E$ are continuous maps with $f_1|_N = f_2|_N$ then N is an isolating neighborhood relative to f_1 if and only if N is an isolating neighborhood relative to f_2. In this case $h(f_1, N) = h(f_2, N)$. The obvious proofs of these assertions are left to the reader.

As in the classical case, the index $h(f, N)$ depends only on the isolated invariant set $\operatorname{Inv}(f, N)$:

Proposition 3.8. *Suppose that $f \in \mathcal{C}$ and S is an isolated invariant set relative to f. If N' and N'' are two isolating neighborhoods of S relative to f then*

$$h(f, N') = h(f, N'').$$

Proof. Suppose $h(f, N') \neq h(f, N'')$. Choose a sequence $(g_n)_{n\in\mathbb{N}}$ of locally Lipschitzian maps in $\mathcal{C}$ such that $|g_n - f|_U \to 0$ as $n \to \infty$. By the definition of the Conley index,

$$h(g_n, N') \neq h(g_n, N'') \quad \text{for all } n \text{ large enough.}$$

Taking a subsequence and exchanging N' with N'' if necessary, we may assume that

$$\operatorname{Inv}(g_n, N') \setminus \operatorname{Inv}(g_n, N'') \neq \emptyset \quad \text{for all } n \in \mathbb{N}.$$

Therefore for every $n \in \mathbb{N}$ there is an $x_n \in \operatorname{Sol}(g_n, N')$ with $x_n(0) \notin \operatorname{Int}_E N''$. An application of Proposition 3.1 yields an $x \in \operatorname{Sol}(f, N')$ with $x(0) \notin \operatorname{Int}_E N''$. Hence $\operatorname{Inv}(f, N') \neq \operatorname{Inv}(f, N'')$, a contradiction. ∎

Thus, whenever $f \in \mathcal{C}$ and S is a compact isolated invariant set relative to f, then we may define the *Conley index $h(f, S)$ of S relative to f* as

$$h(f, S) := h(f, N'),$$

where N' is any isolating neighborhood of S relative to f. In view of Proposition 3.8 this definition is unambiguous.

The index just defined is nontrivial:

Proposition 3.9 (cf. [2]). *If $f \in \mathcal{C}$, N is an isolating neighborhood relative to f and $h(f, N) \neq \overline{0}$, then* $\operatorname{Inv}(f, N) \neq \emptyset$.

Proof. Remember that $\overline{0}$ is the homotopy type of any pointed one-point set. Choose a sequence $(g_n)_{n\in\mathbb{N}}$ of locally Lipschitzian maps in $\mathcal{C}$ such that $|g_n - f|_U \to 0$ as $n \to \infty$. By the definition of the Conley index, $h(g_n, N) \neq \overline{0}$ for all n large enough, so by Conley index theory $\operatorname{Inv}(g_n, N) \neq \emptyset$ for all such n. An application of Proposition 3.1 now shows that $\operatorname{Inv}(f, N) \neq \emptyset$. ∎

We also have the following important property:

PROPOSITION 3.10. *If $f, f' \in \mathcal{C}$, N is an isolating neighborhood relative to f and $|f' - f|_N < \varepsilon(f, N)$ then*

$$h(f', N) = h(f, N).$$

Proof. Since N is an isolating neighborhood relative to f', we know that $\varepsilon(f', N)$ is well defined and positive. Choose a locally Lipschitzian map $g \in \mathcal{C}$ such that

$$|g - f'|_N < \min(\varepsilon(f', N), \varepsilon(f, N) - |f' - f|_N).$$

Then both $|g - f'|_N < \varepsilon(f', N)$ and $|g - f|_N < \varepsilon(f, N)$, so $h(f', N) = h(g, N) = h(f, N)$. ∎

As a corollary we obtain the following version of the *continuation invariance* property:

COROLLARY 3.11 (cf. [2]). *Let (Λ, d) be a metric space and $(f_\lambda)_{\lambda \in \Lambda}$ be a family in $\mathcal{C}$ such that the map*

$$\Lambda \times N \to E, \qquad (\lambda, x) \mapsto f_\lambda(x),$$

is continuous. Assume that for every $\lambda \in \Lambda$ the set N is an isolating neighborhood relative to f_λ. Then the map $\lambda \mapsto h(f_\lambda, N)$ is locally constant. In particular, if Λ is connected then the Conley index $h(f_\lambda, N)$ is independent of $\lambda \in \Lambda$.

Proof. Let $C(N, E)$ be the space of all continuous functions from N to E endowed with the supremum norm. Since N is compact, our hypotheses imply that the map $\Phi : \Lambda \to C(N, E)$, $\lambda \mapsto f_\lambda|_N$, is continuous. Thus, for every $\lambda_0 \in \Lambda$ there is a $\delta \in]0, \infty[$ such that $d(\lambda, \lambda_0) < \delta$ implies $|f_\lambda - f_{\lambda_0}|_N < \varepsilon(f_{\lambda_0}, N)$, i.e., by Proposition 3.10, $h(f_\lambda, N) = h(f_{\lambda_0}, N)$. In other words, the map $\lambda \mapsto h(f_\lambda, N)$ is locally constant. ∎

Finally, the sum formula and the product formula hold for the index:

PROPOSITION 3.12. *Let $f \in \mathcal{C}$ and let $N', N'' \subset U$ be disjoint closed bounded subsets of E. Then $N' \cup N''$ is an isolating neighborhood relative to f if and only if N' and N'' are both isolating neighborhoods relative to f. In this case*

$$h(f, N' \cup N'') = h(f, N') \vee h(f, N'').$$

Proof. Approximate f by an appropriate locally Lipschitzian map and use the sum formula of the classical Conley index theory. ∎

PROPOSITION 3.13. *Suppose there is a direct sum decomposition $E = {}^1E \oplus {}^2E$ where ${}^1E$ and ${}^2E$ are linear subspaces of E. For $i = 1, 2$ let ${}^iU$ be open in ${}^iE$, ${}^iN$ be closed and bounded in ${}^iE$ with ${}^iN \subset {}^iU$ and let ${}^if : {}^iU \to {}^iE$ be a continuous map. Define $f := {}^1f \oplus {}^2f : {}^1U \oplus {}^2U \to {}^1E \oplus {}^2E$, i.e. $f({}^1x + {}^2x) = {}^1f({}^1x) + {}^2f({}^2x)$ for ${}^1x \in {}^1U$ and ${}^2x \in {}^2U$. Then ${}^1N \oplus {}^2N$ is an isolating neighborhood relative to f if and only if ${}^1N$ is an isolating* *neighborhood relative to* ${}^1f$ *and* ${}^2N$ *is an isolating neighborhood relative to* ${}^2f$. *In this case*

$$h(f, {}^1N \oplus {}^2N) = h({}^1f, {}^1N) \wedge h({}^2f, {}^2N).$$

Proof. Approximate ${}^1f$ and ${}^2f$ by locally Lipschitzian maps and use the product formula of the classical Conley index theory. ∎

4. The infinite-dimensional case. In this section we assume that E is infinite-dimensional.

We will now extend the $\mathcal{LS}$-homotopy index theory from [9] to the case of ordinary differential equations of the form

$$\dot{x} = Lx + K(x) \tag{3}$$

where L is a bounded linear operator on E satisfying certain assumptions and $K : U \to E$ is a (*not necessarily locally Lipschitzian*) completely continuous map.

Although the Cauchy problem for equation (3) is (locally) solvable (cf. [6, pp. 21–23]), these (local) solutions are not necessarily unique. Therefore equation (3), in general, does not generate a (local) flow on U, and so the index defined in [9] cannot be applied in such a case.

On the other hand, as in [9], one can use a Galerkin approximation of (3) by a sequence of finite-dimensional ordinary differential equations. These equations have continuous right hand sides and, consequently, one can apply to them the finite-dimensional Conley index defined in Section 3. Thus a Conley-type index (of an isolating neighborhood) of (3) can be defined as a *sequence* of Conley indices (of the corresponding isolating neighborhoods) of certain finite-dimensional approximations of (3).

We prove several properties of this index. In particular, we establish a result (which is new even in the locally Lipschitzian case) stating that the index depends only on the isolated invariant set in question and not on the choice of its isolating neighborhood.

The proofs of the statements presented in this section are given in Section 5.

Let us make the following convenient

Definition 4.1. The quadruple (L, E_{-1}, E_0, E_1) is called a *trichotomy on* E if the following properties are satisfied:

1. $L : E \to E$ is a bounded linear operator.
2. E_j, $j = -1, 0, 1$, are closed L-invariant subspaces of E with $E = E_{-1} \oplus E_0 \oplus E_1$ and E_0 is finite-dimensional.
3. For $j = -1, 0, 1$ let $L_j : E_j \to E_j$ be the restriction of L to E_j. Then there are constants $M \in [0, \infty[$ and $\alpha \in]0, \infty[$ such that

$$\|e^{L_{-1}t}\|_{\mathcal{L}(E_{-1},E_{-1})} \le Me^{-\alpha t} \quad \text{for all } t \in [0, \infty[\tag{4}$$

and

$$\|e^{L_1 t}\|_{\mathcal{L}(E_1,E_1)} \le Me^{\alpha t} \quad \text{for all } t \in \,]-\infty, 0]. \tag{5}$$

The triple (L, E_{-1}, E_1) is called a *dichotomy on* E if the quadruple (L, E_{-1}, E_0, E_1) with $E_0 = \{0\}$ is a trichotomy on E.

Now consider the following

HYPOTHESIS 4.2. *(L, E_{-1}, E_0, E_1) is a given trichotomy on E.*

If $K \in \mathcal{C}$ and L is as in Hypothesis 4.2, then we write $f_K := L|_U + K$. Note that $f_K \in \mathcal{C}$.

We have the following basic result, which is the infinite-dimensional analogue of Proposition 3.1.

PROPOSITION 4.3. *Assume Hypothesis* 4.2. *Let $(K_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{C}$ such that $\bigcup_{n\in\mathbb{N}} K_n(N)$ is relatively compact in E. For every $n \in \mathbb{N}$ set $f_n := f_{K_n}$ and let $x_n \in \mathrm{Sol}(f_n, N)$. Then there is a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ converging, uniformly on compact subsets of $\mathbb{R}$, to a continuous map $x : \mathbb{R} \to E$ with $x(\mathbb{R}) \subset N$.*

If, in addition, $K \in \mathcal{C}$ and $K_n \to K$ as $n \to \infty$ uniformly on compact subsets of N, then $x \in \mathrm{Sol}(f, N)$ where $f := f_K$.

The last result has the following

COROLLARY 4.4. *If $K \in \mathcal{C}$ is such that $K(N)$ is relatively compact in E, then $\mathrm{Inv}(f_K, N)$ is compact. Moreover, whenever $x \in \mathrm{Sol}(f_K, N)$ then $\alpha(x)$ and $\omega(x)$ are nonempty, compact and invariant. Finally, if (f_K, N) is gradient-like with respect to φ and $\mathrm{Inv}(f_K, N)$ is nonempty, then there exists an $a \in N$ with $f_K(a) = 0$.*

REMARK 4.5. The importance of the above corollary lies in the fact that, in applications to variational problems, the space E is frequently a Hilbert space and $f_K = \nabla F$ for a given C^1-function $F : U \to \mathbb{R}$. One is interested in finding critical points of F, i.e. solutions $a \in U$ of the equation

$$f_K(a) = 0. \tag{6}$$

In that case (f_K, N) is gradient-like with respect to the function $\varphi := -F$. Thus, if $\mathrm{Inv}(f_K, N) \neq \emptyset$ (a property that can often be proved by the Conley index theory, cf. Proposition 4.12 below) then Corollary 4.4 implies the existence of a critical point of F contained in N.

Now consider the following additional

HYPOTHESIS 4.6. 1. *$(P^l)_{l\in\mathbb{N}}$ is a sequence of bounded linear operators on E such that $P^l \to \mathrm{Id}$ as $l \to \infty$, uniformly on compact subsets of E. Here, Id is the identity map on E.*

2. *For every $l \in \mathbb{N}$ the space $E^l := P^l(E)$ is L-invariant. Let $L^l : E^l \to E^l$ be the restriction of L to E^l.*

Given $l \in \mathbb{N}$ and an arbitrary function v defined on U and such that $v(U) \subset E^l$ define the function $f_v^l : U \cap E^l \to E^l$ as

$$f_v^l(x) := Lx + v(x) = L^l x + v(x) \quad \text{for all } x \in U \cap E^l.$$

As a corollary of Proposition 4.3 we obtain

Proposition 4.7. *Assume Hypotheses* 4.2 *and* 4.6. *Suppose $K \in \mathcal{C}$ and N is an isolating neighborhood relative to f_K. Let $(K_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{C}$ such that $K_n \to K$ as $n \to \infty$ uniformly on compact subsets of N. Assume also that $K(N)$ and $\bigcup_{n\in\mathbb{N}} K_n(N)$ are relatively compact in E. Then there exist an $l_0 \in \mathbb{N}$ and an $n_0 \in \mathbb{N}$ such that for all $l \geq l_0$, $n \geq n_0$ and $s \in [0,1]$ the set $N \cap E^l$ is an isolating neighborhood relative to $f^l_{P^l\circ(sK+(1-s)K_n)}$.*

Consider the next additional

Hypothesis 4.8. *The subspace E^l is finite-dimensional for all $l \in \mathbb{N}$.*

We can now make the following basic

Definition 4.9. Assume Hypotheses 4.2, 4.6 and 4.8. Let $K \in \mathcal{C}$ be such that the set $K(N)$ is relatively compact in E and N is an isolating neighborhood relative to f_K. Proposition 4.7 with $K_n \equiv K$ implies that there is an $l_0 \in \mathbb{N}$ such that $N \cap E^l$ is an isolating neighborhood relative to $f^l_{P^l\circ K}$ for all $l \geq l_0$. Let $l_0(K,N)$ be the *minimum* of all such l_0.

Then, by the results of Section 3, the Conley index $h(f^l_{P^l\circ K}, N \cap E^l)$ is defined for all $l \geq l_0(K,N)$. We define the *$\mathcal{LS}$-Conley index $h(f_K,N)$ of N relative to f_K* to be the sequence $(h(f_K,N)_l)_{l\geq l_0(K,N)}$, where

$$h(f_K,N)_l := h(f^l_{P^l\circ K}, N \cap E^l) \quad \text{for all } l \geq l_0(K,N).$$

If $H = (H_q)_{q\in\mathbb{Z}}$ is an arbitrary homology or cohomology theory, then the sequence

$$(H(h(f_K,N)_l))_{l\geq l_0(K,N)}$$

of graded groups $H_q(h(f_K,N)_l)$, $q \in \mathbb{Z}$, is called the *(co)homological $\mathcal{LS}$-Conley index of N relative to f_K.*

Remark 4.10. Note that the obvious analogue of Remark 3.7 is valid in the present case. We leave its formulation and proof to the reader.

From now on we assume Hypotheses 4.2, 4.6 and 4.8, unless specified otherwise.

The index just defined depends, in some sense, only on the invariant set isolated by N. More precisely, we have the following

Proposition 4.11. *Let $K \in \mathcal{C}$ and let $N', N'' \subset U$ be closed bounded subsets of E such that $K(N')$ and $K(N'')$ are relatively compact in E. If N' and N'' are both isolating neighborhoods relative to f_K and* $\mathrm{Inv}(f_K, N') =$

$\mathrm{Inv}(f_K, N'')$ *then there is an* l_1 *with* $l_1 \geq l_0(f_K, N')$ *and* $l_1 \geq l_0(f_K, N'')$ *such that*

$$h(f_K, N')_l = h(f_K, N'')_l \quad \text{for all } l \geq l_1.$$

The index is nontrivial in the following sense:

PROPOSITION 4.12. *If* $K \in \mathcal{C}$ *is such that* $K(N)$ *is relatively compact in* E *and* N *is an isolating neighborhood relative to* f_K *with* $\mathrm{Inv}(f_K, N) = \emptyset$, *then there is an* $l_1 \geq l_0(K, N)$ *such that*

$$h(f_K, N)_l = \overline{0} \quad \text{for all } l \geq l_1.$$

The index also enjoys a continuation invariance property, which we state here in the following form:

PROPOSITION 4.13. *Let* (Λ, d) *be a connected metric space and* $(K_\lambda)_{\lambda\in\Lambda}$ *be a family in* $\mathcal{C}$ *such that the map*

$$\Lambda \times N \to E, \qquad (\lambda, x) \mapsto K_\lambda(x),$$

is continuous and $\bigcup_{\lambda\in\Lambda} K_\lambda(N)$ *is relatively compact in* E. *Assume that for every* $\lambda \in \Lambda$ *the set* N *is an isolating neighborhood relative to* f_{K_λ}. *Then the Conley index* $h(f_{K_\lambda}, N)$ *is independent of* $\lambda \in \Lambda$ *in the following sense*: *Whenever* $\lambda', \lambda'' \in \Lambda$ *then there is an* l_1 *with* $l_1 \geq l_0(K_{\lambda'}, N)$ *and* $l_1 \geq l_0(K_{\lambda''}, N)$ *such that*

$$h(f_{K_{\lambda'}}, N)_l = h(f_{K_{\lambda''}}, N)_l \quad \text{for all } l \geq l_1.$$

The sum formula takes the following form:

PROPOSITION 4.14. *Let* $K \in \mathcal{C}$ *and let* $N', N'' \subset U$ *be disjoint closed bounded subsets of* E *such that* $K(N')$ *and* $K(N'')$ *are relatively compact in* E. *Then* $N' \cup N''$ *is an isolating neighborhood relative to* f_K *if and only if* N' *and* N'' *are both isolating neighborhoods relative to* f_K. *In this case there is an* l_1 *with* $l_1 \geq l_0(K, N')$ *and* $l_1 \geq l_0(K, N'')$ *such that*

$$h(f_K, N' \cup N'')_l = h(f_K, N')_l \vee h(f_K, N'')_l \quad \text{for all } l \geq l_1.$$

The product formula takes the following, slightly complicated form:

PROPOSITION 4.15. *Suppose there is a direct sum decomposition* $E = {}^1E \oplus {}^2E$ *where* ${}^1E$ *and* ${}^2E$ *are closed* L-*invariant linear subspaces of* E. *For* $i = 1, 2$ *let* ${}^iL : {}^iE \to {}^iE$ *be the restriction of* L *to* ${}^iE$ *and suppose that there are linear subspaces* ${}^iE_j$, $j = -1, 0, 1$, *of* ${}^iE$ *such that the quadruple* $({}^iL, {}^iE_{-1}, {}^iE_0, {}^iE_1)$ *is a trichotomy on* ${}^iE$. *Suppose that*, *for* $i = 1, 2$, *Hypotheses* 4.6 *and* 4.8 *hold with* E, P^l, E^l, L *and* L^l *replaced by* ${}^iE$, ${}^iP^l$, ${}^iE^l$, ${}^iL$ *and* ${}^iL^l$, *respectively. For* $i = 1$, 2 *let* ${}^iU$ *be open in* ${}^iE$, ${}^iN$ *be closed and bounded in* ${}^iE$ *with* ${}^iN \subset {}^iU$ *and let* ${}^iK : {}^iU \to {}^iE$ *be a continuous map such that* ${}^iK({}^iN)$ *is relatively compact in* ${}^iE$. *Set* ${}^if = {}^iL|_{{}^iU} + {}^iK$. *Define* $f := {}^1f \oplus {}^2f : {}^1U \oplus {}^2U \to {}^1E \oplus {}^2E$, *i.e.* $f({}^1x + {}^2x) = {}^1f({}^1x) + {}^2f({}^2x)$ *for* ${}^1x \in {}^1U$ *and* ${}^2x \in {}^2U$. *Then* ${}^1N \oplus {}^2N$ *is an isolating neighborhood relative to* f *if and* *only if* ${}^1N$ *is an isolating neighborhood relative to* ${}^1f$ *and* ${}^2N$ *is an isolating neighborhood relative to* ${}^2f$. *In this case there is an* l_1 *with* $l_1 \geq l_0({}^1K, {}^1N)$ *and* $l_1 \geq l_0({}^2K, {}^2N)$ *such that*

$$h(f, {}^1N \oplus {}^2N)_l = h({}^1f, {}^1N)_l \wedge h({}^2f, {}^2N)_l \quad \text{for all } l \geq l_1.$$

Now assume the following additional

HYPOTHESIS 4.16. *For every* l *large enough there are linear* L-*invariant subspaces* F^l, F^l_{-1} *and* F^l_1 *of* E *such that* $E^{l+1} = F^l \oplus E^l$ *and the triple* $(L|_{F^l}, F^l_{-1}, F^l_1)$ *is a dichotomy on* F^l. *Let* i_l *denote the dimension of* F^l_1.

REMARK 4.17. Note that Hypotheses 4.2, 4.6, 4.8 and 4.16 are automatically satisfied in the case of equations considered in [9].

We then have the following result:

PROPOSITION 4.18. *Assume that* $K \in \mathcal{C}$ *is such that* $K(N)$ *is relatively compact in* E *and* N *is an isolating neighborhood relative to* f_K. *Then there is an* $l_1 \geq l_0(K, N)$ *such that*

$$h(f_K, N)_{l+1} = \Sigma^{i_l} \wedge h(f_K, N)_l \quad \text{for all } l \geq l_1,$$

where Σ^i *denotes the homotopy type of a pointed* i-*dimensional sphere.*

REMARK 4.19. Proposition 4.18 shows that, whenever Hypotheses 4.2, 4.6, 4.8 and 4.16 hold then we can equivalently define the $\mathcal{LS}$-Conley index in terms of the *spectrum* concept, as in [9]. In particular, we see that whenever the index theory from [9] is applicable then so is the theory presented here and the resulting indices are the same.

5. Proofs of the statements of Section 4. We will now prove the results stated in Section 4. We will use the following simple and well known results:

LEMMA 5.1. *Let* X *and* Y *be metric spaces and* $(f_n)_{n\in\mathbb{N}}$ *be a sequence of functions from* X *to* Y. *Suppose that* f *is a continuous function from* X *to* Y. *Then the following conditions are equivalent*:

1. $f_n \to f$ *as* $n \to \infty$, *uniformly on compact subsets of* X.
2. *Whenever* $a \in X$ *and* $(a_n)_{n\in\mathbb{N}}$ *is a sequence in* X *with* $a_n \to a$ *then* $f_n(a_n) \to f(a)$. ∎

LEMMA 5.2 (Variation-of-constants formula). *Let* $g : \mathbb{R} \to E$ *be a continuous map and* $y : \mathbb{R} \to E$ *be a differentiable map such that*

$$\dot{y}(t) = Ly(t) + g(t) \quad \text{for all } t \in \mathbb{R}.$$

Then for all $t, r \in \mathbb{R}$,

$$y(t) = e^{Lr} y(t-r) + \int_0^r e^{L(r-s)} g(s+t-r)\, ds. \ ∎$$

Proof of Proposition 4.3. Since the right hand side of the equation

$$\dot{x}_n(t) = Lx_n(t) + K_n(x_n(t))$$

has a bound independent of $n \in \mathbb{N}$ and $t \in \mathbb{R}$, the sequence $(x_n)_{n\in\mathbb{N}}$ is uniformly equicontinuous on $\mathbb{R}$. Thus, in order to prove the first part of the proposition, it is enough, by the Arzelà–Ascoli Theorem, to show that for every $t \in \mathbb{R}$, the set $\{x_n(t) \mid n \in \mathbb{N}\}$ is relatively compact in E. For $i = -1, 0, 1$ let Q_i be the projector of E onto E_i along the direct sum decomposition of Hypothesis 4.2. We just have to show that for $i = -1, 0, 1$ the set $\{Q_i(x_n(t)) \mid n \in \mathbb{N}\}$ is relatively compact in E_i. For $i = 0$ this is obvious since E_0 is finite-dimensional. Let $i = -1$. We use the same argument as that used in the proofs of Theorem III.4.4 of [16] and Theorem 3.3 of [4]. Let β be the Kuratowski measure of noncompactness on E_i. Let $\delta > 0$. Then there is an $r = r(\delta) \in]0, \infty[$ such that

$$Me^{-\alpha r} < \delta.$$

By the variation-of-constants formula we have

$$Q_i(x_n(t)) = e^{L_i r} Q_i(x_n(t-r)) + \int_0^r e^{L_i(r-s)} Q_i K_n(x_n(s+t-r))\, ds.$$

There is a compact set Z_1 in E_i such that

$$Q_i K_n(x_n(s+t-r)) \in Z_1 \qquad \text{for all } n \in \mathbb{N} \text{ and all } s \in [0, r].$$

Let $Z_2 := \{e^{L_i(r-s)} v \mid (s, v) \in [0, r] \times Z_1\}$. Of course, Z_2 is compact in E_i. Thus

$$\int_0^r e^{L_i(r-s)} Q_i K_n(x_n(s+t-r))\, ds \in rZ_3 \qquad \text{for all } n \in \mathbb{N}$$

where Z_3 is the closed convex hull of Z_2. Notice that the set Z_3 is compact by Mazur's Theorem so $\beta(rZ_3) = 0$ and so we obtain

$$\begin{aligned}\beta(\{Q_i(x_n(t)) \mid n \in \mathbb{N}\}) &\le \beta(\{e^{L_i r} Q_i(x_n(t-r)) \mid n \in \mathbb{N}\}) + \beta(rZ_3)\\ &\le 2MC_N e^{-\alpha r} \le 2C_N\delta\end{aligned}$$

where $C_N := \sup_{x\in N} \|x\| < \infty$. Since $\delta > 0$ is arbitrary, we obtain

$$\beta(\{Q_i(x_n(t)) \mid n \in \mathbb{N}\}) = 0,$$

which proves our claim. The case $i = 1$ is analogous. This establishes the first part of the proposition. To prove the second part, note that, by Lemma 5.1, $f_{K_{n_k}} \circ x_{n_k} \to f_K \circ x$ as $k \to \infty$, uniformly on compact subsets of $\mathbb{R}$. ∎

Proof of Corollary 4.4. By Proposition 4.3 every sequence of solutions of f_K mapping $\mathbb{R}$ into N has a subsequence which converges, uniformly on compact subsets of $\mathbb{R}$, to a solution of f_K mapping $\mathbb{R}$ into N. This shows that $\mathrm{Inv}(f_K, N)$ is compact. Whenever x is a solution of f_K then for every

$s \in \mathbb{R}$ the function $x(\cdot + s)$ is also a solution of f_K. If, in addition, x maps $\mathbb{R}$ into N and $(t_n)_{n\in\mathbb{N}}$ is an arbitrary sequence in $\mathbb{R}$ then, by Proposition 4.3, the sequence $(x(\cdot + t_n))_{n\in\mathbb{N}}$ has a subsequence converging, uniformly on compact subsets of $\mathbb{R}$, to a solution of f_K mapping $\mathbb{R}$ into N. This proves the assertions of the corollary concerning the sets $\omega(x)$ and $\alpha(x)$. If (f_K, N) is gradient-like with respect to φ and $\mathrm{Inv}(f_K, N) \neq \emptyset$, then there is an $x \in \mathrm{Sol}(f_K, N)$. Since φ is continuous and $\varphi \circ x$ is nonincreasing we deduce that $\varphi(a) = \inf_{t\in[0,\infty[} \varphi(x(t))$ for all $a \in \omega(x)$. Since $\omega(x)$ is nonempty and invariant relative to f_K there are an $a_0 \in \omega(x)$ and a $y \in \mathrm{Sol}(f_K, \omega(x))$ with $y(0) = a_0$. It follows that $\varphi \circ y$ is constant so y is constant. Thus $\dot{y}(t) \equiv 0$ and so $0 = \dot{y}(0) = f_K(y(0)) = f_K(a_0)$. The corollary is proved. ∎

Proof of Proposition 4.7. If the proposition is not true, then there are strictly increasing sequences $(l_k)_{k\in\mathbb{N}}$ and $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ and a sequence $(s_k)_{k\in\mathbb{N}}$ in $[0, 1]$ converging to some s such that, for every $k \in \mathbb{N}$, $N \cap E^{l_k}$ is not an isolating neighborhood relative to $f^{l_k}_{P^{l_k}\circ(s_kK+(1-s_k)K_{n_k})}$. Hence for every $k \in \mathbb{N}$ there is a solution $x_k : \mathbb{R} \to E^{l_k}$ of $f^{l_k}_{P^{l_k}\circ(s_kK+(1-s_k)K_{n_k})}$ with $x_k(\mathbb{R}) \subset N \cap E^{l_k}$ and $x_k(0) \in \partial_{E^{l_k}}(N \cap E^{l_k})$. Thus x_k is differentiable into E and

$$\dot{x}_k(t) = Lx_k(t) + K'_k(x_k(t)) \quad \text{for all } t \in \mathbb{R}$$

where $K'_k : U \to E$ is defined by $K'_k := P^{l_k} \circ (s_kK + (1 - s_k)K_{n_k})$.

We claim that $K'_k \to K$ as $k \to \infty$, uniformly on compact subsets of N. By Lemma 5.1 the claim will be proved if we show that whenever $a_k \to a$ in N as $k \to \infty$, then $K'_k(a_k) \to K(a)$ as $k \to \infty$. Now $K'_k(a_k) = P^{l_k}(b_k)$ where $b_k := s_kK(a_k) + (1 - s_k)K_{n_k}(a_k)$, $k \in \mathbb{N}$. By the assumptions of this proposition together with Lemma 5.1 we have $b_k \to sK(a) + (1 - s)K(a) = K(a)$ as $k \to \infty$ so, by Hypothesis 4.6 and Lemma 5.1, we conclude that $P^{l_k}(b_k) \to K(a)$ as $k \to \infty$. This proves the claim.

We also claim that $\bigcup_{k\in\mathbb{N}} K'_k(N)$ is relatively compact in E. To prove this it is sufficient to show that every sequence in $\bigcup_{k\in\mathbb{N}} K'_k(N)$ has a subsequence which converges in E. Let $(a_m)_{m\in\mathbb{N}}$ be a sequence in $\bigcup_{k\in\mathbb{N}} K'_k(N)$. Then there are sequences $(k_m)_{m\in\mathbb{N}}$ in $\mathbb{N}$ and $(b_m)_{m\in\mathbb{N}}$ in N such that $a_m = K'_{k_m}(b_m)$, $m \in \mathbb{N}$. Thus $a_m = P^{l_{k_m}}(s_{k_m}c_m + (1-s_{k_m})d_m)$, where $c_m = K(b_m)$ and $d_m = K_{n_{k_m}}(b_m)$, $m \in \mathbb{N}$. The assumptions of this proposition imply that $(c_m, d_m)_{m\in\mathbb{N}}$ has a subsequence, denoted again by $(c_m, d_m)_{m\in\mathbb{N}}$, which converges in $E \times E$ to some $(c, d) \in E \times E$.

Suppose first that $(k_m)_{m\in\mathbb{N}}$ is bounded. Then, taking subsequences if necessary, we may assume that $k_m \equiv k$ for some $k \in \mathbb{N}$. Since P^{l_k} is continuous we conclude that

$$a_m = P^{l_k}(s_kc_m + (1 - s_k)d_m) \to P^{l_k}(s_kc + (1 - s_k)d)$$

as $m \to \infty$ and the claim is proved in this case.

Now suppose that $(k_m)_{m\in\mathbb{N}}$ is unbounded. Then we may assume that $k_m \to \infty$ as $m \to \infty$. By Hypothesis 4.6 and Lemma 5.1 we thus find that $a_m \to sc + (1-s)d$ as $m \to \infty$ and the claim is also proved in this case.

Using the above two claims, applying Proposition 4.3 and taking a subsequence if necessary, we may assume that $x_k \to x$ uniformly on compact subsets of $\mathbb{R}$, where $x \in \mathrm{Sol}(f_K, N)$. It follows that $x(\mathbb{R}) \subset V := \mathrm{Int}_E N$. Since $x_k(0) \to x(0) \in V$, it follows that $x_k(0) \in V \cap E^{l_k} \subset \mathrm{Int}_{E^{l_k}}(N \cap E^{l_k})$ for all k large enough, a contradiction, which proves the proposition. ∎

Proof of Proposition 4.11. In view of Proposition 3.8 it is sufficient to prove that there is an l_1 with $l_1 \ge l_0(f_K, N')$ and $l_1 \ge l_0(f_K, N'')$ such that

$$\mathrm{Inv}(f^l_{P^l\circ K}, N' \cap E^l) = \mathrm{Inv}(f^l_{P^l\circ K}, N'' \cap E^l) \quad \text{for all } l \ge l_1.$$

If this is not true, then we may assume that there is a strictly increasing sequence $(l_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ and a sequence $(x_k)_{k\in\mathbb{N}}$ such that for every $k \in \mathbb{N}$ we have $x_k \in \mathrm{Sol}(f^{l_k}_{P^{l_k}\circ K}, N' \cap E^{l_k})$ with $x_k(0) \notin \mathrm{Int}_{E^{l_k}}(N'' \cap E^{l_k})$.

We claim that $K'_k := P^{l_k} \circ K \to K$ uniformly on compact subsets in N'. In fact, this follows immediately from Lemma 5.1 and Hypothesis 4.6.

We also claim that $\bigcup_{k\in\mathbb{N}} K'_k(N')$ is relatively compact in E. To prove this it is sufficient to show that every sequence in $\bigcup_{k\in\mathbb{N}} K'_k(N')$ has a subsequence which converges in E. Let $(a_m)_{m\in\mathbb{N}}$ be a sequence in $\bigcup_{k\in\mathbb{N}} K'_k(N')$. Then there are sequences $(k_m)_{m\in\mathbb{N}}$ in $\mathbb{N}$ and $(b_m)_{m\in\mathbb{N}}$ in N' such that $a_m = K'_{k_m}(b_m)$, $m \in \mathbb{N}$. Thus $a_m = P^{l_{k_m}} K(b_m)$, $m \in \mathbb{N}$. The assumptions of this proposition imply that $(K(b_m))_{m\in\mathbb{N}}$ has a subsequence, denoted again by $(K(b_m))_{m\in\mathbb{N}}$, which converges in E to some $c \in E$.

Suppose first that $(k_m)_{m\in\mathbb{N}}$ is bounded. Then, taking subsequences if necessary, we may assume that $k_m \equiv k$ for some $k \in \mathbb{N}$. Since P^{l_k} is continuous we conclude that $a_m = P^{l_k}(K(b_m)) \to P^{l_k}(c)$ as $m \to \infty$ and the claim is proved in this case.

Now suppose that $(k_m)_{m\in\mathbb{N}}$ is unbounded. Then we may assume that $k_m \to \infty$ as $m \to \infty$. By Hypothesis 4.6 and Lemma 5.1 we find that $a_m \to c$ as $m \to \infty$ and the claim is also proved in this case.

Using the above two claims together with Proposition 4.3 we may assume that $x_k \to x$ as $k \to \infty$, uniformly on compact subsets of $\mathbb{R}$, where $x \in \mathrm{Sol}(f_K, N')$. It follows that $x(0) \in \mathrm{Inv}(f_K, N') = \mathrm{Inv}(f_K, N'') \subset \mathrm{Int}_E N''$ so $x_k(0) \in \mathrm{Int}_{E^{l_k}}(N'' \cap E^{l_k})$ for all k large enough, a contradiction. The proposition is proved. ∎

Proof of Proposition 4.12. In view of Proposition 3.9 it is sufficient to prove that there is an $l_1 \ge l_0(f_K, N)$ such that

$$\mathrm{Inv}(f^l_{P^l\circ K}, N \cap E^l) = \emptyset \qquad \text{for all } l \ge l_1.$$

If this is not true, then we may assume that there is a strictly increasing sequence $(l_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ and a sequence $(x_k)_{k\in\mathbb{N}}$ such that for every $k\in\mathbb{N}$ we have $x_k \in \mathrm{Sol}(f^{l_k}_{P^{l_k}\circ K}, N\cap E^{l_k})$.

Arguing exactly as in the proof of Proposition 4.11 we conclude that $K'_k := P^{l_k}\circ K \to K$ uniformly on compact subsets in N and $\bigcup_{k\in\mathbb{N}} K'_k(N)$ is relatively compact in E. Thus, in view of Proposition 4.3, we may assume that $x_k \to x$ as $k\to\infty$, uniformly on compact subsets of $\mathbb{R}$, where $x\in \mathrm{Sol}(f_K, N)$. It follows that $\mathrm{Inv}(f_K, N)\neq\emptyset$, a contradiction. ∎

The following result is an immediate consequence of Proposition 4.7 and Corollary 3.11:

PROPOSITION 5.3. *Suppose $K\in\mathcal{C}$ and N is an isolating neighborhood relative to f_K. Let $(K_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{C}$ such that $K_n\to K$ as $n\to\infty$ uniformly on compact subsets of N. Assume also that $K(N)$ and $\bigcup_{n\in\mathbb{N}} K_n(N)$ are relatively compact in E. Then there exist an $l_0\in\mathbb{N}$ and an $n_0\in\mathbb{N}$ such that $l_0\ge l_0(K,N)$, $l_0\ge l_0(K_n,N)$ for all $n\ge n_0$ and*

$$h(f_{K_n}, N)_l = h(f_K, N)_l \qquad \textit{for all } n\ge n_0 \textit{ and } l\ge l_0. \ \blacksquare$$

The last result clearly implies the following

PROPOSITION 5.4. *Let (Λ, d) be a metric space and $(K_\lambda)_{\lambda\in\Lambda}$ be a family in $\mathcal{C}$ such that the map*

$$\Lambda\times N\to E, \qquad (\lambda, x)\mapsto K_\lambda(x),$$

is continuous and $\bigcup_{\lambda\in\Lambda} K_\lambda(N)$ is relatively compact in E. Assume that for every $\lambda\in\Lambda$ the set N is an isolating neighborhood relative to f_{K_λ}. Then, for every $\lambda_0\in\Lambda$ there is a $\delta\in\,]0,\infty[$ and an $l_1\in\mathbb{N}$ such that $l_1\ge l_0(f_{K_\lambda}, N)$ and

$$h(f_{K_\lambda}, N)_l = h(f_{K_{\lambda_0}}, N)_l$$

for all λ with $d(\lambda,\lambda_0)<\delta$ and $l\ge l_1$. ∎

Proof of Proposition 4.13. Of course we may assume that Λ is nonempty. Let $\lambda_0\in\Lambda$. Define Λ_0 to be the set of all $\lambda\in\Lambda$ for which there is an l_1 such that $l_1\ge l_0(f_{K_\lambda}, N)$, $l_1\ge l_0(f_{K_{\lambda_0}}, N)$ and

$$h(f_{K_\lambda}, N)_l = h(f_{K_{\lambda_0}}, N)_l \quad \text{for all } l\ge l_1.$$

Proposition 5.4 implies that Λ_0 is both open and closed in Λ. Since $\lambda_0\in\Lambda_0$ it follows that $\Lambda_0=\Lambda$. This proves the proposition. ∎

Proof of Proposition 4.14. The assertion follows easily from Proposition 3.12. ∎

Proof of Proposition 4.15. The assertion follows easily from Proposition 3.13. ∎

Proof of Proposition 4.18. We need the following lemma:

LEMMA 5.5. *There is an l_1 such that $N \cap E^{l+1}$ is an isolating neighborhood relative to $f^{l+1}_{sP^l \circ K+(1-s)P^{l+1}\circ K}$ for all $l \geq l_1$.*

Proof. In fact, if the lemma is not true, then there are a strictly increasing sequence $(l_k)_{k\in\mathbb{N}}$ and a sequence $(s_k)_{k\in\mathbb{N}}$ in $[0,1]$ converging to some s such that, for every $k \in \mathbb{N}$, $N \cap E^{l_k+1}$ is not an isolating neighborhood relative to $f^{l_k+1}_{s_k P^{l_k}\circ K+(1-s_k)P^{l_k+1}\circ K}$. Hence for every $k \in \mathbb{N}$ there is a solution $x_k : \mathbb{R} \to E^{l_k+1}$ of $f^{l_k+1}_{s_k P^{l_k}\circ K+(1-s_k)P^{l_k+1}\circ K}$ with $x_k(\mathbb{R}) \subset N \cap E^{l_k+1}$ and $x_k(0) \in \partial_{E^{l_k+1}}(N \cap E^{l_k+1})$. Thus x_k is differentiable into E and

$$\dot{x}_k(t) = Lx_k(t) + K'_k(x_k(t)) \quad \text{for all } t \in \mathbb{R}$$

where $K'_k : U \to E$ is defined by $K'_k := s_k P^{l_k} \circ K + (1-s_k)P^{l_k+1} \circ K$.

We claim that $K'_k \to K$ as $k \to \infty$, uniformly on compact subsets of N. It is sufficient to show that whenever $a_k \to a$ in N as $k \to \infty$, then $K'_k(a_k) \to K(a)$ as $k \to \infty$. Now $K'_k(a_k) = s_k P^{l_k}(K(a_k)) + (1-s_k)P^{l_k+1}(K(a_k))$. By the continuity of K, Lemma 5.1 and Hypothesis 4.6 we conclude that $K'_k(a_k) \to sK(a) + (1-s)K(a) = K(a)$ as $k \to \infty$. This proves the claim. We also claim that $\bigcup_{k\in\mathbb{N}} K'_k(N)$ is relatively compact in E. It is sufficient to show that every sequence in $\bigcup_{k\in\mathbb{N}} K'_k(N)$ has a subsequence which converges in E. Let $(a_m)_{m\in\mathbb{N}}$ be a sequence in $\bigcup_{k\in\mathbb{N}} K'_k(N)$. Then there are sequences $(k_m)_{m\in\mathbb{N}}$ in $\mathbb{N}$ and $(b_m)_{m\in\mathbb{N}}$ in N such that $a_m = K'_{k_m}(b_m)$, $m \in \mathbb{N}$. Thus $a_m = s_{k_m} P^{l_{k_m}}(K(b_m)) + (1-s_{k_m})P^{l_{k_m}+1}(K(b_m))$.

The assumptions of this proposition imply that $(K(b_m))_{m\in\mathbb{N}}$ has a subsequence, denoted again by $(K(b_m))_{m\in\mathbb{N}}$, which converges in E to some $c \in E$.

Suppose first that $(k_m)_{m\in\mathbb{N}}$ is bounded. Then, taking subsequences if necessary, we may assume that $k_m \equiv k$ for some $k \in \mathbb{N}$. Since P^{l_k} is continuous we conclude that

$$a_m = s_k P^{l_k}(K(b_m)) + (1-s_k)P^{l_k+1}(K(b_m)) \to s_k P^{l_k}(c) + (1-s_k)P^{l_k+1}(c)$$

as $m \to \infty$ and the claim is proved in this case.

Now suppose that $(k_m)_{m\in\mathbb{N}}$ is unbounded. Then we may assume that $k_m \to \infty$ as $m \to \infty$. By Hypothesis 4.6 and Lemma 5.1 we deduce that $a_m \to sc + (1-s)c = c$ as $m \to \infty$ and the claim is also proved in this case.

Using the above two claims, applying Proposition 4.3 and taking a subsequence if necessary, we may assume that $x_k \to x$ uniformly on compact subsets of $\mathbb{R}$, where $x \in \mathrm{Sol}(f_K, N)$. It follows that $x(\mathbb{R}) \subset V := \mathrm{Int}_E N$. Since $x_k(0) \to x(0) \in V$, it follows that $x_k(0) \in V \cap E^{l_k+1} \subset \mathrm{Int}_{E^{l_k+1}}(N \cap E^{l_k+1})$ for all k large enough, a contradiction, which proves the lemma. ∎

Lemma 5.5 and Corollary 3.11 imply that

$$h(f^{l+1}_{P^{l+1}\circ K}, N \cap E^{l+1}) = h(f^{l+1}_{P^l\circ K}, N \cap E^{l+1}) \quad \text{for all } l \geq l_1. \tag{7}$$

We may choose l_1 larger if necessary, to ensure that Hypothesis 4.16 is satisfied for $l \geq l_1$. Let $l \geq l_1$. Note that, by Hypothesis 4.16,

$$h(L|_{F^l}, \{0\}) = \Sigma^{i_l} \tag{8}$$

and

$$\mathrm{Inv}(f^{l+1}_{P^l \circ K}, N \cap E^{l+1}) = S := \mathrm{Inv}(f^l_{P^l \circ K}, N \cap E^l). \tag{9}$$

We may thus choose an open ball U' at zero in F^l, a closed ball $B \subset U'$ at zero in F^l, an open set $U'' \subset U \cap E^l$ in E^l and an isolating neighborhood $N'' \subset U''$ of S relative to $f^l_{P^l \circ K}$ such that $B \oplus N''$ is an isolating neighborhood of S relative to $f^{l+1}_{P^l \circ K}$. Hence, by Proposition 3.8,

$$h(f^{l+1}_{P^l \circ K}, N \cap E^{l+1}) = h(f^{l+1}_{P^l \circ K}, S) = h(f^{l+1}_{P^l \circ K}, B \oplus N''). \tag{10}$$

Write $x = y + z$ where $x \in E^{l+1}$, $y \in F^l$ and $z \in E^l$. For $\theta \in [0,1]$ define $g_\theta : U' \oplus U'' \to E^{l+1}$ by $g_\theta(y+z) = L(y+z) + P^l K(\theta y + z)$. It follows that for all $\theta \in [0,1]$, $B \oplus N''$ is an isolating neighborhood of S relative to g_θ. Thus, by Remark 3.7 and Corollary 3.11, we obtain

$$h(f^{l+1}_{P^l \circ K}, B \oplus N'') = h(g_1, B \oplus N'') = h(g_0, B \oplus N''). \tag{11}$$

Now (8) and the product formula of Proposition 3.13 imply that

$$\begin{aligned} h(g_0, B \oplus N'') &= h(L|_{F^l}, \{0\}) \wedge h(f^l_{P^l \circ K}, N'') \\ &= \Sigma^{i_l} \wedge h(f^l_{P^l \circ K}, N \cap E^l). \end{aligned} \tag{12}$$

Formulas (7), (10), (11) and (12) yield

$$h(f^{l+1}_{P^{l+1} \circ K}, N \cap E^{l+1}) = \Sigma^{i_l} \wedge h(f^l_{P^l \circ K}, N \cap E^l).$$

The proposition is proved. ∎

6. The equivariant case. In this section we will show that the Conley-type index theories constructed in Sections 3 and 4 carry over to the G-equivariant case.

Assume to this end that G is a given compact topological group acting on E by transformations which are isometries with respect to the norm $\|\cdot\|$. Given $S \subset E$ we write $GS := \{gx \mid g \in G,\ x \in S\}$.

Unless otherwise specified we will assume that the set U is G-invariant. We denote by $\mathcal{C}_G$ the subset of $\mathcal{C}$ consisting of the G-equivariant maps.

We will need the following approximation result:

PROPOSITION 6.1. *For every $f \in \mathcal{C}_G$ and every $\varepsilon \in\,]0,\infty[$ there is a locally Lipschitzian map $\widehat{f} \in \mathcal{C}_G$ with*

$$|\widehat{f} - f|_U < \varepsilon.$$

Proof. It is known that there is a locally Lipschitzian map $f_\varepsilon \in \mathcal{C}$ such that

$$|f_\varepsilon - f|_U < \varepsilon.$$

(Cf., e.g., [6, Lemma 1.1, p. 5].) Let μ be the normalized right-invariant Haar measure on G. Define the map $\widehat{f} : U \to E$ by

$$\widehat{f}(x) := \int_G g^{-1} f_\varepsilon(gx)\, d\mu(g) \quad \text{for all } x \in U.$$

It is clear that $\widehat{f}$ is well defined, G-equivariant and $|\widehat{f} - f|_U < \varepsilon$. We only have to show that $\widehat{f}$ is locally Lipschitzian. In fact, we claim that for every x there is an open neighborhood V_x of x such that $f_\varepsilon|_{GV_x}$ is Lipschitzian. This claim together with the definition of $\widehat{f}$ immediately implies that $\widehat{f}|_{V_x}$ is Lipschitzian (with the same Lipschitz constant).

If the claim is not true then for some $x \in U$ there are sequences $(y_n)_{n\in\mathbb{N}}$, $(z_n)_{n\in\mathbb{N}}$ in U with $y_n \to x$, $z_n \to x$ as $n \to \infty$ and there are sequences $(g_n)_{n\in\mathbb{N}}$ and $(h_n)_{n\in\mathbb{N}}$ in G such that

$$\|f_\varepsilon(g_n y_n) - f_\varepsilon(h_n z_n)\| \geq n\|g_n y_n - h_n z_n\| \quad \text{for every } n \in \mathbb{N}. \tag{13}$$

Since in a compact space every net has a convergent subnet (cf. e.g. [7] or [13]) we thus obtain the existence of a directed set $(D, \prec)$ and a function $d \mapsto n_d$ from D to $\mathbb{N}$ with $\lim_{d\in D} n_d = \infty$ such that $\lim_{d\in D} g_{n_d} = g$ and $\lim_{d\in D} h_{n_d} = h$ for some g and $h \in G$. It follows that $\lim_{d\in D} g_{n_d} y_{n_d} = gx$ and $\lim_{d\in D} h_{n_d} z_{n_d} = hx$. If $gx = hx$ then (13) contradicts the fact that f_ε is Lipschitzian in a neighborhood of gx. If $gx \neq hx$, then (13) implies that $\|f_\varepsilon(gx) - f_\varepsilon(hx)\| = \infty$, again a contradiction. ∎

Suppose that E is *finite-dimensional.*

Note that if $f \in \mathcal{C}_G$ is *locally Lipschitzian*, N is a G-invariant isolating neighborhood relative to f and π_f is the local flow generated by f then the G-equivariant Conley index $h_G(f, S) := h_G(\pi_f, S)$ of the set $S := \mathrm{Inv}(f, N)$ is well defined (see Floer's fundamental paper [8]). Since the isolating neighborhood N uniquely determines the isolated invariant set S we also write $h_G(f, N)$ instead of $h_G(f, S)$. This will not lead to confusion.

We can now extend the concept of the G-equivariant Conley index to ODEs with nonunique solutions:

Definition 6.2. Suppose $f \in \mathcal{C}_G$ and N is a G-invariant isolating neighborhood relative to f. Set

$$h_G(f, N) := h_G(\widehat{f}, N)$$

where $\widehat{f} \in \mathcal{C}_G$ is an arbitrary *locally Lipschitzian map* with $|\widehat{f} - f|_N < \varepsilon(f, N)$. The existence of $\widehat{f}$ follows from Proposition 6.1. We call $h_G(f, N)$ the *G-equivariant Conley index of the isolating neighborhood N relative to f*. If $H = (H_q)_{q\in\mathbb{Z}}$ is an arbitrary G-equivariant homology or cohomology theory, then we call the graded group $H(h_G(f, N)) = (H_q(h_G(f, N)))_{q\in\mathbb{Z}}$ the *G-equivariant (co)homological Conley index of the isolating neighborhood N relative to f.*

The analogues of all results of Section 3 (obtained by replacing $h(\cdot,\cdot)$ by $h_G(\cdot,\cdot)$ and assuming that all sets (resp. maps) involved are G-invariant (resp. G-equivariant)) hold with analogous proofs. In particular, the index $h_G(f,N)$ does not depend on the choice of the locally Lipschitzian map $\widehat{f} \in \mathcal{C}_G$ with $|\widehat{f} - f|_N < \varepsilon(f,N)$ and $h_G(f,N)$ only depends on the set $S = \mathrm{Inv}(f,N)$ and not on the choice of a G-invariant isolating neighborhood of S. Thus, for every G-invariant isolated invariant set relative to f we can define the G-equivariant Conley index $h_G(f,S)$ of S relative to f by setting

$$h_G(f,S) := h_G(f,N')$$

where N' is an arbitrary G-invariant isolating neighborhood of S relative to f.

It further follows that the index just defined is nontrivial and enjoys the continuation invariance property.

Finally, the sum and product formulas continue to hold.

Now suppose that E is *infinite-dimensional*.

Consider the following

HYPOTHESIS 6.3.

1. *Hypotheses* 4.2, 4.6 *and* 4.8 *are satisfied.*
2. *The sets* E_j, $j = -1, 0, 1$, *are* G*-invariant.*
3. *The maps* L *and* P^l, $l \in \mathbb{N}$, *are* G*-equivariant.*

We can now make the equivariant version of Definition 4.9:

DEFINITION 6.4. Assume Hypothesis 6.3. Let $K \in \mathcal{C}_G$ be such that $K(N)$ is relatively compact in E and N is a G-invariant isolating neighborhood relative to f_K.

The *G-equivariant $\mathcal{LS}$-Conley index $h_G(f_K,N)$ of N relative to f_K* is the sequence $(h_G(f_K,N)_l)_{l \ge l_0(K,N)}$, where

$$h_G(f_K,N)_l := h_G(f^l_{P^l \circ K}, N \cap E^l) \quad \text{for all } l \ge l_0(K,N).$$

If $H = (H_q)_{q\in\mathbb{Z}}$ is an arbitrary G-equivariant homology or cohomology theory, then the sequence

$$(H(h_G(f_K,N)_l))_{l \ge l_0(K,N)}$$

of graded groups $H_q(h_G(f_K,N)_l)$, $q \in \mathbb{Z}$, is called the *G-equivariant* (*co*)*-homological $\mathcal{LS}$-Conley index of N relative to f_K*.

Under Hypothesis 6.3 one can now easily prove the analogues of Remark 4.10 and Propositions 4.11–4.15 obtained by replacing $h(\cdot,\cdot)$ by $h_G(\cdot,\cdot)$ and assuming that all sets (resp. maps) involved are G-invariant (resp. G-equivariant).

Now consider the additional

Hypothesis 6.5.

1. *Hypothesis* 4.16 *is satisfied.*
2. *For all l large enough the set F^l is G-invariant.*

Note that Hypotheses 6.3 and 6.5 are automatically satisfied in the case of equations considered by the first author in the paper [11].

We now obtain the following analogue of Proposition 4.18:

Proposition 6.6. *Assume Hypotheses* 6.3 *and* 6.5. *Let $K \in \mathcal{C}_G$ be such that $K(N)$ is relatively compact in E and N is a G-invariant isolating neighborhood relative to f_K. Then there is an $l_1 \geq l_0(K, N)$ such that*

$$h_G(f_K, N)_{l+1} = \Sigma_G^{i_l} \wedge h_G(f_K, N)_l \quad \text{for all } l \geq l_1.$$

Here, $\Sigma_G^{i_l}$ is the one-point compactification of the representation F_1^l.

Proof. Note that, by simple arguments,

$$h_G(L|_{F^l}, \{0\}) = \Sigma_G^{i_l}$$

and thus the proof of Proposition 4.18 carries over (mutatis mutandis) to the present equivariant case. ∎

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