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CH and the Sacks property

by

Sandra Quickert (Bonn and Paris)

Abstract. We show the consistency of CH and the statement "no ccc forcing has the Sacks property" and derive some consequences for ccc ω^ω-bounding forcing notions.

In the last few years much progress has been made in studying properties of ccc posets in connection with partition properties. Many of these results deal with the Sacks property: recall that a poset P has the *Sacks property* if for any real r in the generic extension there is a sequence of finite subsets of ω, $\{I_n\}_n \in V$, such that $r \in \prod_{n<\omega} I_n$ and $|I_n| \leq 2^n$ for any n. Notice that when replacing the function $f(n) = 2^n$ by an arbitrary increasing function we get an equivalent formulation. Jensen deduced in [3] the existence of ccc posets with the Sacks property from $\diamondsuit$. Extracting a more general statement about ccc forcings, Veličković showed in [10] that it is possible to have a large continuum and the existence of ccc posets with the Sacks property. On the other hand, Shelah and Veličković showed independently that it is consistent that no ccc forcing has the Sacks property; see [6] and [11]. However, in the models they built the continuum is equal to $\aleph_2$, so the question arises whether it is consistent with CH that no ccc forcing has the Sacks property. It turns out that the answer is yes. In this paper we derive this statement and some consequences for ccc ω^ω-bounding forcing notions from a combinatorial Ramsey-type principle. This principle is known to be consistent with CH, as was proved by Abraham and Todorčević in [1].

The notation we use is standard and might be found in [4] or [2]. If T is a tree in $2^{<\omega}$ and $s \in T$, then we denote by $T[s]$ the subtree of T with stem s, i.e. $T[s] = \{t \in T \mid t \sqsubseteq s \vee s \sqsubseteq t\}$.

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First we introduce some definitions and easy facts which will be used later. In the following, P always denotes a collection of trees in $2^{<\omega}$, ordered by inclusion.

DEFINITION 1. Let $T \in P$, $X \subseteq P$. Then we call

$$\mathrm{tr}(T, X) = \{T' \in X \mid T, T' \text{ are compatible}\}$$

the *trace* of T in X.

$X \subseteq P$ is called *large* iff for any maximal antichain A there is some $T \in X$ which has a finite trace in A. As we will see later, large sets provide us with enough witnesses that a forcing notion P is ω^ω-bounding.

CLAIM 2. *Let $X \subseteq P$ be large, and let A be a maximal antichain. Then*

$$X_A = \{T \in X \mid \mathrm{tr}(T, A) \text{ is finite}\}$$

is also large.

Proof. Let B be a maximal antichain. Select a maximal antichain B' such that any member of B' is below an element of B and below an element of A. Since X is large, there is some $T \in X$ with a finite trace in B'. Then T is clearly in X_A and $\mathrm{tr}(T, B)$ is finite. ∎

CLAIM 3. *Let P be ccc and ω^ω-bounding, and let $P = \bigcup_{n<\omega} P_n$ be a countable partition. Then for some n the set P_n is large.*

Proof. Assume not. For each n choose A_n witnessing the nonlargeness of P_n. By ccc and ω^ω-boundedness, there is some $T \in P$ having a finite trace in any A_n. Therefore, $T \notin P_n$ for any n, a contradiction. ∎

Now we have all we need to prove the main statement, the consistency of CH with "no ccc forcing has the Sacks property". Let $(\star)$ be the following principle:

$(\star)$ Let $\mathcal{I}$ be a p-ideal on $[\kappa]^{\leq\omega}$ where $\kappa \leq 2^{\aleph_0}$. Then exactly one of the two statements holds:

(a) There is an uncountable $X \subseteq \kappa$ such that $[X]^\omega \subseteq \mathcal{I}$.
(b) $\kappa = \bigcup_{n<\omega} X_n$ where each X_n is orthogonal to $\mathcal{I}$.

Recall that a set $X \subseteq \kappa$ is *orthogonal* to an ideal I provided that the intersection of X with any element of I is finite.

THEOREM 4. *Assume the principle $(\star)$ holds. Then no ccc forcing has the Sacks property.*

Proof. Assume the principle $(\star)$. Furthermore, assume that P is a ccc forcing notion having the Sacks property. Since it can be derived from $(\star)$ that there are no Suslin trees (see e.g. [1]), P must add a new real (see [6]).

Fix some P-name $\dot{r}$ for a new element in 2^ω. For any $p \in P$ define

$$T_p = \{s \in 2^{<\omega} \mid \exists q \leq p,\ q \Vdash s \sqsubset \dot{r}\},$$

the tree of possibilities below p. Let $\mathrm{Tree}(P)$ be the forcing notion consisting of these T_p, ordered by inclusion. Note that this collection of trees has the following property: For each tree T and each node $s \in T$ there is a subtree $S \subseteq T[s]$ which belongs to $\mathrm{Tree}(P)$.

Claim 5. *If P has the ccc and the Sacks property, then so does* $\mathrm{Tree}(P)$.

Proof. First, observe that whenever $T_p \perp_{\mathrm{Tree}(P)} T_q$, then $p \perp_P q$, therefore $\mathrm{Tree}(P)$ must have the ccc.

Now we will prove the Sacks property: let $\{A_i \mid i < \omega\}$ be a set of maximal antichains in $\mathrm{Tree}(P)$, and let $x \in \omega^\omega$ be increasing. We have to find some tree T_p which is compatible with at most $x(i)$ members of A_i for any i.

For each $T_{ij} \in A_i$ choose p_{ij} such that $T_{ij} = T_{p_{ij}}$. Let $P_i \supseteq \{p_{ij} \mid j < \omega\}$ be a maximal antichain such that for any $p \in P_i$ the tree T_p is contained in some T_{ij}. Note that this is possible by ω^ω-boundedness: Assume there is some $p \in P$ such that for any $q \leq p$ the tree T_q is not contained in any T_{ij}. Then consider the following name for a real: $\dot{s}(j) = \min\{n \mid \dot{r}{\restriction}n \notin T_{ij}\}$. Fix $q \leq p$ and $h \in \omega^\omega$ such that $q \Vdash \dot{s} \leq h$. Then $T_q \cap T_{ij}$ is finite, which contradicts the maximality of A_i in $\mathrm{Tree}(P)$.

Now choose an enumeration $P_i = \{p^i_j \mid j < \omega\}$. Since P has the Sacks property, there is some $p \in P$ such that for any i, p is compatible with at most $x(i)$ members of P_i, say $\{p_0, \ldots, p_m\}$. Therefore, T_p is covered by the union of the trees T_{p_i} where $i \leq m$. Since each of these trees is included in one tree of the maximal antichain A_i, we may assume for simplicity that T_{p_i} is an element of A_i. But then T_p cannot be compatible with any other tree in A_i, because if we assume some $T \subseteq T_p \cap T_q$ where T_q is incompatible with any T_{p_i}, then there is some $i \leq m$ and some s such that the cone $T[s]$ is included in $T_q \cap T_{p_i}$, which is impossible. Hence, T_p is compatible with at most $x(i)$ members of A_i. ∎

Therefore, it suffices to show the statement for ccc collections of trees in $2^{<\omega}$, ordered by inclusion and having the Sacks property. For the rest of the proof, fix such a forcing notion P.

Now consider the ideal $\mathcal{I} \subseteq [P]^{\leq\omega}$ consisting of all those X for which there is a maximal antichain A such that for any $T \in A$ the trace of T in X is finite.

Claim 6. *$\mathcal{I}$ is a p-ideal.*

Proof. $\mathcal{I}$ is clearly an ideal, so it remains to show that for any countable collection of elements of $\mathcal{I}$ there is some member of $\mathcal{I}$ which almost contains any element of this countable collection.

Let $\{X_i \mid i < \omega\} \subseteq \mathcal{I}$, and let A_i be a maximal antichain witnessing $X_i \in \mathcal{I}$. By ω^ω-boundedness, select a maximal antichain A consisting of trees which have finite traces in any A_i. Therefore, A is a common witness for $X_i \in \mathcal{I}$. Fix an enumeration $A = \{T_i \mid n < \omega\}$. So, if we set $\widetilde{X}_i = X_i \setminus \{\mathrm{tr}(T_j, X_i) \mid j \leq i\}$, then $Y = \bigcup_{i<\omega} \widetilde{X}_i$ fulfills our requirements, since any $T \in A$ has a finite trace in Y. ■

According to the principle ($\star$) we now have two possibilities:

Assume (a) holds, i.e. there is an uncountable $X \subseteq P$ such that any countable subset belongs to $\mathcal{I}$. By ccc, 1_P cannot force that the intersection of the generic filter G with X is countable, because if so, the intersection would be covered by a countable set Y in the ground model and any generic filter G containing an element of $X \setminus Y$ would be a counterexample to this statement. Hence, we can select $T \in P$ forcing that the intersection is uncountable. Again by ccc, choose for $i < \omega$ a countable subset $X_i \subseteq X$ such that

$$T \Vdash \forall i\ \exists T' \in X_i,\ T' \in \dot{G}.$$

Since the intersection of X with the generic filter G will be uncountable whenever $T \in G$, we can choose the X_i in such a way that they are pairwise disjoint. In particular,

$$T \Vdash \dot{G} \cap \bigcup_{i<\omega} X_i \text{ is infinite.}$$

Therefore, any tree below T has an infinite trace in $\bigcup_{i<\omega} X_i$, although the latter set is supposed to be in the ideal by the properties of X, a contradiction.

Assume now (b) holds, i.e. $P = \bigcup_{n<\omega} P_n$ such that for each n the intersection of P_n with any element of $\mathcal{I}$ is finite.

By Claim 3 fix some P_n which is large.

Claim 7. *For any maximal antichain A there is some finite $F_A \subseteq A$ such that each $T \in P_{n,A} = \{T \in P_n \mid \mathrm{tr}(T, A)$ is finite$\}$ has an infinite intersection with $\bigcup F_A$.*

Proof. Assume not. Then there is some maximal antichain A and $\{T_i \mid i < \omega\} \in P_{n,A}$ such that whenever $i \neq j$, then $\mathrm{tr}(T_i, A) \cap \mathrm{tr}(T_j, A) = \emptyset$. In particular, the T_i are pairwise disjoint. Now take some maximal antichain B containing all the T_i. Thus, B witnesses $\{T_i \mid i < \omega\} \in \mathcal{I}$, although no countable subset of P_n is an element of the ideal. ■

If T is a tree in $2^{<\omega}$, then we define the growth function g_T of T as follows:

$$g_T(n) = \min\{m \mid |T \cap 2^m| > n\}.$$

We will use Claim 7 to construct an unbounded family of trees, i.e. a family F such that $\{g_T \mid T \in F\}$ is unbounded in ω^ω.

For each increasing $x \in \omega^\omega$ choose a maximal antichain A_x consisting of trees T where $g_T > x$. Note that this is possible by the Sacks property. By Claim 7, we can choose finite sets $F_x \subseteq A_x$ such that $\bigcup F_x$ intersects every tree in P_{n,A_x}. Then an easy argument shows that $F = \{\bigcup F_x \mid x \in \omega^\omega \text{ increasing}\}$ is an unbounded family of trees: Assume there is a $g \in \omega^\omega$ which bounds the family F. Then define h by $h(i) = i \cdot g(i)$. Clearly, the growth function g_H with $H = \bigcup F_h$ is not bounded by g, which contradicts the choice of g.

Since F is an unbounded family of trees, we find some k such that $\{g_S(k) \mid S \in F\}$ is infinite. We define by recursion:

- $s_0 \subseteq 2$ such that for any l there is some $S \in F$ where $S \cap 2 = s_0$ and $g_S(k) \geq l$.
- $s_j \subseteq 2^{j+1}$ end-extending s_{j-1} such that for any l there is some $S \in F$ with $S \cap 2^{j+1} = s_j$ and $g_S(k) \geq l$.

Therefore, $\bigcup_{j<\omega} s_j$ is a tree with at most k branches, say $[\bigcup_{j<\omega} s_j] = \{r_0, \ldots, r_m\}$. For each j select $S_{x_j} \in F$ such that $S_{x_j} \cap 2^j = s_j$, and let A be a maximal antichain consisting of trees which have finite traces in each A_{x_j}. Thus, $\bigcup_{j<\omega} s_j$ hits each $T \in P_{n,A}$ infinitely many times, i.e. each $T \in P_{n,A}$ has at least one of the r_i as a branch. However, by Claim 2 we know that $P_{n,A}$ is large, so this is not possible: take a maximal antichain B consisting of trees T where $[T] \cap \{r_0, \ldots, r_m\} = \emptyset$. Now select some $T' \in P_{n,A}$ having a finite trace in B. Then T' cannot have any r_i as a branch, since it is covered by finitely many elements of B, and therefore each branch of T' must be a branch of one of the trees in the cover. Therefore, T' contradicts the properties of almost all s_n. ∎

Thus, none of these cases is possible, and no ccc forcing can have the Sacks property. This finishes the proof of Theorem 4. ∎

When assuming CH, the principle $(\star)$ just deals with subsets of ω_1, and this particular instance of $(\star)$ was shown to be consistent with CH by Abraham and Todorčević [1]. Hence:

COROLLARY 8. *CH is consistent with the statement "no ccc forcing has the Sacks property".*

Todorčević [9] also showed that the principle $(\star)$ extended to any p-ideal on countable subsets of an arbitrary set is a consequence of PFA. Thus:

Corollary 9. *Assume PFA holds. Then no ccc forcing has the Sacks property.*

Observe that we needed the Sacks property only in the last part, the rest was done just by ω^ω-boundedness. Moreover, the fact that we deal with trees was only used in part (b). For the first part, we could have defined the trace of a condition p in a set $X \subseteq P$ by $\mathrm{tr}(p, X) = \{q \in X \mid q \text{ is compatible with } p\}$ and just have dealt with the order given by the poset P. Since any antichain belongs to the ideal defined above, we can record the following

Corollary 10. *Assume the principle $(\star)$ holds. Then any ccc ω^ω-bounding forcing notion P satisfies the σ-finite chain condition, i.e. P can be written as $P = \bigcup_{n<\omega} P_n$ where each P_n contains only finite antichains.*

This is very close to the statement that any ccc ω^ω-bounding forcing notion is σ-linked. And indeed, restricting the posets to Suslin forcings and using another partition principle one can show in ZFC:

Theorem 11. *If P is a ccc ω^ω-bounding Suslin forcing, then* $\mathrm{Tree}(P)$ *is a ccc ω^ω-bounding and σ-linked Suslin forcing.*

This is a small hint that the famous question of von Neumann [5] whether random real forcing is minimal among the ccc ω^ω-bounding Suslin forcings, or even is the only such forcing notion example, might be answered affirmatively.

The dichotomy we will now use in order to show Theorem 11 is due to Solecki [8]:

Theorem 12. *Let A be an analytic set and $\mathcal{F}$ be a family of closed sets. Then either A is countably coverable by elements of $\mathcal{F}$, or there is a G_δ-set $G \subseteq A$ such that whenever B is a basic open set having no empty intersection with G, then $G \cap B$ is not countably coverable by elements of $\mathcal{F}$.*

Proof of Theorem 11. Let P be an ω^ω-bounding ccc Suslin forcing. It is well known that P adds a real (see e.g. [7]), therefore we can fix a name $\dot{r}$ for a new element of 2^ω. Let $\mathrm{Tree}(P)$ be as defined above, and fix the following family of closed sets:

$$\mathcal{F} = \{F \mid \forall T \in F\ \forall T' \in F,\ T \cap T' \text{ is infinite}\}.$$

Claim 13. $\mathrm{Tree}(P)$ *is countably coverable by sets from $\mathcal{F}$.*

Proof. On $2^{<\omega}$, we fix the following well-ordering: $s \leq^\star t$ iff $|s| < |t|$ or $|s| = |t|$ and $s(|s|-1) \leq t(|s|-1)$. If $\tau : 2^n \to 2$ for some n, then we denote basic open sets by $N_\tau = \{T \in \mathrm{Tree}(P) \mid \forall s \in \mathrm{dom}(\tau),\ \tau(s) = 1 \text{ iff } s \in T\}$. A node $s \in 2^{<\omega}$ is called an *end-node* of τ if $s \in 2^n$ and $\tau(s) = 1$.

Assume the claim is false, and fix a G_δ-set $G \subseteq \mathrm{Tree}(P)$, $G = \bigcap_{l<\omega} G_l$ with G_l open, such that for any basic open set N_τ which has nonempty intersection with G this intersection $G \cap N_\tau$ is not countably coverable. In particular, G contains two trees whose intersection is finite. Therefore, we find τ_0, τ_1 such that $N_{\tau_k} \cap G$ is nonempty and lies inside G_0, and moreover, no end-node of τ_0 end-extends any end-node in τ_1 and conversely. Hence, $N_{\tau_0} \cap N_{\tau_1} = \emptyset$. Now assume that $\tau_{s^\frown k}$ is defined for $s <^\star t$. Since $N_{\tau_t} \cap G$ is not countably coverable, we find $\tau_{t^\frown 0}$, $\tau_{t^\frown 1}$ extending τ_t such that $N_{\tau_{t^\frown k}} \cap G$ is nonempty, included in $G_{|t|}$, and furthermore, as before, no end-node of $\tau_{t^\frown 0}$ end-extends any end-node in $\tau_{t^\frown 1}$ and vice versa.

Hence, if $x \in 2^\omega$, then $\bigcap_{t \sqsubseteq x} N_{\tau(t)}$ contains a unique element $T_x \in G$, and moreover, if $x_1 \neq x_2$, then $T_{x_1} \cap T_{x_2}$ has finite height, a contradiction to the ccc. ∎

With the aid of Claim 13 we finish the proof of Theorem 11 as follows: Fix $\mathrm{Tree}(P) = \bigcup_{n<\omega} P_n$ such that each $P_n \in \mathcal{F}$. We may assume that each P_n is maximal, i.e. whenever $T \notin P_n$, there is $T' \in P_n$ having a finite intersection with T. For each P_n choose a maximal linked subset X_n. It remains to show that $\bigcup_{n<\omega} X_n = \mathrm{Tree}(P)$. So, assume not, and choose a witness $T \in \mathrm{Tree}(P)$. Since T is not added to any X_n, we can select for each n some $T_n \in X_n$ incompatible with T. Now consider a name $\dot{x}$ for a real such that

$$T \Vdash \dot{x}(n) = m \text{ iff } \dot{r}{\restriction}m \in T_n \wedge \dot{r}{\restriction}m+1 \notin T_n.$$

By ω^ω-boundedness there is some $T' \leq T$ and some $f \in \omega^\omega$ such that $T' \Vdash \dot{x} \leq f$. In particular, $T_n \cap T'$ is finite for any n. However, this is not possible since $T' \in P_n$ for some n, so $T' \cap T_n$ must be infinite. ∎

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Mathematisches Institut
Rheinische Friedrich-Wilhelms Universität Bonn
Beringstr. 6
D-53115 Bonn, Germany
E-mail: sandra@math.uni-bonn.de

Équipe de Logique Mathématique
UFR de Mathématiques (case 7012)
Université Denis Diderot Paris 7
2 place Jussieu
75251 Paris Cedex 05, France
E-mail: quickert@logique.jussieu.fr

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