FUNDAMENTA
MATHEMATICAE
229 (2015)

Hausdorff gaps and towers in $\mathcal{P}(\omega)/\mathrm{Fin}$

by

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Abstract. We define and study two classes of uncountable $\subseteq^*$-chains: Hausdorff towers and Suslin towers. We discuss their existence in various models of set theory. Some of the results and methods are used to provide examples of indestructible gaps not equivalent to a Hausdorff gap. We also indicate possible ways of developing a structure theory for towers based on classification of their Tukey types.

1. Introduction. We say that subsets A, B of ω are in the relation of *almost inclusion* (denoted by $A \subseteq^* B$) if $A \setminus B$ is finite. One of the motivations of this article is the following question:

Question 1. Is there an uncountable well-ordered $\subseteq^*$-chain which consists of pairwise $\subseteq$-incomparable elements?

In a sense this is the question how "far" $\subseteq^*$ is from $\subseteq$.

The answer to Question 1 is positive. We will call well-ordered increasing $\subseteq^*$-chains *towers*. (We do not assume that towers are maximal with respect to end-extension as is often done in the literature, but we treat only uncountable towers.) There are both towers witnessing the positive answer to Question 1 (we will call them *special*) and towers which do not have an uncountable subtower consisting of $\subseteq$-incomparable sets (called *Suslin*). Examples of both sorts are implicitly mentioned in [28].

A tower $(T_\alpha)_{\alpha<\omega_1}$ satisfies *condition* (H) if the set $\{\xi < \alpha : T_\xi \setminus T_\alpha \subseteq n\}$ is finite for each $\alpha < \omega_1$ and $n < \omega$. Although it seems that this notion has not appeared explicitly in the literature, the reader can recognize a resemblance between condition (H) and the well-known Hausdorff condition for gaps (see Section 2). This is not a coincidence: the "left half" of every Hausdorff gap is a Hausdorff tower, i.e. a tower containing a cofinal subtower

2010 *Mathematics Subject Classification*: 03E35, 03E05.
Key words and phrases: Hausdorff gaps, special gaps, towers, oscillations, Suslin trees, Tukey order.

DOI: 10.4064/fm229-3-1

which satisfies condition (H). It turns out that Hausdorff towers are natural examples of special towers. Moreover, by an easy modification of arguments used for analyzing gaps, one can show that under MA_{ω_1} all towers of length ω_1 are Hausdorff. So despite the fact that an object as in Question 1 could seem unusual at first glance, it is quite common. In Section 4 we discuss models in which all ω_1-towers are special. Moreover, we show that for every κ-tower (where $\kappa > \omega$ is regular) there is a ccc forcing making it special in the extension. In fact, under MA_κ each κ-tower is very close to being a $\subseteq$-antichain of size κ (Theorem 21). An analysis of the analogous Luzin condition for almost disjoint systems in $\mathcal{P}(\omega)$ was done by Guzmán and Hrušák. Not surprisingly, many results about Hausdorff towers and Luzin gaps are in direct correspondence (see [13]).

To the best of our knowledge, the first example of a tower which does not contain an uncountable $\subseteq$-antichain was given in [29] under the assumption of CH. More examples are provided by results from [28]. Todorčević proved there a theorem (see Theorem 28 in Section 5 below) which implies that every tower of uncountable cofinality generating a non-meager ideal is Suslin. That is, every tower rich enough (e.g. generating a maximal ideal) cannot be special. There are also Suslin towers generating meager ideals (see Section 5).

The analogy between towers and gaps is strong, at least in the sense that many results about gaps can be easily modified for the case of towers. For instance, under MA_{ω_1} each gap is Hausdorff as well as each tower (of size ω_1) contains a subtower with condition (H). In a model obtained by adding a single Cohen real we can produce a non-special gap and a non-special tower practically in the same manner. However, this analogy breaks down in many ways. Under PID each gap is Hausdorff, but we show that the existence of a non-special tower is consistent with PID (see Section 4). On the other hand, Theorem 23 states that $\mathsf{PID}+\omega_1 < \mathfrak{b}$ is a sufficient condition for all towers to be Hausdorff. It becomes apparent in Section 7 that this result is related to the "only five Tukey types" theorems. We also prove that consistently there is a Hausdorff tower which generates a dense ideal and thus cannot be a half of any gap (Example 32), and a special tower which is equivalent (in the sense of generating the same ideal) to a Suslin tower and thus is not Hausdorff (Example 33). Some of these results are implicitly contained in [28].

The theory of towers is a debtor of the theory of gaps, but it is not an ungrateful one. In fact, the analysis of the property of being a special tower has led us to an example of a gap which is special but not equivalent to a Hausdorff gap (Example 38). Scheepers [19] asked about the existence of such an object and Hirschorn [12] answered this question affirmatively. Our example is of a different sort than Hirschorn's and it has a simpler description. Namely, Hirschorn showed that there is a special gap which does not satisfy a certain condition weaker than being Hausdorff (we call it *left-oriented*). We present an example which is left-oriented but not Hausdorff. In Section 6 we offer other examples of this kind (many of them exist in any model obtained by adding ω_1 Cohen reals). In particular, we prove the consistent existence of a Hausdorff gap $(L_\alpha, R_\alpha)_{\alpha<\omega_1}$ such that $(R_\alpha, L_\alpha)_{\alpha<\omega_1}$ is not Hausdorff (Example 39), a special gap $(L_\alpha, R_\alpha)_{\alpha<\omega_1}$ such that neither $(L_\alpha, R_\alpha)_{\alpha<\omega_1}$ nor $(R_\alpha, L_\alpha)_{\alpha<\omega_1}$ is left-oriented (Example 40), and a gap $(L_\alpha, R_\alpha)_{\alpha<\omega_1}$ which is left-oriented but not Hausdorff and $(R_\alpha, L_\alpha)_{\alpha\in\omega_1}$ is special but not left-oriented (Theorem 41). At the end of Section 6 we come back to towers to construct a special non-Hausdorff tower which is not equivalent to a Suslin tower.

Towers are often used as a combinatorial tool in set theory, set-theoretic topology and functional analysis. For instance, the Stone spaces of Boolean subalgebras of $\mathcal{P}(\omega)$ generated by towers (and $[\omega]^{<\omega}$) are ordered compacta being continuous images of ω^*. Bell [3] used a tower to construct a compact separable space which does not continuously map onto $[0,1]^{\omega_1}$ and which does not have a countable π-base. In [29] a non-special tower generates an L-space and an S-space, both subspaces of $\mathcal{P}(\omega)$ equipped with the Vietoris topology. However, no additional properties of towers are usually needed (with the exception of the last result), except possibly some maximality properties like generating a dense ideal (i.e. such that every infinite subset of ω contains an infinite element of the ideal), or a maximal ideal. Perhaps this is the reason why there were not many attempts to develop a structure theory for towers.

This article can be treated as a modest contribution to the program of filling this gap. Properties of being special or Hausdorff demarcate some dividing lines in the class of towers. In Section 7 we try to examine possible ways to expand this research. We use the Tukey ordering, a tool which has proved its worth in exploring the structure of ultrafilters (see [4]). We show that an ω_1-tower is Hausdorff if and only if it is Tukey top among directed sets of size ω_1. Using results from [4], we observe that consistently there are $2^{\mathfrak{c}}$ pairwise incomparable Tukey types of ω_1-towers.

2. Preliminaries on gaps. It will be convenient to start with definitions and basic facts about gaps. More details can be found in [19] and [31].

Recall that $(L_\alpha, R_\alpha)_{\alpha<\omega_1}$ is a *pre-gap* if $L_\alpha \cap R_\alpha = \emptyset$ for each $\alpha < \omega_1$ and both $(L_\alpha)_{\alpha<\omega_1}$ and $(R_\alpha)_{\alpha<\omega_1}$ are towers. A pre-gap $(L_\alpha, R_\alpha)_{\alpha<\omega_1}$ forms an (ω_1,ω_1)-*gap* if there is no set L interpolating it, i.e. no set L such that $L_\alpha \subseteq^* L$ and $R_\alpha \cap L =^* \emptyset$ for every $\alpha < \omega_1$.

More generally, $(L_\alpha, R_\beta)_{\alpha<\lambda,\beta<\kappa}$ is a (λ,κ)-*gap* if $L_\alpha \cap R_\beta =^* \emptyset$ for every $\alpha < \lambda$ and $\beta < \kappa$, and there is no L such that $L_\alpha \subseteq^* L$ and $L \subsetneq^* R^c_\beta$ for every $\alpha < \lambda$ and $\beta < \kappa$. Notice that the last inequality is slightly more complicated than the equality $L \cap R_\beta =^* \emptyset$ but this setting enables us to consider $(\lambda, 1)$-gaps. In what follows, a *gap* is an (ω_1, ω_1)-gap unless stated otherwise.

We say that a gap $(L_\alpha, R_\alpha)_{\alpha<\omega_1}$ satisfies *condition* (H) if

$$\{\xi < \alpha : L_\xi \cap R_\alpha \subseteq n\} \text{ is finite}$$

for each $\alpha < \omega_1$ and $n < \omega$. Similarly, a (pre-)gap satisfies *condition* (K) if

$$(L_\alpha \cap R_\beta) \cup (L_\beta \cap R_\alpha) \neq \emptyset$$

for each $\alpha < \beta < \omega_1$. Finally, a (pre-)gap satisfies *condition* (O) if

$$L_\alpha \cap R_\beta \neq \emptyset$$

for each $\alpha < \beta < \omega_1$.

Now we are ready to define basic types of gaps (the first two are well-known in the literature).

Definition 2. A *subgap* of a gap $(L_\alpha, R_\alpha)_{\alpha<\kappa}$ is a gap $(L_\alpha, R_\alpha)_{\alpha\in I}$ where I is a cofinal subset of κ. A gap is called *Hausdorff* if it contains a subgap satisfying condition (H). A gap is called *special* (or *indestructible*) if it contains a subgap satisfying condition (K). A gap $(L_\alpha, R_\alpha)_{\alpha<\omega_1}$ is called *left-oriented* (or just *oriented*) if it contains a subgap satisfying condition (O). It is *right-oriented* if $(R_\alpha, L_\alpha)_{\alpha<\omega_1}$ is left-oriented.

The name "indestructible" for special gaps is due to the fact that these are precisely gaps indestructible by ω_1 preserving forcing notions.

Theorem 3 (Kunen, see [19]). *For a gap $\mathcal{G} = (L_\alpha, R_\alpha)_{\alpha<\omega_1}$ the following are equivalent:*

(1) *$\mathcal{G}$ is special;*
(2) *$\mathcal{G}$ is a gap in every ω_1 preserving extension of the universe of sets V;*
(3) *$\mathcal{G}$ is a gap in every generic extension of the universe obtained by a ccc forcing.*

For $i < 2$ consider the gaps $(L^i_\alpha, R^i_\alpha)_{\alpha<\omega_1}$. We say that these two gaps are *equivalent* if $\mathcal{L}^0 = \mathcal{L}^1$ and $\mathcal{R}^0 = \mathcal{R}^1$, where $\mathcal{L}^i$ is the ideal generated by $(L^i_\alpha)_{\alpha<\omega_1}$ (i.e. $\mathcal{L}^i = \{A \subseteq \omega : \exists \alpha < \omega_1\ A \subseteq^* L^i_\alpha\}$), and $\mathcal{R}^i$ is the ideal generated by $(R^i_\alpha)_{\alpha<\omega_1}$.

Lemma 4. *The properties in Definition 2 respect equivalence of gaps.*

Proof. Suppose (L'_α, R'_α) satisfies condition $(\star)$ (where $\star$ is one of H, K, O) and (L_α, R_α) is an equivalent gap. We can find cofinal subgaps

$$(L'_\alpha, R'_\alpha)_{\alpha\in I'} = (O'_\alpha, P'_\alpha)_{\alpha<\omega_1}, \qquad (L_\alpha, R_\alpha)_{\alpha\in I} = (O_\alpha, P_\alpha)_{\alpha<\omega_1}$$

and an integer n such that $O'_\alpha \setminus n \subseteq O_\alpha$, $P'_\alpha \setminus n \subseteq P_\alpha$, and both $O'_\alpha \cap n$ and $P'_\alpha \cap n$ are constant for each $\alpha < \omega_1$. Since $(O'_\alpha, P'_\alpha)_{\alpha<\omega_1}$ satisfies $(\star)$ and for $\alpha, \beta < \omega_1$, $O'_\alpha \cap P'_\beta \subseteq O_\alpha \cap P_\beta$, the gap $(O_\alpha, P_\alpha)_{\alpha<\omega_1}$ satisfies $(\star)$ as well. ∎

The following simple fact reveals the connection between Hausdorff and left-oriented gaps.

FACT 5. *Every Hausdorff gap $\mathcal{G} = (L_\alpha, R_\alpha)$ is a left-oriented (special) gap.*

Proof. Define a set mapping $f\colon \omega_1 \to [\omega_1]^{<\omega}$ by

$$f(\alpha) = \{\xi < \alpha : L_\xi \cap R_\alpha = \emptyset\}.$$

Hajnal's free set theorem (see e.g. [6, Corollary 44.2]) implies that there is an unbounded $X \subseteq \omega_1$ such that $\xi \notin f(\alpha)$ for all $\xi, \alpha \in X$. This means that $L_\xi \cap R_\alpha \neq \emptyset$ for all $\xi < \alpha \in X$. ■

Under MA_{ω_1} or PID (see [1]) every gap is Hausdorff. It is consistent to have special non-Hausdorff gaps; the first example of such a gap was constructed in [12]. In Section 6 we provide a construction of a special non-Hausdorff gap of a quite different nature.

For a given tower $(T_\alpha)_{\alpha<\omega_1}$ it is always possible to construct a Hausdorff gap $(L_\alpha, R_\alpha)_{\alpha<\omega_1}$ such that $L_\alpha \cup R_\alpha = T_\alpha$ for each $\alpha < \omega_1$. It is even possible to construct a large system of such gaps [22, 7, 16].

It is worth mentioning that there is an analogy between gaps and Aronszajn trees in which destructible gaps correspond to Suslin trees (see [1, Section 2.2]). Indeed, if for a given pre-gap $\mathcal{G} = (L_\alpha, R_\alpha)_{\alpha<\omega_1}$ we introduce a compatibility relation on ω_1 in the following way: $\alpha, \beta < \omega_1$ are *compatible* if

$$(L_\alpha \cap R_\beta) \cup (L_\beta \cap R_\alpha) = \emptyset,$$

then $\mathcal{G}$ is a gap if and only if there are no uncountable chains (of pairwise compatible elements) in $\mathcal{P}(\omega_1)$. Moreover, $\mathcal{G}$ is destructible if and only if there are no uncountable antichains (of pairwise incompatible elements) in $\mathcal{P}(\omega_1)$. This remark explains an analogy in results about destructible gaps and Suslin trees. For instance, adding a Cohen real adds both a destructible gap and a Suslin tree; under MA_{ω_1} there are neither Suslin trees nor destructible (ω_1, ω_1)-gaps. We will see that we can add towers to this picture.

3. Basic definitions. We consider towers, i.e. families $(T_\alpha)_{\alpha<\kappa}$ such that $T_\alpha \setminus T_\beta$ is finite if and only if $\alpha \leq \beta$. We do not assume that towers are maximal, κ is always a regular cardinal, and we consider mainly towers of length ω_1. We say that two towers are *equivalent* if they generate (together with Fin) the same ideal in $\mathcal{P}(\omega)$.

We shall define three properties of towers similar to properties used for classification of gaps. It is convenient to reveal some connections between towers and gaps first.

Every gap consists of two towers and every tower is a half of a gap (the other half can be built by induction). Under MA_{ω_1} even more is true: every

ω_1-tower is a half of an (ω_1, ω_1)-gap (see [21] and [20, Remark 2.4]). However, this is not a ZFC theorem. Indeed, if an ω_1-tower is maximal, then it could be only a half of an $(\omega_1, 1)$-gap.

There are also ω_1-towers of different nature which cannot be a half of an (ω_1, ω_1)-gap. If there is an ω_1-*scale* (i.e. a strictly $\leq^*$-increasing sequence $(f_\alpha)_{\alpha<\omega_1}$ of elements of ω^ω eventually dominating all elements of ω^ω), then the tower defined by $T_\alpha = \{(n,m) : m \leq f_\alpha(n)\}$ is not maximal (and its orthogonal is not generated by a single set), but it cannot be a half of an (ω_1, ω_1)-gap. To see this, notice that every set in the orthogonal of $(T_\alpha)_{\alpha<\omega_1}$ is a subset of $n \times \omega$ for some $n \in \omega$. Assume that $(T_\alpha, R_\alpha)_{\alpha<\omega_1}$ forms an (ω_1, ω_1)-pre-gap. Since there are only countably many choices of n, without loss of generality there is a fixed n for which $R_\alpha \subseteq n \times \omega$. Clearly, $n \times \omega$ interpolates $(T_\alpha, R_\alpha)_{\alpha<\omega_1}$.

We say that a tower of length κ satisfies *condition* (K) if $T_\alpha \nsubseteq T_\beta$ for each $\alpha, \beta < \kappa$.

Definition 6. A tower $(T_\alpha)_{\alpha<\kappa}$ is *special* if it contains a cofinal subtower satisfying condition (K). A tower which is not special is called *Suslin*.

The name "Suslin" is justified by the fact that the poset $(\mathcal{T}, \subseteq)$ contains neither uncountable $\subseteq$-chains nor uncountable $\subseteq$-antichains if $\mathcal{T}$ is a Suslin ω_1-tower. We will later see that if we add a tower by forcing, then checking that this forcing is ccc is often the same as checking that the generic tower is Suslin.

We say that a tower $(T_\alpha)_{\alpha<\omega_1}$ satisfies *condition* (H) if

$$\{\xi < \alpha : T_\xi \setminus T_\alpha \subseteq n\} \text{ is finite}$$

for each $\alpha < \omega_1$ and $n < \omega$. (Note that this condition cannot be directly generalized to longer towers.)

Definition 7. A tower $(T_\alpha)_{\alpha<\omega_1}$ is *Hausdorff* if it contains a subtower satisfying condition (H).

The following fact implies that Hausdorff towers are quite common.

Proposition 8. *Let $(L_\alpha, R_\alpha)_{\alpha<\omega_1}$ be a Hausdorff gap. The tower $(L_\alpha)_{\alpha<\omega_1}$ is Hausdorff.*

Proof. Since $L_\alpha \cap R_\alpha = \emptyset$ for each $\alpha < \omega_1$, for every $\alpha < \beta < \omega_1$ if $L_\alpha \cap R_\beta \nsubseteq n$, then $L_\alpha \setminus L_\beta \nsubseteq n$. ∎

Similarly one can prove the following:

Proposition 9. *Let $(L_\alpha, R_\alpha)_{\alpha<\omega_1}$ be a left-oriented gap. The tower $(L_\alpha)_{\alpha<\omega_1}$ is special.*

The proof of the next fact is essentially the same as the proof of Fact 5.

PROPOSITION 10. *If* $(T_\alpha)_{\alpha<\omega_1}$ *satisfies condition* (H), *then there is an unbounded* $X \subseteq \omega_1$ *such that* $T_\alpha \setminus T_\beta \neq \emptyset$ *for all distinct* $\alpha, \beta \in X$.

Proof. Define $f\colon \omega_1 \to [\omega_1]^{<\omega}$ by

$$f(\alpha) = \{\xi < \alpha : T_\xi \subseteq T_\alpha\}.$$

Hajnal's free set theorem implies that there is an unbounded $X \subseteq \omega_1$ such that $\xi \notin f(\alpha)$ for all $\xi, \alpha \in X$. This means that $T_\xi \setminus T_\alpha \neq \emptyset$ for all $\xi < \alpha \in X$. ∎

COROLLARY 11. *Every Hausdorff tower is special.*

In particular, Hausdorff gaps provide examples of uncountable towers which form antichains if ordered by $\subseteq$. Since Hausdorff gaps exist in ZFC, it follows that special towers exist in ZFC.

There are facts indicating that the notion of a Hausdorff tower is more natural than the notion of a special tower. The next proposition shows that this is a "global" property, whereas Example 33 will demonstrate that this is not the case of special towers. (Another fact supporting the statement above is discussed in Section 7.)

PROPOSITION 12. *If a tower* $(T_\alpha)_{\alpha<\omega_1}$ *is equivalent to a Hausdorff tower* $(S_\alpha)_{\alpha<\omega_1}$, *then* $(T_\alpha)_{\alpha<\omega_1}$ *is Hausdorff.*

Proof. We can suppose that $(S_\alpha)_{\alpha<\omega_1}$ satisfies condition (H). There exist some $n < \omega$ and cofinal subtowers $(T'_\alpha)_{\alpha<\omega_1}$ and $(S'_\alpha)_{\alpha<\omega_1}$ such that $S'_\alpha \setminus n \subseteq T'_\alpha \subseteq^* S'_{\alpha+1}$ for each $\alpha < \omega_1$. Suppose that $(T'_\alpha)_{\alpha<\omega_1}$ does not satisfy (H). There are $\beta < \omega_1$ and $m < \omega$ such that $I = \{\xi < \beta : T'_\xi \setminus T'_\beta \subseteq m\}$ is infinite. Fix $k > \max(n,m)$ such that $T'_\beta \subseteq S'_{\beta+1} \cup k$. Now $I \subseteq \{\xi < \beta+1 : S'_\xi \setminus S'_{\beta+1} \subseteq k\}$, and this contradicts (H) of $(S'_\alpha)_{\alpha<\omega_1}$. ∎

The property of being a special tower is invariant under a slightly stronger equivalence relation:

PROPOSITION 13. *Assume* λ *is a cardinal with uncountable cofinality and* $\mathcal{T} = (T_\alpha)_{\alpha<\lambda}$ *is a special tower. If* $\mathcal{T}' = (T'_\alpha)_{\alpha<\lambda}$ *is such that* $T_\alpha =^* T'_\alpha$ *for each* $\alpha < \lambda$, *then* $\mathcal{T}'$ *is special.*

Proof. There is $X \subseteq \lambda$ cofinal in λ such that $(T_\alpha)_{\alpha\in X}$ is a $\subseteq$-antichain. We can find $X' \subseteq X$ cofinal in λ such that both $T'_\alpha \setminus T_\alpha$ and $T_\alpha \setminus T'_\alpha$ are constant for every $\alpha \in X'$. Clearly, $T'_\alpha \nsubseteq T'_\beta$ for all $\alpha < \beta$, $\alpha, \beta \in X'$. ∎

We finish this section by a comment on Proposition 8.

REMARK 14. We are not aware of any way of constructing (in ZFC) a Hausdorff tower without producing a Hausdorff gap (i.e. without implicitly constructing the other half of the gap). However, there are several generic examples of Hausdorff towers which are not halves of Hausdorff gaps. For instance, in every model obtained by forcing with a Suslin tree, there is a Hausdorff tower which is maximal. Let $\mathbb{S}$ be a Suslin tree. Define $\varphi\colon \mathbb{S} \to \mathcal{P}(\omega)$ in such a way that:

(1) $\varphi(s) \cap \varphi(t) =^* \emptyset$ for any incompatible $s, t \in \mathbb{S}$;
(2) $\varphi(s) \subseteq^* \varphi(t)$ if $t \leq s$;
(3) if S is a branch of the tree, then $\{\varphi(s)^c : s \in S\}$ satisfies (H).

Such a φ can be constructed by induction on levels of $\mathbb{S}$, using the fact that all branches of $\mathbb{S}$ are countable and a simplifying assumption that $\mathbb{S}$ does not split at limit levels. Having such a φ, we can see that the $\mathbb{S}$-generic branch through $\varphi''[\mathbb{S}]$ is a tower which is maximal (in principle because $\mathbb{S}$ does not add new subsets of ω, see e.g. [8, Lemma 2]) and Hausdorff.

This provides another example of a family generating a dense ideal which does not realize oscillation 1 (cf. [28, Example 1] and Section 5 below).

4. Special towers. We already know that special towers do exist in ZFC. We will see that consistently there are no other towers of length ω_1. The simplest way to see this is to use OCA. For the formulation of OCA see e.g. [25]; we will only use the following consequence of OCA (see [25, Proposition 8.4]):

Proposition 15 (Todorčević). *Under* OCA *every uncountable subset of $\mathcal{P}(\omega)$ contains an uncountable $\subseteq$-chain or $\subseteq$-antichain.*

A tower is well-ordered by $\subseteq^*$, so it cannot contain an uncountable chain. Hence the following holds:

Proposition 16. *Under* OCA *every ω_1-tower is special.*

It is unclear for us whether OCA implies that all towers of length ω_1 are Hausdorff. However, this is true if we assume MA_{ω_1}.

Lemma 17. *Let $(A_\alpha, B_\alpha)_{\alpha<\omega_1}$ be a sequence such that $A_\alpha \subseteq B_\alpha \subseteq^* A_\beta \subseteq \omega$ and $A_{\alpha+1} \setminus B_\alpha$ is infinite for each for $\alpha < \omega_1$. There exist $\xi < \zeta < \omega_1$ such that $A_\xi \not\subseteq B_\zeta$.*

Proof. Suppose $A_\alpha \subseteq B_\beta$ for each $\alpha \leq \beta < \omega_1$. Set $C_\alpha = \bigcup\{A_\xi : \xi \leq \alpha\}$. We have $C_\alpha \subseteq B_\alpha$, and since $A_{\alpha+1} \setminus B_\alpha$ is infinite, $C_\alpha \neq C_{\alpha+1}$ for all $\alpha < \omega_1$. Thus $(C_\alpha)_{\alpha<\omega_1}$ is an increasing $\subseteq$-chain of type ω_1, a contradiction. ∎

Proposition 18. *Let $\mathcal{T} = (T_\alpha)_{\alpha<\omega_1}$ be a tower. There exists a ccc forcing making $\mathcal{T}$ Hausdorff in the extension.*

Proof. A condition in the desired forcing is a pair $p = (F_p, n_p) \in [\omega_1]^{<\omega} \times \omega$. A condition q is *stronger* than p if $F_p \subseteq F_q$, $n_p \leq n_q$, and for all $\alpha < \beta$, $\alpha \in F_q \setminus F_p$, $\beta \in F_p$, there exists some $m \in (T_\alpha \setminus T_\beta) \setminus n_p$. For each condition p and each ordinal $\alpha < \omega_1$, $F_p < \alpha$, the pair $\langle F_p \cup \{\alpha\}, n\rangle$ (where $n > n_p$) is a condition stronger than p, and thus this forcing adds a subtower cofinal in $\mathcal{T}$ which fulfills condition (H) (provided that ω_1 is preserved).

To prove ccc, let $\{p_\alpha = \langle F_\alpha, n_\alpha\rangle : \alpha < \omega_1\}$ be a set of conditions. We can suppose that $n_\alpha = n$ for each $\alpha < \omega_1$ and that $\{F_\alpha : \alpha < \omega_1\}$ forms a Δ-system with core F. Denote $F'_\alpha = F_\alpha \setminus F$. Assume without loss of generality that $\max F < \min F'_\alpha < \max F'_\alpha < \min F'_\beta$ if $\alpha < \beta$.

For $\alpha < \omega_1$ set $A_\alpha = \bigcap\{T_\xi : \xi \in F'_\alpha\} \setminus n$ and $B_\alpha = \bigcup\{T_\xi : \xi \in F'_\alpha\} \setminus n$. Lemma 17 shows that there are $\alpha < \beta < \omega_1$ such that $A_\alpha \not\subseteq B_\beta$. Note that $\langle F_\alpha \cup F_\beta, n\rangle$ is a condition stronger than both p_α and p_β. ∎

Corollary 19 (MA_{ω_1}). *Every tower of length ω_1 is Hausdorff.*

Since there are no Hausdorff towers of length greater than ω_1, this result does not generalize to higher cardinals. However, the following is still true.

Theorem 20. *Let κ be a regular uncountable cardinal and let $\mathcal{T} = (T_\alpha)_{\alpha<\kappa}$ be a tower. There is a ccc forcing making $\mathcal{T}$ special in the extension.*

Theorem 20 can be proved in a way similar to [29, Theorem 1.4], see also [30, Theorem 4.4]. The forcing consists simply of finite subsets F of $\kappa \setminus \gamma$ (for a suitably chosen $\gamma \in \kappa$) such that $T_\alpha \not\subseteq T_\beta$ if $\alpha \neq \beta$, and $\alpha, \beta \in F$. This forcing is ccc (checking this needs some work but it is not difficult). Therefore there is $\gamma < \kappa$ such that for any condition $F \subseteq \kappa\setminus\gamma$ there are cofinally many β such that $F\cup\{\beta\}$ is a condition (otherwise we could construct an uncountable set of pairwise incompatible conditions). Hence this forcing adds a cofinal subtower satisfying (K). Instead of proving Theorem 20 directly, we show a slightly stronger theorem. Namely, under MA_κ every tower of length $\lambda \leq \kappa$ (with λ of uncountable cofinality) can be modified to a tower with condition (K) by a minor cosmetic operation: it is enough to add at most one integer to each level and to remove at most one integer from each level. Proving ccc for this forcing is similar to proving it for the forcing mentioned above.

Theorem 21. *Let κ be a regular uncountable cardinal and let $\mathcal{T} = (T_\alpha)_{\alpha<\kappa}$ be a tower. There is a ccc forcing $\mathbb{P}$ which generically adds a tower $\mathcal{T}' = (T'_\alpha)_{\alpha<\kappa}$ such that $|T_\alpha \setminus T'_\alpha| \leq 1$ and $|T'_\alpha \setminus T_\alpha| \leq 1$ for each $\alpha < \omega_1$, and $\mathbb{P}$ forces that $\mathcal{T}'$ satisfies condition* (K).

This theorem together with Proposition 13 implies Theorem 20.

Lemma 22. *For $k < \omega$ and each $i < k$ let $\mathcal{T}_i = (T^i_\alpha)_{\alpha<\omega_1}$ be a tower. There exist $\zeta < \xi < \omega_1$ such that $T^i_\zeta \not\subseteq T^i_\xi$ for each $i < k$.*

Proof. We prove the lemma by induction on k. The statement holds true for $k = 1$. At the $(k+1)$th step use the induction hypothesis to find pairs $\zeta_\alpha < \xi_\alpha$ for $\alpha < \omega_1$ such that $T^i_{\zeta_\alpha} \not\subseteq T^i_{\xi_\alpha}$ and $\xi_\alpha < \zeta_\beta$ for each $i < k$ and $\alpha < \beta < \omega_1$.

Claim. *We can moreover assume that $T^i_{\zeta_\alpha} \not\subseteq T^i_{\xi_\beta}$ for all $\alpha, \beta < \omega_1$ and $i < k$.*

Proof. We can first refine the system so that there is $n < \omega$ such that $T^i_{\zeta_\alpha} \cap n \not\subseteq T^i_{\xi_\alpha} \cap n$ for all $i < k$ and $\alpha < \omega_1$. After that refine further to get $T^i_{\zeta_\alpha} \cap n$ and $T^i_{\xi_\alpha} \cap n$ constant for a fixed i. ∎

We are done if $T^k_{\zeta_\alpha} \not\subseteq T^k_{\xi_\alpha}$ for some $\alpha < \omega_1$, so suppose the opposite. Lemma 17 states that there are $\alpha < \beta < \omega_1$ such that $T^k_{\xi_\alpha} \not\subseteq T^k_{\zeta_\beta}$. Thus $\xi = \xi_\alpha$ and $\zeta = \zeta_\beta$ are as required. ∎

Proof of Theorem 21. A condition $p \in \mathbb{P}$ is of the form (F_p, a_p, r_p), where

- $F_p \in [\kappa]^{<\omega}$;
- $a_p \colon F \to \omega$ and $r_p \colon F \to \omega$;
- for every $\alpha < \beta \in F$ we have

$$T_\alpha \cup \{a_p(\alpha)\} \setminus \{r_p(\alpha)\} \not\subseteq T_\beta \cup \{a_p(\beta)\} \setminus \{r_p(\beta)\}.$$

The ordering is given by $q \le p$ if $F_p \subseteq F_q$, $a_q|F_p = a_p$, and $r_q|F_p = r_p$. Notice that for each condition $p \in \mathbb{P}$ and $\alpha < \kappa$ there is $q \in \mathbb{P}$ such that $\alpha \in F_q$ and $q \le p$. Indeed, choose

$$m \notin \bigcup\{T_\xi \cup \{a_p(\xi)\} : \xi \in F_p \setminus \alpha\}$$

and

$$n \in \bigcap\{T_\xi \setminus \{r_p(\xi)\} : \xi \in F_p \cap \alpha\} \setminus \{m\},$$

and define $F_q = F_p \cup \{\alpha\}$, $a_q(\alpha) = m$, $r_q(\alpha) = n$. Let G be a $\mathbb{P}$-generic, $a = \bigcup_{p\in G} a_p$, $r = \bigcup_{p\in G} r_p$. Clearly, the tower defined by

$$T'_\alpha = T_\alpha \cup \{a(\alpha)\} \setminus \{r(\alpha)\}$$

is as desired.

It only remains to show that our forcing is ccc. Let $\{p_\alpha : \alpha < \omega_1\}$ be a set of conditions. We will denote $F_\alpha = F_{p_\alpha}$, $a_\alpha = a_{p_\alpha}$ and $r_\alpha = r_{p_\alpha}$. By thinning out the sequence if necessary, we may assume that $F_\alpha = \{\xi^0_\alpha < \xi^1_\alpha < \cdots < \xi^{k-1}_\alpha\}$ for each $\alpha < \omega_1$ and that $a_\alpha(\xi^i_\alpha)$ and $r_\alpha(\xi^i_\alpha)$ depend only on i. Using the Δ-lemma we further assume that $F_\alpha = F \cup F'_\alpha$ for each $\alpha < \omega_1$, where $(F'_\alpha)_{\alpha<\omega_1}$ is pairwise disjoint and there is $I \subseteq k$ such that $F = \{\xi^i_\alpha : i \in I\}$ for every $\alpha < \omega_1$. So for each $i < k$ the sequence $(\xi^i_\alpha)_{\alpha<\omega_1}$ is either constant or injective. Considering a subsequence once again (if necessary), we may assume that $(\xi^i_\alpha)_{\alpha<\omega_1}$ is either constant or strictly increasing for each $i < k$. We may also assume that there is $l < \omega$ such that the sequence

$$\Big(\big(\big(T_{\xi^i_\alpha} \cup \{a_\alpha(\xi^i_\alpha)\}\big) \setminus \{r_\alpha(\xi^i_\alpha)\}\big) \cap l\Big)_{\alpha<\omega_1}$$

is constant for each $i < k$, where l is such that

$$\big(\big(T_{\xi^i_\alpha} \cup \{a_\alpha(\xi^i_\alpha)\}\big) \setminus \{r_\alpha(\xi^i_\alpha)\}\big) \cap l \quad \text{and} \quad \big(\big(T_{\xi^j_\alpha} \cup \{a_\alpha(\xi^j_\alpha)\}\big) \setminus \{r_\alpha(\xi^j_\alpha)\}\big) \cap l$$

are $\subsetneq$-incompatible for $i \neq j$. Apply Lemma 22 to find α, β such that

$$\big(T_{\xi^i_\alpha} \cup \{a_\alpha(\xi^i_\alpha)\}\big) \setminus \{r_\alpha(\xi^i_\alpha)\} \not\subseteq \big(T_{\xi^i_\beta} \cup \{a_\beta(\xi^i_\beta)\}\big) \setminus \{r_\beta(\xi^i_\beta)\}$$

for each $i \in k \setminus I$. Now $q = (F_\alpha \cup F_\beta, a_\alpha \cup a_\beta, r_\alpha \cup r_\beta)$ is a condition in $\mathbb{P}$, and $q \le p_\alpha$, $q \le p_\beta$. ∎

Proposition 18 is an analogue of the theorem stating that under MA_{ω_1} every gap is Hausdorff. In [1] the authors prove that the same statement holds assuming the P-ideal dichotomy. This is not true for towers. The P-ideal dichotomy is compatible with CH, and under CH Suslin towers do exist. However, if we additionally assume that $\mathfrak{b}$ is big, the P-ideal dichotomy implies that every ω_1-tower is Hausdorff. Recall that $\mathfrak{b}$ is the minimal cardinality of a family in ω^ω which cannot be $\le^*$-dominated by a single function. The *P-ideal dichotomy* (PID) is the assertion: for every P-ideal $\mathcal{I} \subseteq [\omega_1]^\omega$ one of the following holds:

- there is an uncountable $K \subseteq \omega_1$ such that $[K]^\omega \subseteq \mathcal{I}$;
- $\omega_1 = \bigcup_{n<\omega} A_n$ and $A_n \cap I$ is finite for each $n < \omega$ and $I \in \mathcal{I}$.

Notice that if for each uncountable $K \subseteq \omega_1$ there is an infinite $I \subseteq K$, $I \in \mathcal{I}$, then the second alternative cannot hold.

THEOREM 23. *Assume* PID. *Every ω_1-tower is Hausdorff if and only if* $\mathfrak{b} > \omega_1$.

REMARK. A related result with a similar proof was obtained independently in [17]. Namely:

THEOREM. *Assume* PID. *The following are equivalent:*

(1) $\min\{\mathfrak{b}, \mathrm{cof}(\mathcal{F}_\sigma)\} > \omega_1$.
(2) *There are only five Tukey types of directed sets of size at most* ω_1.

For the definition of the cardinal invariant $\mathrm{cof}(\mathcal{F}_\sigma)$ see [17]. The relation of these results becomes apparent in Section 7, where it is shown that Hausdorff towers correspond to the Tukey type $[\omega_1]^{<\omega}$.

Proof of Theorem 23. In the next section we shall prove that a Suslin tower of length $\mathfrak{b}$ always exists (Proposition 26). We prove here only the "if" part of the theorem.

Define an ideal $\mathcal{I} \subseteq [\omega_1]^{\le\omega}$ by

$$I \in \mathcal{I} \text{ iff } C^n_\alpha(I) = \{\xi \in \alpha \cap I : T_\xi \setminus T_\alpha \subseteq n\} \text{ is finite for all } \alpha < \omega_1,\ n < \omega.$$

CLAIM. *If* $\mathfrak{b} > \omega_1$, *then $\mathcal{I}$ is a P-ideal.*

Proof. Consider a sequence $\{I_n : n < \omega\} \in [\mathcal{I}]^\omega$. Assume without loss of generality that $(I_n)_{n<\omega}$ is pairwise disjoint, and fix an enumeration $I_n = \{\xi^n_k : k < \omega\}$ for each n. For every $\alpha < \omega_1$ define a function $f_\alpha \colon \omega \to \omega$ by

$$f_\alpha(n) = \max\{k : T_{\xi^n_k} \setminus T_\alpha \subseteq n\}.$$

Let $g\colon \omega \to \omega$ be a function $\leq^*$-dominating $\{f_\alpha : \alpha < \omega_1\}$. Let

$$I = \bigcup_{n<\omega} I_n \setminus \{\xi^n_k : k \leq g(n)\}.$$

It is straightforward to check that $I \in \mathcal{I}$ and $I_n \subseteq^* I$ for each n. ■

The first alternative of PID for $\mathcal{I}$ gives us a subtower which fulfills condition (H), so we only need to refute the second alternative of PID. We shall show that for each uncountable $K \subseteq \omega_1$ there is $I \in \mathcal{I} \cap [K]^\omega$.

Claim. *There exists $x \in 2^\omega$ such that $x \in \overline{\{T_\alpha : \alpha \in K\}}$ (the closure in the Cantor space) but $x \notin \langle T_\alpha \rangle_{\alpha<\omega_1}$ (the ideal generated by the tower).*

Proof. If $\overline{\{T_\alpha : \alpha \in K\}} \subseteq \langle T_\alpha \rangle_{\alpha<\omega_1}$, then the ideal $\langle T_\alpha \rangle_{\alpha<\omega_1}$ is generated by a closed set and thus it is an analytic P-ideal. On the other hand, an analytic P-ideal which is not countably generated cannot be generated by less than $\mathfrak{d}$ sets [26, Theorem 6]. ■

Fix $I \in [K]^\omega$ such that x is the single accumulation point of $\{T_\alpha : \alpha \in I\}$. To conclude that $I \in \mathcal{I}$, notice that if for some $\beta < \omega_1$ and $n < \omega$ we have $T_\alpha \subseteq T_\beta \cup n$ for infinitely many $\alpha \in I$, there would be an accumulation point of $\{T_\alpha : \alpha \in I\}$ which would be a subset of $T_\beta \cup n$ and hence in $\langle T_\alpha \rangle_{\alpha<\omega_1}$. ■

This seems a convenient moment at which to mention the following two results. Note that none of them directly implies Corollary 19.

Theorem 24 (Shelah [20]). *MA_{ω_1} implies that every ω_1-tower is the right half of a Hausdorff gap.*

Theorem 25 (Spasojević [21]). *$\mathsf{MA}_{\omega_1}(\sigma$-centered$)$ implies that every ω_1-tower is the right half of the left-oriented gap.*

In Section 6 we present ideas behind the proof of the above theorem (see Example 42).

5. Suslin towers. We know that consistently there are no Suslin ω_1-towers. However, Suslin towers, perhaps longer than ω_1, always exist:

Proposition 26. *There is a tower $\mathcal{T} = (T_\alpha)_{\alpha<\mathfrak{b}}$ which is Suslin.*

Proof. Let $\{f_\alpha : \alpha < \mathfrak{b}\} \subseteq \omega^\omega$ be a $\leq^*$-unbounded family which is $\leq^*$-strictly increasing. Define

$$T_\alpha = \{(m,n) : n \leq f_\alpha(m)\}$$

for every α. This is a $\mathfrak{b}$-tower (on $\omega \times \omega$). If $K \subseteq \mathfrak{b}$ is cofinal, then $\{f_\alpha : \alpha \in K\}$ is $\leq^*$-unbounded, and thus there are $\alpha < \beta$, $\alpha, \beta \in K$, such that $f_\alpha(m) \leq f_\beta(m)$ for each $m < \omega$ (see [24]). Therefore $T_\alpha \subseteq T_\beta$ and $(T_\alpha)_{\alpha<\mathfrak{b}}$ is a Suslin tower. ■

The above fact and Theorem 23 may suggest that the existence of a Suslin ω_1-tower is equivalent to $\mathfrak{b} = \omega_1$ in ZFC. This is not the case.

PROPOSITION 27. *Let κ be an uncountable regular cardinal. It is consistent that $\mathfrak{b} = \kappa$ and there is a Suslin ω_1-tower.*

Proof. Start with a model of $\mathfrak{b} = \omega_1$ with a Suslin tower. Then use a finite support iteration of Hechler forcings $\mathbb{H}$ (for adding a dominating real) of length κ. This will make $\mathfrak{b} = \kappa$ in the extension. Hechler forcing is σ-centered and thus has the *Knaster property* (i.e. for every uncountable $X \subseteq \mathbb{H}$ there is an uncountable linked $X_0 \subseteq X$), which is preserved in finite-support iterations.

We will prove that a forcing with the Knaster property does not destroy a Suslin tower $(T_\alpha)_{\alpha<\omega_1}$. (This also follows from the general well-known fact that such forcing preserves ccc-ness of ground model relations.) Suppose that $\mathbb{P}$ is such a forcing, $p \in \mathbb{P}$ is any condition, and $\dot{X}$ is a $\mathbb{P}$-name for an uncountable subset of ω_1. Consider

$$X = \{\alpha < \omega_1 : \exists p_\alpha < p,\, p_\alpha \Vdash \alpha \in \dot{X}\}.$$

There is an uncountable $X_0 \subseteq X$ such that $p_\alpha \parallel p_\beta$ for all $\alpha, \beta \in X_0$. The tower $(T_\alpha)_{\alpha<\omega_1}$ is Suslin, hence there are distinct $\alpha, \beta \in X_0$ such that $T_\alpha \subseteq T_\beta$ and any $q < p_\alpha, p_\beta$ forces that $\alpha, \beta \in \dot{X}$. Therefore the tower remains Suslin in the extension. ∎

The crux of Proposition 26 is Todorčević's result on oscillations of functions. His work on oscillations of subsets of ω in [28] sheds even more light on the existence of Suslin towers. Recall that the *oscillation* of $A, B \subseteq \omega$ (denoted by $\mathrm{osc}(A, B)$) is the cardinality of the set $A \mathbin{\triangle} B/\!\sim$, where $\sim$ is the equivalence relation defined on $A \mathbin{\triangle} B$ by

$$m \sim n \quad \text{iff} \quad [n, m] \cap (A \mathbin{\triangle} B) \subseteq A \setminus B \text{ or } [n, m] \cap (A \mathbin{\triangle} B) \subseteq B \setminus A.$$

(We slightly abuse the notation treating $[n, m]$ as $[m, n]$ for $m < n$.) We say that a family $\mathcal{A} \subseteq \mathcal{P}(\omega)$ *realizes an oscillation* n if there are $A, B \in \mathcal{A}$ such that $\mathrm{osc}(A, B) = n$.

The following is a special case of [28, Corollary 2].

THEOREM 28 (Todorčević [28]). *If a family $\mathcal{A} \subseteq \mathcal{P}(\omega)$ generates a non-meager P-ideal, then it realizes all finite oscillations.*

Notice that if $A \subsetneq^* B$, then $A \subseteq B$ if and only if $\mathrm{osc}(A, B) = 1$. It follows that each tower generating a non-meager ideal is Suslin. We enclose here for the reader's convenience the sketch of the proof of the latter assertion (extracted from [28]):

Proof. We will say that a tower $\mathcal{T} = (T_\alpha)_{\alpha<\kappa}$ has *property* (ξ) if for an arbitrarily large $n < \omega$ there is $t \subseteq n$ such that for each $m > n$ there are arbitrarily large $\beta < \kappa$ with the properties

- $T_\beta \cap n = t$;
- $[n, m) \subseteq T_\beta$.

Claim (ξ). *Let $\mathcal{T}$ be a tower of size κ of uncountable cofinality such that every cofinal subtower of $\mathcal{T}$ has property (ξ). This $\mathcal{T}$ is a Suslin tower.*

Proof. This is basically [28, Lemma 2]. Since $P(\omega)$ is hereditary separable, we can fix a countable set $\mathcal{D} \subseteq \mathcal{T}$ dense in $\mathcal{T}$. There is $\alpha < \kappa$ such that $D \subseteq^* T_\alpha$ for each $D \in \mathcal{D}$. Without loss of generality we can assume that there is $m_0 < \omega$ such that $T_\alpha \setminus m_0 \subseteq T_\beta$ for every $\beta > \alpha$. Using property (ξ) we can fix $m_1 > m_0$ and $t \subseteq m_1$ such that for every $m > m_1$ there is $\beta > \alpha$ such that $T_\beta \cap m_1 = t$ and $[m_1, m) \subseteq T_\beta$.

Pick $D \in [t] \cap \mathcal{D}$. Fix $m > m_1$ such that $D \setminus m \subseteq T_\alpha$, and β such that $[m_1, m) \subseteq T_\beta$. Then

- $D \cap m_1 = t = T_\beta \cap m_1$;
- $D \cap [m_1, m) \subseteq [m_1, m) = T_\beta \cap [m_1, m)$;
- $D \setminus m \subseteq T_\alpha \setminus m_0 \subseteq T_\beta$.

Hence $D \subseteq T_\beta$. ∎

It is enough to show that every tower which generates a non-meager ideal has property (ξ). This is basically the beginning of the proof of [28, Theorem 1] and the proof of [28, Lemma 1]. We may assume that for each finite $F \subseteq \omega$ the set $\{\alpha : F \subseteq T_\alpha\}$ is either empty or cofinal in $\mathcal{T}$. This is standard (since $[\omega]^{<\omega}$ is countable). Then we argue *a contrario*. Subsequently negating (ξ) we obtain an increasing sequence of natural numbers $(n_k)_{k<\omega}$ witnessing the fact that $\mathcal{T}$ generates a meager ideal. ∎

As a corollary we obtain many examples of Suslin towers. For instance, each tower generating a maximal ideal is Suslin.

In a somewhat similar manner (to Claim (ξ)) we can prove that adding a Cohen real adds a Suslin tower. This result is not a surprise; the proof mimics the well-known argument used by Todorčević to show that Cohen reals produce destructible gaps.

Proposition 29. *Let $(T_\alpha)_{\alpha<\omega_1}$ be a tower and let c be a Cohen generic real in an extension. Then $(T_\alpha \cap c)_{\alpha<\omega_1}$ is a Suslin tower.*

Proof. To see that the tower is not eventually constant notice that $c \cap (T_\beta \setminus T_\alpha)$ is infinite for each $\alpha < \beta < \omega_1$.

Let $p \in {}^n2$ be a Cohen condition and $\dot{X}$ be a name for an uncountable subset of ω_1. We can assume that $X = \dot{X}$ belongs to the ground model (by taking a stronger condition if necessary). Consider $\alpha < \beta$ ($\alpha, \beta \in X$) such that $T_\alpha \cap n = T_\beta \cap n$ and fix $m > n$ such that $T_\alpha \subset T_\beta \cup m$. Extend p to $q \in {}^m2$ such that $q^{-1}(1) = p^{-1}(1)$. Now $q \Vdash T_\alpha \cap \dot{c} \subseteq T_\beta \cap \dot{c}$. ∎

This simple example is of some importance, since the resulting Suslin tower will be used in the next section to produce a special non-Hausdorff gap. Notice also that intersecting a Cohen real with a gap gives us a destructible gap with both sides being Suslin towers. So it is possible to have Suslin towers which are far from being non-meager (whose orthogonal is not generated by a single set).

One way to add a tower generically is to use a standard technique inspired by Hechler's work [10]. It allows one to prove (see e.g. [9, Theorem 5.8, Chapter 2]) that whenever $\mathcal{P}$ is a partial order, there is a forcing notion $\mathbb{P}$ such that $\mathbb{P} \Vdash$ "$\check{\mathcal{P}}$ embeds in $\mathcal{P}(\omega)/\mathrm{Fin}$". It seems that whenever $\mathcal{P}$ is a partial order and $\mathcal{C} \subseteq \mathcal{P}$ is an uncountable chain, then in Hechler's extension the embedding of $\mathcal{C}$ into $\mathcal{P}(\omega)/\mathrm{Fin}$ will be Suslin unless we impose some additional restrictions on the conditions of $\mathbb{P}$. We will try to justify this by examples below and in the next section.

Example 30 (The classical Hechler forcing for adding a tower). A condition in $\mathbb{P}$ is a triple $p = (F_p, n_p, A_p)$ where $F_p \in [\omega_1]^{<\omega}$, $n_p < \omega$ and $A_p \subseteq F_p \times n_p$. For two conditions p, q we use the notation $p \cup q = (F_p \cup F_q, n_p \cup n_q, A_p \cup A_q)$. The condition q is stronger than p if $n_p \leq n_q$, $F_p \subseteq F_q$, $A_q \cap (F_p \times n_p) = A_p$, and for all $\alpha, \beta \in F_p$, $\alpha < \beta$, and $i \in [n_p, n_q)$,

$$\text{if } (\alpha, i) \in A_q, \text{ then } (\beta, i) \in A_q.$$

Claim. *$\mathbb{P}$ is ccc.*

Proof. Fix a set of conditions $\{p_\alpha : \alpha < \omega_1\}$. Use the Δ-lemma to find an uncountable set I such that $\{F_{p_\alpha} : \alpha \in I\}$ forms a Δ-system with core Δ, and n_{p_α} is constant for $\alpha \in I$. We can further refine I to an uncountable I' so that $A_{p_\alpha} \cap (\Delta \times n_{p_\alpha})$ is constant. Now for all $\alpha, \beta \in I'$ the conditions p_α and p_β are compatible since $p_\alpha \cup p_\beta$ is their common extension. ∎

Let G be a generic filter. Set $A = \bigcup_{p \in G} A_p$. For $\alpha < \omega_1$ define

$$T_\alpha = \{i < \omega : (\alpha, i) \in A\}.$$

Claim. *$(T_\alpha)_{\alpha<\omega_1}$ is a Suslin tower.*

Proof. It is obvious that $(T_\alpha)_{\alpha<\omega_1}$ is non-constant. Consider a name $\dot{X}$ for an uncountable subset of ω_1 and a condition p. There is an uncountable set

$$X = \{\alpha < \omega_1 : \exists p_\alpha < p,\ \alpha \in F_{p_\alpha},\ p_\alpha \Vdash \alpha \in \dot{X}\}.$$

Now proceed in the same way as in the proof of the previous claim to get an uncountable set $I \subseteq X$. We may further suppose that $\{i < n_{p_\alpha} : (\alpha, i) \in A_{p_\alpha}\}$ is constant for $\alpha \in I$. Hence $p_\alpha \cup p_\beta < p_\alpha, p_\beta$ and

$$p_\alpha \cup p_\beta \Vdash (\alpha, \beta \in \dot{X} \text{ and } T_\alpha \subseteq T_\beta)$$

for $\alpha, \beta \in I$, $\alpha < \beta$. ∎

The forcing in this example is in fact equivalent to the forcing adding ω_1 Cohen reals. In what follows we denote the latter by $\mathbb{C}_{\omega_1}$.

PROPOSITION 31. *$\mathbb{P}$ is equivalent to $\mathbb{C}_{\omega_1}$.*

Proof. Using [14, Main Theorem], it is enough to find a sequence of $(\mathbb{P}_\alpha)_{\alpha<\omega_1}$ such that:

(1) $\mathbb{P}_\gamma = \bigcup_{\alpha<\gamma} \mathbb{P}_\alpha$ for each limit $\gamma \le \omega_1$;
(2) for $\alpha < \beta$, $\mathbb{P}_\alpha$ is a complete suborder of $\mathbb{P}_\beta$;
(3) $\mathbb{P}_{\alpha+1}/\mathbb{P}_\alpha$ is equivalent to Cohen forcing.

For $\alpha < \omega_1$ let $\mathbb{P}_\alpha = \{(F, n, A) \in \mathbb{P} : F \subseteq \alpha\}$. Only checking (2) is non-trivial.

It is enough to show that for $\alpha < \beta \le \omega_1$ there is a pseudo-projection $p\colon \mathbb{P}_\beta \to \mathbb{P}_\alpha$ (see [2, Proposition 2]). That is, we need to define for each $q = (F_q, n_q, A_q) \in \mathbb{P}_\beta$ a condition $p(q) \in \mathbb{P}_\alpha$ such that whenever $r < p(q)$, $r \in \mathbb{P}_\alpha$, then r is compatible with q (in $\mathbb{P}$). It is trivial to check that $p(q) = (F_q \cap \alpha, n_q, A_q \cap (\alpha \times n_q))$ works. ∎

In what follows we will present several other incarnations of $\mathbb{C}_{\omega_1}$ used for producing peculiar towers and gaps.

EXAMPLE 32 (Hechler's forcing with the Hausdorff restriction). Consider a modification of the forcing from Example 30. We add one more requirement for $q < p$. Namely, for each $\alpha \in F_p$ and $\xi \in F_q \setminus F_p$, $\xi < \alpha$, there has to be some $i \ge n_p$ such that $(\xi, i) \in A_q$ and $(\alpha, i) \notin A_q$.

This forcing adds a generic tower $(T_\alpha)_{\alpha<\omega_1}$ satisfying condition (H) in the same way as the forcing from Example 30 adds a Suslin tower. As in Example 30 we can show that this forcing is equivalent to $\mathbb{C}_{\omega_1}$ (and so it is ccc); the same definition of $\mathbb{P}_\alpha$ pseudo-projections works also for this forcing. Notice, however, that checking this is not as trivial as before (but not difficult either).

Notice also that the tower added by this forcing is maximal (and so this is another example of a maximal Hausdorff tower, see Remark 14). Indeed, let $P \subseteq \omega$ be an infinite set from the extension. It is enough to check that P intersects some T_α on an infinite set. Because of ccc, the name for P is guessed at some intermediate step, so we can forget about an initial segment of the tower and assume that P is from the ground model. Then the set

$$D_n = \{p \in \mathbb{P} : 0 \in F_p,\ \exists m > n, (0, m) \in A_p \text{ and } m \in P\}$$

is dense in $\mathbb{P}$ for each n. This proves that $P \cap T_0$ is infinite in the extension.

Probably the most interesting example of this sort is the following one.

EXAMPLE 33 (A special tower equivalent to a Suslin tower). Let κ be an uncountable regular cardinal. We construct a forcing which adds a pair of equivalent towers of length κ, one of them being special and the other one Suslin (in a strong sense).

A condition is a sequence $p=(F_p,n_p,\langle T^\alpha_p,S^\alpha_p\rangle_{\alpha\in F_p})$, where $F_p\in[\kappa]^{<\omega}$, $n_p<\omega$, and $T^\alpha_p,S^\alpha_p\subseteq n_p$ for each $\alpha\in F_p$, and $S^\alpha_p\not\subseteq S^\beta_p$ for $\alpha<\beta\in F_p$.

A condition q is stronger than p if $n_p\le n_q$, $F_p\subseteq F_q$, $T^\alpha_q\cap n_p=T^\alpha_p$, $S^\alpha_q\cap n_p=S^\alpha_p$ for $\alpha\in F_p$, and for all $\alpha,\beta\in F_p$, $\alpha<\beta$, and $i\in[n_p,n_q)$,

$$\text{if } i\in T^\alpha_q\cup S^\alpha_q \text{ then } i\in T^\beta_q\cap S^\beta_q \quad\text{and}\quad \text{if } i\in T^\alpha_q \text{ then } i\in S^\alpha_q.$$

It is easy to see that for each $\alpha<\kappa$ the set $\{p:\alpha\in F_p\}$ is dense, and hence this forcing adds a couple of equivalent towers of length κ^V, $(T_\alpha)_{\alpha<\kappa}$ and $(S_\alpha)_{\alpha<\kappa}$ defined by $T_\alpha=\bigcup_{p\in G}T^\alpha_p$ and $S_\alpha=\bigcup_{p\in G}S^\alpha_p$ for $\alpha<\kappa$.

The tower $(S_\alpha)_{\alpha<\kappa}$ satisfies condition (K). On the other hand $(T_\alpha)_{\alpha<\kappa}$ is far from being special.

Claim. *Every uncountable subtower of $(T_\alpha)_{\alpha<\kappa}$ is Suslin.*

Proof. Consider a name $\dot X$ for an uncountable subset of κ and a condition p. There is an uncountable set

$$X=\{\alpha<\kappa:\exists p_\alpha<p,\ \alpha\in F_{p_\alpha},\ p_\alpha\Vdash\alpha\in\dot X\}.$$

Use the Δ-lemma to find an uncountable set I such that $\{F_{p_\alpha}:\alpha\in I\}$ forms a 'nice' Δ-system with core Δ. Each F_{p_α}, $\alpha\in I$, is split into blocks

$$F_{p_\alpha}=F^0_\alpha\cup\Delta^0\cup F^1_\alpha\cup\Delta^1\cup\cdots\cup F^{k-1}_\alpha\cup\Delta^{k-1},$$

$\Delta=\bigcup\Delta^i$, $\max F^i_\alpha<\min\Delta^i$, $\max\Delta^{i-1}<\min F^i_\alpha$, $\max F^i_\alpha<\min F^i_\beta$, and

$$F^i_\alpha=\{\xi^i_0(\alpha)<\xi^i_1(\alpha)<\cdots<\xi^i_{j(i)-1}(\alpha)\}$$

for any $\alpha<\beta<\omega_1$ $(\alpha,\beta\in I)$ and $i<k$. (F^0_α and some Δ^is may be empty, in that case disregard the required inequalities.)

We may moreover assume that $T^\xi_{p_\alpha}$ and $S^\xi_{p_\alpha}$ are constant for any $\xi\in\Delta$, that n_{p_α}, $T^{\xi^i_m(\alpha)}_{p_\alpha}$, and $S^{\xi^i_m(\alpha)}_{p_\alpha}$ are constant (ranging over $\alpha\in I$) for all $i<k$, $m<j(i)$, and that there are $J,M<\omega$ such that $\alpha=\xi^J_M(\alpha)$ for $\alpha\in I$.

Pick any $\alpha<\beta\in I$. Define a condition q by $F_q=F_{p_\alpha}\cup F_{p_\beta}$, $n_q=n_{p_\alpha}+k+1$, and furthermore:

- for $i<J$ and $\chi\in F^i_\alpha\cup\Delta^i$ let
$$T^\chi_q=T^\chi_{p_\alpha}\cup[n_{p_\alpha},n_{p_\alpha}+i+1) \text{ and } S^\chi_q=S^\chi_{p_\alpha}\cup[n_{p_\alpha},n_{p_\alpha}+i+1);$$
- for $i<J$ and $\chi\in F^i_\beta$ let
$$T^\chi_q=T^\chi_{p_\beta}\cup[n_{p_\beta},n_{p_\beta}+i) \text{ and } S^\chi_q=S^\chi_{p_\beta}\cup[n_{p_\beta},n_{p_\beta}+i);$$
- for $i>J$ and $\chi\in F^i_\alpha\cup\Delta^i$ let
$$T^\chi_q=T^\chi_{p_\alpha}\cup[n_{p_\alpha},n_{p_\alpha}+i+2) \text{ and } S^\chi_q=S^\chi_{p_\alpha}\cup[n_{p_\alpha},n_{p_\alpha}+i+2);$$

- for $i > J$ and $\chi \in F^i_\beta$ let
$$T^\chi_q = T^\chi_{p_\beta} \cup [n_{p_\beta}, n_{p_\beta} + i + 1) \text{ and } S^\chi_q = S^\chi_{p_\beta} \cup [n_{p_\beta}, n_{p_\beta} + i + 1);$$
- for $\chi \in \Delta^J$ let
$$T^\chi_q = T^\chi_{p_\alpha} \cup [n_{p_\alpha}, n_{p_\alpha} + J + 2) \text{ and } S^\chi_q = S^\chi_{p_\alpha} \cup [n_{p_\alpha}, n_{p_\alpha} + J + 2);$$
- for $m < M$ and $\chi = \xi^J_m(\alpha)$ let
$$T^\chi_q = T^\chi_{p_\alpha} \cup [n_{p_\alpha}, n_{p_\alpha} + J + 1) \text{ and } S^\chi_q = S^\chi_{p_\alpha} \cup [n_{p_\alpha}, n_{p_\alpha} + J + 1);$$
- for $m < M$ and $\chi = \xi^J_m(\beta)$ let
$$T^\chi_q = T^\chi_{p_\beta} \cup [n_{p_\beta}, n_{p_\beta} + J) \text{ and } S^\chi_q = S^\chi_{p_\beta} \cup [n_{p_\beta}, n_{p_\beta} + J);$$
- for $m > M$ and $\chi = \xi^J_m(\alpha)$ let
$$T^\chi_q = T^\chi_{p_\alpha} \cup [n_{p_\alpha}, n_{p_\alpha} + J + 2) \text{ and } S^\chi_q = S^\chi_{p_\alpha} \cup [n_{p_\alpha}, n_{p_\alpha} + J + 2);$$
- for $m > M$ and $\chi = \xi^J_m(\beta)$ let
$$T^\chi_q = T^\chi_{p_\beta} \cup [n_{p_\beta}, n_{p_\beta} + J + 1) \text{ and } S^\chi_q = S^\chi_{p_\beta} \cup [n_{p_\beta}, n_{p_\beta} + J + 1);$$
- for $\chi = \xi^J_M(\alpha)$ let
$$T^\chi_q = T^\chi_{p_\alpha} \cup [n_{p_\alpha}, n_{p_\alpha} + J + 1) \text{ and } S^\chi_q = S^\chi_{p_\alpha} \cup [n_{p_\alpha}, n_{p_\alpha} + J + 2);$$
- for $\chi = \xi^J_M(\beta)$ let
$$T^\chi_q = T^\chi_{p_\beta} \cup [n_{p_\beta}, n_{p_\beta} + J + 1) \text{ and } S^\chi_q = S^\chi_{p_\beta} \cup [n_{p_\beta}, n_{p_\beta} + J + 1).$$

To show that q is a condition of $\mathbb{P}$ is straightforward. The condition q is a common extension of both p_α and p_β, $q \Vdash \alpha, \beta \in \dot{X}$, and $q \Vdash \dot{T}_\alpha \subseteq \dot{T}_\beta$. ∎

The proof of the claim also shows that the forcing is ccc. In case $\kappa = \omega_1$ this forcing is equivalent to $\mathbb{C}_{\omega_1}$. To check this, use the same strategy as in the proof of Proposition 31. Define $\mathbb{P}_\alpha = \{q \in \mathbb{P} : F_q \subseteq \omega \cdot \alpha\}$ for $\alpha < \omega_1$. For $\gamma < \beta$ and $q \in \mathbb{P}_\beta$ define a pseudo-projection $p(q) \in \mathbb{P}_\gamma$ as follows. First find a set $F \subseteq \omega \cdot \gamma$ such that $|F| = |F_q \setminus (\omega \cdot \gamma)|$ and $F_q \cap (\omega \cdot \gamma) < F$, and fix an order preserving bijection $b\colon F_q \setminus (\omega \cdot \gamma) \to F$. Define
$$p(q) = \big(F_{p(q)} = (F_q \cap (\omega \cdot \gamma)) \cup F, n_q, \langle T^\alpha_{p(q)}, S^\alpha_{p(q)} \rangle_{\alpha \in F_{p(q)}}\big),$$
where
$$(T^\alpha_{p(q)}, S^\alpha_{p(q)}) = \begin{cases} (T^\alpha_q, S^\alpha_q) & \text{for } \alpha \notin F, \\ (T^\alpha_{p(q)}, S^\alpha_{p(q)}) = (T^{b^{-1}(\alpha)}_q, S^{b^{-1}(\alpha)}_q) & \text{for } \alpha \in F. \end{cases}$$

We will sketch the proof that $p(q)$ is a pseudo-projection. Suppose that $r < p(q)$ and $r \in \mathbb{P}_\gamma$. We want to find $s \in \mathbb{P}_\beta$ such that $s < r$ and $s < q$. Let $F_s = F_r \cup F_q$, $n_s = n_r + 1$. For $\eta \in F_q \setminus (\omega \cdot \gamma)$ set $T^\eta_s = T^{b(\eta)}_r$. If $\xi \leq \max(F_q \cap (\omega \cdot \gamma))$ or $\xi \geq \omega \cdot \gamma$ let $n_r \notin T^\xi_s \cup S^\xi_s$, and for $\xi \in (\max(F_q \cap (\omega \cdot \gamma)), \omega \cdot \gamma)$ let $n_r \in T^\xi_s \cap S^\xi_s$. We have to show that for each $\xi \in F_r$ and $\eta \in F_q \setminus (\omega \cdot \gamma)$ we have $S^\xi_s \nsubseteq S^\eta_s$. If $\xi \in (\max(F_q \cap (\omega \cdot \gamma)), \omega \cdot \gamma)$, then $n_r \in S^\xi_s \setminus S^\eta_s$. If

$\xi \leq \max(F_q \cap (\omega \cdot \gamma))$, then $S_s^\xi \not\subseteq S_s^\eta$ since $S_r^\xi \not\subseteq S_r^{b^{-1}(\eta)}$. Hence $s \in \mathbb{P}_\beta$. It is easy to check that $s < r$ and $s < q$.

This example refutes the natural conjecture that each special tower is in fact Hausdorff (since a Hausdorff tower cannot be equivalent to a Suslin tower). Moreover, it proves that the property of being special, unlike the Hausdorff property, is not invariant under the equivalence of towers (cf. Proposition 12).

In the following section we provide another example of a tower of this kind: a tower which is neither Hausdorff nor equivalent to a Suslin tower.

Notice that most of the examples presented in this section exist in models obtained by adding ω_1 Cohen reals. It seems that the structure of towers is particularly rich in such models. We will show that adding ω_1 Cohen reals produces various interesting gaps.

6. Structure of gaps after adding ω_1 Cohen reals. One of the most natural questions related to destructibility of gaps is asking whether the class of special (ω_1, ω_1)-gaps coincides with the class of Hausdorff gaps. It was posed in [19] as Problem 2. Since we isolated another property lying in between of the above ones, we can ask more specifically:

PROBLEM 34 ([19, Problem 1]). Is every special gap left-oriented?

PROBLEM 35. Is every left-oriented gap equivalent to a Hausdorff gap?

Hirschorn [12] answered Scheepers's problem. More precisely, he gave an example of a left-oriented gap which is not equivalent to any Hausdorff gap, so he answered Problem 35 in the negative. It turns out that the answer to Problem 34 is also negative.

THEOREM 36. *There is a special gap which is not left-oriented.*

First, we give an example which relies only on simple facts and known results. In particular, we need the following theorem due to Roitman:

THEOREM 37 ([18]). *Adding a single Cohen real to a model satisfying* $\mathsf{MA}(\sigma$*-centered*$)$ *preserves* $\mathsf{MA}(\sigma$*-centered*$)$.

EXAMPLE 38 (An inverted Spasojević gap). Work in a model of $\mathsf{MA}_{\omega_1}(\sigma$-centered). Using Proposition 29 and the theorem above, we can add a Cohen real and get a Suslin tower $(R_\alpha)_{\alpha<\omega_1}$ in the extension without loosing $\mathsf{MA}(\sigma$-centered). Of course, the tower cannot be maximal since $\mathfrak{t} > \omega_1$. Theorem 25 now gives a special gap $(L_\alpha, R_\alpha)_{\alpha<\omega_1}$ fulfilling condition (O). Consider the gap $(R_\alpha, L_\alpha)_{\alpha<\omega_1}$. Inverting the sides of an indestructible gap cannot make it destructible, so $(R_\alpha, L_\alpha)_{\alpha<\omega_1}$ is still special. However, it cannot be left-oriented. Indeed, in this case Proposition 9 would imply that $(R_\alpha)_{\alpha<\omega_1}$ is special, but this tower is Suslin.

The reader perhaps wonders if the gap $(L_\alpha, R_\alpha)_{\alpha<\omega_1}$ introduced by the forcing from Theorem 25 is Hausdorff. We will show that it is not. Actually, Example 42 will show that gaps introduced by Spasojević's forcing are left-oriented, but not Hausdorff. Thus to obtain a special non-Hausdorff gap, we do not need to invert the gap in Example 38. As a corollary we observe that left-oriented gaps are not necessarily right-oriented. The following example shows that the Hausdorff condition for gaps is not symmetric either. There is a Hausdorff gap such that the inverted gap is not Hausdorff. We start Hechler's machinery again.

Example 39 (An asymmetric Hausdorff gap). We define a forcing $\mathbb{P}$ consisting of conditions of the form

$$p = (F_p, n_p, \langle L_p^\alpha, R_p^\alpha \rangle_{\alpha \in F_p}),$$

where

(1) $F_p \in [\omega_1]^{<\omega}$;
(2) $n_p < \omega$;
(3) $L_p^\alpha, R_p^\alpha \subseteq n_p$ for each $\alpha \in F_p$;
(4) $L_p^\alpha \cap R_p^\alpha = \emptyset$ for each $\alpha \in F_p$.

A condition q is stronger than p if

(a) $n_p \leq n_q$ and $F_p \subseteq F_q$;
(b) $L_q^\alpha \cap n_p = L_p^\alpha$ and $R_q^\alpha \cap n_p = R_p^\alpha$ for $\alpha \in F_p$;
(c) for any $\alpha, \beta \in F_p$, $\alpha < \beta$, and $i \in [n_p, n_q)$,

$$\text{if } i \in L_q^\alpha \text{ then } i \in L_q^\beta \quad \text{and} \quad \text{if } i \in R_q^\alpha \text{ then } i \in R_q^\beta;$$

(d) for each $\alpha \in F_p$ and $\xi \in F_q \setminus F_p$, $\xi < \alpha$, there is some $i \geq n_p$ such that $i \in L_q^\xi \cap R_q^\alpha$.

It is easy to see that for each $\alpha < \omega_1$ the set $\{p : \alpha \in F_p\}$ is dense. Let G be a $\mathbb{P}$-generic filter, and let $L_\alpha = \bigcup_{p \in G} L_p^\alpha$ and $R_\alpha = \bigcup_{p \in G} R_p^\alpha$ for $\alpha < \omega_1$. Then $(L_\alpha, R_\alpha)_{\alpha<\omega_1}$ is Hausdorff provided $\mathbb{P}$ preserves ω_1.

Claim. *$\mathbb{P}$ is equivalent to adding ω_1 Cohen reals.*

Proof. As in Proposition 31, the forcing $\mathbb{P}_\beta$ consists of the conditions $q = (F_q, n_q, \langle L_q^\alpha, R_q^\alpha \rangle_{\alpha \in F_q})$ with $F_q \subseteq \beta$. The pseudo-projection $p \colon \mathbb{P}_\beta \to \mathbb{P}_\gamma$ is defined by

$$p(q) = (F_q \cap \gamma, n_q, \langle L_q^\alpha, R_q^\alpha \rangle_{\alpha \in F_q \cap \gamma}). \ \blacksquare$$

Claim. *$(R_\alpha)_{\alpha<\omega_1}$ is a Suslin tower.*

Proof. Consider a name $\dot{X}$ for an uncountable subset of ω_1 and a condition $p \in \mathbb{P}$. There is an uncountable set

$$X = \{\alpha < \omega_1 : \exists p_\alpha < p,\ \alpha \in F_{p_\alpha},\ p_\alpha \Vdash \alpha \in \dot{X}\}.$$

We will proceed in the same way as in the examples from the previous section. Use the Δ-lemma to find an uncountable set I such that $\{F_{p_\alpha} : \alpha \in I\}$ forms a Δ-system with core Δ, $\max \Delta < \min F_{p_\alpha} \setminus \Delta$ for $\alpha \in I$, and $n_{p_\alpha} = n^\Delta$ is constant for $\alpha \in I$. We may assume that $\max F_{p_\alpha} < \min F_{p_\beta} \setminus \Delta$ for $\alpha < \beta < \omega_1$. We can further refine I to an uncountable I' so that all $R^\alpha_{p_\alpha}$, $L^\xi_{p_\alpha}$ and $R^\xi_{p_\alpha}$ are constant for all $\xi \in \Delta$, $\alpha \in I'$. Pick any $\alpha < \beta \in I' \setminus \Delta$, and define a condition q by $F_q = F_{p_\alpha} \cup F_{p_\beta}$, $n_q = n^\Delta + 1$,

(i) $L^\xi_q = L^\xi_{p_\alpha}$ and $R^\xi_q = R^\xi_{p_\alpha}$ for $\xi \in \Delta$,
(ii) $L^\xi_q = L^\xi_{p_\alpha} \cup \{n^\Delta\}$ and $R^\xi_q = R^\xi_{p_\alpha}$ for $\xi \in F_{p_\alpha} \setminus \Delta$,
(iii) $L^\xi_q = L^\xi_{p_\beta}$ and $R^\xi_q = R^\xi_{p_\beta} \cup \{n^\Delta\}$ for $\xi \in F_{p_\beta} \setminus \Delta$.

The condition q is a common extension of both p_α and p_β, $q \Vdash \alpha, \beta \in \dot{X}$ and $q \Vdash \dot{R}_\alpha \subseteq \dot{R}_\beta$.

To show that $\mathbb{P}$ is ccc, we do the same reductions for an arbitrary uncountable set of conditions. ∎

We now present another example witnessing the negative answer for Problem 34.

Example 40 (A special gap which is neither left- nor right-oriented). We define a forcing $\mathbb{P}$ similar to the poset from the previous example (and also equivalent to $\mathbb{C}_{\omega_1}$). A condition $p \in \mathbb{P}$ is of the form $p = (F_p, n_p, (L^\alpha_p, R^\alpha_p)_{\alpha \in F_p})$ and it satisfies properties (1)–(4) from Example 39. We impose the following additional restriction:

- $(L^\alpha_p \cap R^\beta_p) \cup (L^\beta_p \cap R^\alpha_p) \neq \emptyset$ for each $\alpha < \beta \in F_p$.

The ordering on $\mathbb{P}$ is defined by conditions (a)–(c) from the previous example.

Let G be a $\mathbb{P}$-generic filter. Set $L_\alpha = \bigcup_{p \in G} L^\alpha_p$ and $R_\alpha = \bigcup_{p \in G} R^\alpha_p$ for $\alpha < \omega_1$. It is clear that $(L_\alpha, R_\alpha)_{\alpha < \omega_1}$ is a special gap.

Claim. *Both $(L_\alpha)_{\alpha<\omega_1}$ and $(R_\alpha)_{\alpha<\omega_1}$ are Suslin towers.*

Proof. We deal with the right side; the proof for the left side is exactly the same. Consider a name $\dot{X}$ for an uncountable subset of ω_1 and a condition p. There is an uncountable set

$$X = \{\alpha < \omega_1 : \exists p_\alpha < p,\ \alpha \in F_{p_\alpha},\ p_\alpha \Vdash \alpha \in \dot{X}\}.$$

Now proceed in the same way as in Example 39 to get an uncountable set $I \subseteq X$. Pick $\alpha < \beta \in I \setminus \Delta$ and define a condition q by $F_q = F_{p_\alpha} \cup F_{p_\beta}$, $n_q = n_{p_\alpha} + 1$, and by (i)–(iii) from the previous example. The condition q is a common extension of both p_α and p_β, $q \Vdash \alpha, \beta \in \dot{X}$ and $q \Vdash \dot{R}_\alpha \subseteq \dot{R}_\beta$. ∎

The proof that this forcing is equivalent to $\mathbb{C}_{\omega_1}$ works in a way similar to Example 33. Let $\mathbb{P}_\beta$ be generated by conditions q such that $F_q \subseteq \omega \cdot \beta$, and define the pseudo-projection in the same way as in Example 33.

We now prove that consistently there is a gap providing answers to both questions from the beginning of this section.

THEOREM 41. *In a model obtained by adding ω_1 Cohen reals there is a gap $(L_\alpha, R_\alpha)_{\alpha<\omega_1}$ such that*

- *$(L_\alpha, R_\alpha)_{\alpha<\omega_1}$ is left-oriented but not Hausdorff;*
- *$(R_\alpha, L_\alpha)_{\alpha<\omega_1}$ is special but not left-oriented.*

Proof. Define a forcing notion equivalent to adding ω_1 Cohen reals which forces the existence of the desired gap. A condition in $\mathbb{P}$ is a sequence

$$p = (F_p, n_p, \langle L_p^\alpha, R_p^\alpha \rangle_{\alpha \in F_p})$$

satisfying properties (1)–(4) of Example 39 and such that additionally

- $L_p^\alpha \cap R_p^\beta \neq \emptyset$ for each $\alpha < \beta \in F_p$.

The ordering of $\mathbb{P}$ is defined by (a)–(c) of Example 39.

As in the previous examples, it is easy to see that $\mathbb{P}$ adds a generic gap which is left-oriented (provided ω_1 is preserved).

CLAIM. *The forcing $\mathbb{P}$ is equivalent to $\mathbb{C}_{\omega_1}$ (and so it is ccc).*

Proof. This is exactly the same proof as in Example 40 (which is in turn similar to the proof from Example 33).

CLAIM. *The tower $(L_\alpha)_{\alpha<\omega_1}$ is not Hausdorff and the tower $(R_\alpha)_{\alpha<\omega_1}$ is Suslin.*

Proof. We prove both statements simultaneously. We need to show that there is no cofinal subtower $(L_\alpha)_{\alpha \in \dot{X}}$ satisfying condition (H). Consider a name $\dot{X}$ for an uncountable subset of ω_1, and suppose that some condition p forces that $(L_\alpha)_{\alpha \in \dot{X}}$ satisfies (H). We show that this leads to a contradiction, and $(L_\alpha)_{\alpha<\omega_1}$ is not Hausdorff. Moreover, we will prove that there are $q < p$ and $\alpha < \beta \in \dot{X}$ such that $q \Vdash \dot{R}_\alpha \subseteq \dot{R}_\beta$, showing that $(R_\alpha)_{\alpha<\omega_1}$ is Suslin.

There is an uncountable set

$$I = \{\alpha < \omega_1 : \exists p_\alpha < p,\ \alpha \in F_{p_\alpha},\ p_\alpha \Vdash \alpha \in \dot{X}\}.$$

Using the Δ-lemma we may assume that $\Delta < F'_\alpha = F_{p_\alpha} \setminus \Delta$, and $n_{p_\alpha} = n$ is constant for $\alpha \in I$. Moreover, $F'_\alpha < F'_\beta$ for $\alpha < \beta$, and $L^\xi_{p_\alpha}$ and $R^\xi_{p_\alpha}$ are constant for all $\xi \in \Delta$. For some $\ell < \omega$ we have $F'_\alpha = \{\xi^0_\alpha < \xi^1_\alpha < \cdots < \xi^{\ell-1}_\alpha\}$ for each α, and $L^{\xi^i_\alpha}_{p_\alpha} = L^{\xi^i_\beta}_{p_\beta}$ and $R^{\xi^i_\alpha}_{p_\alpha} = R^{\xi^i_\beta}_{p_\beta}$ for $\alpha, \beta \in I$, $i \in \ell$. Finally, there is $i' < \ell$ for which $\alpha = \xi^{i'}_{p_\alpha}$ for all $\alpha \in I$.

Let α_0 be the first element of I and let $\beta \in I$ be some ordinal with infinitely many predecessors in I. Define a condition q by $F_q = F_{\alpha_0} \cup F_\beta$, $n_q = n + 1$ and

- $L^\xi_q = L^\xi_{p_\beta}$ for $\xi \in F_{p_\beta}$,

- $L_q^\xi = L_{p_{\alpha_0}}^\xi \cup \{n\}$ for $\xi \in F'_{\alpha_0}$,
- $R_q^\xi = R_{p_{\alpha_0}}^\xi$ for $\xi \in F_{p_{\alpha_0}}$,
- $R_q^\xi = R_{p_\beta}^\xi \cup \{n\}$ for $\xi \in F'_\beta$.

It is straightforward to check that $q \in \mathbb{P}$ and $q < p_{\alpha_0}, p_\beta$. Notice also that $q \Vdash \dot{R}_{\alpha_0} \subseteq \dot{R}_\beta$ (at this point we already know that $(R_\alpha)_{\alpha<\omega_1}$ is Suslin). According to our assumption on $\dot{X}$, there exist some $k < \omega$ and a condition $r < q$ such that

$$r \Vdash |\{\alpha \in \dot{X} \cap \beta : L_\alpha \setminus L_\beta \subseteq n+1\}| < k.$$

Since F_r is finite, we can find $\{\alpha_1 < \cdots < \alpha_k\} \subseteq I \cap \beta$ such that $\alpha_0 < \alpha_1$ and

$$F_r \cap [\min F'_{\alpha_1}, \max F'_{\alpha_k}] = \emptyset.$$

Define a condition s by $F_s = F_r \cup \bigcup_{j \le k} F'_{\alpha_j}$, $n_s = n_r + k$ and

- $L_s^\xi = L_r^\xi$ for $\xi \in F_r$, $\xi \le \max \Delta$,
- $L_s^\xi = L_r^\xi \cup \{n_r\}$ for $\xi \in F_r$, $\max \Delta < \xi < \min F'_{\alpha_1}$,
- $L_s^\xi = L_r^\xi \cup [n_r, n_r + k)$ for $\xi \in F_r$, $\max F'_{\alpha_k} < \xi$,
- $L_s^{\xi^i_{\alpha_j}} = (L_r^{\xi^i_{\alpha_0}} \cup \{n_r + j\}) \cap n_s$ for $i < \ell$, $j \le k$,
- $R_s^\xi = R_r^\xi$ for $\xi \in F_r$,
- $R_s^{\xi^i_{\alpha_j}} = R_r^{\xi^i_{\alpha_0}} \cup [n_r, n_r + j)$ for $i < \ell$, $j \le k$.

It is not difficult to verify that $s \in \mathbb{P}$ and $s < r$, $s < p_{\alpha_j}$ for each $j \le k$. Hence

$$s \Vdash \{\alpha_0, \alpha_1, \dots, \alpha_k, \beta\} \subseteq \dot{X}.$$

Moreover $L_s^{\alpha_j} \setminus L_s^\beta = \{n\}$ for each $j \le k$, and thus $s \Vdash L_{\alpha_j} \setminus L_\beta = \{n\}$. But this contradicts $s < r$. ∎

Notice that Proposition 8 implies that $(L_\alpha, R_\alpha)_{\alpha<\omega_1}$ is not Hausdorff (but it is left-oriented). Moreover, the gap $(R_\alpha, L_\alpha)_{\alpha<\omega_1}$ is still special, but Proposition 9 implies that it cannot be left-oriented. ∎

In fact, by a slight modification of the above proof, we can show that the original (σ-centered) forcing of Spasojević from [21] also produces a left-oriented non-Hausdorff gap.

Example 42 (A left-oriented gap not equivalent to any Hausdorff gap). Let $\mathcal{R} = \{R_\alpha : \alpha < \omega_1\}$ be a given tower. Spasojević introduced ([1]) a σ-centered forcing $\mathbb{P}$ adding a tower $(L_\alpha)_{\alpha<\omega_1}$ such that $(L_\alpha, R_\alpha)_{\alpha<\omega_1}$ is an oriented gap. We show that the tower $(L_\alpha)_{\alpha<\omega_1}$ is not Hausdorff.

([1]) In fact, Spasojević dealt with gaps in $({}^\omega\omega, <^*)$ rather than $([\omega]^\omega, \subset^*)$ but the construction is analogous.

A condition in $\mathbb{P}$ is a triple
$$p = (F_p, n_p, (L_p^\alpha)_{\alpha\in F_p}),$$
where

(1) $F_p \in [\omega_1]^{<\omega}$;
(2) $n_p < \omega$;
(3) $L_p^\alpha \subseteq n_p$ for each $\alpha \in F_p$;
(4) $L_p^\alpha \cap R_\alpha = \emptyset$ and $L_p^\alpha \cap R_\beta \neq \emptyset$ for all $\alpha < \beta \in F_p$;
(5) $R_\alpha \setminus R_\beta \subseteq n_p$ for all $\alpha < \beta \in F_p$.

A condition q is stronger than p if

(a) $n_p \leq n_q$ and $F_p \subseteq F_q$;
(b) $L_q^\alpha \cap n_p = L_p^\alpha$ for $\alpha \in F_p$;
(c) for $\alpha < \beta \in F_p$ we have $L_q^\alpha \cap [n_p, n_q) \subseteq L_q^\beta$.

Lemma 43. *$\mathbb{P}$ is σ-centered.*

Proof. This is proved in [21] in more detail for an analogous forcing. We present a sketch of the argument for the reader's convenience.

For each $\gamma \leq \omega_1$ define a forcing $\mathbb{P}_\gamma$ consisting of the conditions
$$p = (F_p, G_p, n_p, (L_p^\alpha)_{\alpha\in G_p}),$$
where

(2) $F_p \in [\omega_1]^{<\omega}$, $G_p \subseteq F_p \cap \gamma$;
(3) $n_p < \omega$;
(4) $L_p^\alpha \subseteq n_p$ for each $\alpha \in G_p$;
(5) $L_p^\alpha \cap R_\alpha = \emptyset$ and $L_p^\alpha \cap R_\beta \neq \emptyset$ for all $\alpha < \beta$, $\alpha \in G_p$, $\beta \in F_p$.

A condition q is stronger than p if

(a) $n_p \leq n_q$, $F_p \subseteq F_q$ and $G_p \subseteq G_q$;
(b) $L_q^\alpha \cap n_p = L_p^\alpha$ for $\alpha \in G_p$;
(c) for all $\alpha < \beta \in G_p$ we have $L_q^\alpha \cap [n_p, n_q) \subseteq L_q^\beta$;
(d) $L_q^\alpha \cap [n_p, n_q) \cap R_\beta = \emptyset$ for $\alpha \in G_p$, $\beta \in F_p$.

It is easy to check that $\mathbb{P}_{\omega_1}$ is a forcing equivalent to $\mathbb{P}$.

Claim. *$\mathbb{P}_\gamma \subseteq \mathbb{P}_\delta$ is a regular embedding for $\gamma < \delta \leq \omega_1$.*

Proof. The inclusion is an embedding of posets. To show regularity define $\pi\colon \mathbb{P}_\delta \to \mathbb{P}_\gamma$ by
$$(F, G, n, (L^\alpha)_{\alpha\in G}) \mapsto (F, G\cap\gamma, n, (L^\alpha)_{\alpha\in G\cap\gamma}).$$
It is straightforward to check that π is a pseudo-projection from $\mathbb{P}_\delta$ to $\mathbb{P}_\gamma$. ∎

Claim. *$\mathbb{P}_\gamma$ is σ-centered for each $\gamma < \omega_1$.*

Proof. The set of conditions sharing the same G, n and $(L^\alpha)_{\alpha\in G}$ is centered. ∎

To conclude the proof of Lemma 43 notice that $\mathbb{P}_{\omega_1}$ is a direct limit of the sequence of σ-centered posets $\{\mathbb{P}_\gamma : \gamma < \omega_1\}$, and hence is σ-centered. ∎

In the generic extension define $L_\alpha = \bigcup_{p\in G} L_p^\alpha$. Now $(L_\alpha, R_\alpha)_{\alpha<\omega_1}$ is an oriented gap.

CLAIM. *The tower $(L_\alpha)_{\alpha<\omega_1}$ is not Hausdorff. Consequently, the gap $(L_\alpha, R_\alpha)_{\alpha<\omega_1}$ is not equivalent to a Hausdorff gap.*

Proof. For a contradiction, take a name $\dot{X}$ for an uncountable subset of ω_1 and assume that some condition p forces that $(L_\alpha)_{\alpha\in\dot{X}}$ satisfies (H).

There is an uncountable set

$$I = \{\alpha < \omega_1 : \exists p_\alpha < p,\ \alpha \in F_{p_\alpha},\ p_\alpha \Vdash \alpha \in \dot{X}\}.$$

Fix some large enough cardinal θ and countable elementary submodels $M, N \prec H(\theta)$, $I, \mathcal{R} \in N, M$ such that $M \in N$. Notice that $M \subseteq N$ (since M is countable) and $M \neq N$. Fix some $\alpha_0 \in I \cap N \setminus M$.

Work in M. By passing to a subset we can suppose that all conditions p_α for $\alpha \in I$ are isomorphic to p_{α_0} and form a 'nice' Δ-system with core $\Delta = F_{\alpha_0} \cap M$. In particular, we assume that $\Delta < F'_\alpha = F_{p_\alpha} \setminus \Delta$ and $n_{p_\alpha} = n_{p_0} = n$ for $\alpha \in I$. Moreover, $F'_\alpha < F'_\beta$ for $\alpha < \beta$, and $L^\xi_{p_\alpha} \cap n$ and $R_\xi \cap n$ are constant for all $\xi \in \Delta$. For some $\ell < \omega$ we have $F'_\alpha = \{\xi^0_\alpha < \xi^1_\alpha < \cdots < \xi^{\ell-1}_\alpha\}$ for each α, and $L^{\xi^i_\alpha}_{p_\alpha} \cap n = L^{\xi^i_\beta}_{p_\beta} \cap n$, $R_{\xi^i_\alpha} \cap n = R_{\xi^i_\beta} \cap n$ for $\alpha, \beta \in I$, $i < \ell$. Finally, there is $i' < \ell$ for which $\alpha = \xi^{i'}_{p_\alpha}$ for all $\alpha \in I$.

For $\alpha \in I$ denote

$$\overline{R}_\alpha = \bigcup\{R_\xi : \xi \in F'_\alpha\} \quad \text{and} \quad \underline{R}_\alpha = \bigcap\{R_\xi : \xi \in F'_\alpha\}.$$

Fix $\beta \in I \setminus N$. There is some $n_0 > n$, $n_0 \in \underline{R}_\beta$, such that $n_0 \notin \overline{R}_{\alpha_0}$. Define a condition q by $F_q = F_{\alpha_0} \cup F_\beta$, $n_q = \max(n_0, \overline{R}_{\alpha_0} \setminus \underline{R}_\beta) + 1$ and

- $L^\xi_q = L^\xi_{p_\beta}$ for $\xi \in F_\beta$,
- $L^\xi_q = L^\xi_{p_{\alpha_0}} \cup \{n_0\}$ for $\xi \in F'_{\alpha_0}$.

Thus $q < p_{\alpha_0}, p_\beta$. There exist some $k < \omega$ and a condition $r < q$ such that

$$r \Vdash |\{\alpha \in \dot{X} \cap \beta : L_\alpha \setminus L_\beta \subseteq n_q\}| < k.$$

Denote $A = F_r \cap N$, $B = F_r \setminus N$, and let $R_A = \bigcup\{R_\xi : \xi \in A\}$, $R_B = \bigcup\{R_\xi : \xi \in B\}$, $R_r = R_A \cup R_B$.

CLAIM. *There exist a sequence $\{\alpha_1 < \cdots < \alpha_k\} \subseteq I \cap N$, $\max A < \min F'_{\alpha_1}$, such that $R_{\xi^i_{\alpha_j}} \cap n_r = R_{\xi^i_{\alpha_0}} \cap n_r$ for $j \le k$, $i < \ell$, and a sequence $\{n_j : n_j > n_r,\ 0 < j \le k\} \subseteq \omega \setminus R_r$ such that for $0 < j \le k$, $i \le k$ we have $n_j \in \underline{R}_{\alpha_i}$ if $j \le i$ and $n_j \notin \overline{R}_{\alpha_i}$ if $j > i$.*

Proof. Let I_0 be such that $R_{\xi^i_\alpha} \cap n_r = R_{\xi^i_{\alpha_0}} \cap n_r$ for each $\alpha \in I_0$ and $i < \ell$. Notice that $I_0 \in M$ since $I \in M$ and the refinement procedure is definable. Moreover, $|I_0| = \omega_1$. Otherwise, $I_0 \subseteq M$ and, in particular, $\alpha_0 \in M$.

To choose α_1, consider the increasing tower $\{\underline{R}_\alpha \setminus R_A : \alpha \in I_0\} \in N$. This tower is not bounded by the set R_B, hence there exist some $\alpha'_1 \notin N$ and $n_1 > n_r$ such that $n_1 \in \underline{R}_{\alpha'_1} \setminus R_r$. Define $I_1 = \{\alpha \in I_0 : n_1 \in \underline{R}_\alpha\} \in N$. Since $N \prec H(\theta)$ and $\alpha'_1 \notin N$, the set I_1 is uncountable. Pick any $\alpha_1 \in I_1 \cap N$ such that $\max A < \min F'_{\alpha_1}$.

Suppose that $\alpha_j, I_j \in N$ are defined for some $j < k$. Set $Z = R_A \cup \bigcup\{\overline{R}_{\alpha_i} : i \leq j\}$. Consider the tower $\{\underline{R}_\alpha \setminus Z : \alpha \in I_j\} \in N$. This tower is not bounded by R_B, hence there exist some $\alpha'_{j+1} \notin N$ and $n_{j+1} > n_r$ such that $n_{j+1} \in \underline{R}_{\alpha'_{j+1}} \setminus (Z \cup R_r)$. Define

$$I_{j+1} = \{\alpha \in I_j : n_{j+1} \in \underline{R}_\alpha\} \in N.$$

Again, since $N \prec H(\theta)$ and $\alpha'_{j+1} \notin N$, the set I_{j+1} is uncountable. Pick any $\alpha_{j+1} \in I_{j+1} \cap N$, $\alpha_{j+1} > \alpha_j$. ∎

Define a condition s by $F_s = F_r \cup \bigcup_{j \leq k} F'_{\alpha_j}$, $n_s > \max\{n_i : i \leq k\} + n_r$,

- $L^\xi_s = L^\xi_r$ for $\xi \in F_r$, $\xi \leq \max \Delta$,
- $L^\xi_s = L^\xi_r \cup \bigcup\{n_j : 0 < j \leq k\}$ for $\xi \in F_r$, $\max \Delta < \xi$,
- $L^{\xi^i_{\alpha_j}}_s = L^{\xi^i_{\alpha_0}}_r \cup \{n_{j+1}\}$ for $i < \ell$, $0 < j < k$,
- $L^{\xi^i_{\alpha_k}}_s = L^{\xi^i_{\alpha_0}}_r$ for $i < \ell$.

It is not difficult to verify that $s \in \mathbb{P}$ and $s < r$, $s < p_{\alpha_j}$ for each $j \leq k$. Now

$$s \Vdash \{\alpha_0, \alpha_1, \ldots, \alpha_k, \beta\} \subseteq \dot{X}.$$

Moreover $L^{\alpha_j}_s \setminus L^\beta_s = \{n_0\}$ for each $j \leq k$, and so $s \Vdash L_{\alpha_j} \setminus L_\beta = \{n_0\}$. This contradicts $s < r$. ∎

Theorem 41 together with Proposition 9 immediately gives us the corollary promised in the previous section:

COROLLARY 44. *There is an ω_1-tower equivalent neither to a Hausdorff nor to a Suslin tower in models obtained by adding ω_1 Cohen reals.*

REMARK 45. Perhaps the left half of the gap constructed by Hirschorn [12] also has the above property. Hirschorn showed that in the model obtained by adding ω_1 random reals, one can generically add a gap $(L_\alpha, R_\alpha)_{\alpha<\omega_1}$ which is left-oriented but not Hausdorff. Hence $(L_\alpha)_{\alpha<\omega_1}$ cannot be equivalent to a Suslin tower. To show that $(L_\alpha, R_\alpha)_{\alpha<\omega_1}$ is not Hausdorff, Hirschorn used a certain fact based on the Gilles theorem [12, Lemma 5.5]. This fact can be immediately modified for the case of towers in the following way. Assume that $(\mathcal{R}, \lambda)$ is the random algebra with the standard measure and $(\dot{T}_\alpha)_{\alpha<\omega_1}$

is an $\mathcal{R}$-name for a tower. If there is a function $h\colon \omega \to \mathbb{R}^+$ converging to 0 such that

$$\lambda(\|\dot{T}_\alpha \subseteq \dot{T}_\beta \cup n\|) \le h(n)$$

for all $\alpha < \beta < \omega_1$ and $n < \omega$, then $(\dot{T}_\alpha)_{\alpha<\omega_1}$ is not Hausdorff. (Here $\|\varphi\|$ represents the Boolean value of the sentence φ.) However, it does not seem that $(L_\alpha)_{\alpha<\omega_1}$ satisfies this condition for any $h\colon \omega \to \mathbb{R}^+$ converging to 0.

7. Towards a structure theory: Tukey order on towers. Throughout this section we deal only with towers of length ω_1. As we have seen, we can single out several classes of towers defined by their "inclusion structure". It is natural to ask if we can go further in this analysis. A research of this kind was done for ultrafilters in [4], using the classification of Tukey types.

We present here basic facts concerning the Tukey order. See [23, 4] for more details and for the complete bibliography.

Definition 46. Let $\mathcal{D}$ and $\mathcal{E}$ be directed sets. A function $g\colon \mathcal{D} \to \mathcal{E}$ is *Tukey* if the image of every unbounded subset of $\mathcal{D}$ is unbounded in $\mathcal{E}$. In that case, we say that $\mathcal{E}$ is *Tukey above* $\mathcal{D}$ ($\mathcal{D} \le_T \mathcal{E}$). If $\mathcal{D} \ge_T \mathcal{E} \ge_T \mathcal{D}$, then $\mathcal{D}$ and $\mathcal{E}$ are said to be *Tukey equivalent*, written $\mathcal{D} \equiv \mathcal{E}$.

Proposition 47. *If $\mathcal{D}$, $\mathcal{E}$ are directed posets such that $\mathcal{D}$ is a cofinal subset of $\mathcal{E}$, then $\mathcal{D} \equiv \mathcal{E}$.*

Theorem 48 (see [23]). *Let D be a directed poset of size at most ω_1. Then either $D \equiv 1$, or $D \equiv \omega$, or $D \equiv \omega_1$, or $[\omega_1]^{<\omega} \ge_T D \ge_T \omega \times \omega_1$. Moreover, under* PFA *there are no Tukey types in between $\omega\times\omega_1$ and $[\omega_1]^{<\omega}$.*

We have to agree on which emanations of towers we want to examine. Towers ordered by "$\subseteq$" are not satisfactory because we do not really want to pay attention to finite modifications of levels. It is also more convenient to deal with directed sets. Structure theory for non-directed posets is available (see [27]), but seems to be a bit cumbersome. The right structure to study seems to be the ideal generated by the tower (and all finite subsets of ω). As before, we denote it by $\langle\mathcal{T}\rangle$ for a given tower $\mathcal{T}$, this time understanding it as the structure $(\langle\mathcal{T}\rangle, \subseteq)$. The only inconvenience is that $\langle\mathcal{T}\rangle$ has cardinality continuum. For this reason we also consider a cofinal directed subset of $(\langle\mathcal{T}\rangle, \subseteq)$ consisting of finite modifications of elements of $\mathcal{T}$,

$$\langle\mathcal{T}\rangle_* = \{T \cup n : T \in \mathcal{T},\, n \in \omega\}.$$

Definition 49 ([23]). Let $\mathcal{D}$ be a directed poset of cardinality ω_1. We say that $\mathcal{D}$ has *property* (†) if every uncountable subset of $\mathcal{D}$ contains a countable unbounded subset.

It is easy to see that if $\mathcal{D}$ has (†), then $\omega \times \omega_1 <_T \mathcal{D}$.

Theorem 50 ([23]). *Assume* MA_{ω_1}. *If a directed poset* $\mathcal{D}$ *of cardinality* ω_1 *has* (†), *then* $\mathcal{D} \equiv [\omega_1]^{<\omega}$.

Proposition 51. *The poset* $\langle \mathcal{T} \rangle_*$ *has property* (†) *for every tower* $\mathcal{T}$.

Proof. Let $\mathcal{S}$ be an uncountable subset of $\langle \mathcal{T} \rangle_*$. We can assume that $\mathcal{S}$ is an increasing tower cofinal in $\mathcal{T}$, $\mathcal{S} = \{S_\alpha : \alpha < \omega_1\}$. Suppose that for each $\beta < \omega_1$ the set $\{S_\alpha : \alpha < \beta\}$ is bounded by an element of $\langle \mathcal{S} \rangle_*$. In particular, this means that $(\bigcup_{\alpha<\beta} S_\alpha)_{\beta<\omega_1}$ does not stabilize. Hence $(\bigcup_{\alpha<\beta} S_\alpha)_{\beta<\omega_1}$ is an uncountable strictly increasing $\subseteq$-chain, a contradiction. ∎

Theorem 50 now implies that under MA_{ω_1} there is only one Tukey type of ω_1-towers.

Corollary 52. *Every ideal generated by a tower is Tukey top under* MA_{ω_1}.

This should be contrasted with the following.

Theorem 53. *Assume* $2^{\omega_1} > \omega_2$ *and* CH. *There are* $2^{\mathfrak{c}}$ *incomparable Tukey classes represented by tower ideals.*

Proof. According to [4, Corollary 23], if $2^{\omega_1} > \omega_2$, then there are $2^{\mathfrak{c}}$ incomparable Tukey types of P-points. Each P-point is generated by a tower filter (which is its cofinal subset). Now use Proposition 47. ∎

Theorem 54. *A tower* $\mathcal{T}$ *is Hausdorff iff* $\langle \mathcal{T} \rangle \equiv [\omega_1]^{<\omega}$.

Proof. Let $\mathcal{H}$ be a cofinal subtower of $\mathcal{T}$ satisfying (H). We show that each infinite subset of $\mathcal{H}$ is unbounded in $\langle \mathcal{T} \rangle_*$ (and hence any injective map from $[\omega_1]^{<\omega}$ into $\mathcal{H}$ is a Tukey function from $[\omega_1]^{<\omega}$ to $\langle \mathcal{T} \rangle_*$). Pick any countable set $A = \{T_\alpha : \alpha \in I\} \subseteq \mathcal{H}$ and suppose that $X \in \langle \mathcal{T} \rangle_*$ is an upper bound of A. There is some $T_\beta \in \mathcal{H}$, $\sup I < \beta$, and $n < \omega$ such that $X \subseteq T_\beta \cup n$. The set $\{\alpha \in I : T_\alpha \subseteq T_\beta \cup n\}$ is finite since $\mathcal{H}$ satisfies (H). Thus there is some $\alpha \in I$ such that $T_\alpha \nsubseteq T_\beta \cup n$, and hence $T_\alpha \nsubseteq X$, a contradiction.

For the other direction, consider a Tukey map $f\colon [\omega_1]^{<\omega} \to \langle \mathcal{T} \rangle_*$. We may suppose without loss of generality that $f(\{\beta\}) \setminus f(\{\alpha\})$ is infinite iff $\alpha < \beta < \omega_1$. We show that the tower $\mathcal{S} = (f(\{\alpha\}))_{\alpha<\omega_1}$ satisfies condition (H). Suppose that for some $\beta < \omega_1$ and $n < \omega$ the set $A = \{\alpha < \beta : f(\{\alpha\}) \setminus f(\{\beta\}) \subseteq n\}$ is infinite. Then $\{f(\{\alpha\}) : \alpha \in A\}$ is bounded by $f(\{\beta\}) \cup n$ in $\langle \mathcal{T} \rangle_*$, a contradiction. Notice that the towers $\mathcal{T}$ and $\mathcal{S}$ generate the same ideal, so $\mathcal{T}$ is Hausdorff. ∎

Proposition 55. *Consistently, there are Suslin towers* $\mathcal{T}^0$, $\mathcal{T}^1$ *such that* $\langle \mathcal{T}^0 \rangle \times \langle \mathcal{T}^1 \rangle$ *is Tukey top.*

Proof. Consider a Hausdorff tower $\mathcal{T} = (T_\alpha)_{\alpha<\omega_1}$. Then in a model obtained by adding a Cohen real $c \subseteq \omega$ define $T^0_\alpha = T_\alpha \cap c$ and $T^1_\alpha = T_\alpha \setminus c$. By Proposition 29 both of these towers are Suslin. The map $f\colon \langle \mathcal{T} \rangle_* \to \langle \mathcal{T}^0 \rangle_* \times \langle \mathcal{T}^1 \rangle_*$ defined by $T_\alpha \cup n \mapsto ((T_\alpha \cap c) \cup n, (T_\alpha \setminus c) \cup n)$ for $\alpha < \omega_1$, $n < \omega$ is Tukey, so $\langle \mathcal{T}^0 \rangle \times \langle \mathcal{T}^1 \rangle \equiv \langle \mathcal{T} \rangle$. ∎

We do not know if the statement of the above proposition is true when there is a Suslin ω_1-tower.

Notice that by putting together Theorem 54 and Corollary 52, we get

Corollary 56. *If $\mathcal{T}$ is a Suslin tower, then $\langle \mathcal{T} \rangle <_T [\omega_1]^{<\omega}$.*

The last fact is of course an immediate consequence of Theorem 54, but it can be proved directly using the fact that each uncountable subtower of a Suslin tower contains a $\subseteq$-chain of order type $\omega + 1$. Indeed, by the Erdős–Dushnik–Miller theorem ($\omega_1 \to (\omega_1, \omega+1)^2$) (see [6, Theorem 11.3]) we know that either there is an uncountable $\subseteq$-antichain in the subtower or a $\subseteq$-chain of length $\omega + 1$. The first alternative is clearly impossible. It follows that uncountable well-ordered subsets of the ideals generated by Suslin towers have infinite bounded subsets, so they cannot be Tukey equivalent to $[\omega_1]^{<\omega}$.

Theorem 54 gives us one more useful piece of information along these lines. It is not easy to point out the reason why a given tower is not Hausdorff other than the lack of uncountable $\subseteq$-antichains. Consider the following property of a tower $(T_\alpha)_{\alpha<\omega_1}$: *for every uncountable $X \subseteq \omega_1$ there is an infinite $I \subseteq X$ and $\alpha > \sup I$ such that $\bigcup_{\xi\in I} T_\xi \subseteq^* T_\alpha$.* By Theorem 54, this property is equivalent to saying that $(T_\alpha)_{\alpha<\omega_1}$ is not Hausdorff.

Tukey theory harmonizes with the intuition that the Hausdorff property is in a sense more important than the property of having an uncountable $\subseteq$-antichain. It is not clear for us if there are other critical Tukey types of tower ideals.

Remark 57. The above approach has a disadvantage. Generating an ideal can lose the information if the generating tower is Suslin. Instead of examining ideals generated by towers, one can investigate the structure $\langle \{T =^* T_\alpha : \alpha < \omega_1\}, \subseteq, \cup, \cap \rangle$ for a given tower $(T_\alpha)_{\alpha<\omega_1}$. It is easy to see that being Suslin is invariant under isomorphism of such lattices.

8. Questions. In this section we list some questions and open problems related to the topic of this paper.

Problem 58. Is it consistent that each Hausdorff tower is the left half of a Hausdorff gap?

Notice that the standard Hausdorff construction (of a Hausdorff gap) cannot be modified in an obvious way to produce a Hausdorff tower without creating the other half of a Hausdorff gap as a byproduct. In Section 3 we showed a consistent example of a Hausdorff tower which is maximal (see Remark 14), and hence is not a half of any gap.

Problem 59. Is it consistent that all ω_1-towers/gaps are special but there is a non-Hausdorff tower/gap?

In particular, we can ask the following:

Problem 60. Does OCA imply that every ω_1-tower/gap is Hausdorff?

The natural attempt to answer this question in the negative would be to start with a model with a special non-Hausdorff tower/gap and show that forcing OCA preserves its non-Hausdorffness.

Every $\subseteq^*$-descending tower generates a filter in $\mathcal{P}(\omega)/\mathrm{Fin}$, a closed subset of the space of ultrafilters ω^*. It is natural to ask if the closed sets generated by Hausdorff towers have some special properties.

Problem 61. Is there a characterization of the Hausdorff property of towers in topological terms?

Perhaps the next question can be solved using coherent sequences. They produce towers in a nice way, but it is not clear how to analyze the properties of those resulting towers.

Problem 62. Does $\mathfrak{t} = \omega_1$ imply that there is a maximal Hausdorff tower?

Since each Hausdorff tower generates a meager ideal, a positive answer would provide a dense meager ω_1-generated P-ideal. Example 1 in [28] shows that the existence of these objects is in fact equivalent.

Problem 63. Is there a model in which every ideal generated by an (ω_1-)tower is dense only if it is non-meager?

Note that this problem for ω_1-towers is interesting only if we add the requirement $\mathfrak{t} = \omega_1$. The conjecture here is that there is no such model, i.e. a meager dense ω_1-generated P-ideal should be constructible from the assumption $\mathfrak{t} = \omega_1$. Obviously, if $\omega_1 < \mathfrak{b}$, every ω_1-generated ideal is meager. If $\mathrm{non}(\mathcal{M}) = \omega_1$ ($\mathcal{M}$ is the ideal of meager subsets of 2^ω), then there is such an ideal by the following argument due to M. Hrušák.

For a tall ideal $\mathcal{I} \subseteq \mathcal{P}(\omega)$ define

$$\mathrm{cov}^*(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subset \mathcal{I},\ (\forall X \in [\omega]^\omega)(\exists A \in \mathcal{A})(|A \cap X| = \omega)\}.$$

It follows from [11, Propositions 1.5, 3.1, 3.2] that $\mathrm{cov}^*(\mathcal{I}) \leq \mathrm{non}(\mathcal{M})$ for each tall analytic P-ideal $\mathcal{I}$. Thus if $\mathrm{non}(\mathcal{M}) = \omega_1$, for any given tall analytic P-ideal $\mathcal{I}$ there is a tall ω_1-tower which generates an ideal contained in $\mathcal{I}$, hence is meager.

A gap $(f_\alpha, g_\alpha)_{\alpha<\omega_1}$ in $({}^\omega\omega, <^*)$ is *tight* if $(f_\alpha\restriction A, g_\alpha\restriction A)_{\alpha<\omega_1}$ is a gap in $({}^\omega A, <^*)$ for each infinite $A \subseteq \omega$. A positive answer to the following problem would provide a negative answer to Problem 63.

Problem 64. Is the assumption $\mathfrak{t} = \omega_1$ equivalent to the existence of a tight gap in $({}^\omega\omega, <^*)$.

In connection with the previous problem, let us mention that the Borel weak diamond principle $\diamondsuit(2, =)$ of [15] implies the existence of a tight gap (this was suggested by M. Hrušák). In fact, it even implies the existence of a peculiar gap (see [20] for definition). Note also that there are no peculiar gaps in the model from [5], but $\mathfrak{b} = \omega_1$.

We know that there can be Suslin towers generating meager ideals. However, it is unclear whether they are not equivalent to special towers.

Problem 65. Is there a tower generating a tall meager ideal which is not equivalent to a special tower?

In Section 7 we have mentioned that in each Suslin tower there is a $\subseteq$-chain of order type $\omega + 1$. It seems natural to ask the following:

Problem 66. How long $\subseteq$-chains have to exist in Suslin towers?

Acknowledgments. Most of the results presented in this article were obtained during the stay of the authors at Fields Institute in Toronto. We would hereby like to thank the institute for the excellent working conditions. The first author wants to acknowledge the support of the INFTY Research Networking Program (European Science Foundation) via grant 4955. We are grateful to several mathematicians who agreed to devote their time to discuss the subject of this paper, including Grzegorz Plebanek, Michael Hrušák, Jörg Brendle, and Teruyuki Yorioka. We are particularly indebted to Stevo Todorčević for all his invaluable remarks and insightful suggestions. We would also like to thank the anonymous referee for the careful reading of the article and for dozens of valuable comments.

The first author was partially supported by the grant N N201 418939 from the Polish Ministry of Science and Higher Education. The second author was partially supported by the grant IAA100190902 of GA AV ČR and RVO: 67985840.

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Received 30 April 2013;
in revised form 14 September 2014