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Δ_1-Definability of the non-stationary ideal at successor cardinals

by

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Abstract. Assuming $V = L$, for every successor cardinal κ we construct a GCH and cardinal preserving forcing poset $\mathbb{P} \in L$ such that in $L^{\mathbb{P}}$ the ideal of all non-stationary subsets of κ is Δ_1-definable over $H(\kappa^+)$.

1. Introduction. In this paper we prove the following result, which solves in the affirmative a question posed in [8].

THEOREM 1.1. *Let κ be a successor cardinal in L.*

(1) *There exists a GCH and cardinal preserving forcing poset $\mathbb{P} \in L$ such that in $L^{\mathbb{P}}$ the ideal NS_κ of all non-stationary subsets of κ is Δ_1-definable over $H(\kappa^+)$.*
(2) *There exists a cardinal preserving forcing poset $\mathbb{P} \in L$ such that in $L^{\mathbb{P}}$ the ideal NS_κ of all non-stationary subsets of κ is Δ_1-definable over $H(\kappa^+)$, and $2^\kappa = \kappa^{++}$.*

The motivation for Theorem 1.1 comes from *generalized descriptive set theory,* which, roughly speaking, is the study of "nice" subsets of 2^κ for $\kappa > \omega$. Descriptive set theory looks very different in this generalized setting compared to the classical case. For instance, the classical fact that Δ^1_1 sets are Borel is not anymore true. And the non-stationary ideal on κ (possibly restricted to a certain stationary subset) considered in various forcing extensions is an important test space distinguishing various classes of "nice" subsets of 2^κ (see, e.g., [7, Theorem 49] and references therein).

Theorem 1.1 is proved using almost disjoint coding followed by localization, a method invented by David [3] and further developed in works of Friedman and collaborators. This is a new application of this method as the

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previous results regarding the definability of the ideal of non-stationary subsets of κ were mainly achieved using combinatorics related to canary trees (see [13] for the definition). For instance, Mekler and Shelah [13] proved that NS_{ω_1} is Δ_1-definable over $H(\omega_2)$ iff there is a canary tree, and canary trees may or may not exist in models of GCH. The proof presented in [13] had some inaccuracies which were fixed by Hyttinen and Rautila in [10], where they also obtained the result that NS_{κ^+} restricted to the ordinals of cofinality κ can be Δ_1-definable over $H(\kappa^+)$ for any regular κ. The results of [10] were further improved in [7], where it is also shown that NS_κ is not Δ_1-definable in L.

This topic also has connections with large cardinal theory: Using methods similar to those of [7], Friedman and Wu proved [8] that NS_κ restricted to a measure 0 set can be Δ_1-definable for a measurable κ. They also show that the unrestricted NS_κ cannot be Δ_1-definable for a weakly compact κ. Also note that NS_κ is Δ_1-definable if there exists a collection $\mathcal{S}$ of stationary subsets of κ such that $|\mathcal{S}| = \kappa$ and each stationary subset of κ contains some $S \in \mathcal{S}$. For $\kappa = \omega_1$ this is consistent relative to the existence of infinitely many Woodin cardinals (see [14, Section 6.2]).

With the exception of the case $\kappa = \omega_1$, prior results on the Δ_1-definability of NS_κ are limited to restrictions of NS_κ. In the present paper our methods allow us to obtain the Δ_1-definability of the full unrestricted NS_κ for all successor κ.

Throughout this paper we work over the constructible universe L, thus unless otherwise specified $V = L$.

2. Proof of Theorem 1.1. Let γ be the predecessor cardinal of κ, i.e., $\kappa = \gamma^+$. First we prove the first part. At the end we shall indicate how to modify it in order to obtain the proof of the second part.

We say that a transitive ZF$^-$ model M is *suitable* if $\gamma + 1 \subset M$, $(\gamma^{++})^M$ exists and $(\gamma^{++})^M = (\gamma^{++})^{L^M}$. From this it follows, of course, that $(\gamma^+)^M = (\gamma^+)^{L^M}$. We will need an appropriate sequence $\vec{S} = \langle S_\alpha : \alpha < \kappa^+ \rangle$ of stationary subsets of $\kappa^+ \cap \mathrm{Cof}(\kappa)$ such that $(\kappa^+ \cap \mathrm{Cof}(\kappa)) \setminus \bigcup_{\alpha\in\kappa^+} S_\alpha$ is stationary. Let $\langle G_\xi : \xi \in \kappa^+ \cap \mathrm{cof}(\kappa)\rangle$ be a $\diamondsuit_{\kappa^+}(\mathrm{cof}(\kappa))$ sequence which is Σ_1-definable over L_{κ^+}. For every $\alpha < \kappa^+$ let us denote by S_α the set $\{\xi < \kappa^+ : G_\xi = \{\kappa \cdot (\alpha + 1)\}\}$. It follows from the above that S_α's are stationary subsets of $\mathrm{cof}(\kappa) \cap \kappa^+$ which are mutually disjoint and the sequence $\vec{S} = \langle S_\alpha : \alpha < \kappa^+\rangle$ is Σ_1-definable over L_{κ^+}. Moreover, $\bigcup\{S_\alpha : \alpha < \kappa^+\}$ has fat complement because the set $S' = \{\xi < \kappa^+ : G_\xi = \{0\}\}$ is disjoint from the union considered above.

The idea of the proof will be to construct a poset $\mathbb{P}$ such that in $V^{\mathbb{P}}$ we will have the following Σ_1 definition of the complement of NS_κ: $S \subset \kappa$

is stationary iff *there exists* $Y \in [\kappa]^\kappa$ *such that for every suitable model* M *of size* γ *containing* $Y \cap (\gamma^+)^M$, *there is* $\mu < (\gamma^{++})^M$ *such that for all* $\zeta \in T(S) \cap (\gamma^+)^M$ *we have* $M \models$ *"*$S_{\rho\cdot\mu+\zeta}$ *is not stationary"* (where $T(S) = \{2i+1 : i \in S\} \cup \{2i : i \in \kappa \setminus S\}$ and $\rho = \kappa + 3$). In the latter definition by $S_{\rho\cdot\mu+\zeta}$ we mean, of course, its M-version.

We shall force clubs disjoint from certain S_α's by initial segments. This forcing is well-studied, and it is known (see, e.g., [2, Theorem 1]) that under GCH the poset consisting of closed bounded subsets of a stationary subset $S \subset \lambda$, where λ is a successor cardinal, preserves cofinalities, introduces no bounded subsets of λ, and creates a club subset of S if and only if S is *fat* in the sense that for every club $C \subset \lambda$, $C \cap S$ contains closed sets of ordinals of arbitrarily large order-types below λ. Since $\mathrm{Cof}(<\kappa) \cup S$ is easily seen to be fat for any stationary subset $S \subset \mathrm{Cof}(\kappa)$, the posets shooting clubs disjoint from S_α's will have all of these nice properties.

Similarly, but using this time the (κ^+-many) L-least codes for ordinals below κ^+ and a Σ_1-definable $\diamondsuit_\kappa(\mathrm{cof}(\gamma))$ sequence, we can obtain a Σ_1-definable sequence $\vec{A} = \langle A_\zeta : \zeta < \kappa^+ \rangle$ of stationary subsets of $\mathrm{cof}(\gamma) \cap \kappa$ which are mutually almost disjoint (that is, for all $\zeta_0 \neq \zeta_1$ the set $A_{\zeta_0} \cap A_{\zeta_1}$ is bounded in κ).

Let us fix a function $F : \kappa^+ \to L$ and set $\rho = \kappa + 3$. Next, we shall define an iteration $\langle \mathbb{P}_\xi, \dot{\mathbb{Q}}_\xi : \xi < \kappa^+ \rangle$ depending ([1]) on F. Later we will choose a particular F such that the poset associated to it makes NS_κ, the ideal of non-stationary subsets of κ, Δ_1-definable over $H(\kappa^+)$. The choice of this F is made after Corollary 2.12.

Suppose that we have already defined $\mathbb{P}_\xi$ for some $\xi < \kappa^+$. Let us write ξ in the form $\rho \cdot \alpha + \zeta$, where $\zeta < \rho$, and suppose that together with $\mathbb{P}_\xi$ we have also defined a sequence $\langle \dot{Y}_\beta : \beta < \alpha \rangle$ such that $\dot{Y}_\beta$ is a $\mathbb{P}_{\rho\cdot(\beta+1)}$-name for a subset of κ. If $F(\alpha)$ is not a $\mathbb{P}_{\rho\cdot\alpha}$-name for a subset of κ then $\dot{\mathbb{Q}}_\xi$ is trivial. Otherwise let G denote the $\mathbb{P}_\xi$-generic filter. If $F(\alpha)^G$ is not stationary in $V[G\restriction\rho\cdot\alpha]$, then $\mathbb{Q}_\xi = \dot{\mathbb{Q}}_\xi^G$ is trivial. So suppose that $F(\alpha)^G$ is stationary in $V[G\restriction\rho\cdot\alpha]$. Four cases are possible. Before passing to them we shall set the following notation: if A is a subset of κ, then $T(A) = \{2i+1 : i \in A\} \cup \{2i : i \in \kappa \setminus A\}$.

Case 1: $\zeta < \kappa$. If $\zeta \notin T(F(\alpha)^G)$, then $\mathbb{Q}_\xi$ is the trivial poset. Otherwise $\mathbb{Q}_\xi$ is the standard poset shooting a club C_ξ disjoint from S_ξ via initial segments. The $\mathbb{P}_\xi$-name of C_ξ will be denoted by $\dot{C}_\xi$.

Case 2: $\zeta = \kappa$. Before defining $\dot{\mathbb{Q}}_\xi$ we need to fix some notation and introduce some auxiliary objects. Given a set X of ordinals, let *Even*(X)

([1]) Formally we should have written $\langle \mathbb{P}^F_\xi, \dot{\mathbb{Q}}^F_\xi : \xi < \kappa^+ \rangle$ instead of $\langle \mathbb{P}_\xi, \dot{\mathbb{Q}}_\xi : \xi < \kappa^+ \rangle$, but this would only burden the notation.

and $Odd(X)$ be the sets of even and odd ordinals in X, respectively. In the following we treat 0 as a limit ordinal. Let $D_\alpha \subset \kappa^+$ be a set coding the sequences $\langle \dot{Y}^G_\beta : \beta < \alpha \rangle$ and $\langle C_{\rho\cdot\alpha+\zeta} : \zeta < \kappa \rangle$. That is, letting ϕ_l, ϕ_t be the L-minimal injections of $\alpha \times \kappa$ and $\kappa \times \kappa^+$ into $Even(\kappa^+)$ and $Odd(\kappa^+)$, respectively, D_α is such that $Even(D_\alpha) = \phi_l[\{\langle \beta, i \rangle : \beta < \alpha,\ i \in \dot{Y}^G_\beta\}]$ and (2) $Odd(D_\alpha) = \phi_t[\{\langle \zeta, \nu \rangle : \zeta \in T(F(\alpha)^G),\ \nu \in C_\zeta\}]$. Then $\mathbb{Q}_\xi$ adds a subset X^0_α of κ which almost disjointly codes D_α. More precisely, let $\mathbb{Q}_\xi$ be the poset of all pairs $\langle s, s^* \rangle \in [\kappa]^{<\kappa} \times [D_\alpha]^{<\kappa}$, where $\langle t, t^* \rangle$ extends $\langle s, s^* \rangle$ if and only if t end-extends s and $t \backslash s \cap A_\nu = \emptyset$ for every $\nu \in s^*$. Given a $\mathbb{Q}_\xi$-generic filter $G(\xi)$ over $L[G]$, we set $X^0_\alpha = \bigcup \{s : \exists s^* \ (\langle s, s^* \rangle) \in G(\xi)\}$. By genericity we have $D_\alpha = \{\nu : A_\nu \cap X^0_\alpha \text{ is bounded in } \kappa\}$.

Case 3: $\zeta = \kappa + 1$. Let us fix a strictly increasing continuous sequence $\langle N_\nu : \nu < \kappa^+ \rangle$ of elementary submodels of $L_\theta[X^0_\alpha]$ of size κ which contain $\kappa \cup \{X^0_\alpha\}$ as a subset, where θ is a large enough cardinal. Denote by E_α the set $\{(\kappa^+)^{\bar{N}_\nu} : \nu < \kappa^+\}$, where $\bar{N}_\nu$ is the Mostowski collapse of N_ν, and observe that E_α is a club in κ^+. Now choose Z_α to be a subset of κ^+ such that $Even(Z_\alpha) = D_\alpha$, and if $\beta < \kappa^+$ is $(\gamma^{++})^M = (\kappa^+)^M$ for some suitable model M such that $Z_\alpha \cap \beta \in M$, then β belongs to E_α. (This is easily done by placing in Z_α a code for a bijection $\phi : \beta_1 \to \kappa$ on the odd part of the interval $(\beta_0, \beta_0 + \kappa)$ for each adjacent pair $\beta_0 < \beta_1$ from E_α.) Then $\mathbb{Q}_\xi$ adds a subset X^1_α of κ which almost disjointly codes Z_α. More precisely, let $\mathbb{Q}_\xi$ be the poset of all pairs $\langle s, s^* \rangle \in [\kappa]^{<\kappa} \times [Z_\alpha]^{<\kappa}$, where $\langle t, t^* \rangle$ extends $\langle s, s^* \rangle$ if and only if t end-extends s and $t \backslash s \cap A_\nu = \emptyset$ for every $\nu \in s^*$. Given a $\mathbb{Q}_\xi$-generic filter $G(\xi)$ over $L[G]$, we set $X^1_\alpha = \bigcup \{s : \exists s^* \ (\langle s, s^* \rangle) \in G(\xi)\}$. By genericity we find that $Z_\alpha = \{\nu : A_\nu \cap X^1_\alpha \text{ is bounded in } \kappa\}$.

As a result we have:

$(*)_\alpha$ If M is any suitable model such that $\kappa \cup \{X^0_\alpha, X^1_\alpha\} \subset M$ and $(\gamma^{++})^M < \gamma^{++}$, then (3) $M \vDash \psi(\gamma^+, \gamma^{++}, \alpha, F(\alpha)^G, X^0_\alpha)$, where $\psi(\gamma^+, \gamma^{++}, \mu, S, X)$ is the formula "Using the sequence $\vec{A}$, the set X almost disjointly codes a subset D of γ^{++} such that using ϕ_l and ϕ_t, D codes (4) $\mu < \gamma^{++}$, $S \subset \gamma^+$, and a sequence $\langle C_\zeta : \zeta \in T(S) \rangle$, where C_ζ is a club in γ^{++} disjoint from $S_{\rho\cdot\mu+\zeta}$."

The proof of $(*)_\alpha$ is analogous to that of $(*)_\alpha$ in [4]. However, for the sake of completeness we shall present it. Given a suitable model M with

(2) Here we implicitly use that neither κ nor κ^+ is collapsed by $\mathbb{P}_\xi$. This will be proved in Lemmas 2.2 and 2.7. To be formally correct we should have presented this proof simultaneously with the inductive construction of $\mathbb{P}$.

(3) In this case $\kappa = \gamma^+$ in M.

(4) Whenever we verify that $M \vDash \psi(\gamma^+, \gamma^{++}, \mu, T, X)$ for some suitable model M we mean by $\gamma^+, \gamma^{++}, \vec{A}, \phi_t, \phi_l, S_\iota$, as may be expected, their M-versions.

$(\gamma^{++})^M = \beta$ and $\kappa \cup \{X^0_\alpha, X^1_\alpha\} \subset M$, observe that $Z_\alpha \cap \beta \in M$ because $Z_\alpha \cap \beta = \{\nu < \beta : |A_\nu \cap X^1_\alpha| = \kappa\}$ and $\vec{A}^M = \vec{A}^L \restriction \beta$, which yields $\beta \in E_\alpha$ by the construction of Z_α. Also, $D_\alpha \cap \beta \in M$ because $D_\alpha = \mathit{Even}(Z_\alpha)$. Let $\nu < \kappa^+$ be such that $(\gamma^{++})^{\bar{N}_\nu} = \beta$. By the construction we see that $L_\theta[X^0_\alpha] \models \psi(\gamma^+, \gamma^{++}, \alpha, F(\alpha)^G, X^0_\alpha)$, and hence also $\bar{N}_\nu \models \psi(\gamma^+, \gamma^{++}, \alpha, F(\alpha)^G, X^0_\alpha)$ by elementarity. Since the coding apparatus as well as stationary subsets involved in the formula ψ are referring to L, for any two suitable models $M_0, M_1 \supset \{X\}$ we have $M_0 \models \psi(\gamma^+, \gamma^{++}, \mu, S, X)$ iff $M_1 \models \psi(\gamma^+, \gamma^{++}, \mu, S, X)$, provided that $(\gamma^{++})^{M_0} = (\gamma^{++})^{M_1}$. In particular, $M \models \psi(\gamma^+, \gamma^{++}, \alpha, F(\alpha)^G, X^0_\alpha)$ because $\bar{N}_\nu \models \psi(\gamma^+, \gamma^{++}, \alpha, F(\alpha)^G, X^0_\alpha)$ and $(\gamma^{++})^{\bar{N}_\nu} = (\gamma^{++})^M = \beta$, which completes the proof of $(*)_\alpha$.

Case 4: $\zeta = \kappa + 2$. In this case the poset $\mathbb{Q}_\xi$ localizes the property $(*)_\alpha$ of X^0_α in the style of [3]. More precisely, $\mathbb{Q}_\xi$ consists of all functions $r : |r| \to 2$, where the domain $|r|$ of r is a limit ordinal less than κ, such that:

(1) if $\eta < |r|$ then $\eta \in X^0_\alpha$ iff $r(3\eta + 1) = 1$,
(2) if $\eta < |r|$ then $\eta \in X^1_\alpha$ iff $r(3\eta + 2) = 1$,
(3) if $\eta \leq |r|$, M is a suitable model of size γ containing $r \restriction \eta$ as an element and $\eta = (\gamma^+)^M$, then $M \models \psi(\gamma^+, \gamma^{++}, \mu, F(\alpha)^G \cap \eta, X^0_\alpha \cap \eta)$ for some ordinal μ.

The order relation is given by extension. Observe that the poset $\mathbb{Q}_\xi$ produces a generic function from κ into 2, which is the characteristic function of a subset Y_α of κ whose $\mathbb{P}_\xi$-name will be denoted by $\dot{Y}_\alpha$.

Finally, assuming that $\langle \mathbb{P}_\xi, \dot{\mathbb{Q}}_\xi : \xi < \delta \rangle$ has been defined for some limit $\delta < \kappa^+$, we define $\mathbb{P}_\delta$ as follows. Let $\mathbb{S}_\delta$ be the set of all functions p with domain δ such that $p \restriction \xi \in \mathbb{P}_\xi$ for all $\xi < \delta$. For $p \in \mathbb{S}_\delta$ we shall denote the sets

$$\{\xi < \delta : \xi \text{ is of the form } \rho \cdot \alpha + \zeta \text{ for some } \zeta < \kappa \text{ and } p(\xi) \neq 1_{\dot{\mathbb{Q}}_\xi}\}$$

and

$$\{\xi < \delta : \xi \text{ is of the form } \rho \cdot \alpha + \zeta \text{ for some } \zeta \in \{\kappa, \kappa+1, \kappa+2\} \text{ and } p(\xi) \neq 1_{\dot{\mathbb{Q}}_\xi}\}$$

by $\mathrm{supp}_{\kappa^+}(p)$ and $\mathrm{supp}_\kappa(p)$, respectively, and their union will be denoted by $\mathrm{supp}(p)$. The poset $\mathbb{P}_\delta$ consists of those $p \in \mathbb{S}_\delta$ such that $|\mathrm{supp}_\kappa(p)| < \kappa$ and $|\mathrm{supp}_{\kappa^+}(p)| < \kappa^+$. This completes our definition of $\mathbb{P} = \mathbb{P}_{\kappa^+}$ depending on the arbitrary bookkeeping function F.

Even though the following remark has already been used, we isolate it here for future use.

Remark 2.1. Tracing back the statement of the formula ψ as well as the coding apparatus involved, one can see that if N, M are suitable models such that $(\gamma^+)^M = (\gamma^+)^N$, $(\gamma^{++})^M = (\gamma^{++})^N$, and $S, X \subset (\gamma^+)^M$ are elements of $M \cap N$, then $M \models \psi(\gamma^+, \gamma^{++}, \mu, S, X)$ iff $N \models \psi(\gamma^+, \gamma^{++}, \mu, S, X)$ for any $\mu < (\gamma^{++})^M$.

LEMMA 2.2. *The poset* $\mathbb{P}$ *is* $({<}\kappa)$*-distributive.*

Before passing to the proof of Lemma 2.2 we shall introduce some notation. Let $\mathcal{D}_\delta$ be the set of conditions $p \in \mathbb{P}_\delta$ such that

- for all ξ of the form $\rho{\cdot}\alpha{+}\zeta$, where $\zeta \in \{\kappa, \kappa{+}1\}$, we have $p(\xi) = \langle s_\xi, s^*_\xi\rangle\check{}$ for some $s^*_\xi \in [\kappa^+]^{<\kappa}$ and $s_\xi \in [\kappa]^{<\kappa}$;
- for all ξ of the form $\rho \cdot \alpha + \kappa + 2$ we have $p(\xi) = \check{r}$ for some $r : |r| \to 2$;
- $\Vdash_\xi p(\xi) \in \dot{\mathbb{Q}}_\xi$ for all $\xi \in \mathrm{supp}(p)$.

If $\mathbb{Q}$ is a poset, $q \in \mathbb{Q} \in N$, then we say that q is *strongly* $(N, \mathbb{Q})$*-generic* if for every open dense subset O of $\mathbb{Q}$ which is an element of N there exists $p \in O \cap N$ such that $q \leq p$.

Proof of Lemma 2.2. We shall prove by induction on $\xi < \kappa^+$ that $\mathbb{P}_\xi$ has some property which is formally stronger than $({<}\kappa)$-distributivity and that $\mathcal{D}_\xi$ is dense in $\mathbb{P}_\xi$. In order to formulate this property we shall introduce some auxiliary notions.

Let us fix some large enough regular cardinal θ and some large $n \in \omega$. Given a set $X \in L_\theta$, let N_0 be the least Σ_n- elementary submodel of L_θ such that $\{X\} \cup (\gamma + 1) \subset N_0$. The least means here that N_0 is the closure of $\{X\} \cup (\gamma + 1)$ with respect to all Σ_n Skolem functions given by the well-ordering $<_L$ of L_θ. Suppose that for some $\zeta < \kappa$ we have already constructed an increasing chain $\langle N_\xi : \xi < \zeta\rangle$ of Σ_n-elementary submodels of L_θ. If ζ is limit then we set $N_\zeta = \bigcup_{\xi<\zeta} N_\xi$. If $\zeta = \zeta_0 + 1$ let N_ζ be the minimal Σ_n-elementary submodel of L_θ such that $N_{\zeta_0} \in N_\zeta$. This completes the construction of the sequence $\langle N_\zeta : \zeta < \kappa\rangle$ which will be called the *minimal sequence generated by* X throughout the proof ([5]).

By induction on $\xi < \kappa^+$ we shall show that $\mathcal{D}_\xi$ is dense in $\mathbb{P}_\xi$, and

($\dagger_\xi$) for every $q \in \mathbb{P}_\xi$ and $X \in L_\theta$ there exists a condition $q' \leq q$ which is strongly $(N_\zeta, \mathbb{P}_\xi)$-generic for all limit $\zeta \leq \gamma$, where $\langle N_\zeta : \zeta < \kappa\rangle$ is the minimal sequence ([6]) generated by $\{q, X\}$.

Notice that if $X = \langle B_\zeta : \zeta < \gamma\rangle$ is a sequence of open dense subsets of $\mathbb{P}_\xi$, then it follows from the above that $q' \in \bigcap_{\zeta<\gamma} B_\xi$, and hence ($\dagger_\xi$) implies the $({<}\kappa)$-distributivity of $\mathbb{P}_\xi$.

$(\dagger)_0$ is vacuously true. So let us consider three non-trivial cases: ξ is a successor ordinal, ξ is limit of cofinality at most γ, and ξ is limit of co-

([5]) In this proof we will only use the first $\gamma + 1$ elements of minimal sequences. Longer initial segments of minimal sequences will be considered in the proof of Lemma 2.5.

([6]) Here we have $\xi \in N_0$ because $q \in N_0$ and ξ is the domain of q.

finality κ. The latter two cases will be addressed on pages 240 and 241, respectively.

1. $\xi = \xi_0 + 1$. Let us write ξ in the form $\rho \cdot \alpha + \iota$ for some $\iota < \rho$. If $\iota \leq \kappa + 1$ then $\mathbb{Q}_{\xi_0}$ is a $\mathbb{P}_{\xi_0}$-name for a $(<\kappa)$-closed poset, which makes this case straightforward. So let us assume that $\iota = \kappa + 2$, i.e., $\xi = \rho \cdot \alpha + \kappa + 2$.

First we shall prove that $\mathbb{P}_\xi$ is $(<\kappa)$-distributive. Let us denote by μ the ordinal $\rho \cdot \alpha + \kappa$ and fix a collection $X = \{O_{\zeta+1} : \zeta < \gamma\}$ of open dense subsets of $\mathbb{P}_\xi$ and a condition $q \in \mathbb{P}_\xi$. Let also $\langle N_\zeta : \zeta < \kappa \rangle$ be the minimal sequence generated by $\{q, X\}$. We shall show that $1_{\mathbb{P}_\mu}$ forces the poset

$$\dot{\bar{\mathbb{Q}}}_\mu := \dot{\mathbb{Q}}_\mu * \dot{\mathbb{Q}}_{\mu+1} * \dot{\mathbb{Q}}_{\mu+2} = \dot{\mathbb{Q}}_{\rho\cdot\alpha+\kappa} * \dot{\mathbb{Q}}_{\rho\cdot\alpha+\kappa+1} * \dot{\mathbb{Q}}_{\rho\cdot\alpha+\kappa+2}$$

to be $(<\kappa)$-distributive.

Using the inductive assumption we can find a condition $q' \in \mathbb{P}_\mu$ such that $q' \leq q \restriction \mu$ and q' is strongly $(N_\zeta, \mathbb{P}_\mu)$-generic for all limit $\zeta \leq \gamma$. Let G denote a $\mathbb{P}_\mu$-generic filter containing q' and note that $N_\zeta[G] \cap \kappa = N_\zeta \cap \kappa$ for all limit $\zeta < \gamma$. For every (not necessarily limit) $\zeta \leq \gamma$ we shall denote the intersection $N_\zeta \cap \kappa$ by κ_ζ. Since $X, \gamma \in N_0$, there exists an enumeration $\langle O_{\zeta+1} : \zeta < \gamma \rangle \in N_0$ of X. We shall denote $\dot{\mathbb{Q}}_\mu^G$ by $\mathbb{Q}_\mu$ and the $\mathbb{Q}_\mu$-names $\dot{\mathbb{Q}}_{\mu+1}^G$ and $\dot{\mathbb{Q}}_{\mu+2}^G$ by $\underset{\sim}{\mathbb{Q}}_{\mu+1}$ and $\underset{\sim}{\mathbb{Q}}_{\mu+2}$, respectively.

For every $\zeta \leq \gamma$ let us denote by $O'_{\zeta+1}$ the open dense subset $\{\tau^G :$ there exists $u \in G$ such that $\langle u, \tau \rangle \in O_{\zeta+1}\}$ of $\bar{\mathbb{Q}}_\mu = \dot{\bar{\mathbb{Q}}}_\mu^G$. Observe that $\langle O'_{\eta+1} : \eta + 1 \leq \zeta \rangle \in N_0[G]$ for all $\zeta \leq \gamma$. The $(<\kappa)$-distributivity of $\mathbb{P}_\mu$ combined with the $(<\kappa)$-closedness of $\mathbb{Q}_\mu, \underset{\sim}{\mathbb{Q}}_{\mu+1}$ implies that the set U of conditions $r \in \bar{\mathbb{Q}}_\mu$ such that $r(\mu), r(\mu+1), r(\mu+2)$ are of the form $\check{a}$ for some set $a \in L$ of size $< \kappa$, is dense in $\bar{\mathbb{Q}}_\mu$.

Set $p_0 = (q \restriction \{\mu, \mu+1, \mu+2\})^G$. From now on we shall work in $L[G]$. The sequence $\langle N_\zeta[G] : \zeta < \gamma \rangle$ will guide our inductive construction of a decreasing sequence $\langle p_\zeta : \zeta \leq \gamma \rangle$ of conditions in U such that $p_\gamma \in N_{\gamma+1}[G]$ belongs to all $O'_{\zeta+1}$'s as follows. Let $<_G$ be the canonical wellordering of $L[G]$: $x <_G y$ iff $\tau_x <_L \tau_y$, where τ_x, τ_y are the $<_L$-minimal $\mathbb{P}_\mu$-names such that $\tau_x^G = x$ and $\tau_y^G = y$. Suppose that a condition $p_\zeta \in N_{\zeta+1} \cap U$ has already been constructed. Since $\mathbb{Q}_\mu * \underset{\sim}{\mathbb{Q}}_{\mu+1}$ is $(<\kappa)$-closed, we can inductively extend $\langle p_\zeta(\mu), p_\zeta(\mu+1) \rangle$ to a strongly $(N_{\zeta+1}[G], \mathbb{Q}_\mu * \underset{\sim}{\mathbb{Q}}_{\mu+1})$-generic in $L[G]$ condition $\langle p'_\zeta(\mu), p'_\zeta(\mu+1) \rangle \in \mathbb{Q}_\mu * \underset{\sim}{\mathbb{Q}}_{\mu+1}$. We shall additionally assume that $\langle p'_\zeta(\mu), p'_\zeta(\mu+1) \rangle$ is the $<_G$-minimal condition in $\mathbb{Q}_\mu * \underset{\sim}{\mathbb{Q}}_{\mu+1}$ with the properties described above. It follows that we can find $r \in N_{\zeta+1}[G]$ such that $\langle p'_\zeta(\mu), p'_\zeta(\mu+1), \check{r} \rangle \in O'_{\zeta+1}$. In addition, we shall assume that r is the $<_G$-minimal element of $2^{<\kappa}$ with this property. Let $r_{\zeta+1}$ be the $<_G$-minimal extension of r with domain $\kappa_{\zeta+1}$ and such that $r_{\zeta+1} \restriction \big(\{3\eta : \eta < \kappa\}$

$\cap [|r|, |r| + \gamma))$ codes a bijection between $\kappa_{\zeta+1}$ and γ. Letting $p_{\zeta+1}$ be the condition $\langle p'_\zeta(\mu), p'_\zeta(\mu+1), \check{r_{\zeta+1}}\rangle$, by the construction above we conclude that $p_{\zeta+1} \in N_{\zeta+2}[G] \cap U \cap O'_{\zeta+1}$.

If ζ is limit, then we set

$$p_\zeta(\mu) = \Big\langle \widetilde{\bigcup_{\eta<\zeta} s_{\mu,\eta}, \bigcup_{\eta<\zeta} s^*_{\mu,\eta}} \Big\rangle$$

and

$$p_\zeta(\mu+1) = \Big\langle \widetilde{\bigcup_{\eta<\zeta} s_{\mu+1,\eta}, \bigcup_{\eta<\zeta} s^*_{\mu+1,\eta}} \Big\rangle,$$

where $p_\eta(\mu) = \langle \widetilde{s_{\mu,\eta}, s^*_{\mu,\eta}}\rangle$ and $p_\eta(\mu+1) = \langle \widetilde{s_{\mu+1,\eta}, s^*_{\mu+1,\eta}}\rangle$ for all $\eta < \zeta$. In addition, we set $p_\zeta(\mu+2) = \widetilde{\bigcup_{\eta<\zeta} r_\eta}$, where $\check{r}_\eta = p_\eta(\mu+2)$ for all $\eta < \zeta$. Since p_η for $\eta < \zeta$ have been constructed by choosing $<_G$-minimal conditions fulfilling certain requirements, the sequence $\langle p_\eta : \eta < \zeta\rangle$ is an element of $N_{\zeta+1}[G]$, and hence $p_\zeta \in N_{\zeta+1}[G]$ as well.

We claim that $p_\zeta \in \bar{\mathbb{Q}}_\mu$. Observe that $\langle p_\zeta(\mu), p_\zeta(\mu+1)\rangle \in \mathbb{Q}_\mu * \underset{\sim}{\mathbb{Q}}_{\mu+1}$ by the $(<\kappa)$-closeness of the latter poset. It suffices to show that $\langle p_\zeta(\mu), p_\zeta(\mu+1)\rangle \Vdash p_\zeta(\mu+2) \in \underset{\sim}{\mathbb{Q}}_{\mu+2}$. Let $p_\zeta(\mu) = \langle s_{\mu,\zeta}, s^*_{\mu,\zeta}\rangle$, $p_\zeta(\mu+1) = \langle \widetilde{s_{\mu+1,\zeta}, s^*_{\mu+1,\zeta}}\rangle$, and $p_\zeta(\mu+2) = \check{r}_\zeta$. Notice that the condition $\langle p_\zeta(\mu), p_\zeta(\mu+1)\rangle$ is strongly $(N_\zeta[G], \mathbb{Q}_\mu * \underset{\sim}{\mathbb{Q}}_{\mu+1})$-generic in $L[G]$. This means that if $H := H(\mu) * H(\mu+1)$ is a $\mathbb{Q}_\mu * \underset{\sim}{\mathbb{Q}}_{\mu+1}$-generic filter over $L[G]$ containing $\langle p_\zeta(\mu), p_\zeta(\mu+1)\rangle$, then the isomorphism π of the transitive collapse $\bar{N}_\zeta[\bar{g}]$ of $N_\zeta[G]$, onto $N_\zeta[G]$, extends to an elementary embedding from

$$\bar{\bar{N}}_\zeta := \bar{N}_\zeta[\bar{g} * \bar{h}(\bar{\mu}) * \bar{h}(\bar{\mu}+1)]$$

into $L_\theta[G][H]$. Here $\bar{\mu} = \pi^{-1}(\mu)$, $\bar{h}(\bar{\mu})$ is the $\pi^{-1}(\mathbb{Q}_\mu)$-generic filter over $\bar{N}_\zeta[\bar{g}]$ determined by $p_\zeta(\mu)$, i.e., $\bar{h}(\bar{\mu})$ consists of the images under π^{-1} of all conditions in $\mathbb{Q}_\mu$ which are weaker than $p_\zeta(\mu)$ and belong to $N_\zeta[G]$; and $\bar{h}(\bar{\mu}+1)$ is defined in the same way.

By the genericity of H we know that, if we let X^0_α and X^1_α be the unions of the first coordinates of elements of $H(\mu)$ and $H(\mu+1)$, respectively, then property $(*)_\alpha$ holds. By elementarity, $\bar{\bar{N}}_\zeta$ is a suitable model and $\bar{\bar{N}}_\zeta \vDash \psi(\gamma^+, \gamma^{++}, \pi^{-1}(\alpha), \pi^{-1}(F(\alpha)^G), x^0_\alpha)$, where x^0_α and x^1_α are the unions of the first coordinates of elements of $\bar{h}(\bar{\mu})$ and $\bar{h}(\bar{\mu}+1)$ (equivalently, the first coordinates of $p_\zeta(\mu)$ and $p_\zeta(\mu+1)$), respectively. Observe that by the construction of $\mathbb{P}$ we have $\bar{\bar{N}}_\zeta = \bar{N}_\zeta[\bar{g}, x^0_\alpha, x^i_\alpha]$ and hence $\bar{N}_\zeta[\bar{g}, x_0, x_i] \vDash \psi(\gamma^+, \gamma^{++}, \pi^{-1}(\alpha), \pi^{-1}(F(\alpha)^G), x^0_\alpha)$.

Let M be any suitable model containing r_ζ and such that $(\gamma^+)^M = |r_\zeta|$, which is equal to $\kappa \cap N_\zeta = \kappa_\zeta$. We have to show that

$$M \vDash \psi(\gamma^+, \gamma^{++}, \pi^{-1}(\alpha), F(\alpha)^G \cap \kappa_\zeta, x^0_\alpha).$$

Let us denote by ν and λ the intersections $M \cap \mathrm{Ord}$ and $\bar{\bar{\bar{N}}}_\zeta \cap \mathrm{Ord}$, respectively. Three cases are possible.

Case (a): $\nu > \lambda$. Since N_ζ was chosen to be the least sufficiently elementary submodel of $L_\theta[G]$ containing certain objects, it follows that $\kappa_\zeta = (\gamma^+)^M$ is collapsed to γ in L_ν, and hence this case cannot happen.

More precisely, L_ν can compute (and hence contains) the sequence $\langle \pi^{-1}[N_\eta] : \eta < \zeta \rangle$. Indeed, $\bar{N}_\zeta \in L_\nu$ since $\bar{N}_\zeta = L_\xi$, $\pi^{-1}[N_\eta] = \bigcup_{\eta' < \eta} \pi^{-1}[N_{\eta'}]$ for limit $\eta < \zeta$, and $\pi^{-1}[N_{\eta+1}]$ is the closure of $\{\pi^{-1}[N_\eta]\}$ under the Σ_n Skolem functions of L_ξ, and these are elements of L_ν. Therefore L_ν contains the sequence $\langle \bar{N}_\eta : \eta < \zeta \rangle$, where $\bar{N}_\eta$ is the Mostowski collapse of N_η (the Mostowski collapse of N_η coincides with that of $\pi^{-1}[N_\eta]$), and hence $\langle \kappa_\eta = (\gamma^+)^{\bar{N}_\eta} : \eta < \zeta \rangle \in L_\nu$, whereas the latter sequence is cofinal in κ_ζ.

Case (b): $\nu = \xi$. Since $(\gamma^+)^{\bar{N}_\zeta[\bar{g}, x^0_\alpha, x^1_\alpha]} = (\gamma^+)^M$ and $(\gamma^{++})^{\bar{N}_\zeta[\bar{g}, x^0_\alpha, x^1_\alpha]} = (\gamma^{++})^M$ and $\bar{N}_\zeta[\bar{g}, x^0_\alpha, x^1_\alpha] \vDash \psi(\gamma^+, \gamma^{++}, \pi^{-1}(\alpha), F(\alpha)^G \cap \kappa_\zeta, x^0_\alpha)$, we conclude that $M \vDash \psi(\gamma^+, \gamma^{++}, \pi^{-1}(\alpha), F(\alpha)^G \cap \kappa_\zeta, x^0_\alpha)$ (see Remark 2.1).

Case (c): $\nu < \xi$. In this case $M_1 := L_\nu[x^0_\alpha, x^1_\alpha]$ is an element of $\bar{N}_\zeta[\bar{g}, x^0_\alpha, x^1_\alpha]$. Since $L_\theta[G][H]$ satisfies $(*)_\alpha$, by elementarity so does the model $\bar{N}_\zeta[\bar{g}, x^0_\alpha, x^1_\alpha]$ with X^0_α, X^1_α, and α replaced by x^0_α, x^1_α, and $\pi^{-1}(\alpha)$, respectively. In particular, $M_1 \vDash \psi(\gamma^+, \gamma^{++}, \pi^{-1}(\alpha), F(\alpha)^G \cap \kappa_\zeta, x^0_\alpha)$. Applying Remark 2.1 we conclude that $M \vDash \psi(\gamma^+, \gamma^{++}, \pi^{-1}(\alpha), F(\alpha)^G \cap \kappa_\zeta, x^0_\alpha)$, which finishes our proof of $p_\zeta \in \bar{\mathbb{Q}}_\mu$ and hence completes the construction of the sequence $\langle p_\zeta : \zeta \leq \gamma \rangle$.

By the construction we have $p_\gamma \in \bigcap_{\zeta < \gamma} O'_{\zeta+1} \cap N_{\gamma+1}[G]$, and hence $\bar{\mathbb{Q}}_\mu$ as well as $\mathbb{P}_\xi$ are $({<}\kappa)$-distributive. Let τ be a $\mathbb{P}_\mu$-name such that $\tau^G = p_\gamma$ and for every $\zeta < \gamma$ let $q_\zeta \in G$ be such that $q_\zeta \leq q{\restriction}\mu$ and $\langle q_\zeta, \tau \rangle \in O_{\zeta+1}$. Since $\mathbb{P}_\mu$ is $({<}\kappa)$-distributive, there exists $q'' \in G$ such that $q'' \leq q_\zeta$ for all ζ. In addition, we can assume that $q'' \in \mathcal{D}_\mu$ and it forces all coordinates of τ to be equal to certain ground model objects. It follows from the above that $q \geq \langle q'', \tau \rangle \in \bigcap_{\zeta < \gamma} O_{\zeta+1} \cap \mathcal{D}_\xi$, and hence $\mathcal{D}_\xi$ is dense in $\mathbb{P}_\xi$. Combined with the following claim, this implies $(\dagger_\xi)$ and thus completes the successor case.

Claim 2.3. *Let* $\beta < \kappa^+$. *If* $\mathbb{P}_\beta$ *is* $({<}\kappa)$*-distributive and* $\mathcal{D}_\beta$ *is dense, then* $(\dagger_\beta)$ *holds.*

Proof. Let $q \in \mathbb{P}_\beta$, $X \in L_\theta$, and $\langle N_\zeta : \zeta < \kappa \rangle$ be the minimal sequence generated by $\{q, X\}$. We need to find a condition $q' \leq q$ which is strongly $(N_\zeta, \mathbb{P}_\beta)$-generic for all limit $\zeta \leq \gamma$.

Set $p_0 = q$ and assume that conditions $\langle p_\eta : \eta < \zeta \rangle$ have already been defined for some $\zeta \leq \gamma$ so that $p_\eta \in N_{\eta+1} \cap \mathcal{D}_\beta$ for all $\eta < \zeta$. If $\zeta = \eta + 1$, then p_ζ is the $<_L$-minimal condition extending p_η such that $p_\zeta \in \mathcal{D}_\beta$ and it belongs to the intersection of all open dense subsets of $\mathbb{P}_\beta$ which are elements of N_ζ. Since $N_\zeta \in N_{\zeta+1}$, we have $p_\zeta \in N_{\zeta+1}$ as well, as β belongs to N_0. If ζ is limit, then using the fact that $p_\eta \in \mathcal{D}_\beta$ for all $\eta < \zeta$ we can define p_ζ to be the "coordinatewise" union of p_η over $\eta < \zeta$. More precisely, for $\xi \in \bigcup_{\eta<\zeta} \mathrm{supp}_\kappa(p_\eta)$ we set

$$p_\zeta(\xi) = \Big\langle \bigcup_{\eta<\zeta} s_{\xi,\eta}, \bigcup_{\eta<\zeta} s^*_{\xi,\eta} \Big\rangle\check{} \quad \text{and} \quad p_\zeta(\xi) = \bigcup_{\eta<\zeta} \check{r}_{\xi,\eta},$$

where $p_\eta(\xi) = \langle s_{\xi,\eta}, s^*_{\xi,\eta}\rangle\check{}$ for all (7) $\eta < \zeta$ provided that $\xi \in \{\rho\cdot\iota+\kappa, \rho\cdot\iota+\kappa+1\}$ for some ι, and $p_\eta(\xi) = \check{r}_{\xi,\eta}$ for all $\eta < \zeta$ if ξ is of the form $\rho\cdot\iota+\kappa+2$. For $\xi \in \bigcup_{\eta<\zeta} \mathrm{supp}_{\kappa^+}(p_\eta)$ we denote by $p_\zeta(\xi)$ a $\mathbb{P}_\xi$-name τ which is forced by $1_{\mathbb{P}_\xi}$ to be the union of $p_\eta(\xi)$ over all $\eta < \zeta$.

Since p_η for $\eta < \zeta$ have been constructed by choosing $<_G$-minimal conditions fulfilling certain requirements, the sequence $\langle p_\eta : \eta < \zeta\rangle$ is an element of $N_{\zeta+1}$, and hence $p_\zeta \in N_{\zeta+1}$ as well. Thus, once we know that p_ζ is a condition in $\mathbb{P}_\beta$, it is a consequence of its definition that $p_\zeta \in \mathcal{D}_\beta \cap N_{\zeta+1}$. In order to show that $p_\zeta \in \mathbb{P}_\beta$ it is enough to establish by induction on $\xi \in \mathrm{supp}(p_\zeta)$ that $p_\zeta{\restriction}\xi \in \mathbb{P}_\xi$. The only non-trivial case here is when ξ has the form $\rho\cdot\alpha+\kappa+2$. Assuming that $p_\zeta{\restriction}(\rho\cdot\alpha+\kappa+2) \in \mathbb{P}_{\rho\cdot\alpha+\kappa+2}$ for some α, the property $p_\zeta{\restriction}(\rho\cdot\alpha+\kappa+2) \Vdash p_\zeta(\rho\cdot\alpha+\kappa+2) \in \dot{\mathbb{Q}}_{\rho\cdot\alpha+\kappa+2}$ can be established in the same way as above, using the fact that $p_\zeta{\restriction}(\rho\cdot\alpha+\kappa+2)$ is strongly $(N_\eta, \mathbb{P}_{\rho\cdot\alpha+\kappa+2})$-generic for all limit $\eta \le \zeta$ and considering three cases depending on the height of a suitable model under consideration. It suffices to note that $q' = p_\gamma$ is as required. ∎

2. *ξ is a limit ordinal of cofinality $\le \gamma$.* Here we shall work in L. We need the following auxiliary statement.

Claim 2.4. *Suppose that $(\dagger_\beta)$ holds and $\mathcal{D}_\beta$ is dense in $\mathbb{P}_\beta$ for each $\beta < \xi$, where ξ is a limit ordinal of cofinality $\le \gamma$. Then for every $p \in \mathbb{P}_\xi$ and $X_0 \in L_\theta$ there exists an extension $q \in \mathcal{D}_\xi \cap N_{\gamma\cdot\mathrm{cof}(\xi)+1}$ of p such that $q{\restriction}\beta$ is strongly $(N_{\gamma\cdot\mathrm{cof}(\xi)}, \mathbb{P}_\beta)$-generic for all $\beta < \xi$, where $\langle N_\zeta : \zeta < \kappa\rangle$ is the minimal sequence generated by $\{p, X_0\}$.*

Proof. Since $p \in N_0$, we have $\xi \in N_0$, and hence N_0 contains a continuous sequence $\xi_0 < \xi_1 < \cdots$ cofinal in ξ of order type $\mathrm{cof}(\xi)$. Set $p_0 = p{\restriction}\xi_0$ and assume that conditions $\langle p_\eta : \eta < \zeta\rangle$ have already been defined for some $\zeta \le \mathrm{cof}(\xi)$ so that

(i) $p_\eta \in N_{\gamma\cdot\eta+1} \cap \mathcal{D}_{\xi_\eta}$ for all $\eta < \zeta$;
(ii) $p_{\eta_1}{\restriction}\xi_{\eta_0} \le p_{\eta_0}$ for all $\eta_0 < \eta_1 < \zeta$;
(iii) $p_\eta{\restriction}\beta$ is strongly $(N_{\gamma\cdot\eta}, \mathbb{P}_\beta)$-generic for all $\eta < \zeta$ and $\beta \le \xi_\eta$.

(7) We assume here that if $\xi \notin \mathrm{supp}(p)$ then $p(\xi) = \langle\emptyset, \emptyset\rangle\check{}$ provided that $\xi = \rho\cdot\alpha+\zeta$ for some $\zeta \in \{\kappa, \kappa+1\}$ and $p(\xi) = \check{\emptyset}$ otherwise.

Notice that (iii) is vacuous unless β is an element of $N_{\gamma\cdot\eta}$ because otherwise $\mathbb{P}_\beta \notin N_{\gamma\cdot\eta}$. If $\zeta = \eta + 1$, then let p_ζ be the $<_L$-minimal condition extending $p_\eta{}^\frown p_0\restriction[\xi_\eta, \xi_\zeta)$ so that (i)–(iii) hold. Its existence is guaranteed by $(\dagger_{\xi_\zeta})$ applied to $X = N_{\gamma\cdot\eta}$ and the inductive assumption that $\mathcal{D}_{\xi_\zeta}$ is dense in $\mathbb{P}_{\xi_\zeta}$.

If ζ is limit, then we define p_ζ in exactly the same way as in Claim 2.3. In addition, almost literal repetition of the proof there shows that (i)–(iii) are satisfied for all $\eta, \eta_0, \eta_1 \le \zeta$, the essential part here being to prove that $p_\zeta \in \mathbb{P}$. It suffices to set $q = p_{\mathrm{cof}(\xi)}$. ■

We are now in a position to prove the $(<\kappa)$-distributivity of $\mathbb{P}_\xi$. Moreover, the construction below gives a condition in $\mathcal{D}_\xi$ which lies in the intersection of γ open dense subsets of $\mathbb{P}_\xi$, and consequently it establishes that $\mathcal{D}_\xi$ is dense in $\mathbb{P}_\xi$. Combined with Claim 2.3, this will complete the proof that the inductive assumption holds for ξ.

Given $p \in \mathbb{P}_\xi$ and fewer than κ open dense sets $\{O_{\zeta+1} : \zeta < \gamma\}$, let $\langle N_\zeta : \zeta < \kappa\rangle$ be the minimal sequence generated by $\{p, \langle O_{\zeta+1} : \zeta < \gamma\rangle\}$. Set $\gamma' = \gamma \cdot \mathrm{cof}(\xi)$, $p = p_0$, and assume that conditions $\langle p_\eta : \eta < \zeta\rangle$ have already been defined for some $\zeta \le \gamma$ so that

(iv) $p_\eta \in N_{\gamma'\cdot\eta+1} \cap \mathcal{D}_\xi$ for all $\eta < \zeta$;
(v) $p_{\eta_1} \le p_{\eta_0}$ for all $\eta_0 < \eta_1 < \zeta$;
(vi) $p_\eta\restriction\beta$ is strongly $(N_{\gamma'\cdot\eta}, \mathbb{P}_\beta)$-generic for all $\eta < \zeta$ and $\beta < \xi$;
(vii) $p_{\eta+1} \in O_{\eta+1}$ for all $\eta + 1 < \zeta$.

If $\zeta = \eta + 1$, let p_ζ be the $<_L$-minimal condition extending p_η so that (iv)–(vii) hold for all $\eta, \eta_0, \eta_1 \le \zeta$. Its existence is guaranteed by Claim 2.4 applied to $X = N_{\gamma'\cdot\eta}$ and p_η. If ζ is limit, then we define p_ζ in exactly the same way as in Claim 2.3. Once we know that $p_\zeta \in \mathbb{P}_\xi$, the verification of (iv)–(vi) is straightforward, whereas (vii) is vacuous. The verification that $p_\zeta \in \mathbb{P}_\xi$ is exactly the same as in Claim 2.3, which in turn uses of course the ideas from the successor case. It suffices to note that $p_\gamma \in \bigcap_{\zeta<\gamma} O_{\zeta+1}$.

3. *ξ is a limit ordinal of cofinality κ.* Here we shall also work in L. Given $p \in \mathbb{P}_\xi$ and fewer than κ open dense sets $\{O_{\zeta+1} : \zeta < \gamma\}$, let $\langle N_\zeta : \zeta < \kappa\rangle$ be the minimal sequence generated by $\{p, \langle O_{\zeta+1} : \zeta < \gamma\rangle\}$. Set $\xi_\zeta = \sup(N_\zeta \cap \xi)$ for all $\zeta < \kappa$, $p = p_0$, and assume that conditions $\langle p_\eta : \eta < \zeta\rangle$ have already been defined for some $\zeta \le \gamma$ so that

(i) $p_\eta \in N_{\gamma\cdot\eta+1} \cap \mathcal{D}_\xi$ for all $\eta < \zeta$;
(ii) $p_{\eta_1} \le p_{\eta_0}$ for all $\eta_0 < \eta_1 < \zeta$;
(iii) $p_\eta\restriction\beta$ is strongly $(N_{\gamma\cdot\eta}, \mathbb{P}_\beta)$-generic for all $\eta < \zeta$ and $\beta < \xi_{\gamma\cdot\eta}$;
(iv) $p_{\eta+1} \in O_{\eta+1}$ for all $\eta + 1 < \zeta$.

Assume first that $\zeta = \eta+1$. Let $p'_{\eta+1}$ be the $<_L$-minimal condition extending p_η so that $p'_{\eta+1} \in O_{\eta+1}$. Then $p'_{\eta+1} \in N_{\gamma\cdot\eta+1}$. Let $r''_{\eta+1} <_L p'_{\eta+1}\restriction\xi_{\gamma\cdot(\eta+1)}$ be the

$<_L$-minimal element of $\mathcal{D}_{\xi_{\gamma\cdot(\eta+1)}}$ such that $r''_{\eta+1}\restriction\beta$ is strongly $(N_{\gamma\cdot(\eta+1)},\mathbb{P}_\beta)$-generic for all $\beta<\xi_{\gamma\cdot(\eta+1)}$. Its existence follows from the density of $\mathcal{D}_{\xi_{\gamma\cdot(\eta+1)}}$ and $(\dagger_{\xi_{\gamma\cdot(\eta+1)}})$. Note that $r''_{\eta+1}\in N_{\gamma\cdot(\eta+1)+1}$. Now set

$$p_{\eta+1}=r''_{\eta+1}{}^\frown p'_{\eta+1}\restriction[\xi_{\gamma\cdot(\eta+1)},\xi).$$

It is clear that $p_{\eta+1}\in N_{\gamma\cdot(\eta+1)+1}$ and conditions (ii)–(iv) hold. Since $p'_{\eta+1}\in N_{\gamma\cdot\eta+1}$, we have $\operatorname{supp}_\kappa(p'_{\eta+1})\subset N_{\gamma\cdot\eta+1}\cap\xi\subset\xi_{\gamma\cdot(\eta+1)}$. Combining this with $r''_{\eta+1}\in\mathcal{D}_{\xi_{\gamma\cdot(\eta+1)}}$ we conclude that $p_{\eta+1}\in\mathcal{D}_\xi$.

If ζ is limit, then we define p_ζ in exactly the same way as in Claim 2.3. Once we know that $p_\zeta\in\mathbb{P}_\xi$, the verification of (i)–(iii) is straightforward, whereas (iv) is vacuous. The verification that $p_\zeta\in\mathbb{P}_\xi$ is exactly the same as in Claim 2.3. It suffices to note that $p_\gamma\in\mathcal{D}_\xi\cap\bigcap_{\zeta<\gamma}O_{\zeta+1}$.

As in the case of $\operatorname{cof}(\xi)\le\gamma$, we have established the existence of a condition in $\mathcal{D}_\xi$ which lies in the intersection of given γ open dense subsets of $\mathbb{P}_\xi$. Combined with Claim 2.3, this completes the proof that the inductive assumption holds for ξ. $\blacksquare_{\text{Lemma 2.2}}$

Lemma 2.5. *Let $p\in\mathbb{P}_\xi$ for some $\xi<\kappa^+$ and τ be a $\mathbb{P}_\xi$-name. If $p\Vdash_{\mathbb{P}_\xi}$ "τ is a stationary subset of κ", then $p\Vdash_{\mathbb{P}}$ "τ is a stationary subset of κ".*

In other words, every tail of the iteration $\langle\mathbb{P}_\xi,\dot{\mathbb{Q}}_\xi:\xi<\kappa^+\rangle$ preserves stationary subsets of κ.

Proof. In light of Lemma 2.2 we may restrict our attention to conditions $p\in\mathcal{D}_\xi$. Given $p\in\mathcal{D}_\xi$ and $\zeta\in\operatorname{supp}_\kappa(p)$, from now on we shall write simply $p(\zeta)=a$ instead of $p(\zeta)=\check{a}$.

Let $\xi<\kappa^+$ and G be a $\mathbb{P}_\xi$-generic filter over L. Note that $L[G]$ has the same sequences of ordinals of length $<\kappa$ as L. From now on we shall work in $L[G]$. Set $\mathbb{P}'=\mathbb{P}^G_{[\xi,\kappa^+)}$, $\mathcal{D}'=\{p\restriction[\xi,\kappa^+)^G:p\in\mathcal{D}_{\kappa^+},p\restriction\xi\in G\}$, $\mathbb{P}'_\beta=\mathbb{P}^G_{[\xi,\beta)}$, and $\mathcal{D}'_\beta=\{p\restriction[\xi,\beta)^G:p\in\mathcal{D}_\beta,\ p\restriction\xi\in G\}$.

Fix a stationary subset S of κ in $L[G]$. Given any $p\in\mathbb{P}'$ and a $\mathbb{P}'$-name $\dot{C}$ such that $p\Vdash\dot{C}$ is a club in κ, we shall construct $q\in\mathbb{P}'$ stronger than p such that $q\Vdash\dot{C}\cap S\neq\emptyset$.

Let us fix some large enough regular cardinal θ and some large enough n. Given a set $X\in L_\theta[G]$, let N_0 be the least Σ_n-elementary submodel of $L_\theta[G]$ such that $\{X\}\cup(\gamma+1)\subset N_0$. Least means here that N_0 is the closure of $\{X\}\cup(\gamma+1)$ with respect to all Σ_n Skolem functions given by the well-ordering $<_G$ of $L_\theta[G]$. Suppose that for some $\zeta<\kappa$ we have already constructed an increasing chain $\langle N_\epsilon:\epsilon<\zeta\rangle$ of Σ_n-elementary submodels of $L_\theta[G]$. If ζ is limit then we set $N_\zeta=\bigcup_{\epsilon<\zeta}N_\epsilon$. If $\zeta=\zeta_0+1$ we let N_ζ be the minimal Σ_n-elementary submodel of $L_\theta[G]$ such that $(\gamma+1)\cup\{N_{\zeta_0}\}\subset N_\zeta$. This completes the construction of the sequence $\langle N_\zeta:\zeta<\kappa\rangle$ which will be called the *G-minimal sequence generated by X* throughout the proof.

Let $\vec{C} = \langle C_\epsilon : \epsilon \in \mathrm{Lim}(\kappa)\rangle$ be a $\square_\gamma$ sequence and $\langle N_\zeta : \zeta < \kappa\rangle$ be the G-minimal sequence generated by $\{\mathbb{P}', G, S, \dot{C}, \vec{C}, p, \ldots\}$. Set $\kappa_\zeta = N_\zeta \cap \kappa$.

Since S is stationary, we can find a limit ordinal $\zeta < \kappa$ such that $\kappa_\zeta \in S$. We shall find $q \le p$ such that $q \Vdash \kappa_\zeta \in \dot{C}$. Set $\eta = \mathrm{cof}(\zeta)$. Two cases are possible: $\eta > \omega$ and $\eta = \omega$. The latter will be addressed on page 245.

1. $\eta > \omega$. Letting $\langle \kappa_{\zeta_\beta} : \beta \le \eta\rangle$ be the increasing enumeration of $\{\kappa_\zeta\} \cup (\{\kappa_\upsilon : \upsilon < \zeta\} \cap C_{\kappa_\zeta})$, we shall construct a decreasing sequence of conditions $\langle p_\beta : \beta \le \eta\rangle$ such that

(a) $p_{\beta+1} \in \bigcap\{O : O \in N_{\zeta_{\beta+1}}$ is open dense in $\mathbb{P}'\}$ for all $\beta < \eta$;
(b) $p_\beta \in N_{\zeta_\beta+1} \cap \mathcal{D}'$ for all $\beta \le \eta$;
(c) for every $\beta \le \eta$, $\lambda \in \mathrm{supp}(p_\beta)$ of the form $\rho \cdot \alpha + \kappa + 2$, and $\upsilon < \zeta$, if $\kappa_\upsilon \in |p_\beta(\lambda)|$, then $p_\beta(\lambda)(\kappa_\upsilon) = 0$ if and only if $\upsilon \in \{\zeta_\mu : \mu < \eta\}$.

Then as a consequence of (a) and (b) we shall have

(d) $p_{\beta+1} \Vdash [\kappa_{\zeta_\beta}, \kappa_{\zeta_{\beta+1}+1}) \cap \dot{C} \ne \emptyset$ for all $\beta < \eta$.

Let $p_0 = p$ and suppose that for some $\epsilon \le \eta$ we have already constructed a decreasing sequence $\langle p_\beta : \beta < \epsilon\rangle$ satisfying (a)–(c).

If $\epsilon = \beta + 1$ for some β, let $p'_{\beta+1}$ be the $<_G$-least condition $u \le p_\beta$ in $\mathcal{D}'$ such that for every $\lambda \in \mathrm{supp}(u)$ of the form $\rho \cdot \alpha + \kappa + 2$ the following conditions hold:

(e) $\kappa_{\zeta_\beta} \in |u(\lambda)|$;
(f) if $\kappa_\upsilon \in |u(\lambda)|$ for some $\upsilon < \zeta$, then $u(\lambda)(\kappa_\upsilon) = 0$ if and only if $\upsilon \in \{\zeta_\mu : \mu < \eta\}$;
(g) $|u(\lambda)| = |p_\beta(\lambda)| + \gamma$ and $u(\lambda)\restriction([|p_\beta(\lambda)|, |p_\beta(\lambda)| + \gamma) \cap \{3\varepsilon : \varepsilon < \kappa\})$ is the $<_G$-least code for a bijection between γ and $\kappa_{\zeta_{\beta+1}}$.

It is clear that $p'_{\beta+1} \in N_{\zeta_{\beta+1}+1}$. Since (g) makes the third condition of the definition of $\dot{\mathbb{Q}}_\lambda$ for λ of the form $\rho \cdot \alpha + \kappa + 2$ vacuous for ordinals between $|p_\beta(\lambda)|$ and $\kappa_{\zeta_{\beta+1}} + \gamma$, we can find a condition $u \le p'_{\beta+1}$ in $\mathcal{D}' \cap N_{\zeta_{\beta+1}+1}$ such that for every $\lambda \in \mathrm{supp}(u)$ as above the following conditions hold:

(h) $\kappa_{\zeta_{\beta+1}} \in |u(\lambda)|$;
(i) $u(\lambda)(\kappa_\upsilon) = 0$ if and only if $\upsilon \in \{\zeta_\mu : \mu < \eta\}$.

Let $p''_{\beta+1}$ be the $<_G$-least u as above. Then $p''_{\beta+1} \in N_{\zeta_{\beta+1}+1}$. Now let $p_{\beta+1}$ be the $<_G$-least condition $w \in \mathcal{D}'$ below $p''_{\beta+1}$ so that $w \in \bigcap\{O : O \in N_{\zeta_{\beta+1}}$ is open dense in $\mathbb{P}'\}$. It follows that $p_{\beta+1}$ satisfies conditions (a)–(c) (and hence also (d)) with $\beta + 1$ instead of β.

If ϵ is limit then we define p_ϵ to be the "coordinatewise" union of $\{p_\beta : \beta < \epsilon\}$ (see Claim 2.3). It follows from the construction of $\langle p_\beta : \beta < \epsilon\rangle$ that $p_\epsilon \in N_{\zeta_\epsilon+1}$. Indeed, p_ϵ is determined by the sequence $\langle p_\beta : \beta < \epsilon\rangle$ which has been constructed using $C_{\kappa_\zeta} \cap \{\kappa_\upsilon : \upsilon < \zeta_\epsilon\}$ by always choosing $<_G$-minimal conditions with certain properties. Since $C_{\kappa_\zeta} \cap \{\kappa_\upsilon : \upsilon < \zeta_\epsilon\} = C_{\kappa_{\zeta_\epsilon}} \cap \{\kappa_\upsilon : \upsilon < \zeta_\epsilon\} \in N_{\zeta_\epsilon+1}$ by the choice of $\vec{C}$, we conclude that $p_\epsilon \in N_{\zeta_\epsilon+1}$.

In order to show that $p_\epsilon \in \mathbb{P}'$ it is enough to establish by induction on $\lambda \in \operatorname{supp}(p_\epsilon)$ that $p_\epsilon \restriction \lambda \in \mathbb{P}'_\lambda$. The only non-trivial case here is when λ has the form $\rho \cdot (\alpha+1) = \rho \cdot \alpha + \kappa + 3$ for some α. In this case, assuming that $p_\epsilon \restriction (\lambda-1) \in \mathbb{P}'_{\lambda-1}$, the property

$$p_\epsilon \restriction (\lambda-1) \Vdash_{\mathbb{P}'_{\lambda-1}} p_\epsilon(\lambda-1) \in \dot{\mathbb{Q}}_{\lambda-1}$$

can be established as follows: Given a $\mathbb{P}'_{\lambda-3}$-generic filter $R \ni p_\epsilon \restriction (\lambda-3)$ over $L[G]$, the strong $(N_{\zeta_\epsilon}, \mathbb{P}')$-genericity of $p_\epsilon \restriction (\lambda-1)$ (in $L[G]$) by the same argument as in item 1 of the proof of Lemma 2.2 implies that in $L[G * R]$ we have

$$\langle p_\epsilon(\lambda-3), p_\epsilon(\lambda-2)\rangle^{G*R} \Vdash_{(\dot{\mathbb{Q}}_{\lambda-3} * \dot{\mathbb{Q}}_{\lambda-2})^{G*R}} p_\epsilon(\lambda-1)^{G*R} \in \dot{\mathbb{Q}}^{G*R}_{\lambda-1},$$

which yields $p_\epsilon \restriction \lambda \in \mathbb{P}'_\lambda$. The only difference from the proof given in Lemma 2.2 is the case (a) where suitable models M of height $\mathrm{Ord} \cap M > \mathrm{Ord} \cap \bar{N}_{\zeta_\epsilon}$ have to be treated (here $\bar{N}_{\zeta_\epsilon}$ is the Mostowski collapse of N_{ζ_ϵ}). Now the sequence $\langle \kappa_\upsilon : \upsilon < \zeta_\epsilon \rangle$ might have length larger than γ. However, any such suitable model M still has a bijection between γ and $(\gamma^+)^{\bar{N}_{\zeta_\epsilon}}$ by the fact that M contains the sequence $\{\kappa_\upsilon : \upsilon < \zeta_\epsilon\} \cap C_{\kappa_{\zeta_\epsilon}}$ which has length $\leq \gamma$ and is cofinal in κ_{ζ_ϵ}. Since $(\gamma^+)^{\bar{N}_{\zeta_\epsilon}} = (\gamma^+)^M$ for suitable models as above, the latter is impossible, and hence such suitable models M are again ruled out.

The following statement completes the informal argument given above.

Claim 2.6. *Let M be a suitable model of size γ containing $p_\epsilon(\lambda-3)$, $p_\epsilon(\lambda-2)$ and such that $\mathrm{Ord} \cap M > \mathrm{Ord} \cap \bar{N}_{\zeta_\epsilon}$. Then M contains the sequence $\langle \kappa_\upsilon : \upsilon < \zeta_\epsilon \rangle$.*

Proof. Let $H = H(\lambda-3) * H(\lambda-2)$ be a $(\dot{\mathbb{Q}}_{\lambda-3} * \dot{\mathbb{Q}}_{\lambda-2})^{G*R}$-generic filter over $L[G*R]$ containing $\langle p_\epsilon(\lambda-3), p_\epsilon(\lambda-2)\rangle^{G*R}$, and $\pi : N_{\zeta_\epsilon}[R*H] \to \bar{\bar{N}}$ be the Mostowski collapsing function. Observe that by elementarity we have

$$\bar{\bar{N}} = \pi[N_{\zeta_\epsilon}][\pi(R) * \pi(H)] = \pi[N_{\zeta_\epsilon}][x^0_\alpha, x^1_\alpha] = L_{\mathrm{Ord} \cap \bar{\bar{N}}}[x^0_\alpha, x^1_\alpha],$$

where x^0_α and x^1_α are the unions of the first coordinates of all elements of $\pi(H(\lambda-3))$ and $\pi(H(\lambda-2))$ (equivalently, are the first coordinates of $p_\epsilon(\lambda-3)$ and $p_\epsilon(\lambda-2)$), respectively. Indeed, letting X^0_α and X^1_α be the unions of the first coordinates of all elements of $H(\lambda-3)$ and $H(\lambda-2)$, we can easily conclude from the definition of $\mathbb{P}$ that $L[G*R*H] = L[X^0_\alpha, X^1_\alpha]$, and hence also $N_{\zeta_\epsilon}[R*H] = N_{\zeta_\epsilon}[X^0_\alpha, X^1_\alpha] = (N_{\zeta_\epsilon} \cap L)[X^0_\alpha, X^1_\alpha]$.

Since $M \ni p_\epsilon(\lambda-1) = p_\epsilon(\rho\cdot\alpha+\kappa+2)$ and the latter is of the form $\breve{r}$ for some $r : \kappa_{\zeta_\epsilon} \to 2$ such that $r(3\iota+1) = 1$ iff $\iota \in x^0_\alpha$ and $r(3\iota+2) = 1$ iff $\iota \in x^1_\alpha$, we conclude that $x^0_\alpha, x^1_\alpha \in M$, and consequently

$$\pi[N_{\zeta_\epsilon}][\pi(R) * \pi(H)] = L_{\mathrm{Ord}\cap\bar{\bar{N}}}[x^0_\alpha, x^1_\alpha] \in M$$

because $\mathrm{Ord} \cap \bar{\bar{N}} < \mathrm{Ord} \cap M$. In $\pi[N_{\zeta_\epsilon}]$ we find that $\pi[N_{\upsilon+1}]$ is the closure of $\{\pi[N_\upsilon]\}$ under Σ_n Skolem functions of $\pi[N_{\zeta_\epsilon}]$ with respect to $<_{\pi(G)}$. Thus the sequence $\langle \pi[N_\upsilon] : \upsilon < \zeta_\epsilon\rangle$ is definable (as a class) over $\pi[N_{\zeta_\epsilon}]$, and hence the sequence

$$\langle \min(\mathrm{Ord} \setminus \pi[N_\upsilon]) : \upsilon < \zeta_\epsilon\rangle = \langle \kappa_\upsilon : \upsilon < \zeta_\epsilon\rangle$$

is definable over $\pi[N_{\zeta_\epsilon}]$. As a result, this sequence is an element of M. ∎

2. $\eta = \omega$. In this case let $C'_{\kappa_\zeta} \subset \{\kappa_\mu : \mu < \zeta\}$ be a cofinal subset of κ_ζ of order type ω which is an element of $N_{\zeta+1}$. Using C'_{κ_ζ} instead of C_{κ_ζ}, we can repeat the argument from case 1 and construct a decreasing sequence $\langle p_\beta : \beta \le \eta\rangle$ satisfying conditions (a)–(d).

In both of the cases considered above we have $p_\eta \le p_0 = p$ and p_η forces that $\dot{C}$ has non-empty intersection with $[\kappa_{\zeta_\beta}, \kappa_{\zeta_{\beta+1}+1})$ for all $\beta < \eta$, and hence it forces that $\kappa_\zeta = \sup\{\kappa_{\zeta_\beta} : \beta < \eta\}$ is an element of $\dot{C}$. Since $\kappa_\zeta \in S$ this completes our proof. ∎Lemma 2.5

Let us denote by Supp_{κ^+} the set of all $\xi \in \kappa^+$ of the form $\alpha\cdot\rho+\zeta$ for some $\zeta < \kappa$ and set $\mathrm{Supp}_\kappa = \kappa^+ \setminus \mathrm{Supp}_{\kappa^+}$.

Let $p, q \in \mathcal{D}$. We say that $q \le^* p$ if $q \le p$, $\mathrm{supp}_\kappa(p) = \mathrm{supp}_\kappa(q)$ and $q{\restriction}\mathrm{supp}_\kappa(q) = p{\restriction}\mathrm{supp}_\kappa(p)$. Suppose that $q \le p$. We shall define a new condition ${}^q p$ called the *reduction of q to p* by induction as follows. Suppose that ${}^q p{\restriction}\xi$ has already been defined. If $\xi \in \mathrm{Supp}_\kappa$ then $({}^q p)(\xi) = p(\xi)$. If $\xi \in \mathrm{Supp}_{\kappa^+}$ then ${}^q p(\xi)$ is a $\mathbb{P}_\xi$-name τ such that $q{\restriction}\xi \Vdash \tau = q(\xi)$ and $u \Vdash \tau = p(\xi)$ for all $\mathbb{P}_\xi \ni u \le {}^q p{\restriction}\xi$ which are incompatible with $q{\restriction}\xi$. A direct verification shows that ${}^q p \in \mathbb{P}$ and $q \le {}^q p \le^* p$.

For a pair $c = \langle a, b\rangle$ we shall use the following notation: $a = c_0$, $b = c_1$. From now on we shall consider only conditions $p \in \mathcal{D}$ such that $\Vdash_\xi p(\xi) \in \dot{\mathbb{Q}}_\xi$ for all $\xi \in \mathrm{supp}_{\kappa^+}(p)$. It is easy to see that for every $q \in \mathcal{D}$ there exists $p \in \mathcal{D}$ with this property such that $p \le q \le p$.

The next lemma shows, in particular, that $\mathbb{P}$ does not collapse κ^+. Its proof is patterned after that of [5, Proposition 2.3]. Here our choice of the support comes into play.

Lemma 2.7. *Let $p \in \mathbb{P}$ and $\mu < \kappa^+$ be an ordinal of the form $\rho\cdot\alpha+\zeta$ with $\zeta < \kappa$ such that $p \Vdash_{\mathbb{P}} \zeta \notin T(F(\alpha))$. Suppose also that $\dot{C}$ is a $\mathbb{P}$-name for a club in κ^+. Then there exists $q \le p$ such that $q \Vdash_{\mathbb{P}} S_\mu \cap \dot{C} \ne \emptyset$.*

In particular, if G is a $\mathbb{P}$-generic filter such that $\zeta \notin T(F(\alpha))^G$, then S_μ remains stationary in $L[G]$.

Proof. Without loss of generality we may assume that $p \in \mathcal{D}$. Let $\langle M_i : i < \kappa^+\rangle$ be an increasing chain of elementary submodels of L_θ of size κ, where θ is large enough, such that

(i) $M_i \supset [M_i]^\gamma$ for all $i \in \kappa^+$;
(ii) $M_i = \bigcup_{j<i} M_j$ for all $i \in \kappa^+$ of cofinality κ;
(iii) $\kappa \cup \{p, \mathbb{P}, \dot{C}, \alpha, \ldots\} \subset M_0$.

Now a standard Fodor argument yields $i \in \kappa^+$ such that $i = M_i \cap \kappa^+ \in S_\mu$ and $i \notin S_\xi$ for any $\xi \in M_i \setminus \{\mu\}$. Let $\langle\langle O_\upsilon, \phi_\upsilon\rangle : \upsilon < \kappa\rangle \in M_i^\kappa$ be a sequence in which all pairs $\langle O, \phi\rangle \in M_i$ appear cofinally often, where O is an open dense subset of $\mathbb{P}$ and ϕ is a function of size $\leq \gamma$ such that $\mathrm{dom}(\phi) \subset i$, $\phi(\xi) \in [\kappa]^{\leq\gamma} \times [\kappa^+]^{\leq\gamma}$ if ξ is of the form $\rho \cdot \beta + \kappa$ or $\rho \cdot \beta + \kappa + 1$, and $\phi(\xi) \in 2^{<\kappa}$ if ξ is of the form $\rho \cdot \alpha + \kappa + 2$. Let also $\langle i_\upsilon : \upsilon < \kappa\rangle$ be an increasing sequence of ordinals cofinal in i.

Construct by induction on υ a $\leq^*$-decreasing sequence $\langle q^\upsilon : \upsilon \leq \kappa\rangle \in \mathcal{D}^{\kappa+1}$ such that $\langle q^\upsilon : \upsilon < \kappa\rangle \in (\mathcal{D} \cap M_i)^\kappa$ as follows. Set $q^0 = p$ and suppose that $\langle q^\eta : \eta < \upsilon\rangle$ has already been constructed. If υ is limit then we set $q^\upsilon(\xi) = p(\xi)$ if $\xi \in \mathrm{Supp}_\kappa$ and let $q^\upsilon(\xi)$ be a $\mathbb{P}_\xi$-name which is forced by $q^\upsilon\restriction\xi$ to be the union of all $q^\eta(\xi)$, $\eta < \upsilon$, together with its supremum. Since the S_ξ's consist of ordinals of cofinality κ for all $\xi < \kappa^+$, we conclude that $q^\upsilon \in \mathbb{P}$ provided that $\upsilon < \kappa$. Now suppose that $\upsilon = \eta + 1$. Let us first consider the case that there exists a condition $r \in O_\eta \cap \mathcal{D}$ stronger than q^η such that, letting $\psi = r\restriction\mathrm{supp}_\kappa(r)$, the following conditions hold:

(iv) $\mathrm{dom}(\phi_\eta) \subset \mathrm{dom}(\psi)$;
(v) $\Vdash_\xi \psi(\xi) \leq \phi_\eta(\xi)$ for all $\xi \in \mathrm{dom}(\phi_\eta)$;
(vi) $\psi(\xi)_0 = \phi_\eta(\xi)_0$ for all $\xi \in \mathrm{dom}(\phi_\eta)$ of the form $\rho\cdot\beta+\kappa$ or $\rho\cdot\beta+\kappa+1$.

In this case we fix such a condition $r^\eta \in M_i$ and set $q_0^\upsilon = {}^{r^\eta}q^\eta$. If there is no such condition r then we set $q_0^\upsilon = q^\eta$. Now let $q^\upsilon \leq^* q_0^\upsilon$ be such that for all $\xi \in \mathrm{supp}_{\kappa^+}(q_0^\upsilon)$, $\Vdash_\xi$ "$q^\upsilon(\xi) = q_0^\upsilon(\xi) \cup \{\max(q_0^\upsilon(\xi)) + i_\upsilon\}$ if $\varsigma \in T(F(\beta))$ and $q^\upsilon(\xi) = \emptyset$ otherwise", where $\xi = \rho \cdot \beta + \varsigma$.

We claim that $q^\kappa \in \mathbb{P}$ and it is $(M_i, \mathbb{P})$-generic. We shall prove this in several steps.

CLAIM 2.8. *If $\xi \in \mathrm{Supp}_{\kappa^+} \cap M_i$ and $q^\kappa\restriction\xi$ is $(M_i, \mathbb{P}_\xi)$-generic* ([8])*, then $q^\kappa\restriction(\xi+1) \in \mathbb{P}_{\xi+1}$.*

Proof. It suffices to show that $r \Vdash_\xi q^\kappa(\xi) \cap S_\xi = \emptyset$ for every $r \leq q^\kappa\restriction\xi$ which forces $\varsigma \in T(F(\beta))$, where $\xi = \rho \cdot \beta + \varsigma$. Suppose, contrary to our

([8]) In particular, here we assume that $q^\kappa\restriction\xi \in \mathbb{P}_\xi$.

claim, that there exists $r \le q^\kappa \restriction \xi$ such that $r \Vdash_\xi \varsigma \in T(F(\beta))$ but

$$r \Vdash_\xi \Big[\bigcup_{\upsilon<\kappa} q^\upsilon(\xi) \cup \Big\{\sup\Big(\bigcup_{\upsilon<\kappa} q^\upsilon(\xi)\Big)\Big\}\Big] \cap S_\xi \neq \emptyset. \tag{1}$$

Then $\xi \neq \mu$. Indeed, otherwise $r \le q^\kappa \restriction \mu \le p \restriction \mu$, and the latter forces $\zeta \notin T(F(\alpha))$ by our assumptions. Thus $r \Vdash_\mu \zeta \notin T(F(\alpha))$, and hence $r \Vdash_\xi \varsigma \notin T(F(\beta))$ because $\langle \xi, \beta, \varsigma\rangle = \langle \mu, \alpha, \zeta\rangle$, which contradicts the choice of r.

Without loss of generality we may assume that $r \Vdash_\xi \sup(\bigcup_{\upsilon<\kappa} q^\upsilon(\xi)) = j$ for some j. Note that $j \le i$ because r is $(M_i, \mathbb{P}_\xi)$-generic and therefore forces $\max q^\upsilon(\xi) < i$ for each υ. And by the definition of the q^υ's we know that $r \Vdash_\xi \max q^\upsilon(\xi) \ge i_\upsilon$ for all $\upsilon < \kappa$ and therefore $i \le j$, so $i = j$. But (1) is possible only if j belongs to S_ξ, and since ξ belongs to $M_i \setminus \{\mu\}$, we have $j \neq i$ by our choice of i, contradiction. ∎

Claim 2.9. *Suppose that $j \le i$ and $q^\kappa \restriction \xi$ is $(M_i, \mathbb{P}_\xi)$-generic for all $\xi < j$.*

- *If $j < i$, then $q^\kappa \restriction j$ is $(M_i, \mathbb{P}_j)$-generic;*
- *If $j = i$, then $q^\kappa \restriction j = q^\kappa$ is $(M_i, \mathbb{P})$-generic.*

Proof. Let us first consider the case $j < i$. It follows that $q^\kappa \restriction j \in \mathbb{P}_j$, the case of a successor j is handled by Claim 2.8.

Fix an open dense subset $E \in M_i$ of $\mathbb{P}_j$ and $w \le q^\kappa \restriction j$. We need to show that there exists $w_1 \in E \cap M_i$ such that w and w_1 are compatible. Without loss of generality, $w \in \mathcal{D} \cap E$.

Consider the set $K = \mathrm{supp}_\kappa(w) \cap M_i$ and note that $K \in M_i$ and $K \subset j$. For every $\xi \in K$ let $\phi(\xi) = w(\xi)$ if ξ is of the form $\rho \cdot \beta + \kappa + 2$ and $\phi(\xi) = \langle w(\xi)_0, w(\xi)_1 \cap M_i\rangle$ otherwise. Observe that $\phi \in M_i$. Let O be the set of those $r \in \mathbb{P}$ such that $r \restriction j \in E$. Then $O \in M_i$ is an open dense subset of $\mathbb{P}$. Let $\eta < \kappa$ be such that $\langle O, \phi\rangle = \langle O_\eta, \phi_\eta\rangle$ and $\upsilon = \eta + 1$. It follows from the above that we have made the non-trivial choice in the construction of q^υ. More precisely, there exists $r \in O_\eta \cap \mathcal{D}$ (namely w extended by $q^\eta \restriction [j, \kappa^+)$) such that conditions (iv)–(vi) are satisfied. Thus there exists $r^\eta \in O \cap \mathcal{D} \cap M_i$ satisfying (iv)–(vi) such that $q^\upsilon \le^* {}^{r^\eta}p^\eta$. In particular, $w \le q^\kappa \restriction j \le^* {}^{r^\eta}p^\eta \restriction j$ and $r^\eta \restriction j \in E \cap M_i$. We claim that $w_1 = r^\eta \restriction j$ is compatible with w. Let us define a sequence w_2 of length j as follows:

- (vii) $w_2(\xi) = w(\xi)$ if $\xi \in \mathrm{Supp}_{\kappa^+}$;
- (viii) $w_2(\xi) = \langle w(\xi)_0, w(\xi)_1 \cup r^\eta(\xi)_1\rangle$ if [9] $\xi \in \mathrm{supp}_\kappa(w)$ is of the form $\rho \cdot \alpha + \kappa$ or $\rho \cdot \alpha + \kappa + 1$;
- (ix) $w_2(\xi) = w(\xi)$ if [10] ξ is of the form $\rho \cdot \alpha + \kappa + 2$.

[9] $w(\xi)_0 = r^\eta(\xi)_0 = \phi_\eta(\xi)_0$ in this case.

[10] $w(\xi) = r^\eta(\xi) = \phi_\eta(\xi)$ in this case.

We are left with the task of showing that $w_2 \in \mathbb{P}_j$, since then it becomes straightforward that w_2 is a lower bound for w and w_1. We shall show by induction on $\xi < j$ that if $w_2 \restriction \xi \in \mathbb{P}_\xi$ then $w_2 \restriction \xi \Vdash w_2(\xi) \in \dot{\mathbb{Q}}_\xi$. In light of our convention regarding conditions in $\mathcal{D}$ made before Lemma 2.7 we have to consider only the case $\xi \in \operatorname{supp}_\kappa(w)$. By (ix) and $w_2 \restriction \xi \leq w \restriction \xi, w_1 \restriction \xi$ we may further restrict ourselves to ξ's in $\operatorname{supp}_\kappa(w)$ of the form $\rho \cdot \alpha + \kappa$ or $\rho \cdot \alpha + \kappa + 1$. In the latter case $w_2 \restriction \xi$, being a lower bound of $w_1 \restriction \xi = r^\eta \restriction \xi, w \restriction \xi$, forces both $w(\xi)$ and $r^\eta(\xi)$ to be elements of $\dot{\mathbb{Q}}_\xi$. Moreover, $w_2 \restriction \xi$ forces $r^\eta(\xi)$ and $w(\xi)$ to be compatible in $\dot{\mathbb{Q}}_\xi$ (because so are any two conditions in the almost disjoint coding forcing with the same first coordinate), and $w_2(\xi)$ defined as in (viii) to be their largest lower bound. In particular, $w_2 \restriction \xi \Vdash w_2(\xi) \in \dot{\mathbb{Q}}_\xi$, which completes our proof in case of $j < i$.

The case $j = i$ can be proved by almost literal repetition of the above proof: We just have to take $O = E$ and replace most of the instances of j in it by κ^+ (or, alternatively, remove them). However, we shall present this proof for the sake of completeness.

Fix an open dense subset $E \in M_i$ of $\mathbb{P}$ and $w \leq q^\kappa$. We need to show that there exists $w_1 \in E \cap M_i$ such that w and w_1 are compatible. Without loss of generality, $w \in \mathcal{D} \cap E$. Let K, ϕ, η, υ be as in the previous case. It follows from the above that we have made the non-trivial choice in the construction of q^υ. More precisely, there exists $r \in O_\eta \cap \mathcal{D}$ (namely w) such that conditions (iv)–(vi) are satisfied. Thus there exists $r^\eta \in E \cap \mathcal{D} \cap M_i$ satisfying (iv)–(vi) such that $q^\upsilon \leq^* {}^{r^\eta}p^\eta$. In particular, $w \leq q^\kappa \leq^* {}^{r^\eta}p^\eta$ and $r^\eta \in E \cap M_i$. We claim that $w_1 = r^\eta$ is compatible with w. Let us define a sequence w_2 of length κ^+ as follows:

- (vii)′ $w_2(\xi) = w(\xi)$ if $\xi \in \operatorname{Supp}_{\kappa^+}$;
- (viii)′ $w_2(\xi) = \langle w(\xi)_0, w(\xi)_1 \cup r^\eta(\xi)_1 \rangle$ if $\xi \in \operatorname{supp}_\kappa(w)$ is of the form $\rho \cdot \beta + \kappa$ or $\rho \cdot \beta + \kappa + 1$;
- (ix)′ $w_2(\xi) = w(\xi)$ if ξ is of the form $\rho \cdot \beta + \kappa + 2$.

The fact that $w_2 \in \mathbb{P}$ can be verified in exactly the same way as in the case $j < i$, and then it becomes straightforward that w_2 is a lower bound for w and w_1. This completes our proof. ■

By induction on $j \leq i$ using Claim 2.9 we can prove that q^κ is $(M_i, \mathbb{P})$-generic. Since $\dot{C} \in M_i$ this implies $q^\kappa \Vdash i \in \dot{C}$. It remains to note that $i \in S_\mu$ and $q^\kappa \leq p$. ■Lemma 2.7

Let $\mathbb{H}$ be a poset. An $\mathbb{H}$-name $\dot{f}$ is called a *nice name for an element of* κ^κ if $\dot{f} = \bigcup_{\upsilon \in \kappa} \{\langle \langle \upsilon, \check{\eta}^\upsilon_p \rangle, p \rangle : p \in \mathcal{A}_\upsilon(\dot{f})\}$, where $\mathcal{A}_\upsilon(\dot{f})$ is a maximal antichain in $\mathbb{H}$ for all $\upsilon \in \kappa$ and $\eta^\upsilon_p \in \kappa$ for all $p \in \mathcal{A}_\upsilon(\dot{f})$. Then $p \Vdash \dot{f}(\upsilon) = \eta^\upsilon_p$ for all $\upsilon \in \kappa$ and $p \in \mathcal{A}_\upsilon$. From now on we will assume that all names for an element of κ^κ are nice.

Lemma 2.10. *Let* $\dot{f} = \bigcup_{\upsilon\in\kappa}\{\langle\langle\upsilon,\check{\eta}^{\upsilon}_p\rangle, p\rangle : p \in \mathcal{A}_\upsilon\}$ *be a nice name for an element of* κ^κ. *Then for every* $p \in \mathbb{P}$ *there exists* $q \le p$ *and a* $\mathbb{P}$*-name* $\sigma \subset \dot{f}$ *of size* $|\sigma| \le \kappa$ *such that* $q \Vdash \dot{f} = \sigma$.

Proof. Let $\langle M_i : i < \kappa^+\rangle$ be as in Lemma 2.7, where instead of (iii) we require $\kappa \cup \{p, \mathbb{P}, \dot{f}, \ldots\} \subset M_0$. As established in the proof of Lemma 2.7, there exists $i < \kappa^+$ and a $(M_i, \mathbb{P})$-generic condition $q \le^* p$. In particular, $\mathcal{A}_\upsilon \cap M_i$ is predense below q, and hence no elements of $\mathcal{A}_\upsilon \setminus M_i$ are compatible with q. It follows from the above that $q \Vdash \dot{f} = \sigma$, where $\sigma = \bigcup_{\upsilon\in\kappa}\{\langle\langle\upsilon,\check{\eta}^{\upsilon}_p\rangle, p\rangle : p \in \mathcal{A}_\upsilon \cap M_i\}$. ∎

The same proof as above also works for $\mathbb{P}_\xi$ when $\xi < \kappa^+$.

Corollary 2.11. *The poset* $\mathbb{P}_\xi$ *has a dense subset of size* κ^+ *for every* $\xi \le \kappa^+$.

Proof. We shall prove by induction on $\xi \le \kappa^+$ that there exists a $\le^*$-dense subset $\mathcal{D}'_\xi$ of $\mathcal{D}_\xi$ of size κ^+.

The successor case is easily handled by Lemma 2.10. Notice that it is essential here that the generic condition q considered in its proof can be chosen $\le^*$-below the given one.

Suppose that $\mathrm{cof}(\xi) = \eta \le \kappa$ and fix an increasing sequence $\langle \xi_\upsilon : \upsilon < \eta\rangle$ of ordinals, cofinal in ξ, such that $\xi_0 = 0$. Let $p \in \mathcal{D}_\xi$ and M be an elementary submodel of L_θ of size κ, where θ is large enough, such that $M \supset [M]^\gamma$ and $\kappa \cup \{p, \mathbb{P}_\xi, \langle \xi_\upsilon : \upsilon < \eta\rangle, \ldots\} \subset M$. By a standard Fodor argument we may additionally assume that $i \notin S_\mu$ for all $\mu \in M$, where $i = M \cap \kappa^+$: this can be ensured by picking M out of an increasing continuous chain of elementary submodels of L_θ as in Lemma 2.7. Let also $\langle i_\upsilon : \upsilon < \kappa\rangle$ be an increasing sequence of successor ordinals cofinal in i. By the inductive assumption we can construct by induction on υ a sequence $\langle q^\upsilon : \upsilon < \eta\rangle \in M^\eta$ such that the following conditions hold:

(i) $q^\upsilon \in \mathcal{D}'_{\xi_\upsilon}$;
(ii) $q^{\upsilon+1} \le^* q^\upsilon {}^\frown p\restriction[\xi_\upsilon, \xi_{\upsilon+1})$;
(iii) if υ is limit, then q^υ is $\le^*$-below the condition $r^\upsilon \in \mathbb{P}_{\xi_\upsilon}$ defined as follows: for all $\mu \in \mathrm{Supp}_{\kappa^+} \cap \xi_\upsilon$, $\Vdash_\mu$ "$r^\upsilon(\mu) = \bigcup_{\upsilon'<\upsilon} q^{\upsilon'}(\mu) \cup \{\sup(\bigcup_{\upsilon'<\upsilon} q^{\upsilon'}(\mu)) + i_\upsilon\}$ if $\varsigma \in T(F(\beta))$ and $r^\upsilon(\mu) = \emptyset$ otherwise", where $\mu = \rho\cdot\beta + \varsigma$; moreover, $r^\upsilon(\mu) = p(\mu)$ for all $\mu \in \mathrm{Supp}_\kappa \cap \xi_\upsilon$.

Now let r^η be defined as in (iii). Observe that $r^\eta \in \mathcal{D}_\xi$: This is obvious if $\eta < \kappa$ and follows from $i \notin \bigcup_{\mu\in M\cap\xi} S_\mu$ if $\eta = \kappa$. In addition, $r^\eta \le^* p$ by the construction and it is uniquely determined by the sequence $\langle q^\upsilon : \upsilon < \eta\rangle \in \bigcup_{\mu<\xi} \mathcal{D}'_\mu$. Now, it suffices to note that there are at most $(\kappa^+)^\kappa = \kappa^+$ such sequences.

And finally, the case $\xi = \kappa^+$ is straightforward because the supports of conditions have size $\leq \kappa$. ∎

Combining Lemma 2.10 with Corollary 2.11 we conclude that $2^\kappa = \kappa^+$ holds in $V^{\mathbb{P}_\xi}$ for all $\xi \leq \kappa^+$. Recall that our main poset $\mathbb{P}$ depends on a particular bookkeeping function $F : \kappa^+ \to L$, so we may write $\mathbb{P}_F$ instead of $\mathbb{P}$. The following statement is a direct corollary of Lemma 2.10 and Corollary 2.11.

Corollary 2.12. *There exists a bookkeeping function $F : \kappa^+ \to L$ such that for every $\mathbb{P}_F$-name σ for a subset of κ and $p \in \mathbb{P}_F$ there exists $\alpha < \kappa^+$ such that $F(\alpha)$ is a $\mathbb{P}_F$-name, and a condition $q \in \mathbb{P}_F$ below p which forces $\sigma = F(\alpha)$.*

From now on we shall fix a bookkeeping function F_0 with the properties described in Corollary 2.12 and assume that $\mathbb{P} = \mathbb{P}_{F_0}$. Combining Lemmas 2.5 and 2.7 we obtain the following

Corollary 2.13. *Let G be a $\mathbb{P}$-generic filter over L and $\xi < \kappa^+$ be an ordinal of the form $\rho \cdot \alpha + \zeta$ for some $\zeta < \kappa$. Then S_ξ is non-stationary in $L[G]$ iff $F(\alpha)^G$ is a stationary subset of κ and $\zeta \in T(F(\alpha)^G)$.*

The following statement is reminiscent of [4, Lemma 4].

Lemma 2.14. *Let G be $\mathbb{P}$-generic over L and S a subset of κ in $L[G]$. If S is stationary, then there exists $Y \in [\kappa]^\kappa$ such that for every suitable model M of size γ containing $Y \cap (\gamma^+)^M$, the set $S \cap (\gamma^+)^M$ belongs to M and there is $\mu < (\gamma^{++})^M$ such that for all $\zeta \in T(S) \cap (\gamma^+)^M$ we have $M \vDash$ "$S_{\rho\cdot\mu+\zeta}$ is not stationary".*

Proof. Using Corollary 2.12 we may find $\alpha < \kappa^+$ such that $S = F(\alpha)^G$. We claim that Y_α (this is the subset of κ added in Case 4 of the definition of $\mathbb{P}$) is as required. Indeed, let M be a suitable model of size γ containing $Y_\alpha \cap (\gamma^+)^M$. Then by the definition of $\mathbb{Q}_{\rho\cdot\alpha+\kappa+2}$ we know that $S \cap (\gamma^+)^M \in M$ and $M \vDash \psi(\gamma^+, \gamma^{++}, \mu, S \cap (\gamma^+)^M, X^0_\alpha \cap (\gamma^+)^M)$ for some $\mu < (\gamma^{++})^M$, where $X^0_\alpha = \{\upsilon < \kappa : 3\upsilon + 1 \in Y_\alpha\}$. It suffices to analyze the statement of ψ. ∎

The next fact resembles [4, Lemma 5].

Lemma 2.15. *Let G be a $\mathbb{P}$-generic over L and let S be a subset of κ in $L[G]$.* If *there exists $Y \in [\kappa]^\kappa$ such that for every suitable model M of size γ containing $Y \cap (\gamma^+)^M$, there is $\mu < (\gamma^{++})^M$ such that for all $\zeta \in T(S) \cap (\gamma^+)^M$ we have $M \vDash$ "$S_{\rho\cdot\mu+\zeta}$ is not stationary",* then *S is stationary in κ.*

Proof. Let N be an elementary submodel of $L_\theta[G]$ of size γ containing $(\gamma+1) \cup \{S, Y\}$, where θ is a large enough cardinal. Let M be the Mostowski collapse of N and $\pi : N \to M$ be the collapsing function. Then

$$M \models \exists \mu < \pi(\kappa^+)\ \forall \zeta \in \pi(T(S))\ (S_{\rho\cdot\mu+\zeta} \text{ is not stationary in } \pi(\kappa^+)),$$

which implies

$$N \models \exists \alpha < \kappa^+\ \forall \zeta \in T(S)\ (S_{\rho\cdot\alpha+\zeta} \text{ is not stationary in } \kappa^+),$$

and hence in $L[G]$ there exists $\alpha < \kappa^+$ such that for all $\zeta \in T(S)$ the set $S_{\rho\cdot\alpha+\zeta}$ is not stationary in κ^+. This means that $\mathbb{P}$ destroys the stationarity of $S_{\rho\cdot\alpha+\zeta}$ for some ζ, and hence Corollary 2.13 implies that $F(\alpha)^G$ is a stationary subset of κ and $S_{\rho\cdot\alpha+\zeta}$ is non-stationary in $L[G]$ iff $\zeta \in T(F(\alpha)^G)$. It follows from the above that $T(S) \subset T(F(\alpha)^G)$, which gives $S = F(\alpha)^G$ and thus completes our proof. ∎

Theorem 1.1(1) is a direct consequence of Lemmas 2.14 and 2.15, as they easily imply that in $V^{\mathbb{P}}$ we have the Σ_1 definition of the complement of NS_κ presented on page 233.

The proof of Theorem 1.1(2) is completely analogous to that of the first part. In this case we consider the same iteration but proceed until κ^{++}. In order to be able to do this we need a suitable sequence $\langle S_\alpha : \alpha < \kappa^{++}\rangle$ of mutually *almost* disjoint stationary subsets of κ^+. It may be obtained in the same way as in the first part, the only difference being that now we have to use the diamond to "convert" all subsets of κ^+ (previously we restricted ourselves to singletons) into stationary subsets of κ^+. Then we can repeat the same proof with κ^+ replaced with κ^{++} whenever the length of the iteration is concerned. The only new thing here will occur in Corollary 2.11. The same proof shows that it remains true for all $\xi < \kappa^{++}$. The poset $\mathbb{P}_{\kappa^{++}}$ will obviously have size (i.e., a dense subset of size) κ^{++}. By a standard argument it has κ^{++}-c.c. Indeed, in order to prove this it is enough to basically replace ω with κ in the proof of [1, Theorem 2.10], and be a little more careful with the choice of elementary submodels. More precisely, given $\{r_\xi : \xi < \kappa^{++}\} \subset \mathbb{P}_{\kappa^{++}}$, for every ξ choose an elementary submodel $M_\xi \ni r_\xi$ of L_λ of size κ for some large enough λ such that $[M_\xi]^\gamma \cup \{\mathbb{P}_{\kappa^{++}}\} \subset M_\xi$ and there exists an $(M_\xi, \mathbb{P}_{\kappa^{++}})$-generic condition ([11]) below r_ξ, and apply the fact that κ^{++} of these submodels have the same isomorphism type to find $\xi_1 \neq \xi_2$ in κ^{++} such that r_{ξ_1} is compatible with r_{ξ_2}. The existence of the M_ξ's is established in the proof of Lemma 2.7.

3. Final remarks and open problems. In this section we shall consider the set κ^κ with the $({<}\kappa)$-box topology, i.e., the topology with the base $\{[s] : s \in \kappa^{<\kappa}\}$, where $[s] = \{x \in \kappa^\kappa : x \text{ extends } s\}$. Following [9] we say that a subset A of κ^κ is *meager* if it is a union of κ nowhere dense subsets.

([11]) In [1] one can take any $M_\xi \ni r_\xi$ because the poset under consideration is proper.

A is said to have the *Baire property* if $A \triangle O$ is meager for some open subset O of κ^κ. It is well-known [9, Theorem 4.2] (see also [7, Theorem 49]) that NS_κ does not have the Baire property, even though it is Σ^1_1-definable. This is one of the main differences from the classical case $\kappa = \omega$.

One may however hope that there is an analogy between the Baire property of Δ^1_1-definable subsets of κ^κ and that of Δ^1_2-definable subsets of ω^ω: informally, in the uncountable case there is no need for an extra quantifier to express that a relation under consideration is well-founded. It turns out that there is no such analogy, as we can see using the model constructed in the proof of Theorem 1.1 ([12]). Recall that in the classical setting $\kappa = \omega$, the Baire property of all Δ^1_2-definable sets of reals is equivalent to the statement that for every real x there exists a Cohen real y over $L[x]$ (see [11]).

PROPOSITION 3.1. *In the model constructed in the proof of Theorem 1.1(1) for every $X \subset \kappa$ there is $Y \subset \kappa$ which is $Add(\kappa,1)$-generic over $L[X]$, where $Add(\kappa,1) = 2^{<\kappa}$ ordered by extension. Thus the κ-analogue of the condition guaranteeing the Baire property of all Δ^1_2-definable sets does not imply the Baire property of all Δ^1_1-definable subsets of κ^κ, as witnessed by NS_κ.*

Proof. By Corollary 2.12 it is enough to show that posets $\dot{\mathbb{Q}}_\xi$ add $Add(\kappa,1)$ generics over $L^{\mathbb{P}_\xi}$ for cofinally many $\xi \in \kappa^+$. For every $(<\kappa)$-complete filter $\mathcal{F}$ on κ there is a natural poset $\mathbb{M}(\mathcal{F})$ ("M" comes from "Mathias") producing a pseudointersection of $\mathcal{F}$. This poset consists of all pairs $\langle s, F\rangle \in [\kappa]^{<\kappa} \times \mathcal{F}$ where $\langle s', F'\rangle$ extends $\langle s, F\rangle$ if and only if s' end-extends s, $F' \subset F$, and $s' \backslash s \subset F$. Observe that for every ξ of the form $\rho\cdot\alpha+\kappa$, in $V^{\mathbb{P}_\xi}$ the poset $\mathbb{Q}_\xi$ is order isomorphic to $\mathbb{M}(\mathcal{F})$ for the $(<\kappa)$-complete filter on κ generated by $\{\kappa \setminus A_\nu : \nu \in D_\alpha\}$. The following statement may be thought of as folklore. We have learned it from an unpublished manuscript of Brendle.

CLAIM 3.2. *Let $\mathcal{F}$ be a $(<\kappa)$-complete filter on κ such that there exists a function $f : [\kappa]^2 \to 2$ for which $f[[F]^2] = 2$ for all $F \in \mathcal{F}$. Then there exists an $Add(\kappa,1)$-generic filter in $V[\mathbb{M}(\mathcal{F})]$.*

Proof. Let G be a $\mathbb{M}(\mathcal{F})$-generic and $g = \bigcup\{s : \exists F \in \mathcal{F}\ (\langle s, F\rangle \in G)\}$. Set $c(\alpha) = f(\gamma_{2\alpha}, \gamma_{2\alpha+1})$, where $\{\gamma_\alpha : \alpha < \kappa\}$ is the increasing enumeration of g. We claim that $\{c\restriction\alpha : \alpha \in \kappa\}$ is $Add(\kappa,1)$-generic. Indeed, let $D \subset Add(\kappa,1)$ be dense and $\langle s, F\rangle \in \mathbb{M}(\mathcal{F})$ be such that the order type of s equals 2α for some $\alpha \in \kappa$. Thus $\langle s, F\rangle$ determines $c\restriction\alpha$, say $\langle s, F\rangle \Vdash c\restriction\alpha = \sigma$. By the density of D there exists an extension $\tau \in D$ of σ. Since $f[[F\backslash\xi]^2] = 2$ for all $\xi \in \kappa$, we can easily find an end-extension t of s such that $t \setminus s \subset F$,

([12]) We would like to thank Yurii Khomskii for asking us whether such an analogy holds.

order type of t equals 2β, where $\beta = \mathrm{dom}(\tau)$, and $(t, F\setminus \sup t{+}1) \Vdash c\restriction\beta = \tau$. This completes our proof. ∎

In our case κ is a successor cardinal. In particular it is not measurable. It suffices to note that for every $({<}\kappa)$-complete filter $\mathcal{F}$ which is not an ultrafilter there exists a function f as in the claim above. Indeed, take $A \subset \kappa$ such that each element of $\mathcal{F}$ intersects both A and $\kappa\setminus A$ and set $f(\{\alpha,\beta\}) = 1$ iff $\{\alpha,\beta\} \subset A$ or $\{\alpha,\beta\} \subset \kappa \setminus A$. ∎

Instead of arguing as in Proposition 3.1 we could just change the construction of $\mathbb{P}$ by letting $\dot{\mathbb{Q}}_\xi$ be the $\mathbb{P}_\xi$-name for $Add(\kappa, 1)$ for cofinally many $\xi \in \kappa^+$. It is easy to check that this would not interfere with the proof of the Δ_1-definability of NS_κ.

Finally we mention two open questions related to Theorem 1.1 whose solution seems to require essentially different approaches.

Problem 3.3. *Is it consistent that* NS_κ *is* Δ_1*-definable over* $H(\kappa^+)$ *and* $2^\kappa \geq \kappa^{+++}$*?*

Problem 3.4. *Is* $2^\gamma \geq \gamma^{++}$ *together with* NS_{γ^+} *being* Δ_1*-definable over* $H(\gamma^{++})$ *consistent? What if* $\gamma = \omega$*? In the latter case, can we additionally have MA instead of* $\neg CH$*? In case the answer to some of these questions is affirmative, can the corresponding consistency be forced over L?*

Let us note that the existence of a collection $\mathcal{S}$ of stationary subsets of ω_1 such that $|\mathcal{S}| = \omega_1$ and each stationary subset of ω_1 contains some $S \in \mathcal{S}$, which may be thought of as a strong form of the Δ_1-definability of the NS_{ω_1}, implies the existence of a Suslin tree: see, e.g., [6, Theorem 5.28]. Thus it contradicts MA.

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