Partial square at $\omega_1$ is implied by MM
but not by PFA

by

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Abstract. We prove the results stated in the title.

1. Introduction. The square principle, introduced by Jensen [9], and its weak versions, play important roles in Set Theory. Using these square principles, we can construct non-compact objects such as Suslin trees and non-reflecting stationary sets. Thus propositions asserting some compactness tend to imply the failure of square principles. For example, it was shown by Magidor [12] that the stationary reflection principle at $\delta^+$ implies the failure of $\square_\delta$. It is also known, from work of Todorčević [17], that PFA implies the failure of $\square(\delta)$ for any regular $\delta \geq \omega_2$.

In this paper we study consequences of forcing axioms for the partial square principle at $\omega_1$. In particular we study the consequences of Martin’s Maximum, MM, and the Proper Forcing Axiom, PFA. First we recall the partial square principle. Below, for a set $A$ of ordinals, $\text{otp}(A)$ denotes the order type of $A$, and $\text{Lim}(A)$ denotes the set of all limit points in $A$, i.e. $\text{Lim}(A) = \{\alpha \in A \mid \sup(A \cap \alpha) = \alpha\}$.

**Definition 1.1.** Let $\delta$ be an uncountable cardinal. For $S \subseteq \text{Lim}(\delta^+)$ let $\square_\delta(S) \equiv$ there exists a sequence $\langle c_\alpha \mid \alpha \in S \rangle$ such that

(i) $c_\alpha$ is a club in $\alpha$ with $\text{otp}(c_\alpha) \leq \delta$ for each $\alpha \in S$;

(ii) if $\alpha \in S$ and $\beta \in \text{Lim}(c_\alpha)$, then $\beta \in S$ and $c_\beta = c_\alpha \cap \beta$.

A sequence $\langle c_\alpha \mid \alpha \in S \rangle$ satisfying (i) and (ii) is called a $\square_\delta(S)$-sequence.

The above partial square was used in [2], [7], [10], [13], [15], etc.
Note that $\square_\delta(\text{Lim}(\delta^+))$ is equivalent to Jensen’s $\square_\delta$ introduced in [9]. In fact it is easy to see that $\square_\delta$ holds if and only if $\square_\delta(C)$ holds for some club $C \subseteq \text{Lim}(\delta^+)$. On the other hand, it is not hard to see that if $S$ is a nonstationary subset of $\delta^+$, then $\square_\delta(S)$ holds.

As for $\square_\delta(S)$ for a stationary $S \subseteq \text{Lim}(\delta^+)$, the following was shown by Shelah:

**Fact 1.2** (Shelah [15]). Suppose that $\delta$ and $\rho$ are regular cardinals with $\rho < \delta$. Then there exists $S \subseteq \text{Lim}(\delta^+)$ such that

(i) $\square_\delta(S)$ holds,

(ii) the set $\{\alpha \in S \mid \text{cf}(\alpha) = \rho\}$ is stationary in $\delta^+$.

On the other hand, it is known that the following partial square principle $\square^p_\delta$ is independent of ZFC for a regular uncountable cardinal $\delta$ (see [6]):

**Definition 1.3.** For a regular uncountable cardinal $\delta$ let

$\square^p_\delta \equiv$ there exists $S \subseteq \text{Lim}(\delta^+)$ such that

(i) $\square_\delta(S)$ holds,

(ii) the set $\{\alpha \in S \mid \text{cf}(\alpha) = \delta\}$ is stationary in $\delta^+$.

(The superscript “p” in $\square^p_\delta$ stands for “partial”.)

We study consequences of MM and PFA for $\square^p_{\omega_1}$. For simplicity of our notation we omit the subscript $\omega_1$ in $\square_{\omega_1}(S)$ and $\square^p_{\omega_1}$:

**Notation.** Let $\square(S)$ and $\square^p$ denote $\square_{\omega_1}(S)$ and $\square^p_{\omega_1}$, respectively.

It is not hard to see that MM does not imply the failure of $\square^p$ (see Thm. 6.4). Our first result is the following:

**Theorem 1.4.** MM implies $\square^p$.

On the other hand, we also prove that PFA does not imply $\square^p$:

**Theorem 1.5.** If there exists a supercompact cardinal, then there exists a forcing extension in which PFA holds but $\square^p$ fails.

Theorem 1.4 will be proved in §3 and Theorem 1.5 in §5. In §6 we make a remark that $\square^p_\delta$ for a regular cardinal $\delta \geq \omega_2$ is independent of MM.

In §5 and §6 we discuss the consistency of the failure of the partial square. For this we use a strong stationary reflection principle, which was introduced by Magidor [12] and implies the failure of the partial square. In §4 we present facts on this strong stationary reflection principle which we use in §5 and §6.

2. Preliminaries. Here we present our notation and basic facts used in this paper. For those which are not presented below, consult Jech [8].
For a function $f$ and $X \subseteq \text{dom}(f)$ we let $f[X] := \{f(x) \mid x \in X\}$. For a regular cardinal $\delta$ and an ordinal $\kappa > \delta$ let $\mathcal{E}_\delta^\kappa := \{\alpha < \kappa \mid \text{cf}(\alpha) = \delta\}$. Moreover for ordinals $\delta$ and $\kappa$ with $\delta \leq \kappa$ let $\mathcal{E}_{<\delta}^\kappa := \{\alpha < \kappa \mid \text{cf}(\alpha) < \delta\}$. For $i = 0, 1$ let $E_i^2 := \mathcal{E}_{\omega_i}^\omega$.

For a regular cardinal $\delta$ and a set $W$, $[W]^\delta$ denotes the set of all $x \subseteq W$ with $|x| = \delta$. A set $C \subseteq [W]^\delta$ is said to be a club in $[W]^\delta$ if for some function $F : \omega W \to W$, $C$ is the set of all $x \in [W]^\delta$ closed under $F$. A set $X \subseteq [W]^\delta$ is said to be stationary in $[W]^\delta$ if $X \cap C \neq \emptyset$ for any club $C$ in $[W]^\delta$, i.e. for any function $F : \omega W \to W$ there exists $x \in X$ which is closed under $F$.

Let $\mathcal{M}$ be a structure such that there exists a well-ordering of its universe definable over $\mathcal{M}$, and suppose that $x \subseteq \mathcal{M}$. Then $\text{Sk}^\mathcal{M}(x)$ denotes the Skolem hull of $x$ in $\mathcal{M}$, i.e. the smallest $M$ with $x \subseteq M \prec \mathcal{M}$.

For a limit ordinal $\delta$, a set $M$ is said to be internally approachable (i.a.) of length $\delta$ if there exists a $\subseteq$-increasing sequence $\langle M_\xi \mid \xi < \delta \rangle$ such that $\bigcup_{\xi < \delta} M_\xi = M$ and such that $\langle M_\xi \mid \xi < \delta' \rangle \in M$ for any $\delta' < \delta$. A sequence $\langle M_\xi \mid \xi < \delta \rangle$ witnessing the internal approachability of $M$ is called an internally approaching (i.a.) sequence to $M$.

We use an ideal $I[\lambda]$ over a regular cardinal $\lambda \geq \omega_2$, which was introduced by Shelah [14]. First we recall the definition of $I[\lambda]$. Suppose that $\lambda \geq \omega_2$ and that $E \subseteq \lambda$. Then $E \in I[\lambda]$ if and only if there exist a sequence $\langle b_\alpha \mid \alpha < \lambda \rangle$ of bounded subsets of $\lambda$ and a club $C \subseteq \lambda$ such that for any limit ordinal $\alpha \in C \cap E$ we can take an unbounded $b \subseteq \alpha$ with $\text{otp}(b) = \text{cf}(\alpha)$ and $\{b \cap \beta \mid \beta < \alpha\} \subseteq \{b_\beta \mid \beta < \alpha\}$. We use the following fact:

**FACT 2.1 (Shelah [14]).** Let $\delta$ be a regular uncountable cardinal.

1. Suppose that $\lambda$ is a regular cardinal $> \delta$ and that $E$ is a stationary subset of $\mathcal{E}_\omega^\lambda_{<\delta}$ with $E \in I[\lambda]$. Then $E$ remains stationary in $V^\mathbb{P}$ for any $<\delta$-closed poset $\mathbb{P}$.
2. Suppose that $2^{<\delta} = \delta$. Then $\mathcal{E}_\delta^{\delta+} \in I[\delta+]$.

Next we give our notation on forcing.

Let $\mathbb{P}$ be a poset. We also let $\mathbb{P}$ denote the base set of $\mathbb{P}$. The order of $\mathbb{P}$ is denoted as $\leq_\mathbb{P}$, but we usually omit the subscript $\mathbb{P}$. A poset $\mathbb{Q}$ is said to be a suborder of $\mathbb{P}$ if $\mathbb{Q} \subseteq \mathbb{P}$ and $\leq_\mathbb{Q} = \leq_\mathbb{P} \cap (\mathbb{Q} \times \mathbb{Q})$.

A $\mathbb{P}$-name is a set consisting of pairs $(\dot{x}, p)$ such that $\dot{x}$ is a $\mathbb{P}$-name of lower rank and such that $p \in \mathbb{P}$. If $(\dot{x}, p)$ belongs to a $\mathbb{P}$-name $\dot{X}$, then $p$ forces that $\dot{x} \in \dot{X}$. For an ordinal $\kappa$ we say that $\dot{S}$ is a nice $\mathbb{P}$-name for a subset of $\kappa$ if there exists a sequence $\langle A_\alpha \mid \alpha < \kappa \rangle$ of antichains in $\mathbb{P}$ such that $\dot{S} = \{\langle \dot{\alpha}, p \rangle \mid p \in A_\alpha\}$.

For $A \subseteq \mathbb{P}$ and $p \in \mathbb{P}$ we say that $p$ meets $A$ if there is $q \in A$ with $q \geq p$. For $A_0, A_1 \subseteq \mathbb{P}$ we say that $A_0$ refines $A_1$ if all elements of $A_0$ meet $A_1$. 

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\( \mathbb{P} \) is said to be \( \omega_1 \)-stationary preserving if \( \omega_1^{\mathbb{P}} = \omega_1^V \) and every stationary subset of \( \omega_1 \) in \( V \) remains stationary in \( V^{\mathbb{P}} \).

Let \( \delta \) be a regular uncountable cardinal. We say that \( \mathbb{P} \) has the \( \delta \)-chain condition (\( \delta \)-c.c.) if there is no antichain \( A \) in \( \mathbb{P} \) with \( |A| = \delta \).

A subset \( A \subseteq \mathbb{P} \) is said to be directed if for any \( p, q \in A \) there exists \( r \in A \) with \( r \leq p, q \). Furthermore \( \mathbb{P} \) is said to be <\( \delta \)-directed closed if every directed \( A \subseteq \mathbb{P} \) with \( |A| < \delta \) has a lower bound in \( \mathbb{P} \).

\( \mathbb{P} \) is said to be <\( \delta \)-Baire if for any \( p \in \mathbb{P} \) and any family \( \mathcal{A} \) of maximal antichains in \( \mathbb{P} \) with \( |\mathcal{A}| < \delta \), there exists \( p^* \leq p \) which meets all \( A \in \mathcal{A} \). \( \mathbb{P} \) is <\( \delta \)-Baire if and only if a forcing extension by \( \mathbb{P} \) does not add any new sequences of ordinals of length \( \delta \). \( \mathbb{P} \) is said to be \( \sigma \)-Baire if \( \mathbb{P} \) is <\( \omega_1 \)-Baire.

For a regular cardinal \( \delta \) and an ordinal \( \kappa \geq \delta \) let \( \text{Col}(\delta, \kappa) \) denote the poset <\( \delta \kappa \) ordered by reverse inclusion. Moreover let \( \text{Col}(\delta, \kappa) \) be the <\( \delta \)-support product of \( \langle \text{Col}(\delta, \kappa') \mid \delta \leq \kappa' < \kappa \rangle \). Thus if \( \kappa \) is an inaccessible cardinal, then \( \text{Col}(\delta, \kappa) \) is the Lévy collapse forcing \( \kappa \) to be \( \delta^+ \). Furthermore for an ordinal \( \lambda > \kappa \) let \( \text{Col}(\delta, [\kappa, \lambda)) \) be the <\( \delta \)-support product of \( \langle \text{Col}(\delta, \kappa') \mid \kappa \leq \kappa' < \lambda \rangle \).

Next we give our notation and a basic fact on projections between posets. Let \( \mathbb{P} \) and \( \mathbb{Q} \) be posets. A map \( \pi : \mathbb{P} \to \mathbb{Q} \) which has the following properties is called a projection:

(i) \( \pi \) is surjective and order preserving.

(ii) For any \( p \in \mathbb{P} \) and any \( q \in \mathbb{Q} \) with \( q \leq_{\mathbb{Q}} \pi(p) \) there exists \( p^* \in \mathbb{P} \) such that \( p^* \leq_{\mathbb{P}} p \) and \( \pi(p^*) = q \).

Suppose that \( \pi : \mathbb{P} \to \mathbb{Q} \) is a projection and that \( H \) is a \( \mathbb{Q} \)-generic filter. Then, in \( V[H] \), \( \mathbb{P}/H \) denotes the poset obtained by restricting \( \mathbb{P} \) to \( \pi^{-1}[H] \). It is standard that \( \mathbb{Q} \ast (\mathbb{P}/H) \) is forcing equivalent to \( \mathbb{P} \), where \( H \) is the canonical \( \mathbb{Q} \)-name for a \( \mathbb{Q} \)-generic filter. (See Abraham [I] §1.)

Finally we present our notation and a fact on forcing axioms:

For a poset \( \mathbb{P} \) and an uncountable cardinal \( \delta \) let \( \text{FA}_\delta(\mathbb{P}) \) and \( \text{FA}_{\delta^+}(\mathbb{P}) \) be the following forcing axioms:

\[
\text{FA}_\delta(\mathbb{P}) \equiv \text{For any } p \in \mathbb{P} \text{ and any family } \mathcal{A} \text{ of maximal antichains in } \mathbb{P} \text{ with } |\mathcal{A}| \leq \delta \text{ there exists a filter } G \subseteq \mathbb{P} \text{ containing } p \text{ such that } G \cap A \neq \emptyset \text{ for all } A \in \mathcal{A}.
\]

\[
\text{FA}_{\delta^+}(\mathbb{P}) \equiv \text{For any } p \in \mathbb{P}, \text{ any family } \mathcal{A} \text{ of maximal antichains in } \mathbb{P} \text{ with } |\mathcal{A}| \leq \delta \text{ and any family } \mathcal{R} \text{ of } \mathbb{P} \text{-names for stationary subsets of } \delta \text{ with } |\mathcal{R}| \leq \delta \text{ there exists a filter } G \subseteq \mathbb{P} \text{ containing } p \text{ such that } G \cap A \neq \emptyset \text{ for all } A \in \mathcal{A} \text{ and such that } \dot{R}_G := \{ \xi < \delta \mid \exists p \in G, p \vdash "\xi \in \dot{R}\" \} \text{ is stationary in } \delta \text{ for all } \dot{R} \in \mathcal{R}.
\]
Recall that PFA is FA_{\omega_1} for all proper posets and that MM is FA_{\omega_1} for all \omega_1-stationary preserving posets. We let PFA^{++} denote FA^{++}_{\omega_1} for all proper posets. PFA^{++} was introduced by Baumgartner [3], and he observed that if \kappa is a supercompact cardinal, and \langle P_\alpha, Q_\beta | \alpha \leq \kappa, \beta < \kappa \rangle is the standard iteration for PFA, which was also introduced by him (see Devlin [4]), then this iteration in fact forces PFA^{++}.

Let \mathcal{P} be a poset and \mathcal{M} be a set. \mathcal{G} \subseteq \mathcal{P} \cap \mathcal{M} is called an \((\mathcal{M}, \mathcal{P})\)-generic filter if \mathcal{G} is a filter on \mathcal{P} \cap \mathcal{M} such that \mathcal{G} \cap A \neq \emptyset for every maximal antichain \(\mathcal{A} \in \mathcal{M}\) in \(\mathcal{P}\).

We use the following fact. (1) is proved in Woodin [18, proof of Thm. 2.53]:

**Lemma 2.2.** Suppose that \(\mathcal{P}\) is a poset, that \(\delta\) is an uncountable cardinal and that FA_\delta(\mathcal{P}) holds. Let \(p \in \mathcal{P}\), and let \(\theta\) be a regular cardinal \(> \delta\) with \(\mathcal{P} \in \mathcal{H}_\theta\).

1. There are stationary many \(\mathcal{M} \in [\mathcal{H}_\theta]^\delta\) with the following properties:
   i. \(\delta \subseteq \mathcal{M}\) and \(p \in \mathcal{M}\).
   ii. There exists an \((\mathcal{M}, \mathcal{P})\)-generic filter containing \(p\).

2. If \(\mathcal{P}\) is \(<\delta\)-Baire, then there are stationary many \(\mathcal{M} \in [\mathcal{H}_\theta]^\delta\) with the properties (i), (ii) above and the following:
   iii. \(\mathcal{M}\) is internally approachable of length \(\delta\).

3. If FA_\delta^{++}(\mathcal{P}) holds, then there are stationary many \(\mathcal{M} \in [\mathcal{H}_\theta]^\delta\) with the property (i) above and the following:
   iv. There exists an \((\mathcal{M}, \mathcal{P})\)-generic filter \(\mathcal{G}\) containing \(p\) such that \(\mathcal{R}_\mathcal{G} = \{ \xi < \delta | \exists q \in \mathcal{G}, q \Vdash "\xi \in \mathcal{R}" \}\) is stationary in \(\delta\) for any \(\mathcal{P}\)-name \(\mathcal{R} \in \mathcal{M}\) for a stationary subset of \(\delta\).

   If \(\mathcal{P}\) is \(<\delta\)-Baire in addition, then there are stationary many \(\mathcal{M} \in [\mathcal{H}_\theta]^\delta\) with the properties (i), (iii) and (iv).

In the proof of the above lemma we use the following well-known lemma:

**Lemma 2.3 (folklore).** Let \(\theta\) be a regular uncountable cardinal, \(\Delta\) be a well-ordering of \(\mathcal{H}_\theta\), and \(\mathcal{M}\) be a structure obtained by adding countable many constants, functions and predicates to \(\langle \mathcal{H}_\theta, \in, \Delta \rangle\). Suppose that \(\mathcal{M}\) is an elementary submodel of \(\mathcal{M}\) and that \(d \subseteq D \in \mathcal{M}\). Then

\[
\text{Sk}^\mathcal{M}(\mathcal{M} \cup d) = \{ f(b) | b \in {}^{<\omega}d, f : |b| D \to \mathcal{H}_\theta, f \in \mathcal{M} \}.
\]

**Proof.** Let \(N\) be the set on the right side. Then \(\text{Sk}^\mathcal{M}(\mathcal{M} \cup d) \supseteq N\) clearly. We prove the reverse inclusion. Before starting we prepare a notation. For each formula \(\varphi(u, v_0, \ldots, v_{m-1}, w_0, \ldots, w_{n-1})\) of the language for \(\mathcal{M}\) let \(h_\varphi : {}^{m+n}\mathcal{H}_\theta \to \mathcal{H}_\theta\) be the Skolem function for \(\varphi\) in \(\mathcal{M}\). That is, for any \(a = \langle a_0, \ldots, a_{m-1} \rangle \in {}^{m}\mathcal{H}_\theta\) and any \(b = \langle b_0, \ldots, b_{n-1} \rangle \in {}^{n}\mathcal{H}_\theta\), if there
exists $x$ with $M \models \varphi(x, a_0, \ldots, a_{m-1}, b_0, \ldots, b_{n-1})$, then $h_\varphi(a, b)$ is the $\Delta$-least such $x$, otherwise $h_\varphi(a, b) = 0$.

To show that $\text{Sk}^M(M \cup d) \subseteq N$, take an arbitrary $x \in \text{Sk}^M(M \cup d)$. Then there exists a formula $\varphi(u, v_0, \ldots, v_{m-1}, w_1, \ldots, w_{n-1})$, $a^* \in {}^nM$ and $b^* \in {}^n d$ such that $h_\varphi(a^*, b^*) = x$. Let $f : {}^n D \to \mathcal{H}_\theta$ be the function such that $f(b) = h_\varphi(a^*, b)$ for every $b \in {}^n D$. Then $f \in \mathcal{H}_\theta$, and $f$ is definable in $M$ from parameters $D$ and $a^*$, both of which are in $M$. Hence $f \in M$ by the elementarity of $M$. Moreover $x = f(b^*)$. Therefore $x \in N$. 

**Proof of Lemma 2.2.** Let $\Delta$ be a well-ordering of $\mathcal{H}_\theta$, and suppose that $M$ is a structure obtained by adding countably many constants, functions and predicates to $\langle \mathcal{H}_\theta, \in, \Delta, \mathbb{P}, p, \delta \rangle$. For (1) it suffices to find $M \in [\mathcal{H}_\theta]^{\delta}$ with the properties (i) and (ii) such that $M \prec M$. For (2) or (3) it suffices to find such $M$ with (iii) or (iv), respectively.

First we can take $N \in [\mathcal{H}_\theta]^{\delta}$ such that $N$ is i.a. of length $\delta$ and such that $N \prec M$. Here note that $\delta \subseteq N$. Let $A$ be the set of all maximal antichains in $\mathbb{P}$ which belong to $N$, and let $\mathcal{R}$ be the set of all $\mathbb{P}$-names in $N$ for stationary subsets of $\delta$. By $\text{FA}_\delta(\mathbb{P})$ take a filter $G \subseteq \mathbb{P}$ containing $p$ and intersecting all elements of $A$. Here note that if $\text{FA}_\delta^+(\mathbb{P})$ holds, then we can take $G$ so that $\dot{R}_G$ is stationary for all $\dot{R} \in \mathcal{R}$. For each $A \in A$ let $p_A$ be the unique element of $G \cap A$, and let $d := \{p_A : A \in A\}$. Moreover let $M := \text{Sk}^M(N \cup d)$. Clearly $\delta \cup \{p\} \subseteq M \prec M$. It suffices to prove the following:

(a) $g := G \cap M$ is an $(M, \mathbb{P})$-generic filter.
(b) If $\mathbb{P}$ is $<\delta$-Baire, then $M$ is i.a. of length $\delta$.
(c) If $\dot{R}_G$ is stationary in $\delta$ for every $\dot{R} \in \mathcal{R}$, then $\dot{R}_g$ is stationary in $\delta$ for every $\mathbb{P}$-name $\dot{R} \in M$ for a stationary subset of $\delta$.

(a) Let $A^* \in M$ be a maximal antichain in $\mathbb{P}$. We show that $g \cap A^* \neq \emptyset$.

By Lemma 2.3 there exist $b^* = \langle p_0^*, \ldots, p_{n-1}^* \rangle \in {}^{<\omega}d$ and a function $f : {}^n \mathbb{P} \to \mathcal{H}_\theta$ in $N$ such that $f(b^*) = A^*$. We may assume that $f(b)$ is a maximal antichain in $\mathbb{P}$ for every $b \in {}^n \mathbb{P}$.

For each $i < n$ take $A_i \in A$ with $p_i^* = p_{A_i}$. Let $K$ be the set of all $b \in \prod_{i < n} A_i$ which have a lower bound. Here we say that $b = \langle p_0, \ldots, p_{n-1} \rangle$ has a lower bound if $\{p_0, \ldots, p_{n-1}\}$ has a lower bound in $\mathbb{P}$. Note that if $b, b' \in K$ and $b \neq b'$, then $b$ and $b'$ have no common lower bound. This is because each $A_i$ is an antichain.

For each $b \in K$ let $A_b$ be the $\Delta$-least maximal antichain below $b$ which refines $f(b)$. Let $A^\circ := \bigcup_{b \in K} A_b$. Then it is easy to see that $A^\circ$ is a maximal antichain in $\mathbb{P}$ and that $A^\circ \in N$. That is, $A^\circ \in \mathcal{A}$.

Here note that $p_{A^\circ}$ must be in $A^\circ$. Otherwise $p_{A^\circ}$ is incompatible with at least one of $p_0^*, \ldots, p_{n-1}^*$, and this contradicts that all $p_{A^\circ}, p_0^*, \ldots, p_{n-1}^*$
belong to the filter $G$. Moreover recall that $A_{b^*}$ refines $f(b^*) = A^*$. Let $p^*$ be the unique element of $A^*$ with $p^* \geq p_{A^*}$.

Then $p^* \in G$ because $p^* \geq p_{A^*} \in G$. Moreover $p^* \in M$ because $p^*$ is definable from $p_{A^*}, A^* \in M$. Therefore $p^* \in g \cap A^* \neq \emptyset$.

(b) Let $\langle N_\xi \mid \xi < \delta \rangle$ be an i.a. sequence to $N$. We may assume that $|N_\xi| < \delta$ for each $\xi < \delta$. For each $\xi < \delta$ let $A_\xi := A \cap N_\xi, d_\xi := \{p_A \mid A \in A_\xi\}$ and

$$M_\xi := \{f(b) \mid b \in {<^\omega} d_\xi, f : [b]_P \rightarrow \mathcal{H}_\theta, f \in N_\xi\}.$$ 

We show that $\langle M_\xi \mid \xi < \delta \rangle$ is an i.a. sequence to $M$.

Clearly $\langle M_\xi \mid \xi < \omega_1 \rangle$ is $\subseteq$-increasing, and $\bigcup_{\xi<\omega_1} M_\xi = M$ by Lemma \[2.3\].

Thus it suffices to show that $\langle M_\xi \mid \xi < \zeta \rangle \in M$ for every $\zeta < \delta$. Here note that $\langle M_\xi \mid \xi < \zeta \rangle$ is definable in $\langle H_\theta, \in \rangle$ from parameters $P, \langle N_\xi \mid \xi < \zeta \rangle$ and $\langle d_\xi \mid \xi < \zeta \rangle$. Moreover $P, \langle N_\xi \mid \xi < \zeta \rangle \in N \subseteq M$. Therefore all we have to show is that $\langle d_\xi \mid \xi < \zeta \rangle \in M$ for every $\zeta < \delta$.

Fix $\zeta < \delta$. Because $P$ is $<\delta$-Baire, there exists a maximal antichain $A^*$ in $P$ which refines all maximal antichains in $A_\zeta$. We can take such $A^*$ in $N$ because $A_\zeta \in N$. Then for each $A \in A_\zeta, p_A$ is the unique $p \in A$ with $p \geq p_{A^*}$. Hence $d_\xi = \{p \in \bigcup A_\xi \mid p \geq p_A\}$ for each $\xi < \zeta$. Then $\langle d_\xi \mid \xi < \zeta \rangle \in M$ because $p_{A^*}, \langle A_\xi \mid \xi < \zeta \rangle \in M \setminus \langle H_\theta, \in \rangle$.

(c) Suppose that $\hat{R}_G$ is stationary in $\delta$ for all $\hat{R} \in R$. Take an arbitrary $P$-name $\hat{R}^* \in M$ for stationary subsets of $\delta$. We show that $\hat{R}_g^*$ is stationary in $\delta$.

By Lemma \[2.3\] there exist $b^* = \langle p_0^*, \ldots, p_{n-1}^* \rangle \in {^n}d$ and a function $f : {^n}P \rightarrow \mathcal{H}_\theta$ in $N$ such that $f(b^*) = \hat{R}^*$. We may assume that $f(b)$ is a $P$-name for a stationary subset of $\delta$ for every $b \in {^n}d$. Moreover take $A_i, i < n$, and $K$ as in the proof of (a) above.

Then we can take a $P$-name $\hat{R}^o \in N$ such that for any $b \in K$ all lower bounds of $b$ force that $\hat{R}^o = f(b)$. Recall that $f(b^*) = \hat{R}^*$, that $b^* = \langle p_0^*, \ldots, p_{n-1}^* \rangle$ and that $p_0^*, \ldots, p_{n-1}^* \in G$. Then it is easy to see that $\hat{R}_G^o = \hat{R}_g^*$. Moreover $\hat{R}_G^o$ is stationary in $\delta$ because $\hat{R}^o \in R$. Therefore $\hat{R}_g^*$ is stationary in $\delta$. ■

3. MM implies $\Box^P$. In this section we prove

**THEOREM [1.4]** MM implies $\Box^P$.

This will be done in §3.3 In the preceding subsections, we make preliminaries for the proof.

3.1. $\omega_1$-stationary preserving $\sigma$-Baire posets. In the proof of Theorem \[1.4\] we will construct an $\omega_1$-stationary preserving $\sigma$-Baire poset and apply MM to it. Here we present a sufficient condition for a poset to be
ω₁-stationary preserving and σ-Baire. For this we use the notions of projectively stationary sets and of strongly generic conditions:

**Definition 3.1 (Feng–Jech [5])**. Let \( W \) be a set with \( \omega_1 \subseteq W \). Then \( X \subseteq [W]^{\omega_1} \) is said to be **projectively stationary** if the set \{ \( x \in X \mid x \cap \omega_1 \in R \) \} is stationary in \([W]^{\omega_1}\) for every stationary \( R \subseteq \omega_1 \).

**Definition 3.2.** Let \( P \) be a poset and \( M \) be a set. Then \( p \in P \) is called a **strongly \((M, P)\)-generic condition** if \{ \( q \in P \cap M \mid q \geq p \) \} is an \((M, P)\)-generic filter.

Note that if \( p \) is a strongly \((M, P)\)-generic condition, then \( p \) meets \( A \cap M \) for every maximal antichain \( A \in M \) in \( P \).

Now we give a sufficient condition:

**Lemma 3.3.** Let \( P \) be a poset. Suppose that \( P \) satisfies the following:

\((\ast)\) For every sufficiently large regular cardinal \( \theta \) and every \( p \in P \) the following set is projectively stationary:

\[ \{ M \in [H_\theta]^{\omega_2} \mid \text{a strongly (}M, P\text{)-generic condition below } p \text{ exists} \}. \]

Then \( P \) is \( \omega_1 \)-stationary preserving and σ-Baire.

**Proof.** Assume \((\ast)\). Let \( \theta \) be a sufficiently large regular cardinal.

To show that \( P \) is σ-Baire, suppose that \( p \in P \) and that \( A \) is a countable family of maximal antichains in \( P \). By \((\ast)\) we can take \( M \in [H_\theta]^{\omega_1} \) and \( p^* \leq p \) such that \( A \cup \{ p \} \subseteq M \prec \langle H_\theta, \in \rangle \) and such that \( p^* \) is a strongly \((M, P)\)-generic condition. Then \( p^* \leq p \), and \( p^* \) meets all elements of \( A \). This completes the proof of the σ-Baireness.

Next, to prove that \( P \) is \( \omega_1 \)-stationary preserving, arbitrarily take \( p \in P \), a stationary \( R \subseteq \omega_1 \) and a \( \mathbb{P} \)-name \( \dot{C} \) for a club in \( \omega_1^V \). It suffices to find \( p^* \leq p \) and \( \xi \in R \) such that \( p^* \vdash \“ \xi \in \dot{C} \” \).

By \((\ast)\) we can take \( M \in [H_\theta]^{\omega_1} \) and \( p^* \leq p \) such that \( P, p, R, \dot{C} \in M \prec \langle H_\theta, \in \rangle \), such that \( M \cap \omega_1 \in R \) and such that \( p^* \) is a strongly \((M, \mathbb{P})\)-generic condition. Let \( \xi := M \cap \omega_1 \). Then \( \xi \in R \), and \( p^* \vdash \“ \xi \in \dot{C} \” \) by the standard argument.

**3.2. Variant of diamond principle in \([\omega_2]^{\omega_1}\).** In the proof of Theorem 1.4 we use a certain diamond principle in \([\omega_2]^{\omega_1}\). Here we prove that \( \text{MM} \) implies it.

For \( X \subseteq [\omega_2]^{\omega_1} \) we say that \( \sup \downarrow X \text{ is injective} \) if \( \sup x \neq \sup y \) for any distinct \( x, y \in X \).

**Lemma 3.4.** Assume \( \text{MM} \). Let \( S \) be a stationary subset of \( E_0^2 \). Then there are \( X \subseteq [\omega_2]^{\omega_1} \) and a sequence \( \langle B_x \mid x \in X \rangle \) with the following properties:

(i) \( \sup x \notin x \) for each \( x \in X \), \( \{ \sup x \mid x \in X \} = S \), and \( \sup \downarrow X \) is injective.
(ii) $B_x$ is a countable family of subsets of $x$ for each $x \in X$.

(iii) For every sufficiently large regular cardinal $\theta$, the set of all $M \in [\mathcal{H}_\theta]^\omega$ such that

- $M \cap \omega_2 \in X$,
- $B_{M \cap \omega_2} = \{B \cap M \mid B \in \mathcal{P}(\omega_2) \cap M\}$

is projectively stationary.

Lemma 3.4 follows from Facts 3.5, 3.6 and Lemma 3.7 below:

**FACT 3.5** (Foreman–Magidor–Shelah [6]). MM implies that $2^{\omega_1} = \omega_2$.

**FACT 3.6** (Shelah [16]). If $2^{\omega_1} = \omega_2$, then $\diamondsuit_{\omega_2}(S)$ holds for every stationary $S \subseteq \mathcal{E}_0^3$.

**LEMMA 3.7.** Suppose that $S$ is a stationary subset of $\mathcal{E}_0^3$ and that $\diamondsuit_{\omega_2}(S)$ holds. Then there exist $X$ and $\langle B_x \mid x \in X \rangle$ satisfying (i)–(iii) in Lemma 3.4.

We prove Lemma 3.7. For this we need the following result:

**LEMMA 3.8.** Suppose that $S$ is a stationary subset of $\mathcal{E}_0^3$ and that $\diamondsuit_{\omega_2}(S)$ holds. Then there exist $X \subseteq [\omega_2]^\omega$ and a sequence $\langle b_x \mid x \in X \rangle$ such that:

(i) $\sup x \notin x$ for each $x \in X$, $\{\sup x \mid x \in X\} = S$, and $\sup \{x \mid x \in X\}$ is injective.

(ii) $b_x \subseteq x$ for each $x \in X$.

(iii) For every $B \subseteq \omega_2$ the set $\{x \in X \mid b_x = B \cap x\}$ is projectively stationary.

**Proof.** We may assume that $S \subseteq \mathcal{E}_0^3 \setminus \omega_1$. By $\diamondsuit_{\omega_2}(S)$ there exists a sequence $\langle R_\alpha, f_\alpha, b'_\alpha \mid \alpha \in S \rangle$ with the following properties:

- For each $\alpha \in S$, $R_\alpha$ is a stationary subset of $\omega_1$, $f_\alpha$ is a function from $\vartriangleleft_{\omega_1} \alpha$ to $\alpha$, and $b'_\alpha \subseteq \alpha$.

- If $R$ is a stationary subset of $\omega_1$, $F$ is a function from $\vartriangleleft_{\omega_2} \omega_2$ to $\omega_2$, and $B \subseteq \omega_2$, then there exists $\alpha \in S$ such that $R_\alpha = R$, $f_\alpha = F|_{\vartriangleleft_{\omega_1} \alpha}$ and $b'_\alpha = B \cap \alpha$.

For each $\alpha \in S$, take $x_\alpha \in [\alpha]^\omega$ such that $\sup x_\alpha = \alpha$, $x_\alpha \cap \omega_1 \in R_\alpha$ and $x_\alpha$ is closed under $f_\alpha$. We can take such $x_\alpha$ because $\alpha \in \mathcal{E}_0^3 \setminus \omega_1$ and $R_\alpha$ is stationary. Let $X := \{x_\alpha \mid \alpha \in S\}$. Moreover let $b_x := b'_{\sup x} \cap x$ for each $x \in X$. (Hence $b_{x_\alpha} = b'_\alpha \cap x_\alpha$ for each $\alpha \in S$.)

We show that these $X$ and $\langle b_x \mid x \in X \rangle$ witness the lemma. Clearly they satisfy (i) and (ii). We check (iii).

Fix $B \subseteq \omega_2$. It suffices to show that for every stationary $R \subseteq \omega_1$ and every function $F : \vartriangleleft_{\omega_2} \omega_2 \to \omega_2$ there exists $x \in X$ such that $x \cap \omega_1 \in R$, $x$ is closed under $F$ and $b_x = B \cap x$.

Take an arbitrary stationary $R \subseteq \omega_1$ and an arbitrary function $F : \vartriangleleft_{\omega_2} \omega_2 \to \omega_2$. Then there exists $\alpha \in S$ with $R_\alpha = R$, $f_\alpha = F|_{\vartriangleleft_{\omega_1} \alpha}$ and
Thus by the choice of \( x_\alpha \cap \omega_1 \in R, x_\alpha \) is closed under \( F \), and \( b_{x_\alpha} = b'_\alpha \cap x_\alpha = B \cap x_\alpha \) by the choice of \( x_\alpha \). Moreover \( x_\alpha \in X \). Hence \( x_\alpha \) is as desired. □

**Proof of Lemma 3.7.** Before starting we prepare a notation. For each \( D \subseteq \text{On} \times \text{On} \) and each \( \gamma \in \text{On} \), let \( D(\gamma) \) denote the set \( \{ \beta \in \text{On} \mid \langle \gamma, \beta \rangle \in D \} \).

Now we start the proof. By Lemma 3.8 we can take \( X \subseteq [\omega_2]^\omega \) and a sequence \( \langle d_x \mid x \in X \rangle \) such that:

(i) \( \sup x \neq x \) for each \( x \in X \), \( \{ \sup x \mid x \in X \} = S \), and \( \sup \}X \) is injective.

(ii) \( d_x \subseteq x \times x \).

(iii) For every \( D \subseteq \omega_2 \times \omega_2 \) the set \( \{ x \in X \mid d_x = D \cap (x \times x) \} \) is projectively stationary.

For each \( x \in X \) let \( B_x = \{ d_x(\gamma) \mid \gamma \in \} \}. \) We show that \( X \) and \( \langle B_x \mid x \in X \rangle \) witness Lemma 3.7. Clearly (i) and (ii) in Lemma 3.4 hold. We check (iii).

Let \( \theta \) be a sufficiently large regular cardinal. Take an arbitrary stationary \( R \subseteq \omega_1 \) and an arbitrary function \( F : [\omega_1]^{\omega} \rightarrow \mathcal{H}_\theta \). It suffices to find \( M \in [\mathcal{H}_\theta]^{\omega} \) such that \( M \cap \omega_1 \in R, M \) is closed under \( F \), \( M \cap \omega_2 \in X \) and \( \mathcal{B}_M \cap \omega_2 = \{ B \cap M \mid B \in \mathcal{P}(\omega_2) \cap M \} \).

First take \( N \subseteq \mathcal{H}_\theta \) such that \( |N| = \omega_2 \subseteq N, \mathcal{P}(\omega_2) \cap N \neq \emptyset \) and \( N \) is closed under \( F \). Moreover take an enumeration \( \langle B_\gamma \mid \gamma \in \omega_2 \rangle \) of \( \mathcal{P}(\omega_2) \cap N \). For each \( x \in [\omega_2]^\omega \) let

\[
M_x := \text{cl}_F(x \cup \{ B_\gamma \mid \gamma \in x \}) \subseteq N,
\]

where \( \text{cl}_F(a) \) denotes the closure of \( a \) under \( F \). Then let \( C \) be the set of all \( x \in [\omega_2]^\omega \) with \( M_x \cap \omega_2 = x \) and \( \mathcal{P}(\omega_2) \cap M_x = \{ B_\gamma \mid \gamma \in x \} \). Finally let \( D \) be a subset of \( \omega_2 \times \omega_2 \) such that \( D(\gamma) = B_\gamma \) for each \( \gamma \in \omega_2 \).

Note that \( C \) is a club in \( [\omega_2]^\omega \). Hence, by (iii'), there exists \( x \in X \cap C \) such that \( x \cap \omega_1 \in R \) and \( d_x = D \cap (x \times x) \). Then \( M_x \in [\mathcal{H}_\theta]^{\omega}, M_x \cap \omega_2 = x \in X, M_x \cap \omega_1 = x \cap \omega_1 \in R \), and \( M_x \) is closed under \( F \). Moreover

\[
\mathcal{B}_{M_x \cap \omega_2} = \mathcal{B}_x = \{ d_x(\gamma) \mid \gamma \in x \} = \{ D(\gamma) \cap x \mid \gamma \in x \} = \{ B_\gamma \cap x \mid B \in \mathcal{P}(\omega_2) \cap M_x \} = \{ B \cap M_x \mid B \in \mathcal{P}(\omega_2) \cap M_x \}.
\]

Thus \( M_x \) is as desired. □

**3.3. Proof of Theorem 1.4.** Before proving the theorem we present a poset to which we apply \( \text{MM} \):

**Definition 3.9.** Suppose that \( S \subseteq E_0^2 \) and that \( \vec{c} = \langle c_\alpha \mid \alpha \in S \rangle \) is a \( \square(S) \)-sequence. \((S \text{ may be bounded in } \omega_2. \) Then let \( \mathcal{P}(\vec{c}) \) be the following poset:
\begin{itemize}
\item $\mathbb{P}(\vec{c}) = S$.
\item $\alpha \leq_{\mathbb{P}(\vec{c})} \beta$ if and only if $\beta \in \text{Lim}(c_\alpha) \cup \{\alpha\}$ for each $\alpha, \beta \in S$.
\end{itemize}

For $g \subseteq \mathbb{P}(\vec{c})$ let

$$c_g := \bigcup_{\alpha \in g} c_\alpha.$$

Note that $\alpha \leq_{\mathbb{P}(\vec{c})} \beta$ if and only if $c_\beta$ is an initial segment of $c_\alpha$. The following lemma can be easily proved. The proof is left to the reader.

**Lemma 3.10.** Let $S$ be a subset of $E_0^2$ and $\vec{c} = \langle c_\alpha \mid \alpha \in S \rangle$ be a $\square(S)$-sequence.

1. Suppose that $g$ is a filter on $\mathbb{P}(\vec{c})$. Then $c_g$ is a club in $\sup c_g$ of order type $\leq \omega_1$. Moreover, if $\beta \in \text{Lim}(c_g)$, then $\beta \in S$, and $c_\beta = c_g \cap \beta$.
2. Suppose that the following (**) holds:

   $$(**) \mathbb{P}(\vec{c}) \setminus \gamma \text{ is dense in } \mathbb{P}(\vec{c}) \text{ for every } \gamma < \omega_2.$$

   Let $\theta$ be a sufficiently large regular cardinal and $M$ be an elementary submodel of $\langle \mathcal{H}_\theta, \in, \vec{c} \rangle$. Suppose also that $g$ is an $(M, \mathbb{P}(\vec{c}))$-generic filter. Then $\sup c_g = \sup(M \cap \omega_2)$.

**Proof of Theorem 1.4.** Assume MM. We want to prove that $\square(S)$ holds for some $S \subseteq \text{Lim}(\omega_2)$ with $S \cap E_1^2$ stationary.

Our proof is composed of two steps. First we construct a $\square(E_0^2)$-sequence $\vec{c} = \langle c_\alpha \mid \alpha \in E_0^2 \rangle$ so that $\mathbb{P}(\vec{c})$ satisfies (*) in Lemma 3.3 and (**) in Lemma 3.10. Then, using Lemma 2.2, we show that $\vec{c}$ can be extended to a $\square(S)$-sequence for some $S \subseteq \text{Lim}(\omega_2)$ with $S \cap E_1^2$ stationary.

**Step 1:** Construction of $\vec{c} = \langle c_\alpha \mid \alpha \in E_0^2 \rangle$. First take a stationary partition $\langle T_\beta \mid \beta \in E_0^2 \rangle$ of $E_0^2$, i.e. $\langle T_\beta \mid \beta \in E_0^2 \rangle$ is a pairwise disjoint sequence of stationary subsets of $E_0^2$ such that $\bigcup \{T_\beta \mid \beta \in E_0^2\} = E_0^2$. By Lemma 3.4, for each $\beta \in E_0^2$ we can take $X_\beta \subseteq [\omega_2]^{\omega}$ and $\langle B^\beta_x \mid x \in X_\beta \rangle$ with the following properties:

1. $\sup x \notin x$ for each $x \in X_\beta$, $\{\sup x \mid x \in X_\beta\} = T_\beta$, and $\sup X_\beta$ is injective.
2. $B^\beta_x$ is a countable family of subsets of $x$ for each $x \in X_\beta$.
3. For every sufficiently large regular cardinal $\theta$ the set of all $M \in [\mathcal{H}_\theta]^{\omega}$ such that
   - $M \cap \omega_2 \in X_\beta$,
   - $B^\beta_{M \cap \omega_2} = \{B \cap M \mid B \in \mathcal{P}(\omega_2) \cap M\}$
   is projectively stationary.

By induction on $\alpha \in E_0^2$ we construct a $\square(E_0^2)$-sequence $\vec{c} = \langle c_\alpha \mid \alpha \in E_0^2 \rangle$. Suppose that $\alpha \in E_0^2$ and that $\langle c_\beta \mid \beta \in E_0^2 \cup \alpha \rangle$ has been
defined to be a □(E^2_0 ∩ α)-sequence. Then take c_α as follows: First let
\[ β_α := \text{the unique element of } E^2_0 \text{ with } α ∈ T_βα, \]
x_α := the unique element of X_βα with sup x_α = α.
If β_α ∉ x_α or there exists β ∈ E^2_0 ∩ x_α with Lim(c_β) ∉ x_α, then let c_α be an
arbitrary unbounded subset of α of order type ω.
Suppose that β_α ∈ x_α and that Lim(c_β) ⊆ x_α for every β ∈ E^2_0 ∩ x_α.
Then note that \( \langle c_β | β ∈ E^2_0 ∩ x_α \rangle \) is a □(E^2_0 ∩ x_α)-sequence. Let
\[ P_α := P(\langle c_β | β ∈ E^2_0 ∩ x_α \rangle). \]
Note also that β_α ∈ P_α ⊆ x_α.
Recall that B^{β_α}_x is a countable family of subsets of x_α. Hence we can take
a filter g_α on P_α such that:
(iv) β_α ∈ g_α.
(v) g_α ∩ b ≠ ∅ for every b ∈ B^{β_α}_x, which is a maximal antichain in P_α.
If sup c_α = α, then let c_α := c_α. Otherwise, take an unbounded c ⊆ α such
that otp(c) = ω and β_α = min c, and let c_α := c_β ∪ c.
This completes the choice of c_α. Using Lemma 3.10(1), it is easy to check
that \( \langle c_β | β ∈ E^2_0 ∩ α + 1 \rangle \) is a □(E^2_0 ∩ α + 1)-sequence. Note that if β_α ∈ x_α
and Lim(c_β) ⊆ x_α for every β ∈ E^2_0 ∩ x_α, then β_α ∈ Lim(c_α).
Now we have constructed a □(E^2_0)-sequence \( \vec{c} = \langle c_α | α ∈ E^2_0 \rangle \).
We show that P(\vec{c}) satisfies (\*) and (\**):

CLAIM 1. P(\vec{c}) satisfies (\**) in Lemma 3.10.

Proof. Take an arbitrary β* ∈ E^2_0 and an arbitrary γ* < ω_2. We must
find α* ∈ E^2_0 \ γ* with α* ≤ P(\vec{c}) β*.
Let θ be a sufficiently large regular cardinal. Because X_β* is stationary in
[ω_2]_θ, we can take M ⊥ \( \mathcal{H}_θ, ε, \vec{c} \) such that β*, γ* ∈ M and M \cap ω_2 ∈ X_β*.
Let α* := sup(M \cap ω_2). Clearly α* ∈ E^2_0 \ γ*.
Note that α* ∈ T_β* so β_α* = β*. Note also that x_α* = M \cap ω_2. Hence
β_α* ∈ x_α* by the choice of M. Moreover Lim(c_β) ⊆ x_α* for every β ∈ E^2_0 ∩ x_α*
because M ⊥ \( \mathcal{H}_θ, ε, \vec{c} \) and each c_β is a countable set. Then
β* = β_α* ∈ Lim(c_α*) by the choice of c_α*. Thus α* ≤ P(\vec{c}) β*.

CLAIM 2. P(\vec{c}) satisfies (\*) in Lemma 3.3.

Proof. Suppose that θ is a sufficiently large regular cardinal and that
β* ∈ E^2_0 = P(\vec{c}). We prove that there are projectively stationary many
M ∈ [\mathcal{H}_θ]_θ such that a strongly (M, P(\vec{c}))-generic condition below β* exists.
Let Y be the set of all M ∈ [\mathcal{H}_θ]_θ such that:
(vi) β*, \vec{c} ∈ M ⊥ \( \mathcal{H}_θ, ε \).
(vii) M \cap ω_2 ∈ X_β*.
(viii) B^{β}_M = \{ B \cap M | B ∈ P(ω_2) \cap M \}.
Then $Y$ is projectively stationary in $[\mathcal{H}_\theta]^\omega$ by (iii). It suffices to show that $\sup(M \cap \omega_2)$ is a strongly $(M, \mathbb{P}(\bar{c}))$-condition below $\beta^*$ for each $M \in Y$.

Fix $M \in Y$, and let $\alpha^* := \sup(M \cap \omega_2)$. Then $\beta_{\alpha^*} = \beta^*$, and $x_{\alpha^*} = M \cap \omega_2$. Hence $\beta_{\alpha^*} \in x_{\alpha^*}$, and $\text{Lim}(c_\beta) \subseteq x_{\alpha^*}$ for each $\beta \in E_0^2 \cap x_{\alpha^*}$. Note also that $\mathbb{P}(\alpha^*) = \mathbb{P}(\bar{c}) \cap M$. So $g_{\alpha^*}$ is a filter on $\mathbb{P}(\bar{c}) \cap M$ containing $\beta^*$ by (iv). Moreover $g_{\alpha^*}$ is an $(M, \mathbb{P}(\bar{c}))$-generic filter by (v) and (viii).

Here note that $sup(c_{\alpha^*}) = sup(M \cap \omega_2) = \alpha^*$ by Lemma 3.10(1) and Claim 1. Hence $c_{\alpha^*} = c_{g_{\alpha^*}}$, and so $g_{\alpha^*} = \{ \beta \in \mathbb{P}(\bar{c}) \cap M \mid \beta \geq_{\mathbb{P}(\bar{c})} \alpha^* \}$. Therefore $\alpha^*$ is a strongly $(M, \mathbb{P}(\bar{c}))$-generic condition below $\beta^*$. \[\boxed{\text{Claim 2}}\]

**Step 2:** Extension of $\bar{c}$. Let $\theta$ be a sufficiently large regular cardinal, and let $Z$ be the set of all $N \in [\mathcal{H}_\theta]^\omega_1$ such that:

- (ix) $N \prec \langle \mathcal{H}_\theta, \in, \bar{c} \rangle$.
- (x) $N$ is i.a. of length $\omega_1$.
- (xi) There exists an $(N, \mathbb{P}(\bar{c}))$-generic filter.

By Claim 2 and Lemma 3.3, $\mathbb{P}(\bar{c})$ is $\omega_1$-stationary preserving and $\sigma$-Baire. Hence $Z$ is stationary in $[\mathcal{H}_\theta]^\omega_1$ by MM and Lemma 2.2.

Note that $sup(N \cap \omega_2) \in E_1^2$ for each $N \in Z$ because $N$ is i.a. of length $\omega_1$. Hence $S' := \{ sup(N \cap \omega_2) \mid N \in Z \}$ is a stationary subset of $E_1^2$.

For each $\alpha \in S'$ choose $N_\alpha \in Z$ with $sup(N_\alpha \cap \omega_2) = \alpha$ and an $(N_\alpha, \mathbb{P}(\bar{c}))$-generic filter $g_\alpha$. Moreover let $c_\alpha := c_{g_\alpha}$ for each $\alpha \in S'$. Note that $sup(c_\alpha) = \alpha$ by Claim 1 and Lemma 3.10(2). Then, by Lemma 3.10(1), $c_\alpha$ is a club of $\alpha$ of order type $\omega_1$, and if $\beta \in \text{Lim}(c_\alpha)$, then $\beta \in E_0^2$, and $c_\beta = c_\alpha \cap \beta$.

Now let $S := E_0^2 \cup S'$. Then $S \cap E_1^1 = S'$ is stationary. Moreover $\langle c_\alpha \mid \alpha \in S \rangle$ is a $\Box(S)$-sequence. \[\boxed{\text{Step 2}}\]

This completes the proof of Theorem 1.4. \[\boxed{\text{4. Strong stationary reflection principle.}}\]

In §5 and §6 we will discuss the consistency of the failure of the partial square. For this we will use the following stationary reflection principle, which implies the failure of the partial square:

**Definition 4.1.** Let $\kappa$ be a successor cardinal of some regular uncountable cardinal $\delta$. Then let

$\text{OSR}^*_\kappa \equiv \text{For any stationary } S \subseteq \mathcal{E}_\kappa^\delta \text{ there exists a club } C \subseteq \kappa \text{ such that } S \cap \alpha \text{ is stationary in } \alpha \text{ for any } \alpha \in C \cap \mathcal{E}_\delta^\kappa.$

$\text{OSR}^*_\omega_2$ was introduced by Magidor [12], and $\text{OSR}^*_\kappa$ is its straightforward generalization.

First we prove that $\text{OSR}^*_\delta \equiv$ implies the failure of $\Box^0_\delta$. This can be shown by the same argument as the fact that the usual stationary reflection principle
for stationary subsets of \(E^2_0\) implies the failure of \(\square_{\omega_1}\), which is also shown in [12]:

**Lemma 4.2.** Let \(\delta\) be a regular uncountable cardinal, and let \(\kappa := \delta^+\). Then \(\text{OSR}^*_\kappa\) implies the failure of \(\square^p_\delta\).

**Proof.** Assume \(\square^p_\delta\). We prove that \(\text{OSR}^*_\kappa\) fails.

Let \(\langle c_\alpha \mid \alpha \in S \rangle\) be a witness of \(\square^p_\delta\). First note that \(S \cap E^\kappa_\delta\) is stationary in \(\kappa\) because \(S \cap E^\kappa_\delta\) is stationary in \(\kappa\), and \(S \cap \alpha\) contains a club \(c_\alpha \subseteq \alpha\) for any \(\alpha \in S \cap E^\kappa_\delta\). Moreover the correspondence \(\beta \mapsto \text{otp}(c_\beta)\) is regressive on \(S \cap E^\kappa_\delta\). Hence by Fodor’s lemma there exist \(\xi^*\) and a stationary \(S^* \subseteq S \cap E^\kappa_\delta\) such that \(\text{otp}(c_\beta) = \xi^*\) for all \(\beta \in S^*\).

Note that \(\text{Lim}(c_\alpha) \cap S^*\) contains at most one element, the \(\xi^*\)th element of \(c_\alpha\), for any \(\alpha \in S \cap E^\kappa_\delta\). Thus \(S^* \cap \alpha\) is nonstationary in \(\alpha\) for all \(\alpha \in S \cap E^\kappa_\delta\).

Therefore \(S^*\) witnesses the failure of \(\text{OSR}^*_\kappa\).

In the rest of this section we discuss the construction of models of \(\text{OSR}^*_\kappa\).

Magidor [12] proved that, after a weakly compact cardinal is Lévy collapsed to \(\omega_2\), there exists a \(<\omega_2\)-Baire \(\omega_3\)-c.c. poset forcing \(\text{OSR}^*_\omega_2\). If a weakly compact cardinal is Lévy collapsed to \(\omega_2\), then \(\text{FA}^{++}\) holds for all \(\sigma\)-closed posets of size \(\leq \omega_2 = 2^{\omega_1}\) (see Lemma 6.2 below). In fact this forcing axiom implies the existence of a \(<\omega_2\)-Baire \(\omega_3\)-c.c. poset forcing \(\text{OSR}^*_\omega_2\). Here we prove the following proposition which generalizes this fact:

**Proposition 4.3.** Let \(\delta\) be a regular uncountable cardinal. Assume that \(\text{FA}^{++}\) holds for all \(<\delta\)-directed closed posets of size \(\leq 2^\delta\) and that \(E^\delta_{<\delta} \in I[\delta^+]\). Then there is a \((2^\delta)^+-\)c.c. \(<\delta^+-\)Baire \(<\delta\)-directed closed poset which forces \(\text{OSR}^*_\delta^+\).

Note that if \(\delta = \omega_1\), then \(E_{<\delta}^\delta = E_0^2 \in I[\omega_2] = I[\delta^+]\). Thus the second assumption of the proposition holds if \(\delta = \omega_1\).

The rest of this section is devoted to the proof of Proposition 4.3. Fix a regular uncountable cardinal \(\delta\), and let \(\kappa := \delta^+\).

The poset forcing \(\text{OSR}^*_\kappa\) will be one of the following \(\text{OSR}^*_\kappa\)-iterations, which are essentially \(\delta\)-support iterations of club shootings through \(\kappa\):

**Definition 4.4.** We say that \(\langle B_\mu, \dot{S}_\nu \mid \mu \leq \nu, \nu < \nu \rangle\) is an \(\text{OSR}^*_\kappa\)-iteration of length \(v\) if it satisfies the following:

(i) Each \(B_\mu\) is a poset, and each \(\dot{S}_\nu\) is a nice \(B_\nu\)-name for a stationary subset of \((E_{<\delta}^\kappa)^V\).

(ii) Each \(B_\mu\) is the poset of all partial functions \(p\) on \(\mu\) such that:

- \(|\text{dom}(p)| < \kappa\).
- \(p|\nu \in B_\nu\) for all \(\nu < \mu\).
Partial square is implied by MM but not by PFA

- If $\nu \in \text{dom}(p)$, then $p(\nu)$ is a closed bounded subset of $\kappa$, and
  $p|\nu \models " \hat{S}_\nu \cap \alpha \text{ is stationary} "$ for any $\alpha \in p(\nu) \cap \mathcal{E}_\delta^\kappa$.

(iii) For $p,q \in \mathbb{B}_\mu$, $p \leq q$ if and only if $\text{dom}(p) \supseteq \text{dom}(q)$, and $p(\nu)$ end-extends $q(\nu)$ for every $\nu \in \text{dom}(q)$.

If $\langle \mathbb{B}_\mu, \hat{S}_\nu \mid \mu \leq \nu, \nu < \nu \rangle$ is an OSR\textsubscript{$\mu$}-iteration, then $\mathbb{B}_\nu$ is also called an OSR\textsubscript{$\kappa$}-iteration.

First we present basic properties of OSR\textsubscript{$\kappa$}-iterations:

**Lemma 4.5.** All OSR\textsubscript{$\kappa$}-iterations have the $(2^\delta)^+\text{-c.c.}$

*Proof.* Suppose that $A$ is a subset of $\mathbb{B}_\nu$ with $|A| = (2^\delta)^+$. We find distinct $p,q \in A$ which are compatible.

By the $\Delta$-system lemma we may assume that there is $e \subseteq \nu$ of size $\leq \delta$ such that $\text{dom}(p) \cap \text{dom}(q) = e$ for all distinct $p,q \in A$. Note that $|\{p|e \mid p \in A\}| \leq 2^\delta < |A|$. Hence we can take distinct $p,q \in A$ such that $p|e = q|e$. Then $p \cup q$ is a common extension of $p$ and $q$. ■

Next we examine the closure property of OSR\textsubscript{$\kappa$}-iterations. It is easy to see that OSR\textsubscript{$\kappa$}-iterations are $<\delta$-directed closed. Below we prove a more general fact for the later use. More precisely, we prove that appropriate complete suborders of OSR\textsubscript{$\kappa$}-iterations and their quotients of OSR\textsubscript{$\kappa$}-iterations are both $<\delta$-directed closed:

**Definition 4.6.** Suppose that $\mathbb{H} := \langle \mathbb{B}_\mu, \hat{S}_\nu \mid \mu \leq \nu, \nu < \nu \rangle$ is an OSR\textsubscript{$\mu$}-iteration. $U \subseteq \nu$ is said to be a $\mathbb{H}$-complete subset of $\nu$ if for any $\nu \in U$ and any $(\check{\alpha}, p) \in \hat{S}_\nu$ we have $\text{dom}(p) \subseteq U$. For a $\mathbb{B}$-complete $U \subseteq \nu$ and each $\mu \leq \nu$ let $\mathbb{B}_{\mu,U}$ be the suborder of $\mathbb{B}_\mu$ such that

$$\mathbb{B}_{\mu,U} = \{ p \in \mathbb{B}_\mu \mid \text{dom}(p) \subseteq U \}.$$ 

**Lemma 4.7.** Suppose that $\mathbb{H} := \langle \mathbb{B}_\mu, \hat{S}_\nu \mid \mu \leq \nu, \nu < \nu \rangle$ is an OSR\textsubscript{$\kappa$}-iteration and that $U$ is a $\mathbb{H}$-complete subset of $\nu$. Then the following hold for every $\mu \leq \nu$:

1. $p|U \in \mathbb{B}_{\mu,U}$ for every $p \in \mathbb{B}_\mu$.
2. The map $p \mapsto p|U$ is a projection from $\mathbb{B}_\mu$ to $\mathbb{B}_{\mu,U}$.
3. $\mathbb{B}_{\mu,U}$ is $<\delta$-directed closed.
4. $\|\mathbb{B}_{\mu,U} \quad \text{"}\mathbb{B}_\mu/\hat{H}_{\mu,U} \text{ is } <\delta\text{-directed closed"", where } \hat{H}_{\mu,U} \text{ is the canonical } \mathbb{B}_{\mu,U}\text{-name for a } \mathbb{B}_{\mu,U}\text{-generic filter.}$

*Proof.* We show the lemma by induction on $\mu \leq \nu$. Suppose that $\mu \leq \nu$ and that the conclusion holds for all $\nu < \mu$. We prove it for $\mu$.

1. Take an arbitrary $p \in \mathbb{B}_\mu$. It suffices to show that $r := p|U \in \mathbb{B}_\mu$.

First suppose that $\mu$ is a limit ordinal. To see that $r \in \mathbb{B}_\mu$, it suffices to show that $r|\nu \in \mathbb{B}_\nu$ for all $\nu < \mu$ and that $|\text{dom}(r)| < \kappa$. The latter is clear.
because $r$ is a restriction of $p$ which belongs to $\mathbb{B}_\mu$. The former easily follows from the induction hypothesis (1) for $\nu < \mu$.

Next suppose that $\mu$ is a successor ordinal, and let $\nu := \mu - 1$. Then note that $r|\nu \in \mathbb{B}_\nu$. This implies that $r \in \mathbb{B}_\mu$ if $\nu \notin \text{dom}(r)$. So suppose that $\nu \in \text{dom}(r)$. It suffices to show that $r|\nu \models_{\mathbb{B}_\nu} \ " \hat{S}_\nu \cap \alpha \text{ is stationary} "$ for all $\alpha \in r(\nu) \cap \mathcal{E}_\delta^\kappa$.

Fix $\alpha \in r(\nu) \cap \mathcal{E}_\delta^\kappa$. First note that $\hat{S}_\nu$ is a $\mathbb{B}_{\nu,U}$-name because $\nu \in U$, and $U$ is $\mathbb{B}$-complete. Moreover $p|\nu \models_{\mathbb{B}_\nu} \ " \hat{S}_\nu \cap \alpha \text{ is stationary} "$ because $p \in \mathbb{B}_\mu$. Then

$$r|\nu = (p|\nu)|U \models_{\mathbb{B}_{\nu,U}} \ " \hat{S}_\nu \cap \alpha \text{ is stationary} "$$

because the restriction to $U$ is a projection from $\mathbb{B}_\nu$ to $\mathbb{B}_{\nu,U}$ by the induction hypothesis (2). But $\text{cf}(\alpha) = \delta$, and $V^{\mathbb{B}_\nu}$ is a $<\delta$-closed forcing extension of $V^{\mathbb{B}_{\nu,U}}$ by the induction hypothesis (4) for $\nu$. Therefore $r|\nu \models_{\mathbb{B}_\nu} \ " \hat{S}_\nu \cap \alpha \text{ is stationary} "$.

(2) Clearly the map $p \mapsto p|U$ from $\mathbb{B}_\mu$ to $\mathbb{B}_{\mu,U}$ is order preserving and surjective. Suppose that $p \in \mathbb{B}_\mu$ and $q \leq p|U$ in $\mathbb{B}_{\mu,U}$. We show that there exists $p^* \leq p$ in $\mathbb{B}_\mu$ with $p^*|U = q$.

Let $p^* := q \cup (p|\mu \setminus U)$. Then by induction on $\nu \leq \mu$ we can easily prove that $p^*|\nu \in \mathbb{B}_\nu$ and that $p^*|\nu \leq p|\nu,q|\nu$ in $\mathbb{B}_\nu$. Thus $p^*$ is as desired.

(3) Suppose that $A \subseteq \mathbb{B}_{\mu,U}$ is directed and that $|A| < \delta$. We must find a lower bound $p^*$ of $A$.

Let $c_\nu := \bigcup\{p(\nu) \mid p \in A \land \nu \in \text{dom}(p)\}$ for each $\nu \in \bigcup\{\text{dom}(p) \mid p \in A\}$, and let $p^*$ be a function on $\bigcup\{\text{dom}(p) \mid p \in A\}$ such that $p^*(\nu) = c_\nu \cup \{\sup c_\nu\}$. First note that each $p^*(\nu)$ is a closed bounded subset of $\kappa$. Note also that if $\sup c_\nu \notin c_\nu$, then $\sup c_\nu \notin \mathcal{E}_\delta^\kappa$ because $|A| < \delta$. Then by induction on $\nu \leq \mu$ we can easily prove that $p^*|\nu \in \mathbb{B}_\nu$ and that $p^*|\nu$ is a lower bound of $\{p|\nu \mid p \in A\}$. Therefore $p^*$ is as desired.

(4) Suppose that $H_{\mu,U}$ is a $\mathbb{B}_{\mu,U}$-generic filter over $V$. In $V[H_{\mu,U}]$ suppose that $A \subseteq \mathbb{B}_{\mu}/H_{\mu,U}$ is directed and that $|A| < \delta$. We find a lower bound $p^*$ of $A$ in $\mathbb{B}_{\mu}/H_{\mu,U}$. Here recall that $\mathbb{B}_{\mu}/H_{\mu,U} = \{p \in \mathbb{B}_\mu \mid p|U \in H_{\mu,U}\}$.

First note that $A$ is directed in $\mathbb{B}_\mu$ by the definition of $\mathbb{B}_{\mu}/H_{\mu,U}$. Note also that $A \subseteq V$ because $\mathbb{B}_{\mu,U}$ is $<\delta$-closed by (3). In $V$ construct $p^*$ from $A$ as in the proof of (3). That is, let $c_\nu := \bigcup\{p(\nu) \mid p \in A \land \nu \in \text{dom}(p)\}$ for each $\nu \in \bigcup\{\text{dom}(p) \mid p \in A\}$, and let $p^*$ be a function on $\bigcup\{\text{dom}(p) \mid p \in A\}$ such that $p^*(\nu) = c_\nu \cup \{\sup c_\nu\}$. Then $p^*$ is a lower bound of $A$ in $\mathbb{B}_\mu$. It suffices to show that $p^* \in \mathbb{B}_{\mu}/H_{\mu,U}$.

Note that $p^*|U$ is the greatest lower bound of $\{p|U \mid p \in A\}$ in $\mathbb{B}_{\mu,U}$. Moreover $\{p|U \mid p \in A\} \subseteq V$, and $\{p|U \mid p \in A\} \subseteq H_{\mu,U}$ because $A \subseteq \mathbb{B}_{\mu}/H_{\mu,U}$. Therefore $p^*|U \in H_{\mu,U}$ by the genericity of $H_{\mu,U}$. That is, $p^* \in \mathbb{B}_{\mu}/H_{\mu,U}$. ■
Note that if \( \vec{B} = \langle B_\mu, \dot{S}_\nu \mid \mu \leq v, \nu < v \rangle \) is an \( \text{OSR}^*_\kappa \)-iteration, then every \( \mu \leq v \) is a \( \vec{B} \)-complete subset of \( v \). Thus \( B_\mu \) is \( < \delta \)-directed closed for every \( \mu \leq v \) by Lemma 4.7(3). Moreover, if we let \( \dot{H}_\mu \) be the canonical \( B_\mu \)-name for a \( B_\mu \)-generic filter, then \( B_\mu \) forces that \( B_\nu / \dot{H}_\mu \) is \( < \delta \)-directed closed by Lemma 4.7(4).

Hence \( \text{OSR}^*_\kappa \)-iterations preserve all cardinals \( \leq \delta \). They also preserve all cardinals \( \geq (2^\delta)^+ \) by Lemma 4.5. But in general \( \text{OSR}^*_\kappa \)-iterations do not preserve the cardinality of \( \kappa \). We prove that under the assumption of Proposition 4.3 they are \( < \kappa \)-Baire and so preserve the cardinality of \( \kappa \):

**Lemma 4.8.** Assume that \( \text{FA}_\delta^{++} \) holds for all \( < \delta \)-directed closed posets of size \( \leq 2^\delta \) and that \( E^\kappa_{<\delta} \in I[\kappa] \). Then all \( \text{OSR}^*_\kappa \)-iterations are \( < \kappa \)-Baire.

This easily follows from Lemmas 4.9 and 4.10 below:

**Lemma 4.9.** Let \( P \) be a poset which preserves all cofinalities \( \leq \delta \) and all stationary subsets of \( E^\kappa_{<\delta} \). Assume that \( \text{FA}_\delta^{++}(\mathbb{P} \ast \text{Col}((\delta, \kappa))) \) holds and that \( E^\kappa_{<\delta} \in I[\kappa] \). Then for any sufficiently large regular cardinal \( \theta \) and \( p \in \mathbb{P} \) there are stationarily many \( M \in [H_\theta]^{<\delta} \) with the following properties:

(i) \( M \cap \kappa \in \kappa \), and \( p \in M \).

(ii) There is an \((M, \mathbb{P})\)-generic filter \( g \) containing \( p \) such that 
\[
\dot{S}_{g,M} := \{ \alpha < M \cap \kappa \mid \exists p' \in g, p' \Vdash \text{“} \alpha \in \dot{S} \text{”} \}
\]

is stationary in \( M \cap \kappa \) for any \( \mathbb{P} \)-name \( \dot{S} \in M \) for a stationary subset of \( E^\kappa_{<\delta} \).

**Proof.** Suppose that \( \theta \) is a sufficiently large regular cardinal and that \( p \in \mathbb{P} \). Let \( M \) be a structure obtained by adding countably many constants, functions and predicates to \( \langle H_\theta, \in, \mathbb{P}, p \rangle \). It suffices to find \( M \in [H_\theta]^{<\delta} \) such that \( M \prec M \) and such that \( M \) satisfies (i) and (ii).

By Lemma 2.2 we can take \( M \in [H_\theta]^{<\delta} \) and an \((M, \mathbb{P} \ast \text{Col}(\delta, \kappa))\)-generic filter \( \vec{g} \) containing \( p \ast \vec{0} \) such that \( \delta \subseteq M \prec M \) and such that \( \dot{R}_{\vec{g}} \) is stationary in \( \delta \) for any \( \mathbb{P} \ast \text{Col}(\delta, \kappa) \)-name \( \dot{R} \in M \) for a stationary subset of \( \delta \). Note that \( M \cap \kappa \in \kappa \) because \( \delta \subseteq M \prec M \). So \( M \) satisfies (i).

We show that \( g := \{ q \in \mathbb{P} \cap M \mid q \ast \vec{0} \in \vec{g} \} \) witnesses property (ii) for \( M \). Note that \( g \) is an \((M, \mathbb{P})\)-generic filter containing \( p \). Take an arbitrary \( \mathbb{P} \)-name \( \dot{S} \in M \) for a stationary subset of \( E^\kappa_{<\delta} \). We show that \( \dot{S}_{g,M} \) is stationary.

First note that \( \text{cf}(\kappa) = \delta \in V^{\mathbb{P} \ast \text{Col}(\delta, \kappa)} \). Let \( \dot{f} \in M \) be a \( \mathbb{P} \ast \text{Col}(\delta, \kappa) \)-name for an increasing continuous cofinal map from \( \delta \) to \( \kappa \), and let \( \dot{R} \in M \) be a \( \mathbb{P} \ast \text{Col}(\delta, \kappa) \)-name for \( \dot{f}^{-1}[\dot{S}] \).

Here note that \( \dot{S} \) remains stationary in \( V^{\mathbb{P} \ast \text{Col}(\delta, \kappa)} \) by the \( < \delta \)-closure of \( \text{Col}(\delta, \kappa) \) and the fact that \( E^\kappa_{<\delta} \in I[\kappa] \) in \( V^{\mathbb{P}} \) (see Fact 2.1). Hence \( \dot{R} \) is a
\(P \ast \text{Col}(\delta, \kappa)\)-name for a stationary subset of \(\delta\). Then \(\hat{R}_g\) is stationary in \(\delta\) by the choice of \(M\) and \(g\). Moreover \(f_g := \{\langle \xi, \alpha \rangle \in \delta \times \kappa \mid \exists r \in g, r \vdash \text{“} f(\xi) = \alpha \text{”} \}\) is an increasing continuous cofinal map from \(\delta\) to \(M \cap \kappa\), and \(f_g[\hat{R}_g] = \hat{S}_{g,M}\). Therefore \(\hat{S}_{g,M}\) is stationary in \(M \cap \kappa\).

**Lemma 4.10.** Assume that \(\mathcal{E}^\kappa_{<\delta} \in I[\kappa]\). Let \(\mathbb{B} = \langle \mathbb{B}_\mu, \hat{S}_\nu \mid \mu < \nu, \nu < \nu \rangle\) be an \(\text{OSR}^*_\nu\)-iteration, \(U\) be a \(\mathbb{B}\)-complete subset of \(\nu\) and \(p\) be a condition in \(\mathbb{B}_{\nu,U}\). Moreover let \(\theta\) be a sufficiently large regular cardinal. Assume that \(M \in [\mathcal{H}_\theta]^{\delta}\), that \(M \prec (\mathcal{H}_\theta, \in, \mathbb{B}, U, p)\) and that \(M\) satisfies conditions (i) and (ii) in Lemma 4.9 for \(P = \mathbb{B}_{\nu,U}\). Let \(g\) be an \((M, \mathbb{B}_{\nu,U})\)-generic filter witnessing (ii). Then \(g\) has a lower bound in \(\mathbb{B}_{\nu,U}\).

**Proof.** For each \(\nu \in M \cap U\) let \(c_\nu := \bigcup \{q(\nu) \mid q \in g\}\). Then \(c_\nu\) is club in \(M \cap \kappa\) for every \(\nu \in M \cap U\) because \(g\) is an \((M, \mathbb{B}_{\nu,U})\)-generic filter. Let \(\gamma^* := M \cap \kappa\), and let \(p^*\) be a function on \(M \cap U\) such that \(p^*(\nu) = c_\nu \cup \{\gamma^*\}\). It suffices to show that \(p^* \in \mathbb{B}_\nu\). (Then \(p^* \in \mathbb{B}_{\nu,U}\), and clearly \(p^*\) is a lower bound of \(g\).)

By induction on \(\mu \leq \nu\) we prove that \(p^*\mid \mu \in \mathbb{B}_\mu\). Suppose that \(\mu \leq \nu\) and that \(p^*\mid \nu \in \mathbb{B}_\nu\) for every \(\nu < \mu\). We show that \(p^*\mid \mu \in \mathbb{B}_\mu\). If \(\mu\) is a limit ordinal, then this follows from the fact that \(|\text{dom}(p^*)| \leq |M| = \delta\).

Suppose that \(\mu\) is a successor ordinal, and let \(\nu := \mu - 1\). If \(\nu \notin M \cap U\), then \(p^*\mid \mu \in \mathbb{B}_\nu\) clearly. Thus we also assume that \(\nu \in M \cap U\). It suffices to show that if \(\alpha \in p^*(\nu) \cap \mathcal{E}^\kappa_{<\delta}\), then \(p^*\mid \nu \vdash \text{“} \hat{S}_\nu \cap \alpha\text{ is stationary”}\). Take an arbitrary \(\alpha \in p^*(\nu) \cap \mathcal{E}^\kappa_{<\delta}\).

First suppose that \(\alpha \in c_\nu\). So there exists \(q \in g\) such that \(\alpha \in q(\nu)\). Then \(q(\nu) \vdash \text{“} \hat{S}_\nu \cap \alpha\text{ is stationary”}\) because \(q \in \mathbb{B}_\nu\). But \(p^*\mid \nu \leq q(\nu)\). Hence \(p^*\mid \nu \vdash \text{“} \hat{S}_\nu \cap \alpha\text{ is stationary”}\).

Next suppose that \(\alpha = \gamma^*\). Note that \(\hat{S}_\nu\) can be seen as a \(\mathbb{B}_{\nu,U}\)-name because \(\nu \in U\). Moreover \(\hat{S}_\nu\) is stationary in \(\mathcal{E}^\kappa_{<\delta}\) in \(V^\mathbb{B}_{\nu,U}\) by Lemma 4.7 Fact 2.1 and the fact that \(\mathcal{E}^\kappa_{<\delta} \in I[\kappa]\) (in \(V^\mathbb{B}_\nu\)). Then \(s := (\hat{S}_\nu)_{g,M}\) is stationary in \(\gamma^*\) by the assumption on \(M\) and \(g\). Note that \(s\) is the set of all \(\beta < \gamma^*\) such that \(q(\nu) \vdash \text{“} \beta \in \hat{S}_\nu\text{”}\) for some \(q \in g\). Moreover \(p^*\mid \nu\) is a lower bound of \(\{q(\nu) \mid q \in g\}\). Hence \(p^*\mid \nu \vdash \text{“} \hat{S}_\nu \cap \gamma^* \supseteq s\text{”}\). Here note that \(\mathbb{B}_\nu\) preserves stationary subsets of \(\gamma^*\) because \(\mathbb{B}_\nu\) is \(<\delta\)-closed, and \(\text{cf}(\gamma^*) \leq \delta\). So \(s\) remains stationary in \(V^\mathbb{B}_\nu\). Therefore \(p^*\mid \nu \vdash \text{“} \hat{S}_\nu \cap \gamma^*\text{ is stationary”}\).

**Proof of Lemma 4.8.** Let \(\mathbb{B} = \langle \mathbb{B}_\mu, \hat{S}_\nu \mid \mu \leq \nu, \nu < \nu \rangle\) be an \(\text{OSR}^*_\nu\)-iteration. Suppose that \(p \in \mathbb{B}_\nu\) and that \(A\) is a family of maximal antichains in \(\mathbb{B}_\nu\) with \(|A| \leq \delta\). We will find \(p^* \leq p\) meeting all maximal antichains in \(A\).

First take a \(\mathbb{B}\)-complete \(U \subseteq \nu\) such that \(|U| \leq 2^\delta\) and such that \(\{p\} \cup \bigcup A \subseteq \mathbb{B}_{\nu,U}\). We can easily find such \(U\) by the \((2^\delta)^+\)-c.c. of \(\mathbb{B}_\nu\). Then \(|\mathbb{B}_{\nu,U}| \leq 2^\delta\), and \(\mathbb{B}_{\nu,U}\) is \(<\delta\)-directed closed by Lemma 4.7. Then, by the as-
Partial square is implied by MM but not by PFA

Thus we can take \( M \in |\mathcal{H}_\theta|^\delta \) such that \( \mathcal{A} \subseteq M \prec (\mathcal{H}_\theta, \in, \mathbb{B}, U, p) \) and such that \( M \) satisfies conditions (i) and (ii) in Lemma 4.9 for \( P = \mathbb{B}_{v,U} \). Let \( g \) be an \((M, \mathbb{B}_{v,U})\)-generic filter witnessing (ii). Then there is a lower bound \( p^* \) of \( g \) in \( \mathbb{B}_{v,U} \) by Lemma 4.10.

Here note that \( p^* \) is a lower bound of \( g \) also in \( \mathbb{B}_v \). Hence \( p^* \leq p \) because \( p \in g \). Note also that each \( A \in \mathcal{A} \) is a maximal antichain in \( \mathbb{B}_{v,U} \) which belongs to \( M \). Thus \( p^* \) meets all \( A \in \mathcal{A} \) by the \((M, \mathbb{B}_{v,U})\)-genericity of \( g \). Therefore \( p^* \) is as desired.

Now Proposition 4.3 easily follows from what we have proved:

**Proof of Proposition 4.3.** Let \( \nu := 2^\kappa \cdot (2^\delta)^+ \). Then by the \((2^\delta)^+\)-c.c. of \( \text{OSR}^*_\kappa \)-iterations we can construct an \( \text{OSR}^*_\kappa \)-iteration \( \langle \mathbb{B}_{\mu}, \dot{S}_\nu \mid \mu \leq \nu, \nu < \nu \rangle \) such that for any \( \mathbb{B}_v \)-name \( \dot{S} \) for a stationary subset of \( \mathcal{E}_{<\delta} \) there exists \( \nu < \nu \) with \( \models_{\mathbb{B}_v} \dot{S} = \dot{S}_\nu \). Now \( \mathbb{B}_v \) is \(<\delta\)-directed closed by Lemma 4.7 and \(<\kappa\)-Baire by Lemma 4.8. Moreover \( \mathbb{B}_v \) forces \( \text{OSR}^*_\kappa \) by the construction of \( \mathbb{B}_v \).


5. PFA does not imply \( \square^p \). In this section we prove

**Theorem 1.5.** If there exists a supercompact cardinal, then there exists a forcing extension in which PFA holds but \( \square^p \) fails.

In fact we prove the following result which implies the above theorem by Lemma 4.2:

**Theorem 5.1.** PFA is consistent with \( \text{OSR}^*_\omega_2 \). More precisely, if there exists a supercompact cardinal, then there exists a forcing extension in which both PFA and \( \text{OSR}^*_\omega_2 \) hold.

Recall that if \( \kappa \) is a supercompact cardinal, and \( \langle \mathbb{P}_\alpha, \dot{Q}_\beta \mid \alpha \leq \kappa, \beta < \kappa \rangle \) is the standard iteration for PFA, then \( \text{PFA}^{++} \) and \( 2^{\omega_1} = \kappa = \omega_2 \) hold in \( V^{\mathbb{P}_\kappa} \). Hence, by results in the previous section, in \( V^{\mathbb{P}_\kappa} \) we can construct an \( \text{OSR}^*_\omega_2 \)-iteration forcing \( \text{OSR}^*_\omega_2 \). Thus the following lemma implies Theorem 5.1.

**Lemma 5.2.** If \( \text{PFA}^{++} \) holds, then \( \mathbb{B}_v \) forces PFA for any \( \text{OSR}^*_\omega_2 \)-iteration \( \langle \mathbb{B}_\mu, \dot{S}_\nu \mid \mu \leq \nu, \nu < \nu \rangle \).

**Proof.** Assume that \( \text{PFA}^{++} \) holds and that \( \mathbb{B} = \langle \mathbb{B}_\mu, \dot{S}_\nu \mid \mu \leq \nu, \nu < \nu \rangle \) is an \( \text{OSR}^*_\omega_2 \)-iteration. Suppose that \( p \in \mathbb{B}_v \), that \( \dot{Q} \) is a \( \mathbb{B}_v \)-name for a proper poset and that \( \mathcal{A} \) is a family of \( \mathbb{B}_v \)-names for maximal antichains in \( \dot{Q} \) with \( |\mathcal{A}| \leq \omega_1 \). It suffices to find \( p^* \leq p \) and a \( \mathbb{B}_v \)-name \( \dot{H} \) such that \( p^* \) forces that \( \dot{H} \) is a filter on \( \dot{Q} \) and that \( \dot{H} \cap \dot{A} \neq \emptyset \) for all \( \dot{A} \in \mathcal{A} \). We work in \( V \).
Take a sufficiently large regular cardinal \( \theta \). Note that \( \mathbb{B}_v * \hat{Q} \) is proper and that \( \text{FA}^{++} \) holds for \( \mathbb{B}_v * \hat{Q} * \text{Col}(\omega_1, \omega_2 V) \) by \( \text{PFA}^{++} \). Note also that \( \mathcal{E}_{<\omega_1}^{\omega_2} = E_0^2 \in I[\omega_2] \). Then by Lemma 4.9 we can take \( M \in [\mathcal{H}_\theta]^{\omega_1} \) and an \((M, \mathbb{B}_v * \hat{Q})\)-generic filter \( k \) such that:

(i) \( M \cap \omega_2 \in \omega_2 \), and \( A \subseteq M \prec \langle \mathcal{H}_\theta, \in, \mathbb{B}, p \rangle \).

(ii) \( p * 1 \notin \hat{Q} \in k \).

(iii) \( \hat{S}_{k,M} \) is stationary in \( M \cap \omega_2 \) for any \( \mathbb{B}_v * \hat{Q} \)-name \( \hat{S} \in M \) for a stationary subset of \( (E_0^2)^V \).

Let \( g := \{ p' \in \mathbb{B}_v \cap M \mid p' * 1 \notin \hat{Q} \in k \} \). Then \( g \) is an \((M, \mathbb{B}_v)\)-generic filter, and \( \hat{S}_{g,M} \) is stationary in \( M \cap \omega_2 \) for any \( \mathbb{B}_v \)-name \( \hat{S} \in M \) for a stationary subset of \( (E_0^2)^V \). (For the latter note that if \( \hat{S} \) is a \( \mathbb{B}_v \)-name for a stationary subset of \( (E_0^2)^V \), then \( \mathbb{B}_v * \hat{Q} \) forces \( \hat{S} \) to remain stationary because \( \hat{Q} \) is proper.) So by Lemma 4.10 we can take a lower bound \( p^* \in \mathbb{B}_v \) of \( g \).

Let \( h := \{ (\hat{q}, p') \mid p' * \hat{q} \in k \} \). Then \( h \) is a \( \mathbb{B}_v \)-name for a subset of \( \hat{Q} \). Moreover it is easy to see that \( p^* \) forces \( h \) to be an \((M[G], \hat{Q})\)-generic filter, where \( \hat{G} \) is the canonical name for a \( \mathbb{B}_v \)-generic filter, and \( M[G] \) denotes the set \( \{ \hat{x}_G \mid \hat{x} \) is a \( \mathbb{B}_v \)-name in \( M \} \) for a \( \mathbb{B}_v \)-generic filter \( G \). In particular, \( p^* \) forces that \( \hat{h} \cap \hat{A} \neq 0 \) for all \( \hat{A} \in \hat{A} \). Let \( \hat{H} \) be a \( \mathbb{B}_v \)-name for the filter on \( \hat{Q} \) generated by \( \hat{h} \). Then \( p^* \) and \( \hat{H} \) as desired.

6. \( \Box^p_\delta \) for regular \( \delta \geq \omega_2 \). To end this paper we make a remark that \( \Box^p_\delta \) for a regular \( \delta \geq \omega_2 \) is independent of \( \text{MM} \).

First we prove that \( \text{MM} \) is consistent with the failure of \( \Box^p_\delta \). In fact we prove the following stronger fact:

**Theorem 6.1.** \( \text{MM} \) is consistent with \( \text{OSR}_{\delta^+}^* \) for a regular \( \delta \geq \omega_2 \). More precisely: In \( V \) suppose that \( \text{MM} \) holds, that \( \delta \) is a regular cardinal \( \geq \omega_2 \) and that there exists a weakly compact cardinal \( > \delta \). Then there exists a \( <\delta^+-\text{directed closed} <\delta^+-\text{Baire forcing extension in which both} \text{MM} \) and \( \text{OSR}_{\delta^+}^* \) hold.

For this we use the following well-known lemma:

**Lemma 6.2** (folklore). Suppose that \( \delta \) is a regular uncountable cardinal and that \( \kappa \) is a weakly compact cardinal \( > \delta \). Then \( \text{FA}_{\delta^+}^{++} \) for all \( <\delta \)-closed posets of size \( \leq 2^\delta \) holds in \( V^{\text{Col}(\delta, <\kappa)} \).

**Proof.** Let \( H \) be a \( \text{Col}(\delta, <\kappa) \)-generic filter over \( V \). In \( V[H] \) suppose that \( \mathbb{P} \) is a \( <\delta \)-closed poset of size \( \leq 2^\delta \), that \( \mathcal{A} \) is a family of maximal antichains in \( \mathbb{P} \) with \( |\mathcal{A}| \leq \delta \) and that \( \mathcal{R} \) is a family of \( \mathbb{P} \)-names for stationary subsets of \( \delta \) with \( |\mathcal{R}| \leq \delta \). In \( V[H] \) we find a filter \( G^* \) on \( \mathbb{P} \) such that \( G^* \cap \mathcal{A} \neq \emptyset \) for
any $A \in \mathcal{A}$ and such that $\hat{R}_{G^*}$ is stationary in $\delta$ for all $\hat{R} \in \mathcal{R}$. Note that $2^\delta = \kappa$ in $V[H]$. So we may assume that $\mathbb{P} \subseteq \kappa$.

Let $\mathbb{P}$, $\mathbb{A}$ and $\mathcal{R}$ be $\text{Col}(\delta, <\kappa)$-names for $\mathbb{P}$, $\mathbb{A}$ and $\mathcal{R}$, respectively. We may assume that $\mathbb{P}$, $\mathbb{A}$, $\mathcal{R} \in \mathcal{H}_{\kappa^+}$ in $V$. Then in $V$ we can take a transitive $M \prec (\mathcal{H}_{\kappa^+}, \in, \delta, \kappa, \mathbb{P}, \mathbb{A}, \mathcal{R})$ of size $\kappa$. Moreover, in $V$ we can take a transitive $N$ and an elementary embedding $j : M \rightarrow N$ whose critical point is $\kappa$. This is because $\kappa$ is weakly compact in $V$.

Note that $M \subseteq N$ because $\mathcal{P}(\kappa) \cap M \subseteq N$. Hence $\mathbb{P}, \mathbb{A}, \mathcal{R} \in M[H] \subseteq N[H]$. Moreover in $N[H]$ it is easy to see that $\mathbb{P}$ is $<\delta$-closed, that $\mathbb{A}$ is a family of maximal antichains in $\mathbb{P}$ with $|\mathbb{A}| \leq \delta$ and that $\mathcal{R}$ is a family of $\mathbb{P}$-names for stationary subsets of $\delta$ with $|\mathcal{R}| \leq \delta$.

Here take a $\mathbb{P} \times \text{Col}(\delta, [j(\kappa)])^{N[H]}$-generic filter $G \times I$ over $V[H]$. Below we work in $V[H][G \times I]$.

Note that $\mathbb{P} \times \text{Col}(\delta, [j(\kappa)])^{N[H]}$ is isomorphic to $\text{Col}(\delta, [\kappa, j(\kappa)])^{N[H]}$ in $N[H]$ because $\mathbb{P}$ is a $<\delta$-closed poset of size $\leq \kappa$. Thus $H* (G \times I)$ can be seen as a $\text{Col}(\delta, <j(\kappa))$-generic filter over $N$. Hence $j : M \rightarrow N$ can be naturally extended to an elementary embedding $j : M[H] \rightarrow N[H][G][I]$. In $N[H][G][I]$ let $\bar{G}$ be a filter on $\mathcal{P}$ generated by $j[G] = G \in N[H][G][I]$.

Then $\bar{G} \cap j(\mathbb{A}) \neq \emptyset$ for all $A \in \mathcal{A}$. Moreover for each $\hat{R} \in \mathcal{R}$, $j(\hat{R})_G = \hat{R}_G$, and $\hat{R}_G$ is stationary in $\delta$ in $N[H][G][I]$. The latter is because $\hat{R}$ is a $\mathbb{P}$-name for a stationary subset of $\delta$ in $N[H]$, and $N[H][G][I]$ is a $<\delta$-closed forcing extension of $N[H][G]$. Note also that $j(A) = j[\mathbb{A}]$ and $j(\mathcal{R}) = j[\mathcal{R}]$ because $|\mathbb{A}| = |\mathcal{R}| \leq \delta < \kappa$ in $M[H]$. Therefore in $N[H][G][I]$ we see that $\bar{G} \cap A' \neq \emptyset$ for all $A' \in j(\mathbb{A})$ and that $\hat{R}_G'$ is stationary in $\delta$ for all $\hat{R}' \in j(\mathcal{R})$.

Then, by the elementarity of $j$, in $M[H]$ we can take a filter $G^*$ on $\mathbb{P}$ such that $G^* \cap A \neq \emptyset$ for all $A \in \mathcal{A}$ and such that $\hat{R}_{G^*}$ is stationary in $\delta$ for all $\hat{R} \in \mathcal{R}$. Here note that $\mathcal{P}(\delta) \cap V[H] \subseteq M[H]$. Hence $\hat{R}_{G^*}$ is stationary also in $V[H]$ for each $\hat{R} \in \mathcal{R}$. Thus $G^*$ is as desired. ■

In the proof of Theorem 6.1 we also use the following folklore. Its proof is found in Larson [11]:

**FACT 6.3 (folklore).** $\text{MM}$ is preserved by $<\omega_2$-directed closed forcing extensions.

Now we can easily prove Theorem 6.1. In $V$ assume that $\text{MM}$ holds, that $\delta$ is a regular cardinal $\geq \omega_2$ and that $\kappa$ is a weakly compact cardinal $> \delta$.

Let $G$ be a $\text{Col}(\delta, <\kappa)$-generic filter over $V$. Then $\kappa = \delta^+$ in $V[G]$. Moreover $\text{FA}^{++}$ holds for all $<\delta$-directed closed posets of size $\leq 2^\delta$ in $V[G]$ by Lemma 6.2. Furthermore $\mathcal{E}^{\kappa}_< \delta \in I[\kappa]$ in $V[G]$ because $2^\delta = \delta$ (See Fact 2.1). Then by Proposition 4.3 in $V[G]$ there exists a $<\kappa$-Baire $<\delta$-directed closed poset $\mathbb{B}$ forcing $\text{OSR}^\kappa$. Let $H$ be a $\mathbb{B}$-generic filter over $V[G]$.  

Then $\text{OSR}^*_\delta$ holds in $V[G][H]$. Moreover $\text{MM}$ remains to hold in $V[G][H]$ by Fact 6.3 and the fact that $V[G][H]$ is a $<\delta$-directed closed forcing extension of $V$. ■

From Theorem 6.1 and Lemma 4.2 it follows that $\text{MM}$ is consistent with the failure of $\Box^\mathbb{P}_\delta$ for a regular $\delta \geq \omega_2$. Next we prove that $\text{MM}$ is consistent with $\Box^\mathbb{P}_\delta$:

**Theorem 6.4.** $\text{MM}$ is consistent with $\Box^\mathbb{P}_\delta$ for an uncountable cardinal $\delta$. More precisely: In $V$ suppose that $\text{MM}$ holds and that $\delta$ is an uncountable cardinal. Then there exists a $<\delta^+$-directed closed forcing extension in which both $\text{MM}$ and $\Box^\mathbb{P}_\delta$ hold.

This follows from Fact 6.3 and the following lemma:

**Lemma 6.5.** Let $\delta$ be an uncountable cardinal. Then there exists a $<\delta^+$-directed closed forcing extension in which $\Box^\mathbb{P}_\delta$ holds.

**Proof.** Let $\mathbb{P}$ be the following poset:

- $\mathbb{P} := \{p \mid p$ is a $\Box^\mathbb{P}_\delta(s)$-sequence for some bounded $s \subseteq \text{Lim}(\delta^+)\}$.
- For $p = \langle c_\alpha \mid \alpha \in s \rangle$ and $p' = \langle c'_\alpha \mid \alpha \in s' \rangle$ in $\mathbb{P}$, $p \leq p'$ if $p$ is an end-extension of $p'$, that is, $s' = s \cap \text{sup} \{\alpha + 1 \mid \alpha \in s'\}$, and $c_\alpha = c'_\alpha$ for all $\alpha \in s'$.

If $p = \langle c_\alpha \mid \alpha \in s \rangle \in \mathbb{P}$, then $c_\alpha$ and $s$ are denoted as $c_{p,\alpha}$ and $s_p$, respectively.

It is easy to see that $\mathbb{P}$ is $<\delta^+$-directed closed. We show that $\models_{\mathbb{P}} \Box^\mathbb{P}_\delta$.

Note that if $G$ is a $\mathbb{P}$-generic filter, then clearly $\bigcup G$ is a $\Box^\mathbb{P}_\delta(S_G)$-sequence, where $S_G = \bigcup_{p \in G} s_p$. Thus all we have to show is that $S_G \cap \mathcal{E}^\delta_\delta$ is stationary in $V[G]$.

In $V$ take an arbitrary $p \in \mathbb{P}$ and an arbitrary $\mathbb{P}$-name $\hat{C}$ for a club subset of $\delta^+$. It suffices to find $p^* \leq p$ and $\alpha^* \in \mathcal{E}^\delta_\delta$ such that $\alpha^* \in s_{p^*}$ and such that $p^* \models “\alpha^* \in \hat{C}”$. We work in $V$.

By induction on $\xi$ we can easily construct a strictly descending sequence $\langle p_\xi \mid \xi \leq \delta \rangle$ below $p$ so that

- $s_{p_\xi}$ has the greatest element $\alpha_\xi$ for each $\xi$,
- if $\xi$ is successor, then $p_\xi \models “\hat{C} \cap [\alpha_{\xi-1}, \alpha_\xi) \neq \emptyset”$,
- if $\xi$ is limit, then $\alpha_\xi = \text{sup}_{\eta < \xi} \alpha_\eta$, and $c_{p_\xi,\alpha_\xi} = \{\alpha_\eta \mid \eta < \xi\}$.

Then it is easy to see that $p^* := p_\delta$ and $\alpha^* := \alpha_\delta$ are as desired. ■

**References**


Partial square is implied by MM but not by PFA


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