

Free trees and the optimal bound in Wehrung's theorem

by

Pavel Růžička (Praha)

Abstract. We prove that there is a distributive $(\vee, 0, 1)$ -semilattice \mathcal{G} of size \aleph_2 such that there is no weakly distributive $(\vee, 0)$ -homomorphism from $\text{Con}_c A$ to \mathcal{G} with 1 in its range, for any algebra A with either a congruence-compatible structure of a $(\vee, 1)$ -semilattice or a congruence-compatible structure of a lattice. In particular, \mathcal{G} is not isomorphic to the $(\vee, 0)$ -semilattice of compact congruences of any lattice. This improves Wehrung's solution of Dilworth's Congruence Lattice Problem, by giving the best cardinality bound possible. The main ingredient of our proof is the modification of Kuratowski's Free Set Theorem, which involves what we call *free trees*.

1. Introduction. Congruence lattices of universal algebras correspond to algebraic lattices. By the theorem of N. Funayama and T. Nakayama [2], the congruence lattice of a lattice is, in addition, distributive (see also [3, II. 3. Theorem 11]). On the other hand, R. P. Dilworth proved that every finite distributive lattice is isomorphic to the congruence lattice of a finite lattice (first published in [5]) and he conjectured that every distributive algebraic lattice is isomorphic to the congruence lattice of a lattice (see again [5]). This conjecture, referred to as the *Congruence Lattice Problem*, despite many attempts (see surveys [3, Appendix C] and [12]), remained open for over sixty years until, recently, F. Wehrung disproved it in [17].

Wehrung's solution involves a combination of new ideas (see, in particular, Lemmas 4.4, 5.1, and 6.2 in [17]) and methods developed in earlier papers, which originated in [14] and were pursued further in [9, 10, 11, 13, 15]. In these papers, counterexamples to various problems related to the Congruence Lattice Problem were obtained. The optimal cardinality bound for all these counterexamples is \aleph_2 , but Wehrung's argument requires an algebraic distributive lattice with at least $\aleph_{\omega+1}$ compact elements. In the present

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paper, we improve Wehrung's result by proving that there is a counterexample of size \aleph_2 . As in the related cases, \aleph_2 turns out to be the optimal cardinality bound for a negative solution of the Congruence Lattice Problem. Our proof closely follows Wehrung's ideas. The main difference consists in an enhancement of Kuratowski's Free Set Theorem by a new combinatorial principle which involves finite trees.

Wehrung's construction in [17] uses a "free" distributive extension of a $(\vee, 0)$ -semilattice, a functor that assigns to every $(\vee, 0)$ -semilattice a distributive $(\vee, 0)$ -semilattice, constructed previously by M. Ploščica and J. Tůma in [9]. The main features of this construction for the refutation of the Congruence Lattice Problem are extracted in the so-called Evaporation Lemma [17, Lemma 4.4]. We generalize this idea by defining a *diluting functor* whose properties are sufficient to prove the Evaporation Lemma, and we prove that the free distributive extension of a $(\vee, 0)$ -semilattice is, indeed, a diluting functor.

Further, we modify Kuratowski's Free Set Theorem, the combinatorial essence of the above mentioned counterexamples. Given a set Ω and a map $\Phi: [\Omega]^{<\omega} \rightarrow [\Omega]^{<\omega}$, we define a *free k -tree (with respect to Φ)*, for every positive integer k , which is a k -ary tree with some combinatorial properties derived from Kuratowski's Free Set Theorem. We prove that a free k -tree exists whenever the cardinality of the set Ω is at least \aleph_{k-1} , and we apply the existence of a free 3-tree in every set of cardinality at least \aleph_2 to attain the optimal cardinality bound in the Wehrung's result.

2. Basic concepts. A $(\vee, 0)$ -semilattice S is *distributive* if for any $\mathbf{a}, \mathbf{b}, \mathbf{c} \in S$ satisfying $\mathbf{c} \leq \mathbf{a} \vee \mathbf{b}$, there are $\mathbf{a}' \leq \mathbf{a}$ and $\mathbf{b}' \leq \mathbf{b}$ such that $\mathbf{a}' \vee \mathbf{b}' = \mathbf{c}$. A homomorphism $\mu: S \rightarrow T$ of join-semilattices is called *weakly distributive* at $\mathbf{x} \in S$ if for all $\mathbf{y}_0, \mathbf{y}_1 \in T$ such that $\mu(\mathbf{x}) \leq \mathbf{y}_0 \vee \mathbf{y}_1$, there are $\mathbf{x}_0, \mathbf{x}_1 \in S$ such that $\mathbf{x} \leq \mathbf{x}_0 \vee \mathbf{x}_1$ and $\mu(\mathbf{x}_i) \leq \mathbf{y}_i$ for all $i < 2$ (see [17]). The homomorphism μ is *weakly distributive* if it is weakly distributive at every element of S .

Let A be an algebra. We say that an n -ary operation f on A is *congruence-compatible* (see [8, 17]) if for every congruence θ of A , $(x_i, y_i) \in \theta$ for $i = 0, \dots, n-1$ implies that $(f(x_0, \dots, x_{n-1}), f(y_0, \dots, y_{n-1})) \in \theta$. In particular, a semilattice operation \vee , resp. \wedge , on A is congruence-compatible providing that $(x, y) \in \theta$ implies $(x \vee z, y \vee z) \in \theta$, resp. $(x \wedge z, y \wedge z) \in \theta$, for all $x, y, z \in A$ and $\theta \in \text{Con } A$.

Given an algebra A and elements $x, y \in A$, we denote by $\Theta_A(x, y)$ the smallest congruence (i.e., intersection of all the congruences) of A identifying x and y . We denote by $\text{Con } A$, resp. $\text{Con}_c A$ the lattice of all congruences of A , resp. the join-semilattice of all compact congruences of A . We say that A has *permutable congruences* if $\mathbf{a} \vee \mathbf{b} = \mathbf{a} \circ \mathbf{b} = \mathbf{b} \circ \mathbf{a}$ for all $\mathbf{a}, \mathbf{b} \in \text{Con } A$.

We will use the standard set-theoretic notation and terminology. We identify each ordinal number with the set of its predecessors, in particular, $n = \{0, \dots, n-1\}$ for each positive integer n . We denote by ω the first infinite ordinal, and by ω_n the first ordinal of size \aleph_n , for every positive integer n . For a set X , we denote by $[X]^{<\omega}$ the set of all finite subsets of X and by $[X]^n$ the set of all its n -element subset, for every natural number n . We denote by $|X|$ the cardinality of a set X . As in [17], we put $\varepsilon(n) = n \bmod 2$ for every integer n .

3. Diluting functors. Denote by \mathbf{S} the category of $(\vee, 0)$ -semilattices (with $(\vee, 0)$ -homomorphisms).

DEFINITION 1. An *expanding functor* on \mathbf{S} is a pair (\mathcal{F}, ι) , where \mathcal{F} is an endofunctor on \mathbf{S} and ι is a natural transformation from the identity to \mathcal{F} such that $\iota_S: S \rightarrow \mathcal{F}(S)$ is an embedding for every $(\vee, 0)$ -semilattice S . We shall denote the expanding functor above by \mathcal{F} once the natural transformation ι is understood, and we shall identify $\iota_S(\mathbf{x})$ with \mathbf{x} for all $\mathbf{x} \in S$.

An expanding functor \mathcal{F} on \mathbf{S} is a *diluting functor* if for all $(\vee, 0)$ -semilattices S and T and every $(\vee, 0)$ -homomorphism $f: S \rightarrow T$, the following property is satisfied: for all $\mathbf{v} \in \mathcal{F}(S)$ and $\mathbf{u}_0, \mathbf{u}_1 \in \mathcal{F}(T)$, if $\mathcal{F}(f)(\mathbf{v}) \leq \mathbf{u}_0 \vee \mathbf{u}_1$, then there are $\mathbf{x}_0, \mathbf{x}_1 \in \mathcal{F}(S)$ and $\mathbf{y} \in S$ such that

$$f(\mathbf{y}) \leq \mathbf{u}_0 \vee \mathbf{u}_1, \quad \mathcal{F}(f)(\mathbf{x}_i) \leq \mathbf{u}_i \quad \text{for all } i < 2, \quad \mathbf{v} \leq \mathbf{x}_0 \vee \mathbf{x}_1 \vee \mathbf{y}.$$

Given a $(\vee, 0)$ -semilattice S and subsets U, V of S , we shall use the notation

$$U \vee V = \{\mathbf{u} \vee \mathbf{v} \mid \mathbf{u} \in U \text{ and } \mathbf{v} \in V\}.$$

LEMMA 3.1. *Let \mathcal{F} be an expanding functor on \mathbf{S} . Let S be a $(\vee, 0)$ -semilattice and let $S_i, i < 2$, be $(\vee, 0)$ -subsemilattices of S such that $S = S_0 \vee S_1$ and there are retractions $r_i: S \rightarrow S_i$ for $i < 2$. Put $s_i = \mathcal{F}(r_i)$ for every $i < 2$. Let $\mathbf{u}_i \in \mathcal{F}(S_i), i < 2$, be such that $s_i(\mathbf{u}_{1-i}) = 0$ for all $i < 2$. Then for every $\mathbf{y} \in S$ such that $\mathbf{y} \leq \mathbf{u}_0 \vee \mathbf{u}_1$, there are $\mathbf{y}_i \in S_i, i < 2$, such that $\mathbf{y} \leq \mathbf{y}_0 \vee \mathbf{y}_1$ and $\mathbf{y}_i \leq \mathbf{u}_i$ for all $i < 2$.*

Proof. Put $\mathbf{y}_i = r_i(\mathbf{y})$ for every $i < 2$. Since $S = S_0 \vee S_1$, there are $\mathbf{y}'_i \in S_i$ for $i < 2$ such that $\mathbf{y} = \mathbf{y}'_0 \vee \mathbf{y}'_1$. Since the maps $r_i, i < 2$, are retractions, $\mathbf{y}'_i = r_i(\mathbf{y}'_i) \leq r_i(\mathbf{y}_i) = \mathbf{y}_i$ for all $i < 2$, whence $\mathbf{y} \leq \mathbf{y}_0 \vee \mathbf{y}_1$.

It remains to prove that $\mathbf{y}_i \leq \mathbf{u}_i$ for all $i < 2$. Fix $i < 2$. Since $s_i \upharpoonright S = r_i$ and $s_i: \mathcal{F}(S) \rightarrow \mathcal{F}(S_i)$ is a retraction, $s_i(\mathbf{u}_i) = \mathbf{u}_i$. Since, by the assumptions, $s_i(\mathbf{u}_{1-i}) = 0$, we conclude that

$$(3.1) \quad \mathbf{y}_i = s_i(\mathbf{y}) \leq s_i(\mathbf{u}_0 \vee \mathbf{u}_1) = s_i(\mathbf{u}_0) \vee s_i(\mathbf{u}_1) = s_i(\mathbf{u}_i) = \mathbf{u}_i. \quad \blacksquare$$

Given an expanding functor \mathcal{F} , define \mathcal{F}^0 to be the identity functor and, inductively, $\mathcal{F}^{n+1} = \mathcal{F} \circ \mathcal{F}^n$ for every natural number n . By our assumption, the inclusion map defines a natural transformation from the identity functor

on \mathbf{S} to \mathcal{F} , therefore we can define $\mathcal{F}^\infty(S) = \bigcup_{n \in \omega} \mathcal{F}^n(S)$, resp. $\mathcal{F}^\infty(f) = \bigcup_{n \in \omega} \mathcal{F}^n(f)$, for every $(\vee, 0)$ -semilattice S , resp. every $(\vee, 0)$ -homomorphism $f: S \rightarrow T$, and again the inclusion map defines a natural transformation from the identity functor on \mathbf{S} to \mathcal{F}^∞ . In particular, if \mathcal{F} is an expanding functor on \mathbf{S} , then \mathcal{F}^∞ is expanding as well.

LEMMA 3.2. *Let \mathcal{F} be a diluting functor on \mathbf{S} . Then the functor \mathcal{F}^∞ is diluting as well.*

Proof. Let S and T be $(\vee, 0)$ -semilattices and let $f: S \rightarrow T$ be a $(\vee, 0)$ -homomorphism. Let $\mathbf{v} \in \mathcal{F}^\infty(S)$ and let $\mathbf{u}_0, \mathbf{u}_1 \in \mathcal{F}^\infty(T)$ be such that $\mathcal{F}^\infty(f)(\mathbf{v}) \leq \mathbf{u}_0 \vee \mathbf{u}_1$. We are looking for $\mathbf{x}_0, \mathbf{x}_1 \in \mathcal{F}^\infty(S)$ and $\mathbf{y} \in S$ such that

$$f(\mathbf{y}) \leq \mathbf{u}_0 \vee \mathbf{u}_1, \quad \mathcal{F}^\infty(f)(\mathbf{x}_i) \leq \mathbf{u}_i \quad \text{for all } i < 2, \quad \mathbf{v} \leq \mathbf{x}_0 \vee \mathbf{x}_1 \vee \mathbf{y}.$$

We shall argue by induction on the least natural number n such that $\mathbf{v} \in \mathcal{F}^n(S)$. If $n = 0$, we put $\mathbf{x}_0 = \mathbf{x}_1 = 0$, $\mathbf{y} = \mathbf{v}$, and we are done. Suppose that $\mathbf{v} \in \mathcal{F}^{n+1}(S)$ for some natural number n , and that the property is proved at stage n . Let $k \geq n$ be a natural number such that $\mathbf{u}_0, \mathbf{u}_1 \in \mathcal{F}^{k+1}(T)$. Denote by g the composition of the $(\vee, 0)$ -homomorphism $\mathcal{F}^n(f)$ and the inclusion map from $\mathcal{F}^n(T)$ to $\mathcal{F}^k(T)$. By applying the assumption that \mathcal{F} is a diluting functor to the $(\vee, 0)$ -homomorphism $g: \mathcal{F}^n(S) \rightarrow \mathcal{F}^k(T)$, we obtain elements $\mathbf{x}'_0, \mathbf{x}'_1 \in \mathcal{F}^{n+1}(S)$ and $\mathbf{y}' \in \mathcal{F}^n(S)$ such that

$$g(\mathbf{y}') \leq \mathbf{u}_0 \vee \mathbf{u}_1, \quad \mathcal{F}(g)(\mathbf{x}'_i) \leq \mathbf{u}_i \quad \text{for all } i < 2, \quad \mathbf{v} \leq \mathbf{x}'_0 \vee \mathbf{x}'_1 \vee \mathbf{y}'.$$

Since $g(\mathbf{y}') \leq \mathbf{u}_0 \vee \mathbf{u}_1$ implies $\mathcal{F}^\infty(f)(\mathbf{y}') \leq \mathbf{u}_0 \vee \mathbf{u}_1$, there are, by the induction hypothesis, elements $\mathbf{x}''_0, \mathbf{x}''_1 \in \mathcal{F}^\infty(S)$ and $\mathbf{y} \in S$ such that

$$f(\mathbf{y}) \leq \mathbf{u}_0 \vee \mathbf{u}_1, \quad \mathcal{F}^\infty(f)(\mathbf{x}''_i) \leq \mathbf{u}_i \quad \text{for all } i < 2, \quad \mathbf{y}' \leq \mathbf{x}''_0 \vee \mathbf{x}''_1 \vee \mathbf{y}.$$

Now it is easy to conclude that $\mathbf{x}_i = \mathbf{x}'_i \vee \mathbf{x}''_i$ for $i < 2$ and \mathbf{y} are the desired elements. ■

As in [17], denote by \mathcal{L} the functor from the category of sets to \mathbf{S} which assigns to a set Ω the $(\vee, 0, 1)$ -semilattice $\mathcal{L}(\Omega)$ defined by generators 1 and $\mathbf{a}_0^\xi, \mathbf{a}_1^\xi$ for $\xi \in \Omega$, subjected to the relations

$$(3.2) \quad \mathbf{a}_0^\xi \vee \mathbf{a}_1^\xi = 1 \quad \text{for all } \xi \in \Omega,$$

and which assigns to a map $f: X \rightarrow Y$ the unique $(\vee, 0, 1)$ -homomorphism $\mathcal{L}(f): \mathcal{L}(X) \rightarrow \mathcal{L}(Y)$ such that $\mathcal{L}(f)(\mathbf{a}_i^\xi) = \mathbf{a}_i^{f(\xi)}$ for all $\xi \in X$ and all $i < 2$.

Given a finite subset A of Ω and a map $\varphi: A \rightarrow 2$, we put $\mathbf{a}_\varphi^A = \bigvee_{\alpha \in A} \mathbf{a}_{\varphi(\alpha)}^\alpha$. By Corollary 4.2 below, the following lemma is a generalization of Wehrung’s original “Evaporation Lemma” [17, Lemma 4.4].

LEMMA 3.3. *Let \mathcal{F} be a diluting functor on \mathbf{S} . Define $\mathcal{G} = \mathcal{F} \circ \mathcal{L}$. Let Ω be a set, let A_0, A_1 be finite disjoint subsets of Ω , and let $\delta \in \Omega \setminus (A_0 \cup A_1)$.*

Let $\mathbf{v} \in \mathcal{G}(\Omega \setminus \{\delta\})$, let $\varphi_i: A_i \rightarrow 2$, and let $\mathbf{u}_i \in \mathcal{G}(\Omega \setminus A_{1-i})$ for $i < 2$. Then

$$\mathbf{v} \leq \mathbf{u}_0 \vee \mathbf{u}_1 \quad \text{and} \quad \mathbf{u}_i \leq \mathbf{a}_{\varphi_i}^{A_i}, \mathbf{a}_i^\delta \quad \text{for all } i < 2$$

implies that $\mathbf{v} = 0$.

Proof. Denote by f the inclusion map from $\Omega \setminus \{\delta\}$ to Ω , and observe that $\mathcal{L}(f)$ corresponds to the inclusion $\mathcal{L}(\Omega \setminus \{\delta\}) \subseteq \mathcal{L}(\Omega)$. Since \mathcal{F} is diluting, there are elements $\mathbf{x}_0, \mathbf{x}_1 \in \mathcal{G}(\Omega \setminus \{\delta\})$ and $\mathbf{y} \in \mathcal{L}(\Omega \setminus \{\delta\})$ such that

$$\mathbf{y} \leq \mathbf{u}_0 \vee \mathbf{u}_1, \quad \mathcal{G}(f)(\mathbf{x}_i) \leq \mathbf{u}_i \quad \text{for all } i < 2, \quad \mathbf{v} \leq \mathbf{x}_0 \vee \mathbf{x}_1 \vee \mathbf{y}.$$

Fix $i < 2$. There is a unique retraction $p_i: \mathcal{L}(\Omega) \rightarrow \mathcal{L}(\Omega \setminus \{\delta\})$ satisfying $p_i(\mathbf{a}_i^\delta) = 0$ and $p_i(\alpha_{1-i}^\delta) = 1$. Observe that $q_i = \mathcal{F}(p_i): \mathcal{G}(\Omega) \rightarrow \mathcal{G}(\Omega \setminus \{\delta\})$ is a retraction with respect to $\mathcal{G}(f)$. Since $\mathbf{x}_i \in \mathcal{G}(\Omega \setminus \{\delta\})$ we have $q_i(\mathcal{G}(f)(\mathbf{x}_i)) = \mathbf{x}_i$, while, by our assumptions, $q_i(\mathbf{a}_i^\delta) = 0$. Since $\mathcal{G}(f)(\mathbf{x}_i) \leq \mathbf{u}_i \leq \mathbf{a}_i^\delta$, we conclude that $\mathbf{x}_i = 0$.

Let $r_i: \mathcal{L}(\Omega) \rightarrow \mathcal{L}(\Omega \setminus A_{1-i})$ be a unique retraction such that $r_i(\mathbf{a}_{\varphi_{1-i}}^{A_{1-i}}) = 0$, and put $s_i = \mathcal{F}(r_i)$. From $\mathbf{u}_{1-i} \leq \mathbf{a}_{\varphi_{1-i}}^{A_{1-i}}$ it follows that $s_i(\mathbf{u}_{1-i}) = 0$. By Lemma 3.1, there are $\mathbf{y}_j \in \mathcal{L}(\Omega \setminus A_{1-j})$ with $\mathbf{y}_j \leq \mathbf{u}_j$ for all $j < 2$ such that $\mathbf{y} \leq \mathbf{y}_0 \vee \mathbf{y}_1$. Since $\mathbf{y}_j \leq \mathbf{u}_j \leq \mathbf{a}_{\varphi_j}^{A_j}, \mathbf{a}_j^\delta$ and $\delta \notin A_j$, we conclude that $\mathbf{y}_j = 0$ for all $j < 2$. ■

4. Free distributive extension is diluting. We summarize the main properties of the construction of the extension $\mathcal{R}(S)$ of a $(\vee, 0)$ -semilattice S (see [9, Section 2]) referring to the outline in [17, Sections 3,4]. We shall prove that the functor \mathcal{R} is diluting. For a $(\vee, 0)$ -semilattice S , we shall put $\mathcal{C}(S) = \{(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in S^3 \mid \mathbf{c} \leq \mathbf{a} \vee \mathbf{b}\}$. We say that a finite subset \mathbf{v} of $\mathcal{C}(S)$ is *reduced* if the following properties are satisfied:

- (1) the set \mathbf{v} contains exactly one triple of the form $(\mathbf{a}, \mathbf{a}, \mathbf{a})$; we define $\pi(\mathbf{v}) = \mathbf{a}$ and $\mathbf{v}^* = \mathbf{v} \setminus \{(\mathbf{a}, \mathbf{a}, \mathbf{a})\}$.
- (2) $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathbf{v}$ and $(\mathbf{b}, \mathbf{a}, \mathbf{c}) \in \mathbf{v}$ implies that $\mathbf{a} = \mathbf{b} = \mathbf{c}$, for all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in S$.
- (3) if $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathbf{v}^*$, then $\mathbf{a}, \mathbf{b}, \mathbf{c} \not\leq \pi(\mathbf{v})$, for all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in S$.

Observe that if \mathbf{v} is a reduced subset of $\mathcal{C}(S)$ and $\mathbf{u} \subseteq \mathbf{v}^*$, then $\mathbf{u} \cup \{(0, 0, 0)\}$ is a reduced subset as well.

We denote by $\mathcal{R}(S)$ the set of all reduced subsets of $\mathcal{C}(S)$. By [9, Lemma 2.1] (see also [17, Corollary 3.2]), $\mathcal{R}(S)$ is a $(\vee, 0)$ -semilattice with respect to the partial ordering \leq defined by

$$(4.1) \quad \mathbf{v} \leq \mathbf{u} \quad \text{iff} \quad \text{for all } (\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathbf{v} \setminus \mathbf{u} \text{ either } \mathbf{a} \leq \pi(\mathbf{u}) \text{ or } \mathbf{c} \leq \pi(\mathbf{u}),$$

and the assignment $\mathbf{v} \mapsto \{(\mathbf{v}, \mathbf{v}, \mathbf{v})\}$ is a $(\vee, 0)$ -embedding from S into $\mathcal{R}(S)$.

As in [17], we use the symbol \bowtie_S to denote the elements of $\mathcal{R}(S)$ defined as

$$\bowtie_S(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \begin{cases} \mathbf{c} & \text{if either } \mathbf{a} = \mathbf{b} \text{ or } \mathbf{b} = 0 \text{ or } \mathbf{c} = 0, \\ 0 & \text{if } \mathbf{a} = 0, \\ \{(0, 0, 0), (\mathbf{a}, \mathbf{b}, \mathbf{c})\} & \text{otherwise,} \end{cases}$$

for all $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathcal{C}(S)$. Recall that by formula (3.3) in [17],

$$(4.2) \quad \mathbf{x} = \bigvee \{ \bowtie_S(\mathbf{a}, \mathbf{b}, \mathbf{c}) \mid (\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathbf{x} \} \quad \text{for all } \mathbf{x} \in \mathcal{R}(S).$$

By [17, Proposition 3.5], every $(\vee, 0)$ -homomorphism $f: S \rightarrow T$ extends to a unique $(\vee, 0)$ -homomorphism $\mathcal{R}(f): S \rightarrow T$ such that

$$(4.3) \quad \mathcal{R}(f)(\bowtie_S(\mathbf{a}, \mathbf{b}, \mathbf{c})) = \bowtie_T(f(\mathbf{a}), f(\mathbf{b}), f(\mathbf{c})) \quad \text{for all } (\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathcal{C}(S),$$

and the assignment $S \mapsto \mathcal{R}(S)$, $f \mapsto \mathcal{R}(f)$, is a functor on the category \mathbf{S} . It follows that if $f: S \rightarrow T$ is a $(\vee, 0)$ -homomorphism, $\mathbf{v} \in S$, and $\mathbf{u} \in T$, then

$$(4.4) \quad \mathcal{R}(f)(\mathbf{v}) \leq \mathbf{u} \quad \text{iff} \quad \bowtie_T(f(\mathbf{a}), f(\mathbf{b}), f(\mathbf{c})) \leq \mathbf{u} \quad \text{for all } (\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathbf{v}.$$

LEMMA 4.1. *The functor \mathcal{R} is diluting.*

Proof. Let S and T be $(\vee, 0)$ -semilattices and let $f: S \rightarrow T$ be a $(\vee, 0)$ -homomorphism. We have to verify that for every $\mathbf{v} \in \mathcal{R}(S)$ and $\mathbf{u}_0, \mathbf{u}_1 \in \mathcal{R}(T)$ such that $\mathcal{R}(f)(\mathbf{v}) \leq \mathbf{u}_0 \vee \mathbf{u}_1$, there are $\mathbf{x}_0, \mathbf{x}_1 \in \mathcal{R}(S)$ and $\mathbf{y} \in S$ such that

$$f(\mathbf{y}) \leq \mathbf{u}_0 \vee \mathbf{u}_1, \quad \mathcal{R}(f)(\mathbf{x}_i) \leq \mathbf{u}_i \quad \text{for all } i < 2, \quad \mathbf{v} \leq \mathbf{x}_0 \vee \mathbf{x}_1 \vee \mathbf{y}.$$

For all $i < 2$ define

$$\mathbf{x}_i = \{ (\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathbf{v} \mid (f(\mathbf{a}), f(\mathbf{b}), f(\mathbf{c})) \in \mathbf{u}_i^* \} \cup \{(0, 0, 0)\},$$

and observe that $\mathbf{x}_0, \mathbf{x}_1$, as subsets of $\mathbf{v}^* \cup \{(0, 0, 0)\}$, are reduced, that is, $\mathbf{x}_0, \mathbf{x}_1 \in \mathcal{R}(S)$. It follows from (4.3) that $\mathcal{R}(f)(\mathbf{x}_i) \leq \mathbf{u}_i$ for all $i < 2$. An easy application of [17, Lemma 3.1] yields $(\mathbf{u}_0 \vee \mathbf{u}_1)^* \subseteq \mathbf{u}_0^* \cup \mathbf{u}_1^*$, and so $\bowtie_S(\mathbf{a}, \mathbf{b}, \mathbf{c}) \leq \mathbf{x}_0 \vee \mathbf{x}_1$ for every $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathbf{v}$ such that $(f(\mathbf{a}), f(\mathbf{b}), f(\mathbf{c})) \in (\mathbf{u}_0 \vee \mathbf{u}_1)^*$.

For all $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathbf{v}$ define

$$\varrho((\mathbf{a}, \mathbf{b}, \mathbf{c})) = \begin{cases} \mathbf{a} & \text{if } f(\mathbf{a}) \leq \pi(\mathbf{u}_0 \vee \mathbf{u}_1), \\ \mathbf{c} & \text{otherwise,} \end{cases}$$

and put

$$\mathbf{y} = \bigvee \{ \varrho((\mathbf{a}, \mathbf{b}, \mathbf{c})) \mid (\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathbf{v} \text{ and } (f(\mathbf{a}), f(\mathbf{b}), f(\mathbf{c})) \notin (\mathbf{u}_0 \vee \mathbf{u}_1)^* \}.$$

Clearly, $\mathbf{y} \in S$, and, by (4.1), $\bowtie_S(\mathbf{a}, \mathbf{b}, \mathbf{c}) \leq \mathbf{y}$ for all $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathbf{v}$ such that $(f(\mathbf{a}), f(\mathbf{b}), f(\mathbf{c})) \notin (\mathbf{u}_0 \vee \mathbf{u}_1)^*$. Overall, we have proved that $\bowtie_S(\mathbf{a}, \mathbf{b}, \mathbf{c}) \leq \mathbf{x}_0 \vee \mathbf{x}_1 \vee \mathbf{y}$ for all $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathbf{v}$, and so, by (4.2), $\mathbf{v} \leq \mathbf{x}_0 \vee \mathbf{x}_1 \vee \mathbf{y}$.

Finally, since $\mathcal{R}(f)(\mathbf{v}) \leq \mathbf{u}_0 \vee \mathbf{u}_1$, it follows from (4.1) that $f(\varrho((\mathbf{a}, \mathbf{b}, \mathbf{c}))) \leq \pi(\mathbf{u}_0 \vee \mathbf{u}_1)$ for every $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathbf{v}$ such that $(f(\mathbf{a}), f(\mathbf{b}), f(\mathbf{c})) \notin (\mathbf{u}_0 \vee \mathbf{u}_1)^*$, whence $f(\mathbf{y}) \leq \mathbf{u}_0 \vee \mathbf{u}_1$. ■

Observe that $\mathcal{R}(S)$ is distributive “relative to” the $(\vee, 0)$ -semilattice S , that is, for any $\mathbf{a}, \mathbf{b}, \mathbf{c} \in S$ with $\mathbf{c} \leq \mathbf{a} \vee \mathbf{b}$, there are $\mathbf{a}' \leq \mathbf{a}$, $\mathbf{b}' \leq \mathbf{b}$ in $\mathcal{R}(S)$ such that $\mathbf{c} = \mathbf{a}' \vee \mathbf{b}'$. It follows that the $(\vee, 0)$ -semilattice $\mathcal{R}^\infty(S)$ is distributive. By Lemma 3.2, we obtain

COROLLARY 4.2. *The functor \mathcal{R}^∞ is diluting. Moreover, $\mathcal{R}^\infty(S)$ is a distributive $(\vee, 0)$ -semilattice for every $(\vee, 0)$ -semilattice S .*

Observe that the functor \mathcal{R}^∞ is denoted by \mathcal{D} in [17].

5. Free trees. Let k be a positive integer and let Ω be a set. Given a map $\Psi: [\Omega]^{k-1} \rightarrow [\Omega]^{<\omega}$, we say that a k -element subset B of Ω is *free* (with respect to Ψ) if $b \notin \Psi(B \setminus \{b\})$ for all $b \in B$.

KURATOWSKI'S FREE SET THEOREM [7]. *Let k be a positive integer, let Ω be a set, and let $\Psi: [\Omega]^{k-1} \rightarrow [\Omega]^{<\omega}$ be any map. If $|\Omega| \geq \aleph_{k-1}$, then there is a k -element free subset of Ω .*

Notation. Let k and n be natural numbers with $k > 0$. Given a natural number $m \leq n$ and a map $g: \{m, \dots, n-1\} \rightarrow k$, we shall put

$$T_{n,k}(g) = \{f: n \rightarrow k \mid f \text{ extends } g\}$$

Given a natural number $m < n$, a map $g: \{m+1, \dots, n-1\} \rightarrow k$, and $i < k$, we shall use the notation

$$\begin{aligned} T_{n,k}(g, i) &= \{f \in T_{n,k}(g) \mid f(m) = i\}, \\ T_{n,k}(g, -i) &= \{f \in T_{n,k}(g) \mid f(m) \neq i\}. \end{aligned}$$

DEFINITION 2. Let Ω be a set and let $\Phi: [\Omega]^{<\omega} \rightarrow [\Omega]^{<\omega}$ be a map. Let k and n be natural numbers with $k > 0$. We say that a family $\mathcal{T} = (\alpha(f) \mid f: n \rightarrow k)$ of distinct elements of Ω is a *free k -tree of height n* (with respect to Φ) if

$$(5.1) \quad \{\alpha(f) \mid f \in T_{n,k}(g, i)\} \cap \Phi(\{\alpha(f) \mid f \in T_{n,k}(g, -i)\}) = \emptyset$$

for all $m < n$, all maps $g: \{m+1, \dots, n-1\} \rightarrow k$, and all $i < k$. We will call the set $\text{rng } \mathcal{T} = \{\alpha(f) \mid f: n \rightarrow k\}$ the *range* of \mathcal{T} .

LEMMA 5.1. *Let Ω be a set and let $\Phi: [\Omega]^{<\omega} \rightarrow [\Omega]^{<\omega}$ be a map. Let k be a positive integer. Every subset X of Ω of cardinality at least \aleph_{k-1} contains the range of a free k -tree of height n , for every natural number n .*

Proof. We shall argue by induction on n . If $n = 0$, pick $\mathcal{T} = \{\alpha_\emptyset\}$, where α_\emptyset is an arbitrary element of X . Let n be natural number and suppose that the statement holds for n . We shall prove that X contains a free k -tree, \mathcal{T} ,

of height $n + 1$. Decompose X into a union of pairwise disjoint subsets X_ξ for $\xi < \omega_{k-1}$ of cardinality at least \aleph_{k-1} . By the induction hypothesis, each X_ξ contains the range of a free tree $\mathcal{T}_\xi = (\alpha_\xi(f) \mid f: n \rightarrow k)$ of height n . Define a map $\Psi: [\omega_{k-1}]^{k-1} \rightarrow [\omega_{k-1}]^{<\omega}$ by

$$(5.2) \quad \Psi(F) = \left\{ \nu < \omega_{k-1} \mid \text{rng } \mathcal{T}_\nu \cap \Phi \left(\bigcup_{\xi \in F} \text{rng } \mathcal{T}_\xi \right) \neq \emptyset \right\}$$

for all $F \in [\omega_{k-1}]^{k-1}$ (since the sets $\text{rng } \mathcal{T}_\xi$ are pairwise disjoint and finite, $\Psi(F)$ is finite for every $F \in [\omega_{k-1}]^{k-1}$). By Kuratowski's Free Set Theorem, there is a k -element free subset, $B = \{\xi_0, \dots, \xi_{k-1}\}$, of X with respect to Ψ . Put $\alpha(f) = \alpha_{\xi_{f(n)}}(f \upharpoonright n)$ for all maps $f: (n + 1) \rightarrow k$. We claim that $\mathcal{T} = (\alpha(f) \mid f: (n + 1) \rightarrow k)$ is a free k -tree with respect to Φ . Let $m < n + 1$ and fix a map $g: \{m + 1, \dots, n\} \rightarrow k$. If $m = n$, the only possibility is $g = \emptyset$. Then

$$\begin{aligned} \{\alpha(f) \mid f \in T_{n+1,k}(g, i)\} &= \text{rng } \mathcal{T}_{\xi_i}, \\ \{\alpha(f) \mid f \in T_{n+1,k}(g, \neg i)\} &= \bigcup_{j < k, j \neq i} \text{rng } \mathcal{T}_{\xi_j}, \end{aligned}$$

for all $i < k$. Since B is a free set with respect to Ψ ,

$$\text{rng } \mathcal{T}_{\xi_i} \cap \Phi \left(\bigcup_{j < k, j \neq i} \text{rng } \mathcal{T}_{\xi_j} \right) = \emptyset,$$

by (5.2). Let $m < n$ and $i < k$. Put $g' = g \upharpoonright \{m + 1, \dots, n - 1\}$. Then

$$\begin{aligned} \{\alpha(f) \mid f \in T_{n+1,k}(g, i)\} &= \{\alpha_{\xi_{g(n)}}(f) \mid f \in T_{n,k}(g', i)\}, \\ \{\alpha(f) \mid f \in T_{n+1,k}(g, \neg i)\} &= \{\alpha_{\xi_{g(n)}}(f) \mid f \in T_{n,k}(g', \neg i)\}. \end{aligned}$$

Since $\mathcal{T}_{\alpha_{\xi_{g(n)}}$ is a free k -tree with respect to Φ ,

$$\{\alpha_{\xi_{g(n)}}(f) \mid f \in T_{n,k}(g', i)\} \cap \Phi(\{\alpha_{\xi_{g(n)}}(f) \mid f \in T_{n,k}(g', \neg i)\}) = \emptyset,$$

by (5.1). ■

6. The optimal bound in Wehrung's theorem. Let \mathcal{F} be an expanding functor on \mathbf{S} satisfying the following properties: For every $(\vee, 0)$ -semilattice S and every family $(S_i \mid i \in I)$ of $(\vee, 0)$ -subsemilattices of S ,

$$(6.1) \quad \bigcap_{i \in I} \mathcal{F}(S_i) = \mathcal{F} \left(\bigcap_{i \in I} S_i \right).$$

For a nonempty upwards directed poset P and every family $(S_p \mid p \in P)$ of $(\vee, 0)$ -semilattices such that S_p is a $(\vee, 0)$ -subsemilattice of S_q whenever $p \leq q$ in P ,

$$(6.2) \quad \bigcup_{p \in P} \mathcal{F}(S_p) = \mathcal{F} \left(\bigcup_{p \in P} S_p \right).$$

Put $\mathcal{G} = \mathcal{F} \circ \mathcal{L}$. Then for every set Ω and every family $(A_i \mid i \in I)$ of subsets of Ω ,

$$\bigcap_{i \in I} \mathcal{G}(A_i) = \mathcal{G}\left(\bigcap_{i \in I} A_i\right),$$

and for a nonempty upwards directed poset P and every family $(A_p \mid p \in P)$ of sets such that $A_p \subseteq A_q$ whenever $p \leq q$ in P ,

$$\bigcup_{p \in P} \mathcal{G}(A_p) = \mathcal{G}\left(\bigcup_{p \in P} A_p\right).$$

It follows that, given a set Ω and an element $\mathbf{a} \in \mathcal{G}(\Omega)$, there is a smallest finite subset F of Ω such that $\mathbf{a} \in \mathcal{G}(F)$. We shall call the subset F the *support* of \mathbf{a} . We denote the support of \mathbf{a} by $\text{Supp}(\mathbf{a})$ (see [17]). Now we rephrase [17, Theorem 6.1].

THEOREM 6.1. *Let Ω be a set of cardinality at least \aleph_2 and let \mathcal{F} be a diluting functor satisfying (6.1) and (6.2). Let A be an algebra with either a congruence-compatible structure of a $(\vee, 1)$ -semilattice or a congruence compatible structure of a lattice. Then there does not exist a weakly distributive $(\vee, 0)$ -homomorphism from $\text{Con}_c A$ to $\mathcal{G}(\Omega)$ containing 1 in its range.*

Proof. Assume for contradiction that there exists a weakly distributive $(\vee, 0)$ -homomorphism $\mu: \text{Con}_c A \rightarrow \mathcal{G}(\Omega)$ with 1 in its range. Since 1 is in the range of μ , there is a finite subset T of Ω such that

$$\mu\left(\bigvee\{\Theta_A(s, t) \mid s, t \in T\}\right) = 1.$$

Put $s = 1$ (the largest element of A) if A has a congruence-compatible structure of a $(\vee, 1)$ -semilattice, and $s = \bigvee T$ if A has a congruence compatible structure of a lattice. Then

$$\mu\left(\bigvee_{t \in T} \Theta_A(s, t)\right) = \bigvee_{t \in T} \mu(\Theta_A(s, t)) = 1.$$

Let $\xi \in \Omega$. Since the homomorphism μ is weakly distributive, there are $\theta_0^\xi, \theta_1^\xi \in \text{Con}_c A$ such that $\bigvee_{t \in T} \Theta_A(s, t) \leq \theta_0^\xi \vee \theta_1^\xi$ and $\mu(\theta_i^\xi) \leq \mathbf{a}_i^\xi$ for $i = 0, 1$. In particular, $\Theta_A(s, t) \leq \theta_0^\xi \vee \theta_1^\xi$ for every $t \in T$. Fix $t \in T$. Since $\Theta_A(s, t) \leq \theta_0^\xi \vee \theta_1^\xi$, there are a positive integer n_ξ and elements $t = z_0^\xi, z_1^\xi, \dots, z_{n_\xi}^\xi = s$ in A such that

$$(6.3) \quad \mu\Theta_A(z_i^\xi, z_{i+1}^\xi) \leq \mathbf{a}_{\varepsilon(i)}^\xi \quad \text{for all } i < n_\xi,$$

for all $\xi \in \Omega$. (Recall that $\varepsilon(i) = i \bmod 2$.)

If A has a congruence-compatible structure of a $(\vee, 1)$ -semilattice, we replace each z_i^ξ with $\bigvee_{j=0}^i z_j^\xi$. If A has a congruence-compatible structure of a lattice, we replace each z_i^ξ with $s \wedge (\bigvee_{j=0}^i z_j^\xi)$. In both cases we obtain an

increasing chain $t = z_0^\xi \leq z_1^\xi \leq \dots \leq z_{n_\xi}^\xi = s$ in A such that (6.3) remains satisfied.

As in [17, Section 6], we denote by $\text{Con}_c^U A$ the $(\vee, 0)$ -subsemilattice of $\text{Con}_c A$ generated by all principal congruences $\Theta_A(u, v)$, where $u, v \in U$, for every subset U of A . Further, we denote by $S(F)$ the $(\vee, 0)$ -subsemilattice of A generated by $\{z_i^\xi \mid \xi \in F \text{ and } 0 \leq i \leq n_\xi\}$, and we put

$$(6.4) \quad \Phi(F) = \bigcup \{ \text{Supp}(\mu\theta) \mid \theta \in \text{Con}_c^{S(F)} A \}$$

for every subset F of Ω . Observe that if the subset F is finite, then both $S(F)$ and $\Phi(F)$ are finite.

Since the size of Ω is at least \aleph_2 , there are a positive integer n and a subset X of Ω of cardinality at least \aleph_2 such that $n_\xi = n$ for all $\xi \in X$. The following crucial claim is analogous to [17, Lemma 6.2], giving another illustration of the “erosion method”.

CLAIM 1. *Let $\mathcal{T} = (\alpha(f) \mid f: n \rightarrow 3)$ be a free 3-tree with respect to Φ with $\text{rng } \mathcal{T} \subseteq X$. Then*

$$(6.5) \quad \Theta_A\left(s, \bigvee_{f \in T_{n,2}(g)} z_{n-m}^{\alpha(f)}\right) = 0$$

for every nonnegative integer $m \leq n$ and every map $g: \{m, \dots, n-1\} \rightarrow 2$.

Proof of Claim. We shall argue by induction on m . If $m = 0$, then the equality (6.5) is trivially satisfied for every map $g: \{m, \dots, n-1\} \rightarrow 2$. Let $m < n$, let $g: \{m+1, \dots, n-1\} \rightarrow 2$ be a map, and suppose that (6.5) is satisfied at stage m . Put

$$x_i = \bigvee_{f \in T_{n,2}(g,i)} z_{n-m-1}^{\alpha(f)} \quad \text{for all } i < 2.$$

Fix $i < 2$. Clearly,

$$\mu\Theta_A(s, x_i) \leq \left(\bigvee_{f \in T_{n,2}(g,i)} \mu\Theta_A(z_{n-m-1}^{\alpha(f)}, z_{n-m}^{\alpha(f)}) \right) \vee \mu\Theta_A\left(s, \bigvee_{f \in T_{n,2}(g,i)} z_{n-m}^{\alpha(f)}\right).$$

Put

$$v = \mu\Theta_A\left(s, \bigvee_{f \in T_{n,2}(g)} z_{n-m-1}^{\alpha(f)}\right) = \mu\Theta_A(s, x_0 \vee x_1).$$

Put $A_i = \{\alpha(f) \mid f \in T_{n,2}(g, i)\}$, and let $\varphi_i: A_i \rightarrow 2$ be the constant map with the value $\varepsilon(n-m-1)$. By the induction hypothesis

$$\mu\Theta_A\left(s, \bigvee_{f \in T_{n,2}(g,i)} z_{n-m}^{\alpha(f)}\right) = 0,$$

and, by (6.3), $\mu\Theta_A(z_{n-m-1}^{\alpha(f)}, z_{n-m}^{\alpha(f)}) \leq \mathbf{a}_{\varepsilon(n-m-1)}^{\alpha(f)}$ for all $f \in T_{n,2}(g, i)$. Thus

$$\mu\Theta_A(s, x_i) \leq \bigvee_{f \in T_{n,2}(g,i)} \mathbf{a}_{\varepsilon(n-m-1)}^{\alpha(f)} = \mathbf{a}_{\varphi_i}^{A_i}.$$

Let δ be any element of X . By the Erosion Lemma [17, Lemma 5.1], there are $\mathbf{u}_j \in \text{Conc}_c^{S(A_j \cup \{\delta\})} A$ such that $\mathbf{v} \leq \mathbf{u}_0 \vee \mathbf{u}_1$ and $\mathbf{u}_j \leq \mathbf{a}_{\varepsilon(j)}^\delta, \mu\Theta_A(s, x_j)$ (and so $\mathbf{u}_j \leq \mathbf{a}_{\varepsilon(j)}^\delta, \mathbf{a}_{\varphi_j}^{A_j}$) for all $j < 2$.

Now let $\delta = \alpha(f)$ for some $f \in T_{n,3}(g, 2)$. By (6.4), $\text{Supp}(\mathbf{v}) \subseteq \Phi(\{\alpha(f) \mid f \in T_{n,2}(g)\}) = \Phi(A_0 \cup A_1)$ and $\text{Supp}(\mathbf{u}_j) \subseteq \Phi(A_j \cup \{\delta\})$ for all $j < 2$. Since \mathcal{T} is a free 3-tree with respect to Φ , we have $\delta \notin \Phi(A_0 \cup A_1)$ and $A_{1-j} \cap \Phi(A_j \cup \{\delta\}) = \emptyset$ for all $j < 2$. It follows that $\mathbf{v} \in \mathcal{G}(\Omega \setminus \{\delta\})$ and $\mathbf{u}_j \in \mathcal{G}(\Omega \setminus A_{1-j})$ for all $j < 2$. Since the functor \mathcal{F} is diluting, applying Lemma 3.3, we conclude that $\mathbf{v} = 0$ as desired. $\blacksquare_{\text{Claim 1}}$

Now we finish the proof of Theorem 6.1. By Lemma 5.1, the set X contains the range of a free 3-tree $\mathcal{T} = (\alpha(f) \mid f : n \rightarrow 3)$ of height n . By Claim 1,

$$\Theta_A\left(s, \bigvee_{f \in T_{n,2}(g)} z_{n-m}^{\alpha(f)}\right) = 0$$

for every nonnegative integer $m \leq n$ and every map $g : \{m, \dots, n-1\} \rightarrow 2$. For $m = n$ and $g = \emptyset$ we have

$$\bigvee_{f \in T_{n,2}(g)} z_{n-m}^{\alpha(f)} = \bigvee_{f : n \rightarrow 2} z_0^{\alpha(f)} = t,$$

whence $\Theta_A(s, t) = 0$ for every $t \in T$. Consequently, $\bigvee_{t \in T} \Theta_A(s, t) = 0$, which leads to a contradiction. \blacksquare

By Corollary 4.2, the functor \mathcal{R}^∞ is diluting, and by [17, Lemma 3.6], it satisfies both (6.1) and (6.2). Let us denote by \mathcal{G} the composition $\mathcal{R}^\infty \circ \mathcal{L}$ (it is the same \mathcal{G} as the one considered in [17]). Since $\mathcal{R}^\infty(S)$ is distributive for every $(\vee, 0)$ -semilattice S , we obtain the following corollary.

COROLLARY 6.2. *Let Ω be a set of cardinality at most \aleph_2 . Then there is no lattice L such that $\mathcal{G}(\Omega)$ is isomorphic to $\text{Conc}_c L$.*

A. P. Huhn [6] (see also [3, Theorem 13 in Appendix C]) proved that every distributive $(\vee, 0)$ -semilattice of size at most \aleph_1 is isomorphic to $\text{Conc}_c L$ for some lattice L . Moreover, the lattice L can be taken sectionally complemented and modular [16, Corollary 5.3], or relatively complemented, locally finite, and with zero [4]. In particular, in all these cases, the lattice L has permutable congruences [1].

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Department of Algebra
 Faculty of Mathematics and Physics
 Charles University
 Sokolovská 83
 186 75 Praha 8, Czech Republic
 E-mail: ruzicka@karlin.mff.cuni.cz

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