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Rainbow Ramsey theorems for colorings establishing negative partition relations

by

András Hajnal (Budapest)

Abstract. Given a function f, a subset of its domain is a rainbow subset for f if f is one-to-one on it. We start with an old Erdős problem: Assume f is a coloring of the pairs of ω_1 with three colors such that every subset A of ω_1 of size ω_1 contains a pair of each color. Does there exist a rainbow triangle? We investigate rainbow problems and results of this style for colorings of pairs establishing negative "square bracket" relations.

1. Introduction and history. Anti-Ramsey theorems appeared probably for the first time in the 1973 paper [9] of Richard Rado, claiming the existence of subsets with elements of different colors of the domain of a given coloring. Later in the game, the more expressive name of rainbow subset was coined. In this paper we will mostly consider 2-partitions, i.e. colorings f of unordered pairs of a set. A subset of pairs will be called a *rainbow subset* (for f) if f is one-to-one on it. Our starting point will be a problem of Paul Erdős, stated long before any of these names were coined:

Assume $f : [\omega_1]^2 \to 3$ is a 2-partition of ω_1 with three colors such that each subset $A \subseteq \omega_1$ of size ω_1 contains a pair of each color. Does there exist a rainbow triangle for f?

This is Problem 68 of [3] written in 1967. We restate it in the jargon of partition relations developed in [5]:

Problem 1.1. *Assume $f : [\omega_1]^2 \to 3$ establishes $\omega_1 \not\to [\omega_1]^2_3$. Does there exist a rainbow triangle for f?*

We knew that the answer is affirmative under some stronger conditions e.g.

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Fact 1.2. *Assume* $f : [\omega_1]^2 \to 3$ *establishes* $\omega_1 \not\to [(\omega, \omega_1)]^2_3$ (*i.e. for* $A \in [\omega_1]^\omega$ *and* $B \in [\omega_1]^{\omega_1}$, f *takes all three values on* $[A, B]^{1,1}$). *Then there exists a rainbow triangle for* f.

However, in those early days, we could only construct an f satisfying the condition of 1.2 using CH.

Definition 1.3. For a coloring $d : [k]^2 \to \omega_1$, $k \le \omega$ we write $d \Rightarrow f$ if there is a one-to-one map $\Phi : k \to \omega_1$ such that

$$d(\{n, m\}) = f(\{\Phi(n), \Phi(m)\}) \quad \text{for } n, m \in k.$$

We could generalize 1.2 to

Fact 1.4. *Assume* $f : [\omega_1]^2 \to \omega_1$ *establishes* $\omega_1 \not\to [(\omega, \omega_1)]^2_{\omega_1}$. *Then* $d \Rightarrow f$ *for an arbitrary* $d : [\omega]^2 \to \omega_1$.

As already mentioned, we were not able to verify in ZFC that this does not hold vacuously and it bothered us that we could not lift it e.g. replacing ω, ω_1 by ω_1, ω_2 respectively. The next steps were taken in a paper of Shelah [10] written in 1975. He proved

Theorem 1.5 (Shelah [10]).

1. *CH implies that* 1.1 *fails for some* f *with* ω *colors.*
2. $\diamond$ *implies that* 1.1 *fails for an* f *with* ω_1 *colors.*

Shelah also showed in [10] that a possible "lifting" of Fact 1.4 is consistently false say adding one Cohen real to a model of GCH. In more detail, he constructed a graph of size ω_1 from the Cohen real which does not embed isomorphically into any graph of the ground model. Then any graph of the ground model establishing the partition relation $\omega_2 \not\to [(\omega_2, \omega_1)]^2_{\omega_1}$ satisfies the same relation in the new model, and we have a graph of size ω_1 that does not embed into it.

Knowing all this, in our 1978 paper [2] we stated implicitly a generalization of 1.4.

Theorem 1.6 ([2]). *Assume that* f *establishes* $\omega_1 \not\to [(\omega_1; \omega_1)]^2_\omega$. *Then* $d \Rightarrow f$ *for an arbitrary* $d : [\omega]^2 \to \omega$.

The symbol with the semi-colon ";" means that all ω_1 by ω_1 "half-graphs" are totally multicolored, i.e. for all $A, B \subseteq \omega_1$ with $|A| = |B| = \omega_1$ and $n < \omega$ there are $\alpha \in A$ and $\beta \in B$ with $\alpha < \beta$ such that $f(\{\alpha, \beta\}) = n$. I want to mention that [2] seems to be the first paper in print where this important concept was used. I think it was invented (discovered) by Fred Galvin. The following was proved 37 years later by Justin Moore:

Theorem 1.7 (Moore [7]). (*ZFC*) *There is an* f *establishing*

$$\omega_1 \not\to [(\omega_1; \omega_1)]^2_{\omega_1}.$$

This is a byproduct of Moore's result [7] showing the existence of L-spaces in ZFC. All the above justifies revisiting the old Problem 1.1.

2. $\not\Rightarrow$ relations. First we remark that we still do not know if the conclusions of either clauses of Theorem 1.5 can be proved under weaker conditions. Next we want to show that a Theorem 1.7 type generalization cannot hold if we only assume that each $[A]^2$ with $|A| = \omega_1$ is totally multicolored.

THEOREM 2.1. *There exist a rainbow $d : [4]^2 \to 6$ and an $f : [\omega_1]^2 \to 6$ establishing $\omega_1 \not\to [\omega_1]^2_6$ such that*

$$d \not\Rightarrow f.$$

Proof. First we define $e : [4]^2 \to W$ and $g : [\omega_1]^2 \to W$ where

$$W = \{(+,+),(+,-),(-,+),(-,-)\}.$$

Let

$$e(\{0,1\}) = (+,-), \quad e(\{1,2\}) = (-,+), \quad e(\{2,3\}) = (+,-),$$
$$e(\{0,3\}) = (-,+), \quad e(\{0,2\}) = (+,+), \quad e(\{1,3\}) = (-,-).$$

Let $<_R$ and $<_A$ be real and Aronszajn type orderings of ω_1. For $\alpha < \beta < \omega_1$ let $g(\alpha,\beta) = (u,v)$ with $u,v \in \{+,-\}$, where $u = +$ iff $\alpha <_A \beta$, and $v = +$ iff $\alpha <_R \beta$.

It is a well known property of these orderings that for all $B \in [\omega_1]^{\omega_1}$ there are $C,D,E,F \in [B]^{\omega_1}$ such that $C <_A D$, $C <_R D$, $E <_A F$ and $F <_R E$. This implies that each $B \in [\omega_1]^{\omega_1}$ contains a complete ω_1 by ω_1 half-graph for g in each of the colors in W.

It is an easy exercise to see that $e \not\Rightarrow g$. Let now h be as in Moore's Theorem 1.7. Then $k = (g,h)$ establishes $\omega_1 \not\to [\omega_1]^2_{\omega_1}$. Using k and e it is a matter of easy calculation to get f and d as required in the theorem. ∎

Next we are going to investigate the cases when f establishes

$$\omega_1 \not\to [(\omega_1,\omega_1)]^2_\gamma,$$

i.e. all ω_1 by ω_1 subgraphs are totally multicolored for some γ.

FACT 2.2. *Assume f establishes $\omega_1 \not\to [(\omega_1,\omega_1)]^2_3$. Let $d : [3]^2 \to 3$ be one-to-one. Then $d \Rightarrow f$, i.e. all possible rainbow triangles exist.*

Proof. The assumption implies that for some $\alpha \in \omega_1$ both sets

$$\{\beta \in \omega_1 : f(\alpha,\beta) = d(0,1)\}, \quad \{\gamma \in \omega_1 : f(\alpha,\gamma) = d(0,2)\}$$

are of cardinality ω_1. ∎

FACT 2.3. *There exist a rainbow $d : [5]^2 \to 10$ and an $f : [\omega_1]^2 \to 10$ establishing $\omega_1 \not\to [(\omega_1,\omega_1)]^2_{10}$ such that*

$$d \not\Rightarrow f.$$

Proof (outline). Define $e : [5]^2 \to 2$ by the stipulation

$$e(\{i,j\}) = 0 \quad \text{for } i < 5 \text{ and } j \equiv i+1 \bmod 5.$$

That is, e is a "pentagon without a diagonal". Let $d : [5]^2 \to 10$ be one-to-one such that $d(\{i,i+1\}) < 5$ iff $e(\{i,,i+1\}) = 0$. Let $<_R$ be a real type ordering of ω_1. Let $g(\alpha,\beta) : [\omega_1]^2 \to 2$ be the "Sierpiński" partition, that is, $g(\alpha,\beta) = 0$ iff $\alpha <_R \beta$ for $\alpha < \beta < \omega_1$. It is well known that every complete bipartite ω_1 by ω_1 half-graph contains a complete bipartite ω_1 by ω_1 half-graph in both colors for g. Again by Moore's theorem, we can take an h establishing $\omega_1 \not\to [(\omega_1;\omega_1)]\,^2_5$. Set $f = g \cdot 5 + h$. Then f establishes $\omega_1 \not\to [(\omega_1,\omega_1)]^2_{10}$ and $d \Rightarrow f$ would imply $e \Rightarrow g$, which is known to be false. ∎

PROBLEM 2.4. *Can we improve* 2.3 *to have a* $d : [4]^2 \to 6$ *and an* f *establishing* $\omega_1 \not\to [(\omega_1,\omega_1)]^2_6$*?*

3. Rainbow theorems

THEOREM 3.1. *Assume* $f : [\omega_1]^2 \to \omega$ *establishes* $\omega_1 \not\to [(\omega_1,\omega_1)]\,^2_\omega$. *Then there exists an infinite rainbow set.*

Proof. We use $A, B, C, \ldots$ to denote subsets of ω_1 of size ω_1, and $N, M, \ldots$ to denote infinite subsets of ω; moreover, we set

$$f_j(x) = \{y \in \omega_1 : f(x,y) = f(\{x,y\}) = j\}$$

for $j < \omega$ and $x \in \omega_1$.

3.1.1. *Assume* $B \cap C = \emptyset$ *and*

$$\forall n \in M\ \forall x \in B\ (|f_n(x) \cap C| \le \omega).$$

Then

$$\forall n \in M\ \forall C' \subseteq C\ \exists y \in C'\ (|f_n(y) \cap B| = \omega_1).$$

Otherwise we could pick, by transfinite induction, a pair (B', C'') omitting the color n. ∎

Let $(*)(A,N)$ be the following property of A and N: There are $B, C \subseteq A$ and $M \subseteq N$ such that

$$\forall B' \subseteq B\ \forall C' \subseteq C\ \forall m \in M\ \exists x \in B'\ (|f_m(x) \cap C'| = \omega_1).$$

When $(*)(A,N)$ holds we denote by

$$B(A,N),\ C(A,N),\ M(A,N)$$

the relevant sets B, C, M respectively, with $B \cap C = \emptyset$.

3.1.2. *Assume that for some* A_0, N_0, $(*)(A,N)$ *holds for all* $A \subseteq A_0$ *and* $N \subseteq N_0$. *Then there is an infinite rainbow subset.*

Define A_k, B_k, N_k by induction on $k < \omega$. Assume A_k, N_k are defined. Let $B_k = B(A_k, N_k)$, $A_{k+1} = C(A_k, N_k)$, $N_{k+1} = N(A_k, N_k)$. Let $\{N'_k : k < \omega\}$ be a disjoint refinement of $\{N_k : k < \omega\}$ and let

$$N'_k = \{n^k_i : i < \omega\}$$

be a one-to-one enumeration of N'_k for $k < \omega$. It is now easy to pick $x_i \in A_i$ for $i < \omega$ in such a way that $c(x_i, x_j) = n^i_j$ for $i < j < \omega$. This proves 3.1.2, as $\{x_i : i < \omega\}$ is an infinite rainbow set. ∎

Hence to finish the proof of Theorem 3.1 it is sufficient to prove

3.1.3. *Assume $(*)(A, N)$ is false for some A and N. Then A has an infinite rainbow subset.*

Let $N = \bigcup_{k<\omega} N_k$, $A = \bigcup_{k<\omega} A_k$ be disjoint partitions. To prove 3.1.3 we first prove

3.1.4. *There are $x \in A_0$ and $\{n_i \in N_0 : 1 \leq i < \omega\}$ one-to-one such that*

$$|f_{n_i}(x) \cap A_i| = \omega_1 \quad \text{for } 1 \leq i < \omega.$$

For an $x \in A_0$ we try to choose n_i, $1 \leq i < \omega$, by induction on i. Assume we have chosen n_k, $1 \leq k \leq i$, with $|f_{n_k}(x) \cap A_k| = \omega_1$. If there is always an n such that

$$(+) \qquad n \in N_0 \setminus \{n_k : 1 \leq k \leq i\} \quad \text{and} \quad |f_n(x) \cap A_{i+1}| = \omega_1$$

we can choose n_{i+1} to be the smallest of these and 3.1.3 is true. If not, let $i(x)$ be the smallest i for which $(+)$ fails. If $(+)$ fails for all $x \in A_0$ then for some $1 \leq i < \omega$ and $M = N_0 \setminus \{n_k : 1 \leq i\}$,

$$C = \{x \in A_0 : i(x) = i\}$$

has cardinality ω_1. Choosing $B = A_{i+1}$ we find that

$$|f_n(x) \cap B| \leq \omega \quad \text{for } n \in M \text{ and } x \in C.$$

But then, by 3.1.2, for all $n \in M$ there is $x \in B$ with $|f_n(x) \cap C| = \omega_1$, a contradiction to the assumption that $(*)(A, N)$ is false. This shows 3.1.4. To finish the proof of 3.1.3 and Theorem 3.1, we can use 3.1.4 inductively. ∎

Here is a problem that has not been looked at very thoroughly:

Problem 3.2. *Under the conditions of* 3.1, *is there a rainbow set containing all the colors?*

Theorem 3.3. *For every $1 < k < \omega$ there is an $n \in \omega$ with $\binom{k}{2} \leq n$ such that every f satisfying $\omega_1 \not\to [(\omega_1, \omega_1)]^2_n$ has a rainbow set of size k.*

Proof. We prove the following statement by induction on $2 \leq k < \omega$. There is an $n < \omega$ such that if $\mathrm{Dom}(f) \subseteq [\omega_1]^2$ satisfies $\omega_1 \not\to [(\omega_1, \omega_1)]^2_n$ (note that this means that for all $A, B \subseteq \omega_1$ with $|A| = |B| = \omega_1$ and for all $i < n$ there are $\alpha \in A$ and $\beta \in B$ with $\{\alpha, \beta\} \in \mathrm{Dom}(f)$ such that

$f(\{\alpha, \beta\}) = i$) and $\{A_i : i < n\}$ are pairwise disjoint subsets of ω_1 of size ω_1, then there is a rainbow partial transversal P ($[P]^2 \subseteq \mathrm{Dom}(f)$) of size k for these sets. Just as in the proof of 3.1, put

$$f_j(x) = \{y \in \omega_1 : f(x, y) = f(\{x, y\}) = j\}$$

for $j < \omega$ and $x \in \omega_1$. Assume n is good for k and $A_0, \dots, A_{2n-1}$ are pairwise disjoint subsets of ω_1 of size ω_1 with union A.

Let $(*)$ denote the following statement: There are x, i_x, N_x, φ_x such that $x \in A_{i_x}$, $N_x \subseteq 2n \setminus \{i_x\}$, $\varphi_x : N_x \to 2n$ is one-to-one,

$$|f_{\varphi_x(j)}(x) \cap A_j| = \omega_1 \quad \text{for } j \in N_x,$$

and $|N_x| = n$. If $(*)$ holds for an x then applying the induction hypothesis for the sets

$$f_{\varphi_x(j)}(x) \cap A_j, \quad j \in N_x,$$

and for the color set $2n \setminus \varphi[N_x]$ we get a rainbow partial transversal of size k for these sets, and adding x to it we get a rainbow transversal of size $k+1$ for the sets $A_0, \dots, A_{2n-1}$.

If $(*)$ is false, choosing an N_x of maximal size for $x \in A$ we will have $|N_x| \leq n - 1$ for $x \in A$. By thinning out, we get sets $B_i \subseteq A_i$, $i < 2n$, of size ω_1 and $N_i, M_i \subseteq 2n$, $i < 2n$, such that $N_x = N_i$ and $M_i = \varphi_x[N_i]$ for $x \in B_i$ for $i < 2n$.

Then $i \mapsto N_i$ is a set mapping of order at most $n-1$ on $2n$. By a theorem of de Bruijn and Erdős, from 1951, there are $i \neq j$ such that $i \notin N_j$ and $j \notin N_i$. As $|M_i \cup M_j| < 2n$ we can choose $l \notin M_i \cup M_j$. By the maximality of N_i we know that $|f_l(x) \cap B_j| \leq \omega$ for $x \in B_i$ and likewise $|f_l(x) \cap B_i| \leq \omega$ for $x \in B_j$. We could then pick, by an easy transfinite induction, sets $C_i \subseteq B_i$ and $C_j \subseteq B_j$, both of size ω_1, such that the color l is missing from the bipartite (ω_1, ω_1) determined by C_i and Cj. This contradicts the assumption. ∎

Corollary 3.4. *In Theorem 3.3, n can be chosen to be 2^{k-2} for $2 \leq k < \omega$.*

Problem 3.5. *Can n be taken to be $\binom{k}{2}$ in Theorem 3.3?*

4. Resurrecting the problem for larger cardinals. We explained in Section 1 how Shelah's example described in 1.5 forced us to consider problems only for underlying sets of size at most ω_1. In [2] written in 1978 we tried to ask if we can get every graph of size ω_1 as an induced subgraph provided the graph shows $\omega_2 \not\to [(\omega_1, \omega)]^2_{\omega_1}$, a stronger assumption that one can only make consistent. Recently Soukup showed that the simple method of adding one Cohen real gives a negative answer as well. Working through the material of this paper I realized that this trick only kills questions of $\Rightarrow$ type. The following is probably the simplest problem I cannot solve:

Problem 4.1. *Assume GCH and let f establish*

$$\omega_2 \not\to [(\omega_1, \omega_2)]^2_{\omega_1}.$$

Does there exist a rainbow subset of size ω_1 for f?

In fact, we do not know a single case where for some $\kappa > \lambda > \omega$ some $f : [\kappa]^2 \to \lambda$ establishes $\kappa \not\to [(\kappa, \kappa)]^2_\lambda$ and for all such f there is an uncountable rainbow set.

5. Finitary problems. In our paper [4] we considered finitary Ramsey problems and proved in 1989

Theorem 5.1 (Erdős–Hajnal [4, Theorem 1.3]). *Assume $2 \le k, s < \omega$ and $d : [k]^2 \to s$. Then there are n_0 and a real number $r > 0$ such that for all $f : [n]^2 \to s$ establishing*

$$n \not\to [e^{r\sqrt{\log n}}]^2_s,$$

$d \Rightarrow f$ holds.

In fact, we only wrote down the proof of this result for $s = 2$. Janos Pach kindly communicated to us that he can prove a much stronger result for a great many cases. Most relevant to this paper, he can prove:

Theorem 5.2 (Fox–Pach [6]). *There are n_0 and $\varepsilon > 0$ such that for any $n > n_0$ and f establishing*

$$n \not\to [n^\varepsilon]^2_3$$

there is a rainbow triangle for f.

References

[1] P. Erdős, F. Galvin and A. Hajnal, *On set-systems having large chromatic number and not containing prescribed subsystems*, in: Colloq. Math. Soc. J. Bolyai 10, North-Holland, 1975, 425–513.

[2] P. Erdős and A. Hajnal, *Embedding theorems for graphs establishing negative partition relations*, Period. Math. Hungar. 9 (1978), 205–230.

[3] —, —, *Unsolved problems in set theory*, in: Proc. Sympos. Pure Math. 13, Part I, Amer. Math. Soc., Providence, RI, 1971, 17–48.

[4] —, —, *Ramsey type theorems*, Discrete Appl. Math. 25 (1989), 39–52.

[5] P. Erdős, A. Hajnal, A. Máté and R. Rado, *Combinatorial Set Theory: Partition Relations for Cardinals*, Studies Logic Found. Math. 106, Akadémiai Kiadó and North-Holland, Budapest and Amsterdam, 1984.

[6] J. Fox and J. Pach, *Erdős–Hajnal-type results on intersection patterns of geometric objects*, Israel J. Math., to appear.

[7] J. T. Moore, *A solution to the L space problem and related ZFC constructions*, preprint, 2005.

[8] —, *A solution to the L space problem*, J. Amer. Math. Soc. 9 (2006), 717–736.

[9] R. Rado, *Anti-Ramsey theorems*, in: Colloq. Math. Soc. J. Bolyai 10, Vol. III, North-Holland, 1975, 1159–1168.

[10] S. Shelah, *Colouring without triangles and partition relations*, Israel J. Math. 20 (1975), 1–12.

Rényi Institute
Reáltanoda u. 13–15
1053 Budapest, Hungary
E-mail: ahajnal@renyi.hu

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