Axiomatic theory of divergent series
and cohomological equations

by

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Abstract. A general theory of summation of divergent series based on the Hardy–Kolmogorov axioms is developed. A class of functional series is investigated by means of ergodic theory. The results are formulated in terms of solvability of some cohomological equations, all solutions to which are nonmeasurable. In particular, this realizes a construction of a nonmeasurable function as first conjectured by Kolmogorov.

1. Introduction and general theorems. A natural axiomatic framework for the summation of divergent series already appeared in Hardy's early papers (see [7]) and also in Kolmogorov's short note [10]. Hardy reproduced the axioms in the book [6] (Section 1.3) and stated that most of the known summation methods meet them. For instance, this is the case for the classical Cesàro method (C, k) of any order k.

In [14] the Hardy–Kolmogorov axioms were translated into the language of functional analysis, and a brief sketch of their main consequences was presented without proofs. Now we give a developed exposition with applications to some functional series generated by dynamical systems. The "sums" of such series satisfy some functional (cohomological) equations and, for this reason, they happen to be nonmeasurable. Under the summability assumption, for lacunary trigonometric series this phenomenon was discovered by Kolmogorov [10] and justified by Zygmund [17]. We prove that functional series in a wide class, including Kolmogorov's, are indeed summable. Namely, by using the Birkhoff–Khinchin ergodic theorem, we verify that our general criterion of summability (Theorem 1.8) is applicable. Thus, summation of divergent series is a source of nonmeasurable functions, as conjectured by Kolmogorov [10].

The following general definition of a summation method is that of [14].

DEFINITION 1. Let s be the linear space of all sequences $x = (\xi_n \in \mathbb{C} : n \geq 0)$, let T be the shift operator on s, i.e. $T(\xi_n) = (\xi_{n+1})$, and, finally, let $L \subset s$ be a T-invariant subspace. A linear functional $\sigma : L \to \mathbb{C}$ is called a *summation method* (a *summation*, for short) on L if

$$(1.0) \qquad\qquad \sigma(x) = \xi_0 + \sigma(Tx), \quad x \in L.$$

(In fact, we identify the sequence (ξ_n) with the corresponding series $\xi_0 + \xi_1 + \cdots$).

We do not assume that s is provided with a topology.

If there exists a summation σ on a T-invariant subspace L then we say that L *admits summation*, and we call L the *domain* of σ. In this case for any T-invariant subspace $M \subset L$ the restriction $\sigma|M$ is a summation on M. Moreover, if $L = M \oplus N$ where N is also a T-invariant subspace then L admits summation if and only if there are some summations on M and N.

A series x is called *summable* if it belongs to a subspace L admitting summation. If the summation is σ, we say that x is σ-*summable*.

It is very instructive to rewrite (1.0) in the "cohomological" form

$$(1.1) \qquad\qquad \sigma(x) - \sigma(Tx) = \xi_0(x), \quad x \in L.$$

By iteration of T we obtain

$$(1.2) \qquad\qquad \sigma(x) - \sigma(T^n x) = s_n(x) \equiv \sum_{k=0}^{n-1} \xi_k, \quad n \geq 0.$$

(To include the case $n = 0$ we set $s_0(x) = 0$.) By linearity of σ the formula (1.2) can be extended to

$$(1.3) \qquad\qquad \phi(1)\sigma(x) - \sigma(\phi(T)x) = \sum_{n=1}^{m} a_n s_n(x)$$

where

$$\phi(\lambda) = \sum_{n=0}^{m} a_n \lambda^n.$$

A series x is called *finite of length $l(x) = l$* if either $l = 0$ (i.e., $x = 0$) or $\xi_n = 0$ for $n \geq l > 0$, but $\xi_{l-1} \neq 0$. The set F_m of finite series x of length $l(x) \leq m$ is a T-invariant linear subspace. From (1.2) it follows that the functional

$$(1.4) \qquad\qquad \sigma_F(x) = \sum_{k=0}^{l(x)-1} \xi_k$$

is a unique summation on the space F of all finite series.

On the space c^0 of convergent series we have the *standard summation*

$$(1.5) \qquad \sigma^0(x) = \sum_{k=0}^{\infty} \xi_k = \lim_{n \to \infty} s_n(x).$$

However, *there are infinitely many other summations on* c^0. We show this after a short consideration of the general uniqueness problem.

Now we rewrite (1.1) as $\sigma(\delta x) = \xi_0(x)$, where $\delta = \mathbf{1} - T$ and $\mathbf{1}$ is the identity operator. This δ is the classical difference operator: $\delta(\xi_n) = (\xi_n - \xi_{n+1})$. For every T-invariant subspace L we introduce its *derivative subspace* $L' = \operatorname{Im} \delta_L$, where $\delta_L : L \to L$ is the restriction of δ to L. Obviously, L' is also T-invariant.

LEMMA 1.1. *If L admits a summation σ_0 then the set of summations on L consists of all linear extensions of $\sigma_0|L'$ to L.*

Proof. The equation (1.1) is equivalent to $\sigma(\delta x) = \sigma_0(\delta x)$, i.e. to $\sigma|L' = \sigma_0|L'$. ∎

As a consequence, we obtain

THEOREM 1.2. *A summation on L is unique if and only if $L' = L$, i.e. the operator δ_L is surjective, in other words, the equation $\delta x = y$ has a solution $x \in L$ for every $y \in L$.*

REMARK 1.3. *In the whole space s the operator δ is surjective, i.e. $s' = s$.* Indeed, for $x = (\xi_n)$ and $y = (\eta_n)$ the equation $\delta x = y$ is actually $\xi_n - \xi_{n+1} = \eta_n$. Its general solution is $\xi_n = \xi_0 - s_n(y)$ with an arbitrary ξ_0, like indefinite integral.

REMARK 1.4. Using Lemma 1.1 one can explicitly describe all summations σ on L. Namely, for a fixed direct sum decomposition $L = L' \oplus R$ we have $\sigma = \sigma_0 \oplus \chi$, where χ is an arbitrary linear functional on R. The independent parameters of this description are the values of χ on a basis B of the subspace R. We get a one-to-one correspondence between summations σ on L and complex-valued functions on B. As a result, *if a summation on L is not unique then the set of all summations on L is infinite.*

Returning to the space c^0 of convergent series we consider the closely related space $c_0 = \{(\xi_n) : \lim_{n \to \infty} \xi_n = 0\}$ and prove

LEMMA 1.5. $c_0' = c^0$, *moreover, for every $y = (\eta_n) \in c^0$ its unique δ-preimage in c_0 is*

$$\widehat{y} = \Big(\sum_{k=n}^{\infty} \eta_k \Big).$$

Proof. If $x = (\xi_n) \in c_0$ then $y = \delta x$ belongs to c^0 since $s_n(y) = \xi_0 - \xi_n \to \xi_0$ as $n \to \infty$. Conversely, if $y \in c^0$ and $x = \widehat{y}$ then $x \in c_0$ and $\delta x = y$. This x is unique since if $\delta x = 0$ then all $\xi_n = \xi_0$, hence $x = 0$ by letting $n \to \infty$. $\blacksquare$

COROLLARY 1.6. *The derivative subspace* $(c^0)'$ *is* $c^{00} = \{y \in c^0 : \widehat{y} \in c^0\}$.

Proof. Let $y \in (c^0)'$, i.e $y = \delta x$, $x \in c^0$. Since $(c^0)' \subset c^0 \subset c_0$, we have $y \in c^0$ and $x \in c_0$. By Lemma 1.5, $\widehat{y} = x$, thus $y \in c^{00}$. Conversely, let $y \in c^{00}$, i.e. $y \in c^0$ and $\widehat{y} \in c^0$. Since $y = \delta \widehat{y}$, we have $y \in (c^0)'$. $\blacksquare$

The existence of nonstandard summations on c^0 follows from Theorem 1.2 and Corollary 1.6. Indeed, the set $c^0 \setminus c^{00}$ is not empty. For instance, it contains any series

$$\zeta_\alpha = ((n+1)^{-\alpha}), \quad 1 < \alpha \le 2.$$

Now we proceed to the general existence problem.

LEMMA 1.7. *The series* $\pi_0 : 1 + 1 + \cdots$ *is not summable.*

Proof. Let π_0 be σ-summable. Then $\sigma(\pi_0) - \sigma(T\pi_0) = 0$ since $T\pi_0 = \pi_0$. On the other hand, $\xi(\pi_0) = 1$. $\blacksquare$

The series π_0 generates the 1-dimensional subspace $\Pi_0 = \ker \delta$ of *constant series*, $\xi_0 + \xi_0 + \cdots$. This is a subspace of the space Π_∞ of the series $(\pi(n))$, where π runs over all complex-valued polynomials. Obviously, Π_∞ is T-invariant as also is every subspace $\Pi_m = \{\pi : \deg \pi \le m\}$, $m \ge 0$. Moreover, $\Pi'_m = \Pi_{m-1}$, $m \ge 1$.

THEOREM 1.8. *Let* L *be a* T-invariant subspace. The following statements are equivalent.

(1) *L admits summation.*
(2) *π_0 does not belong to L, i.e. $L \cap \Pi_0 = 0$.*
(3) *There is no nonzero polynomial series in L, i.e. $L \cap \Pi_\infty = 0$.*

Proof. (1)$\Rightarrow$(2) since π_0 is not summable. Conversely, (2)$\Rightarrow$(1): we have $L \cap \ker \delta = 0$, hence δ_L is injective, so left invertible. Let $i : L \to L$ be a left inverse to δ_L. (This is not unique if $L' \ne L$.) Then the linear functional $\sigma_0(x) = \xi_0(ix)$ is a summation on L since $\sigma_0(\delta x) = \xi_0(x)$.

Obviously, (3)$\Rightarrow$(2). Conversely, (2)$\Rightarrow$(3): let $\pi \in L \cap \Pi_\infty$, and let $\pi \ne 0$, $\deg \pi = m$. Then $\delta^m \pi = \gamma \pi_0$ where $\gamma = \mathrm{const} \ne 0$. Hence $\pi_0 \in L$, contrary to (2). $\blacksquare$

COROLLARY 1.9. *A* T-invariant subspace L *admits summation if and only if the operator* δ_L *is injective.*

Combining this result with Theorem 1.2 we obtain

COROLLARY 1.10. *A* T-invariant subspace L *admits a unique summation if and only if the operator* δ_L *is bijective.*

COROLLARY 1.11. *If a finite-dimensional T-invariant subspace admits summation then the summation is unique.*

Proof. In this case all injective operators are surjective. ∎

REMARK 1.12. Informally speaking, the subspaces with a unique summation are just those where the "integration" becomes definite. Indeed, δ_L is bijective if and only if it is invertible.

REMARK 1.13. For any T-invariant subspace L consider the sequence of derivative subspaces $L \supset L' \supset L'' \supset \cdots$. Let N be their intersection. One can prove that *if L admits summation but it is not unique then the nonuniqueness is kept on any derivative subspace, while there is uniqueness on N.*

The summability problem can be "localized" as follows.

THEOREM 1.14. *Let L be a T-invariant subspace. Then L admits summation if and only if every $x \in L$ is summable.*

Proof. The "only if" part is trivial. The "if" part follows from Theorem 1.8 since π_0 is not summable, thus $\pi_0 \notin L$. ∎

Now for every $x \in s$ we consider the smallest T-invariant subspace L_x containing x. This is

$$L_x = \mathrm{Span}(T^n x) = \{\phi(T)x\}$$

where ϕ runs over all polynomials of one variable. Obviously, x is summable if and only if L_x admits summation. Combining this fact with Theorem 1.14 we obtain a "local" version of Theorem 1.8.

THEOREM 1.15. *A T-invariant subspace L admits summation if and only if for every $x \in L$ the subspace L_x does not contain π_0, or equivalently, $L_x \cap \Pi_\infty = 0$.*

We conclude this section with a few examples.

EXAMPLE 1.16. The subspace c_0 admits summation since $\pi_0 \notin c_0$.

EXAMPLE 1.17. The subspace $m = \{x : \sup_n |\xi_n| < \infty\}$ does not admit summation since $\pi_0 \in m$. However, given a Banach limit (an invariant mean) on m, the T-invariant subspace $m_0 = \{x \in m : B\text{-}\lim_{n\to\infty} \xi_n = 0\}$ admits summation since $\pi_0 \notin m_0$. Note that $m_0 \supset c_0$ since the Banach limit coincides with the standard limit for all convergent sequences.

EXAMPLE 1.18. The formula $\sigma(x) = B\text{-}\lim_{n\to\infty} s_n(x)$ determines a summation on the subspace $\widehat{m} = \{x : \sup_n |s_n(x)| < \infty\}$. We call it the *Banach summation*. Note that $\widehat{m} \subset m_0$ since $\xi_n = s_{n+1} - s_n$ and the Banach limit is T-invariant.

EXAMPLE 1.19. A summation method going back to Euler uses an analytic continuation of the *generating function*

$$(1.6) \qquad g(t;x) = \sum_{n=0}^{\infty} \xi_n t^n, \qquad x = (\xi_n),$$

where t is a complex variable. This function is defined and analytic in the disk $|t| < r_x$ if

$$(1.7) \qquad r_x \equiv (\limsup_{n\to\infty} |\xi_n|^{1/n})^{-1} > 0.$$

Assume that all series from a T-invariant subspace L satisfy (1.7), so we have a linear space $\mathcal{A}_L$ of analytic germs $g(t;x)$ at $t = 0$. Let $G \subset \mathbb{C}$ be an open connected set containing $t = 0, 1$, and let each $g \in \mathcal{A}_L$ be the Taylor germ of a function $\widetilde{g}(t;x)$ that is analytic on $G_x = G \setminus \Gamma_x$ where Γ_x is a finite set, $1 \notin \Gamma_x$. Then $\epsilon(x) = \widetilde{g}(1;x)$ is a summation on L (*Euler's summation*). Indeed, from (1.6) it follows that

$$g(t;x) - tg(t;Tx) = \xi_0(x), \qquad |t| < \min(r_x, r_{Tx}).$$

By uniqueness of analytic continuation, $\widetilde{g}(t;x)$ is a linear functional of x and

$$\widetilde{g}(t;x) - t\widetilde{g}(t;Tx) = \xi_0(x), \qquad t \in G_x \cap G_{Tx}.$$

In particular, $\epsilon(x) - \epsilon(Tx) = \widetilde{g}(1;x) - \widetilde{g}(1;Tx) = \xi_0(x)$.

The best known example of Euler's sum is $\epsilon(((-1)^n)) = 1/2$. More generally,

$$\epsilon((\lambda^n)) = (1-\lambda)^{-1}, \qquad \lambda \in \mathbb{C} \setminus \{1\}.$$

2. Quasiexponential series. The geometric progression (or *exponential series*) (λ^n) is an eigenvector of T for the eigenvalue $\lambda \in \mathbb{C}$. (For $\lambda = 0$ we set $0^0 = 1$.) The corresponding eigenspace $E_\lambda = \ker(T - \lambda\mathbf{1})$ is 1-dimensional. This is the first member of the increasing sequence of the *root subspaces* $E_{\lambda,m} = \ker(T - \lambda\mathbf{1})^m$, $m = 1, 2, \ldots$; their union is denoted by $E_{\lambda,\infty}$. We have $E_{0,m} = F_m$, the space of all finite series of length $\leq m$, so $E_{0,\infty} = F$. If $\lambda \neq 0$ then $E_{\lambda,m}$ consists of all series $(\pi(n)\lambda^n)_0^\infty$, $\pi \in \Pi_{m-1}$. In particular, $E_{1,m} = \Pi_{m-1}$, $E_{1,\infty} = \Pi_\infty$. Note that $\dim E_{\lambda,m} = m$ in any case.

Now let $\phi(\lambda)$ be a nonconstant polynomial, i.e.

$$\phi(\lambda) = \lambda^{m_0} \prod_{k=1}^{\nu} (\lambda - \lambda_k)^{m_k},$$

where $m_0 \geq 0$, and if $\nu > 0$ then λ_k are nonzero pairwise distinct roots with multiplicities $m_k \geq 1$. Then

$$\ker \phi(T) = E_{0,m_0} \oplus E_{\lambda_1,m_1} \oplus \cdots \oplus E_{\lambda_\nu,m_\nu}$$

according to the Jordan form of $T|\ker\phi(T)$. In other words, $\phi(T)x = 0$ *if and only if*

$$(2.1) \qquad \xi_n = \sum_{k\,:\,\lambda_k \neq 0} \pi_k(n)\lambda_k^n + \zeta_n,$$

where $\pi_k \in \Pi_{m_k-1}$ and $(\zeta_n) \in F$, $\zeta_n = 0$ for $n \geq m_0$. The decomposition (2.1) is unique. By the way, $\phi(T)x = 0$ is nothing but a homogeneous linear difference equation with constant coefficients, and (2.1) is its general solution. In the case $m_0 = 0$ this formula turns into the classical one concerning two-sided sequences.

Any series x with members of the form (2.1) is called *quasiexponential*. The complex linear space of all quasiexponential series will be denoted by Q. This is a T-invariant subspace of s. Letting $Z_{m_0} = \{0\}$ for $m_0 > 0$ and $Z_{m_0} = \emptyset$ for $m_0 = 0$, we call the set $\{\lambda_k : \pi_k \neq 0\} \cup Z_{m_0}$ the *spectrum* of $x \in Q$ and denote it by $\mathrm{spec}(x)$. Obviously, $\mathrm{spec}(x) \neq \emptyset$ if $x \neq 0$ but $\mathrm{spec}(0) = \emptyset$. Let $x \neq 0$. Then x is finite or polynomial if and only if $\mathrm{spec}(x) = \{0\}$ or $\{1\}$, respectively. In general, $\mathrm{spec}(x)$ coincides with the set of roots of a minimal polynomial $\phi_x(\lambda)$ such that $\phi_x(T)x = 0$. (As usual, the minimality means that $\deg\phi_x$ is minimal. This polynomial is unique up to a constant factor.)

The following theorems show the importance of quasiexponential series for general summation theory.

THEOREM 2.1. *A series x is not summable if and only if $x \in Q$ and $1 \in \mathrm{spec}(x)$.*

Proof. By Theorem 1.15, x is not summable if and only if $\pi_0 \in L_x$, i.e. $L_x \cap \ker(\mathbf{1} - T) \neq 0$ or, equivalently, there is a polynomial ψ such that

$$(2.2) \qquad \psi(T)x \neq 0, \quad (\mathbf{1} - T)\psi(T)x = 0.$$

From (2.2) it follows that $x \in Q$ and $(1-\lambda)\psi(\lambda)$ is divisible by the minimal polynomial $\phi_x(\lambda)$. Hence, $\phi_x(1) = 0$, otherwise, ψ is divisible by ϕ_x, so $\psi(T)x = 0$. Conversely, if $x \in Q$ and $\phi_x(1) = 0$ then $\phi_x(\lambda) = (1-\lambda)\psi(\lambda)$, so (2.2) is valid since $\deg\psi < \deg\phi_x$. ∎

In view of Theorem 2.1 let us introduce

$$Q_1 = \{x \in Q : 1 \notin \mathrm{spec}(x)\},$$

so $x \in Q_1$ if and only if $\phi(T)x = 0$ for a polynomial ϕ such that $\phi(1) \neq 0$. The subspace Q_1 is T-invariant, and $Q = Q_1 \oplus \Pi_\infty$ according to (2.1).

Now note that $x \in Q$ if and only if the set $\{T^n x\}_{n=0}^\infty$ is linearly dependent, i.e. the subspace L_x is finite-dimensional. Its basis is $\{T^n x\}_{n=0}^{\nu-1}$, where ν is the degree of the related minimal polynomial, thus, $\dim L_x = \nu$. If $x \notin Q$ then L_x is infinite-dimensional since $\{T^n x\}_{n=0}^\infty$ is its basis.

THEOREM 2.2. *Let x be a summable series. Then the summation on L_x is unique if and only if $x \in Q_1$.*

Proof. By Theorem 2.1 either $x \in Q_1$, or $x \notin Q$. In the first case the summation is unique by Corollary 1.11. In the second case the values $\sigma(T^n x)$, $n \geq 1$, are determined by (1.2), while $\sigma(x)$ remains arbitrary. ∎

COROLLARY 2.3. *Every subspace $L \subset Q_1$ admits a unique summation.*

Proof. Let $x \in L$ and let σ be a summation on L. Then $\sigma(x) = (\sigma|L_x)(x)$. By Theorem 2.2 the summation $\sigma|L_x$ does not depend on the choice of σ. ∎

We denote by ϵ_1 the unique summation on Q_1. By Corollary 2.3 the unique summation on any subspace $L \subset Q_1$ is $\epsilon_1|L$.

EXAMPLE 2.4. Let $Q_0 = Q \cap c^0$, i.e. Q_0 is the subspace of convergent quasiexponential series. Using (2.1) one can prove that $x \in Q_0$ if and only if $x \in Q$ and $\operatorname{spec}(x) \subset \{\lambda \in \mathbb{C} : |\lambda| < 1\}$. In particular, $1 \notin \operatorname{spec}(x)$ for $x \in Q_0$, so $Q_0 = Q_1 \cap c^0$. Therefore, on Q_0 the summation ϵ_1 coincides with the standard summation σ^0.

Now we prove that *the summation ϵ_1 coincides with the restriction of Euler's summation ϵ to the subspace Q_1.*

LEMMA 2.5. *For $x \in Q$ the generating function $g(t; x)$ is well-defined, and $\widetilde{g}(t; x)$ is a rational function of t. The set of its poles is*

$$(2.3) \qquad \left\{ t = \lambda^{-1} : \lambda \in \operatorname{spec}(x), \ \lambda \neq 0 \right\}.$$

Proof. It suffices to consider the case of a single-point spectrum. If $\operatorname{spec}(x) = \{0\}$ then x is finite, so $\widetilde{g}(t; x)$ is a polynomial in t. On the other hand, the set (2.3) is empty in this case. Now let $\operatorname{spec}(x) = \{\lambda\}$, $\lambda \neq 0$. Then $x = (\pi(n)\lambda^n)$ where $\pi \in \Pi_\infty$. Accordingly,

$$g(t; x) = \sum_{n=0}^{\infty} \pi(n)\lambda^n t^n, \qquad |t| < |\lambda|^{-1}.$$

If $\deg \pi = \nu - 1$ then π can be represented as

$$\pi(n) = \sum_{k=0}^{\nu-1} c_k \binom{k+n}{k}, \qquad c_{\nu-1} \neq 0.$$

This yields

$$(2.4) \qquad \widetilde{g}(t; x) = \sum_{k=0}^{\nu-1} \frac{c_k}{(1 - \lambda t)^{k+1}}. \qquad ∎$$

By the way, any rational function $g(t)$ which is regular at $t = 0$ is the generating function of a series $x \in Q$. According to (2.4), this x can be obtained from the decomposition of $g(t)$ into partial fractions.

COROLLARY 2.6. *For $x \in Q_1$ the function $\widetilde{g}(t; x)$ is rational and regular at $t = 1$.*

This means that Q_1 is a subspace of the domain of Euler's summation. Therefore, $\epsilon_1 = \epsilon | Q_1$ by uniqueness of summation on Q_1.

For $x \in Q_1$ an explicit expression of $\epsilon_1 | L_x$ follows from the formula (1.3). Namely, *if*

$$\phi_x(\lambda) = \sum_{n=0}^{\nu} a_n \lambda^n$$

is a corresponding minimal polynomial then

$$\epsilon_1(z) = \frac{1}{\phi_x(1)} \sum_{n=1}^{\nu} a_n s_n(z), \quad z \in L_x.$$

Indeed, $\phi_x(T)z = 0$ for all $z \in L_x$. It remains to substitute $x = z$ and $\phi = \phi_x$ into (1.3). Actually, we see that the minimal polynomial ϕ_x can be changed to any polynomial ϕ such that $\phi(T)x = 0$ and $\phi(1) \neq 0$.

EXAMPLE 2.7. Let x be $(l+1)$-periodic, i.e. $T^{l+1}x = x$, and let $1 \notin \operatorname{spec}(x)$ or equivalently, $s_{l+1}(x) = 0$. Then

$$\epsilon_1(x) = \frac{1}{l+1} \sum_{n=1}^{l} s_n(x),$$

so $\epsilon_1(x)$ coincides with the Cesàro sum of order 1. This is not an accidental fact. The point is that a quasipolynomial series x is $(C, 1)$-summable if and only if $\operatorname{spec}(x) \subset \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$ and the roots of the minimal polynomial lying on the unit circle are simple. If, in addition, $1 \notin \operatorname{spec}(x)$ then the Cesàro sum of x coincides with $\epsilon_1(x)$ by uniqueness.

3. Extension theory. In the spirit of the classical definition (see e.g. [6, Section 4.3]) we say that a summation τ is *stronger* than σ and write $\tau \succ \sigma$ if τ is an extension of σ. For instance, $\sigma^0 \succ \sigma_F$ (see (1.4) and (1.5)). In turn, σ is called *regular* if $\sigma \succ \sigma^0$. The Banach summation on $\widehat{m}$ (Example 1.18) is regular by definition of the Banach limit. The Cesàro methods of all orders are regular, while Euler's method is not. Indeed, $r_x \geq 1$ for $x \in c^0$ but for some x's the function $g(t; x)$ cannot be analytically continued to $t = 1$.

A summation μ is called *maximal* if there are no summations $\tau \succ \mu$, $\tau \neq \mu$.

THEOREM 3.1. *For every summation σ there exists a maximal $\mu \succ \sigma$.*

Proof. We use the Zorn lemma. The relation "$\succ$" is a partial order on the set of all summations, a fortiori, on the subset $\{\tau : \tau \succ \sigma\}$. This order is inductive: there is a majorant τ for any linearly ordered subset $\{\tau^\alpha \succ \sigma\}$.

Indeed, let L^α be the domain of τ^α, and let $L = \bigcup_\alpha L^\alpha$. Then τ is well-defined on L as $\tau x = \tau^{\alpha_x} x$ where α_x is any index such that $x \in L^{\alpha_x}$. ∎

COROLLARY 3.2. *There exists a regular maximal summation.*

Actually, any extension can be realized as a sequence (transfinite, in general) of minimal steps. Every such step extends the domain L of a summation σ to $L[x] = L + L_x$ with some $x \notin L$. To analyze this situation we consider the set $I_{x,L}$ of polynomials $\phi(\lambda)$ such that $\phi(T)x \in L$. Since L is T-invariant, the $I_{x,L}$ is an ideal of the ring of all polynomials of λ. Though $0 \in I_{x,L}$, the nonzero constants do not belong to $I_{x,L}$ as long as $x \notin L$. Obviously, $I_{x,K} \subset I_{x,L}$ if $K \subset L$, in particular, $I_{x,0} \subset I_{x,L}$. Implicitly, we already dealt with the ideal $I_{x,0}$ in Section 2.

LEMMA 3.3. *If $I_{x,L} = 0$ then any summation σ on L extends to $L[x]$.*

Proof. In this case $L \cap L_x = 0$ and $x \notin Q$. Thus, $L[x] = L \oplus L_x$ and L_x admits summation by Theorem 2.1. ∎

REMARK 3.4. In Lemma 3.3 the set of extensions is infinite by Theorem 2.2.

Now we assume $I_{x,L} \neq 0$ and introduce

$$\theta_{x,L}(\lambda) = \sum_{n=0}^{\nu} c_n \lambda^n,$$

a minimal polynomial in $I_{x,L}$. This is a greatest common divisor of all $\phi \in I_{x,L}$. Below we use the simplified notation $\theta(\lambda) \equiv \theta_{x,L}(\lambda)$. It is convenient to normalize this polynomial so that $c_\nu = 1$, i.e.

$$(3.1) \qquad \theta(\lambda) = \lambda^\nu + \sum_{n=0}^{\nu-1} c_n \lambda^n.$$

The trivial case $\theta(\lambda) \equiv 1$ (i.e. $\nu = 0$) is formally included in this setting.

LEMMA 3.5. *A summation σ on L extends to $L[x]$ if and only if either $\theta(1) \neq 0$, and then σ is arbitrary, or $\theta(1) = 0$, and then σ is such that*

$$(3.2) \qquad \sigma(\theta(T)x) + \sum_{n=1}^{\nu} c_n s_n(x) = 0.$$

The extension is unique if and only if $\theta(1) \neq 0$.

Proof. Let τ be an extension of σ to $L[x]$. Then

$$(3.3) \qquad \theta(1)\tau(x) = \tau(\theta(T)x) + \sum_{n=1}^{\nu} c_n s_n(x)$$

according to (1.3). However, $\tau(\theta(T)x) = \sigma(\theta(T)x)$ since $\theta(T)x \in L$ and $\tau|L = \sigma$. Thus, if $\theta(1) = 0$ then (3.3) turns into (3.2).

In the converse direction we start with a value $\tau(x)$ such that

$$(3.4) \qquad \theta(1)\tau(x) = \sigma(\theta(T)x) + \sum_{n=1}^{\nu} c_n s_n(x).$$

This value does exist under our conditions (and is uniquely determined if $\theta(1) \neq 0$, and arbitrary otherwise). Setting

$$(3.5) \qquad \tau(T^n x) = \tau(x) - s_n(x), \qquad 1 \leq n \leq \nu - 1,$$

we determine a linear extension τ of σ to

$$L[x] = L \oplus R, \qquad R = \mathrm{Span}\{T^n x\}_{n=0}^{\nu-1}.$$

To prove that τ is a summation it remains to verify the equality

$$(3.6) \qquad \tau(T^\nu x) = \tau(x) - s_\nu(x).$$

Note that, as a rule, $T^\nu x \notin R$, so the space R is not T-invariant. Indeed, by (3.1) we have

$$(3.7) \qquad T^\nu x = \theta(T)x \oplus (T^\nu - \theta(T))x,$$

so $T^\nu x \notin R$ as long as $\theta(T)x \neq 0$. According to (3.7),

$$\tau(T^\nu x) = \sigma(\theta(T)x) + \tau((T^\nu - \theta(T))x) = \sigma(\theta(T)x) - \sum_{n=0}^{\nu-1} c_n \tau(T^n x).$$

By substitution from (3.5) we obtain

$$\tau(T^\nu x) = \sigma(\theta(T)x) + \sum_{n=0}^{\nu-1} c_n s_n(x) - \tau(x) \sum_{n=0}^{\nu-1} c_n.$$

(Recall that $s_0(x) = 0$.) This yields (3.6) because of (3.4) and the relation

$$\theta(1) - \sum_{n=0}^{\nu-1} c_n = c_\nu = 1. \quad \blacksquare$$

COROLLARY 3.6. *Every maximal summation μ is stronger than ϵ_1.*

Proof. By the uniqueness of the summation ϵ_1 on Q_1 we only have to show that the domain M of μ contains Q_1. Let $x \in Q_1$, so $\phi(T)x = 0$ where ϕ is a polynomial with $\phi(1) \neq 0$. Thus, $\phi \in I_{x,M}$, so $\theta_{x,M}$ is a divisor of ϕ. Therefore, $\theta_{x,M}(1) \neq 0$. By Lemma 3.5 and maximality of μ we obtain $x \in M$. $\blacksquare$

Lemma 3.5 shows that the only obstacle to extension of a summation σ from L to $L[x]$ is the inequality

$$\sigma(\theta(T)x) + \sum_{n=1}^{\nu} c_n s_n(x) \neq 0$$

in the case $\theta(1) = 0$. However, this can be removed by a "polynomial regularization" of x.

LEMMA 3.7. *Let $\theta_{x,L}(1) = 0$ and let m be the multiplicity of this root of $\theta_{x,L}(\lambda)$. Then there exists a polynomial series π of degree $\leq m - 1$ such that any summation σ extends from L to $L[x - \pi]$.*

Proof. We start with the case $\theta_{x,L}(\lambda) = (\lambda - 1)^m$. For every $y = x - p$ with $p \in \Pi_{m_x - 1}$, we have $\theta_{x,L}(T)y = \theta_{x,L}(T)x \in L$, so $\theta_{x,L} \in I_{x,L}$. Hence, $\theta_{x,L}$ is divisible by $\theta_{y,L}$, so $\theta_{y,L}(\lambda) = (\lambda - 1)^{m_y}$ with some $m_y \leq m$. We choose the summand p in y to make m_y minimal. If $m_y = 0$ then $\theta_{y,L} = 1$, hence $y \in L$. Thus, we have a trivial extension $L[x - \pi] = L$ with $\pi = p$.

Let $m_y \geq 1$. Then we consider $z = y - q$ with $q \in \Pi_{m_y - 1}$, so that $z = x - \pi$ where $\pi = p + q \in \Pi_{m-1}$. As before, $\theta_{z,L}(\lambda) = (\lambda - 1)^{m_z}$ where $m_z \leq m_y$. Finally, $m_z = m_y$ by minimality of the latter. Thus, $\theta_{z,L} = \theta_{y,L}$, and accordingly,

$$(3.8) \qquad \sigma(\theta_{z,L}(T)x) + \sum_{n=1}^{m_y} c_n s_n(z) = \sigma(\theta_{y,L}(T)y) + \sum_{n=1}^{m_y} c_n s_n(y) - \sum_{n=1}^{m_y} c_n s_n(q).$$

The corresponding obstacle to extension of σ to $L[z]$ disappears if, for instance,

$$q(n) = \alpha \binom{n}{m_y - 1}$$

with a suitable $\alpha \in \mathbb{C}$. Indeed, for this q the last sum in (3.8) reduces to α.

In general, $\theta_{x,L}(\lambda) = \phi(\lambda)(\lambda - 1)^m$, where $\phi(1) \neq 0$. With $u = (T - 1)^m x$ we have $\phi(T)u = \theta_{x,L}(T)x \in L$. Therefore, ϕ is divisible by $\theta_{u,L}$, thus $\theta_{u,L}(1) \neq 0$. By Lemma 3.5, σ extends to a summation τ on $L[u]$. In turn, $\theta_{x,L[u]}(\lambda) = (\lambda - 1)^l$ with $l \leq m$. Hence, there exists $\pi \in \Pi_{l-1} \subset \Pi_{m-1}$ such that τ extends to $(L[u])[x - \pi]$, and, a fortiori, to $L[x - \pi]$. ∎

Combining Lemmas 3.3, 3.5 and 3.7 we obtain the following general

THEOREM 3.8. *Suppose that a subspace L admits summation. Then for every $x \in s$ there exists a polynomial series π such that any summation σ extends from L to $L[x - \pi]$.*

As an important consequence we obtain

THEOREM 3.9. *A T-invariant subspace M is the domain of a maximal summation if and only if*

$$(3.9) \qquad\qquad M \oplus \Pi_\infty = s,$$

i.e. M is a T-invariant direct complement in s of the subspace of polynomial series.

Thus, every maximal summation is applicable to all series up to a polynomial regularization. In this sense, the maximal summations are *universal*.

Proof. "If": M admits summation, since $M \cap \Pi_\infty = 0$. Any summation on M is maximal since any nontrivial extension of M intersects Π_∞.

"Only if": We apply Theorem 3.8 to $L = M$. By maximality of M the extension $M[x - \pi]$ is trivial, i.e. $x - \pi \in M$. Thus, $M + \Pi_\infty = s$. Moreover, $M \cap \Pi_\infty = 0$ since M admits summation. ∎

COROLLARY 3.10. *Every maximal summation is unique on its domain.*

Proof. The operator $\delta = \mathbf{1} - T$ is surjective on the whole space s (see Remark 1.3). Since in (3.9) both summands are T-invariant, the restriction δ_M is also surjective. Thus, Theorem 1.2 is applicable. ∎

As a result, we have *a 1-1 correspondence between maximal summations and T-invariant direct complements of Π_∞ in s.*

4. Orbital series. A *functional series on a set* $A \neq \emptyset$ is a mapping $X : A \to s$, i.e. for every $\alpha \in A$ we have a numerical series $X(\alpha) = (\xi_n(\alpha))$. Given a summation σ with a domain L, we say that X is σ-*summable* if so are all series $X(\alpha)$, i.e. $\operatorname{Im} X \subset L$, and

$$(4.1) \qquad \sigma(X(\alpha)) - \sigma((TX)(\alpha)) = \xi_0(\alpha), \qquad \alpha \in A,$$

where $(TX)(\alpha) = T(X(\alpha)) = (\xi_{n+1}(\alpha))$.

A functional series X is called *summable* if there exists a summation σ such that $X(\alpha)$ is σ-summable for every α. For example, every trigonometric series whose coefficients tend to zero is summable. Moreover, there is a common summation for all these series, namely, any summation on c_0.

An important class of functional series is

$$(4.2) \qquad X(\alpha) = (\xi_0(f^n \alpha)), \qquad \alpha \in A,$$

where f is a mapping $A \to A$. The sequence $(f^n \alpha)$ is the f-*orbit* of the point α, therefore, we call the functional series (4.2) *orbital*. In this case the subspace

$$L_X = \operatorname{Span}(\operatorname{Im} X) \subset s$$

is T-invariant since

$$(4.3) \qquad (TX)(\alpha) = X(f\alpha).$$

Hence, *an orbital series X is summable if and only if there exists a summation on L_X.* Combining (4.3) and (4.1) we obtain

PROPOSITION 4.1. *If an orbital series X is σ-summable then the function $\psi(\alpha) = \sigma(X(\alpha))$ satisfies the cohomological equation* (c.e.)

$$(4.4) \qquad \psi(\alpha) - \psi(f\alpha) = \xi_0(\alpha), \qquad \alpha \in A.$$

This is a bridge between summations and functional equations playing a considerable role in the modern theory of dynamical systems and group representation theory (see e.g. [1], [5], [8], [9]). In standard terms related to the dynamical system (A, f), any function $\psi : A \to \mathbb{C}$ is a 0-cochain, and its coboundary is the 1-cochain

$$\theta(n, \alpha) = \psi(\alpha) - \psi(f^n \alpha), \quad n \geq 0, \, \alpha \in A.$$

A 1-cochain $\omega(n, \alpha)$ is a cocycle if

$$\omega(n, f^m \alpha) - \omega(n + m, \alpha) + \omega(m, \alpha) = 0 \quad (n, m \geq 0).$$

Every coboundary is a cocycle but, in general, the converse is not true, i.e. not every cocycle is "cohomologically trivial". For any 0-cochain ξ_0 the 1-cochain

$$s(n, \alpha) \equiv s_n(\alpha) = \sum_{k=0}^{n-1} \xi_0(f^k \alpha)$$

is a cocycle. This cocycle is a coboundary if and only if the c.e. (4.4) is solvable.

In the context of summations we have a dynamical system (L, T), where L is a T-invariant subspace of s, and deal with the cocycle

$$(4.5) \qquad s(n, x) \equiv s_n(x) = \sum_{k=0}^{n-1} \xi_0(T^k x), \quad x \in L.$$

A linear functional σ on L is a summation if and only if the cocycle (4.5) is the coboundary of σ (see (1.2)). Accordingly, L admits summation if and only if the cocycle (4.5) is cohomologically trivial in the class of linear cochains.

Later on we assume that A is provided with a measure $d\alpha$, mes $A = 1$, and f is a measure preserving transformation of A into itself. In this setting all cochains are assumed measurable, and accordingly, two cochains which coincide almost everywhere (a.e.) can be identified. (This is not necessary for our purposes.)

The following lemma can be extracted from [16] (see also [15, Section 5]).

LEMMA 4.2. *Let $\psi(\alpha)$, $\alpha \in A$, be a measurable function and let $\varepsilon > 0$. Then there exist $M > 0$ and a sequence of subsets $A_n \subset A$ such that mes $A_n > 1 - \varepsilon$ and*

$$(4.6) \qquad |\psi(\alpha) - \psi(f^n \alpha)| \leq M, \quad \alpha \in A_n.$$

Proof. There is a subset D such that $|\psi(\alpha)| \leq M/2$, $\alpha \in D$ and mes$(A \setminus D) < \varepsilon/2$. The inequality (4.6) is valid on $A_n := D \cap f^{-n} D$. On the other hand, mes $A_n > 1 - \varepsilon$ since

$$\text{mes}(A \setminus A_n) \leq \text{mes}(A \setminus f^{-n} D) + \text{mes}(A \setminus D) = 2 \, \text{mes}(A \setminus D) < \varepsilon. \quad \blacksquare$$

THEOREM 4.3. *Suppose that there exists a sequence of subsets $B_n \subset A$ with $\inf_n(\operatorname{mes} B_n) > 0$ and*

$$\text{(4.7)} \qquad \inf_{\alpha \in B_n} \left| \sum_{k=0}^{n-1} \xi_0(f^k \alpha) \right| \to \infty, \qquad n \to \infty.$$

Then the c.e. (4.4) has no measurable solutions.

Proof. We use Lemma 4.2 with $\varepsilon < \inf_n(\operatorname{mes} B_n)$; then $A_n \cap B_n \neq \emptyset$. For $\alpha \in A_n \cap B_n$ the equality

$$\sum_{k=0}^{n-1} \xi_0(f^k \alpha) = \psi(\alpha) - \psi(f^n \alpha)$$

yields

$$\inf_{\alpha \in B_n} \left| \sum_{k=0}^{n-1} \xi_0(f^k \alpha) \right| \leq M,$$

contrary to (4.7). ∎

Now we consider the space $L_1(A, d\alpha)$ of Lebesgue integrable complex-valued functions. In this setting the following Birkhoff–Khinchin ergodic theorem (see e.g. [12, Ch. 1]) is our main tool.

THEOREM 4.4. *Let $\phi \in L_1(A, d\alpha)$. Then the limit*

$$\widetilde{\phi}(\alpha) = \lim_{m \to \infty} \frac{1}{m} \sum_{n=0}^{m-1} \phi(f^n \alpha)$$

exists for $\alpha \in A_\phi$ where A_ϕ is an f-invariant subset of A with $\operatorname{mes}(A \setminus A_\phi) = 0$. The limit function $\widetilde{\phi}$ is f-invariant, it belongs to $L_1(A, d\alpha)$, and

$$\int \widetilde{\phi} \, d\alpha = \int \phi \, d\alpha.$$

Recall that f is said to be *ergodic* if every f-invariant measurable function is constant a.e. In this case

$$\text{(4.8)} \qquad \lim_{m \to \infty} \frac{1}{m} \sum_{n=0}^{m-1} \phi(f^n \alpha) = \int \phi \, d\alpha, \qquad \alpha \in A_\phi,$$

where A_ϕ may not be the same as before, but has the same properties. Later on we deal with the A_ϕ from (4.8).

THEOREM 4.5. *Let f be ergodic, and let $\xi_0 \in L_1(A, d\alpha)$ be such that*

$$\text{(4.9)} \qquad \int \xi_0 \, d\alpha = 0.$$

Then the orbital series (4.2) is summable on A_{ξ_0}, hence a.e.

Proof. By Theorem 1.8 it suffices to show that $\pi_0 \notin L_X$. Suppose otherwise. Then

$$(4.10) \qquad \sum_k \lambda_k \xi_0(f^n \alpha_k) = 1, \quad n \geq 0,$$

for a finite set $\{(\alpha_1, \lambda_1), (\alpha_2, \lambda_2), \ldots\}$ with $\alpha_k \in A_{\xi_0}$, $\lambda_k \in \mathbb{C}$. This contradicts (4.8) with $\phi = \xi_0$. Indeed, by (4.9) the averaging (in the sense of (4.8)) over n in (4.10) yields 0 on the left hand side, and 1 on the right. ∎

REMARK 4.6. Obviously, for any measure preserving f the L_1-solvability of (4.4) implies that ξ_0 belongs to $L_1(A, d\alpha)$ and satisfies (4.9). Moreover, the latter is necessary for the existence of a measurable solution to (4.4) (see [1]). However, it is not sufficient. For the irrational rotations of the circle and continuous ξ_0 this was shown in [1] with the references to some dynamical constructions due to Neumann and Kolmogorov. (For another construction see [4].) In [13] the nonexistence of measurable solutions was established by means of the Banach closed graph theorem. (See [2] for a generalization.) Also note that the measurable solutions may not be Lebesgue integrable [1], [11].

REMARK 4.7. For a multiplicative version of c.e. the absence of measurable solutions was proven in [3] assuming that the known function in the equation is not homotopic to a constant. For this reason the problem for the additive equation cannot be reduced to the result of [3] by exponentiating.

In [10] Kolmogorov claimed (without any proof or heuristics) that if the trigonometric series

$$(4.11) \qquad \sin t + \sin 3t + \cdots + \sin 3^n t + \cdots, \quad t \in \mathbb{R},$$

is summable, then *one can construct an effective example of a Lebesgue nonmeasurable function.* Formally, the last sentence sounds as "the sum (in the sense of a summation) of the series (4.11) is nonmeasurable". This property was proven by Zygmund ([17, Ch. 5, Problem 26]) for the series

$$(4.12) \qquad \cos t + \cos 2t + \cdots + \cos 2^n t + \cdots, \quad t \in \mathbb{R}.$$

Our general theory allows us to prove Kolmogorov's conjecture in the form: *the series* (4.11) *is summable a.e., and its sum is nonmeasurable.* The same is true for the series (4.12). (It is interesting that (4.12) turns into the nonsummable series π_0 at $t = 0$.) Moreover, we prove

THEOREM 4.8. *Let q be an integer*, $q \geq 2$. *Then*

(1) *For any 2π-periodic function $\theta \in L_1(0, 2\pi)$ with zero mean value the series*

$$(4.13) \qquad \theta(t) + \theta(qt) + \cdots + \theta(q^n t) + \cdots$$

is summable a.e. to a function $\psi(t)$.

(2) $\psi(t)$ *satisfies the c.e.*

$$(4.14) \qquad \psi(t) - \psi(qt) = \theta(t).$$

(3) *Let θ be a trigonometric polynomial,*

$$(4.15) \qquad \theta(t) = \sum_{i=1}^{m}(a_i \cos \nu_i t + b_i \sin \nu_i t),$$

and suppose that no ratio ν_i/ν_j $(i > j)$ is a power of q. Then all solutions to the equation (4.14) are nonmeasurable.

In particular, in (3), $\theta(t)$ can be any trigonometric polynomial of degree $< q$.

Proof. The transformation $f_q : t \mapsto qt \pmod{2\pi}$ is ergodic. Hence, (1) follows from Theorem 4.5. Then Proposition 4.1 implies (2). To prove (3) we use Theorem 4.3.

Consider the sequence of trigonometric polynomials

$$(4.16) \qquad \theta_n(t) = \sum_{k=0}^{n-1} \theta(q^k t), \quad n \geq 1.$$

The Fourier spectrum Ω_n of $\theta_n(t)$ is the union of the pairwise disjoint sets $\{q^k \nu_i\}_{i=1}^m, 0 \leq k \leq n-1$. Accordingly, the summands in (4.16) are pairwise orthogonal. Moreover, they have the same L_2-norm, say τ. Therefore, the L_2-norm of $\theta_n(t)$ is equal to $\tau\sqrt{n}$. On the other hand, the sets Ω_n are uniformly lacunar: there is $\kappa > 1$ independent of n such that $\omega' \geq \kappa\omega$ for all $\omega', \omega \in \Omega_n$ with $\omega' > \omega$. Indeed, let $\omega' = q^k \nu_i$ and $\omega = q^l \nu_j$. Then either $\omega' \geq 2\omega$, or $q^{k-l} < 2\max\{\nu_j/\nu_i : 1 \leq i, j \leq m\}$. In the second case the set of all possible differences $k - l$ is finite since, in addition, $q^{k-l} > \min\{\nu_j/\nu_i : 1 \leq i, j \leq m\}$. Hence, the latter inequality can be strengthened by inserting a factor of $\kappa > 1$ on the right hand side. This yields $\omega' > \kappa\omega$. (Obviously, $\kappa < 2$ if the second case is nonempty, otherwise, $\kappa = 2$.)

By the established properties of $\theta_n(t)$ there are $\gamma, \delta > 0$ (depending on κ only) such that the measure of every set

$$B_n = \left\{ t : \left| \sum_{k=0}^{n-1} \theta_n(t) \right| \geq \gamma\sqrt{n} \right\}$$

is greater than δ (see [17, Ch.5, Th. 8.25]). Thus, Theorem 4.3 is applicable. ∎

COROLLARY 4.9. *If θ is a trigonometric polynomial such that the c.e. (4.14) has a measurable solution ψ, then $\psi(t)$ is a trigonometric polynomial a.e.*

Proof. By Theorem 4.8 there is $\nu_i \equiv 0 \pmod{q}$ in (4.15). Let $\nu = \nu_l = q\mu$ be the maximum of such ν_i. We will argue by induction on ν. Consider

$$\widetilde{\theta}(t) = \theta(t) + a_l(\cos \mu t - \cos \nu t) + b_l(\sin \mu t - \sin \nu t).$$

Accordingly, we introduce

$$\widetilde{\psi}(t) = \psi(t) + a_l \cos \mu t + b_l \sin \mu t,$$

so that $\widetilde{\psi}(t) - \widetilde{\psi}(qt) = \widetilde{\theta}(t)$. If $\widetilde{\theta} \not= 0$ then $\widetilde{\psi}$ is a trigonometric polynomial a.e. since there is $\widetilde{\nu} < \nu$ in the role of ν for $\widetilde{\theta}$. If $\widetilde{\theta} = 0$ then $\widetilde{\psi}$ is a constant a.e. by ergodicity. As a result, ψ is a trigonometric polynomial a.e. in any case. ∎

Now we can explicitly describe all the "trigonometric coboundaries" θ.

THEOREM 4.10. *A general form of the trigonometric coboundaries is*

$$(4.17) \qquad \theta(t) = \sum_{p \in I_d} \sum_{i=0}^{i_{p,d}} (a_{p,i} \cos pq^i t + b_{p,i} \sin pq^i t)$$

where $d \geq 1$, $I_d = \{p : 1 \leq p \leq d,\ p \not\equiv 0 \pmod{q}\}$, $i_{p,d} = \min\{i : pq^i > d\}$, *and the coefficients satisfy*

$$(4.18) \qquad \sum_{i=0}^{i_{p,d}} a_{p,i} = 0, \qquad \sum_{i=0}^{i_{p,d}} b_{p,i} = 0.$$

Proof. By substitution of

$$\psi(t) = \sum_{j=1}^{d} (h_j \cos jt + g_j \sin jt)$$

into (4.14) we obtain (4.17) with

$$(4.19) \qquad a_{p,0} = h_p, \qquad a_{p,i} = h_{pq^i} - h_{pq^{i-1}} \quad (1 \leq i \leq i_{p,d} - 1),$$
$$a_{p,i_{p,d}} = -h_{pq^{i_{p,d}-1}}$$

and similar formulas for $b_{p,i}$. The relations (4.18) follow from (4.19) by summation. This calculation is invertible since the representation $j = pq^i$ with $p \in I_{p,d}$ and $0 \leq i \leq i_{p,d} - 1$ is unique for every j, $1 \leq j \leq d$. ∎

In conclusion we return to Proposition 4.1 and inverse it as follows.

THEOREM 4.11. *Let f be ergodic, and let $\psi \in L_1(A, d\alpha)$ be a solution of the c.e. (4.4) for $\alpha \in A^0$ where A^0 is an f-invariant subset of A with* $\mathrm{mes}(A \setminus A^0) = 0$. *Then the formula*

$$(4.20) \qquad \sigma(X(\alpha)) = \psi(\alpha) - \int \psi\, d\alpha, \qquad \alpha \in A_1 = A^0 \cap A_\psi,$$

determines a summation σ of the orbital series (4.2).

Let us emphasize that the set A_1 is f-invariant and $\mathrm{mes}(A \setminus A_1) = 0$, so the series (4.2) is summable a.e.

Proof. For any constant c the function $\psi + c$ is also a solution of (4.4) on A^0. In particular, so is

$$\widehat{\psi}(\alpha) = \psi(\alpha) - \int \psi \, d\alpha,$$

so (4.20) can be rewritten as

(4.21) $$\sigma(X(\alpha)) = \widehat{\psi}(\alpha), \quad \alpha \in A_1,$$

with

(4.22) $$\int \widehat{\psi} \, d\alpha = 0.$$

Formula (4.21) correctly defines $\sigma(X(\alpha))$, $\alpha \in A_1$, if

$$X(\alpha_1) = X(\alpha_2) \;\Rightarrow\; \widehat{\psi}(\alpha_1) = \widehat{\psi}(\alpha_2).$$

Moreover, it can be extended linearly as long as

(4.23) $$\sum_k \lambda_k X(\alpha_k) = 0 \;\Rightarrow\; \sum_k \lambda_k \widehat{\psi}(\alpha_k) = 0$$

for all finite sets $\{(\alpha_1, \lambda_1), (\alpha_2, \lambda_2), \dots\}$ with $\alpha_k \in A_1$, $\lambda_k \in \mathbb{C}$. The resulting σ is indeed a summation of $X(\alpha)$ on A_1 since

$$\sigma(X(\alpha)) - \sigma(T(X(\alpha))) = \widehat{\psi}(\alpha) - \widehat{\psi}(f\alpha) = \xi_0(\alpha).$$

It remains to prove the implication (4.23).

The hypothesis in (4.23) can be rewritten as

$$\sum_k \lambda_k \xi_0(f^l \alpha_k) = 0, \quad l \geq 0,$$

or equivalently as

$$\sum_k \lambda_k \widehat{\psi}(f^l \alpha_k) - \sum_k \lambda_k \widehat{\psi}(f^{l+1} \alpha_k) = 0, \quad l \geq 0.$$

The sum of these equalities over $0 \leq l \leq n - 1$ yields

(4.24) $$\sum_k \lambda_k \widehat{\psi}(\alpha_k) = \sum_k \lambda_k \widehat{\psi}(f^n \alpha_k), \quad n \geq 0.$$

By (4.22) the averaging over n in (4.24) yields the conclusion in (4.23). ∎

REMARK 4.12. Without any assumption on ψ the c.e. (4.4) is solvable if and only if $s_n(\alpha) = 0$ for all $\alpha \in A$, $n \geq 1$, such that $f^n \alpha = \alpha$. The necessity of this condition is obvious. For the converse we introduce an equivalence relation on A via $f^m \beta = f^n \alpha$ for some m, n (depending on α, β). It suffices to solve (4.4) separately on each equivalence class, say, the class of an α. To this end we determine $\psi(f^n \alpha) = \psi(\alpha) - s_n(\alpha)$, $n \geq 1$, and then $\psi(\beta) = \psi(f^n \alpha) + s_m(\beta)$ as long as $f^m \beta = f^n \alpha$. It is easy to show that ψ is correctly defined and satisfies (4.4). For preperiodic f an explicit solution has been given in [2].

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