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Lifting of homeomorphisms to branched coverings of a disk

by

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Abstract. We consider a simple, possibly disconnected, d-sheeted branched covering π of a closed 2-dimensional disk D by a surface X. The isotopy classes of homeomorphisms of D which are pointwise fixed on the boundary of D and permute the branch values, form the braid group $\mathbf{B}_n$, where n is the number of branch values. Some of these homeomorphisms can be lifted to homeomorphisms of X which fix pointwise the fiber over the base point. They form a subgroup L^π of finite index in $\mathbf{B}_n$. For each equivalence class of simple, d-sheeted coverings π of D with n branch values we find an explicit small set generating L^π. The generators are powers of half-twists.

1. Introduction. Let $\pi : X \to D$ be a simple, possibly disconnected, d-sheeted branched covering of a closed 2-dimensional disk D. *Simple* means that over each point of D there are either d simple points of X or $d-2$ simple points and one "double" point, a *branch point*. The image $A = \pi(B)$ of a branch point B is called a *branch value*. The isotopy classes of homeomorphisms of D which fix the boundary of D pointwise and permute the branch values form the braid group $\mathbf{B}_n$, where n is the number of branch points. Some of these homeomorphisms can be lifted to homeomorphisms of X which fix pointwise the fiber over the base point. They form a subgroup L^π of finite index in $\mathbf{B}_n$. The group L^π is finitely generated. The purpose of this paper is to find an explicit small set generating L^π for all equivalence classes of d-sheeted coverings π of D with n branch points. Theorem 31 solves the problem for connected coverings and Theorem 33 shows how to construct a set of generators for any covering from the generators for connected coverings.

Basic information on braids, braid groups and half-twists can be found in [B]. Branched coverings of a disk and their equivalence classes were studied by Hurwitz [H] and by Berstein and Edmonds [BE]. Equivalence classes of branched coverings of surfaces of any genus were studied by Gabai and

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Kazez [GK]. Lifting of homeomorphisms was considered in [BW] for 3-sheeted coverings and in [CW] and [MP] for n-sheeted coverings of a disk by a disk. Mulazzani and Piergallini [MP] also considered arbitrary, simple, n-sheeted coverings and proved that L^π is always generated by powers of half-twists. Apostolakis [A] considered 4-sheeted coverings and found generators for a certain quotient of the group L^π. In [WW] a small finite set of generators of L^π was found for every simple 4-sheeted covering of a disk.

2. Preliminaries and notation. We consider a simple d-sheeted, possibly disconnected branched covering $\pi : X \to D$ of a disk D with n branch values $A_1, \ldots, A_n$. We choose a base point A_0 on the boundary of D. Let $\pi^{-1}(A_0) = \{B_1, \ldots, B_d\} \subset X$. Let σ be a closed loop in D which starts at A_0 and misses the branch values. When we lift σ to X from any point B_i, we end up at some point B_j. This defines a permutation $\mu(\sigma)$ in the symmetric group Σ_d, which depends only on the homotopy class of σ in the complement of the branch values. In this way we get the *monodromy homomorphism* μ from the fundamental group of $D - \{A_1, \ldots, A_n\}$ based at A_0 to the group Σ_d. We compose loops from left to right and we compose permutations from left to right, but homeomorphisms are composed from right to left. The monodromy of the boundary ∂D, oriented clockwise, is called the *total monodromy* of the covering π. We say that coverings $\pi_1 : X_1 \to D_1$ and $\pi_2 : X_2 \to D_2$ are *equivalent* if there exist orientation preserving homeomorphisms $h : D_1 \to D_2$ and $\phi : X_1 \to X_2$ such that $hp_1 = p_2\phi$.

Basic terminology and notation:

- A *curve* is a simple path connecting A_0 to some A_i and avoiding the other branch values.
- An *arc* is a simple path connecting two branch values and avoiding the other branch values.
- Curves are called *disjoint* if they meet only at A_0.
- $D(\gamma)$ is the complement of a regular neighborhood of a curve γ in D; $D(\gamma_1, \ldots, \gamma_k)$ is the complement of a regular neighborhood of the union of pairwise-disjoint curves.
- $\pi|D_1$ is the restriction $\pi : \pi^{-1}(D_1) \to D_1$ of the covering π.
- $\pi(\gamma) = \pi|D(\gamma)$.
- $X(\gamma) = \pi^{-1}(D(\gamma))$.
- $\hat{\gamma}$ is a simple loop surrounding the curve γ clockwise, based at A_0.
- $\mu(\gamma)$ is the monodromy of the loop $\hat{\gamma}$.
- L^π is the group of homeomorphisms of D which lift to X through π and leave the points $B_1, \ldots, B_d$ fixed.
- $L(\gamma)$ is the subgroup of L^π which leaves the curve γ invariant.
- For a subgroup K of L^π we let $K(\gamma) = K \cap L(\gamma)$.

- A curve γ is K-*admissible* if $K(\gamma) = L(\gamma)$.
- An arc z is K-*admissible* if the liftable powers of the half-twist z belong to K.
- Two curves (or arcs) are K-*equivalent* if there exists an element of K which takes one to the other.
- τ denotes the total monodromy of π.
- The *excess* of π is described at the end of Definition 15.
- Arcs of *type* 1, 2 and 3 are described in Remark 14.
- The *index* of a curve or of an arc β is described in Definition 24.

Definition 1. We say that the covering π is *connected* if X is connected. A covering π is *quasi-connected* if X has several components, one component covers D non-trivially and the other components cover D trivially. By the *genus* of a connected surface X we mean the genus of the surface obtained by gluing a disk to each component of the boundary of X. In the quasi-connected case the genus of X means the genus of the non-trivial component of X. If D_1 is a subdisk of D then $\pi|D_1$ denotes the restriction of π to $\pi^{-1}(D_1)$.

The following result was proven in [BE] and again in [MP].

Proposition 2. *Connected simple coverings π_1 and π_2 are equivalent if and only if they have the same degree d (number of sheets), the same number of branch points, and the total monodromy of π_1 is conjugate to the total monodromy of π_2 in the symmetric group Σ_d.*

Definition 3. A *curve* in D is a simple path which begins at A_0 and ends at some branch value and avoids the other branch values. Curves are defined up to isotopy relative to the branch values. We say that curves are disjoint if they meet only at A_0. By the *monodromy* $\mu(\alpha)$ of a curve α we mean the monodromy of a closed path $\hat{\alpha}$ which goes along α to a point very close to its end point, a branch value A_i, then goes clockwise around A_i along a small circle and then comes back along α; $\mu(\alpha)$ is always a transposition in Σ_d.

Definition 4. Following [MP] we say that curves $\gamma_1, \dots, \gamma_k$ form a *system of curves* if $\gamma_i \cap \gamma_j = \{A_0\}$ for any $i \neq j$ and the curves meet at A_0 in this clockwise order. If $\gamma_1, \dots, \gamma_k$ form a system of curves then the sequence of transpositions $(\mu(\alpha_1), \dots, \mu(\alpha_k))$ is called the *monodromy sequence* of the system. A maximal system of curves, consisting of n curves, is called a (geometric) *basis*.

Definition 5. We say that a homeomorphism h of D which keeps the boundary ∂D pointwise fixed and permutes the branch values is *liftable* if there exists a homeomorphism ϕ of X such that $hp = p\phi$ and ϕ fixes the points $B_1, \dots, B_d$.

Lemma 6. *Let $\pi : X \to D$ be a simple, possibly disconnected, covering of a disk D. A homeomorphism h of D is liftable if and only if it preserves the monodromy of some geometric basis. If h is liftable then it preserves the monodromy of every curve. If we have two geometric bases with the same monodromy sequences, then there exists a liftable homeomorphism which takes one basis to the other.*

Proof. Suppose h lifts to ψ which fixes the points B_k. If γ is a curve then there exist two points, say B_i and B_j, such that the liftings of γ from them meet over the end point of γ, and then $\mu(\gamma) = (i, j)$. Since ψ fixes B_i and B_j, the curve $h(\gamma)$ has the same monodromy as γ. In particular h preserves the monodromy of every basis. Conversely, if h preserves the monodromy of a basis, then it preserves the monodromy of every loop and we can construct its lifting to X which fixes the points B_i. For any two bases there exists a homeomorphism h of D which is pointwise fixed on ∂D and takes one basis to the other preserving the order. If the bases have the same monodromy sequences then h is liftable. ∎

If $\gamma_1, \ldots, \gamma_k$ is a system of curves then $D(\gamma_1, \ldots, \gamma_k)$ denotes the complement of a regular neighborhood of their union which does not contain branch values other than the end points of the curves $\gamma_1, \ldots, \gamma_k$. We denote by $X(\gamma_1, \ldots, \gamma_k)$ the preimage $\pi^{-1}(D(\gamma_1, \ldots, \gamma_k))$ and by $\pi(\gamma_1, \ldots, \gamma_k)$ the restriction of π to $X(\gamma_1, \ldots, \gamma_k)$. We need to choose a new base point in $D(\gamma_1, \ldots, \gamma_k)$. There are two natural choices: a point A_0' on ∂D a little to the right of A_0 (before A_0 in the clockwise order along ∂D) or a point A_0'' on ∂D to the left of A_0. This defines curves and their monodromies in $D(\gamma_1, \ldots, \gamma_k)$ on the right side of the system $\gamma_1, \ldots, \gamma_k$ and on the left side of the system, because the points above A_0' and A_0'' have a numbering induced by the numbering of sheets above A_0. If we have a curve β which meets the system of curves $\gamma_1, \ldots, \gamma_k$ only at A_0 on the left side of γ_1 (respectively on the right side of γ_k), then we can slide the beginning of β to the left, to A_0'' (respectively to the right, to A_0'), and get a curve in $D(\gamma_1, \ldots, \gamma_k)$ on the left side or on the right side, which we also denote by β. If a curve β is disjoint from a curve γ and we consider β in $D(\gamma)$, we mean $D(\gamma)$ on the same side of γ as β.

Definition 7. A curve γ is *non-separating* if $X(\gamma)$ has the same number of connected components as X. If curves β, γ are disjoint, then we say that γ *does not separate* $X(\beta)$ or that γ *is non-separating in the complement of* β (in $D(\beta)$) if the number of connected components of $X(\beta, \gamma)$ is the same as the number of connected components of $X(\beta)$.

Remark 8. If curves β and γ are disjoint and γ is non-separating and β is non-separating in the complement of γ, then β is non-separating and γ is non-separating in the complement of β. Indeed if we remove both curves in the correct order then the number of connected components of X does not change, therefore it does not change if we remove the curves in the opposite order.

DEFINITION 9. An *arc* in D is a simple path which connects two branch values and is disjoint from the other branch values and from the boundary of D. Arcs are defined up to isotopy relative to the branch values. A closed regular neighborhood of an arc x can be identified with the closed unit disk U in the complex plane $\mathbb{C}$ with the arc x corresponding to the subarc $y = [-1/2, 1/2]$ of the real axis. A *half-twist around* x is the isotopy class of a homeomorphism of D obtained by extending by the identity the following homeomorphism T of U: the homeomorphism T rotates the disc $\{z : |z| \le 1/2\}$ counterclockwise around 0 by 180 degrees and the rotation is damped out to the identity at the boundary of U. We denote the half-twist around x again by x.

DEFINITION 10. Let β and γ be disjoint curves and let β be on the left side of γ. The arc x which is isotopic to the path $\beta^{-1}\gamma$ by an isotopy which fixes the end points of β and γ and does not pass through the other branch values is called the *arc corresponding to the pair* β, γ. The curve $\beta' = x(\gamma)$ is disjoint from γ, lies on the right side of γ and ends at the end point of β. We call β' the result of *jumping with* β *to the right over* γ. In a similar way $\gamma' = x^{-1}(\beta)$ is the result of jumping with γ to the left over β (see Figure 1).

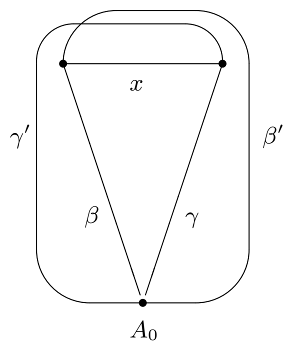

Fig. 1. Jumping with one curve over another one

LEMMA 11. *Let β and γ be disjoint curves. Let β' be the result of jumping with β over γ. Then β is non-separating in the complement of γ if and only if β' is non-separating in the complement of γ.*

Proof. The subdisk $D(\beta, \gamma)$ is isotopic to $D(\beta', \gamma)$ by an isotopy fixed on the branch values. The isotopy can be lifted to an isotopy of $X(\beta, \gamma)$ onto

$X(\beta', \gamma)$, therefore the number of connected components of these spaces is the same. ■

DEFINITION 12. A *sequence of arcs* consists of arcs $x_1, \dots, x_{k-1}$ such that x_i meets x_{i+1} at one of its end points and there are no other intersections between x_i and x_j for $1 \le i < j \le k-1$. We associate a sequence of arcs $x_1, \dots, x_{k-1}$ with any system of curves $\alpha_1, \dots, \alpha_k$. The arc x_i corresponds to the pair of curves α_i, α_{i+1} (see Definition 10). The sequence of arcs associated with a basis is called a *basic sequence of arcs.*

2.1. Hurwitz action and Hurwitz moves. Consider n-tuples $(\tau_1, \dots, \tau_n)$ of transpositions belonging to Σ_d. The *Hurwitz action* of the braid group $\mathbf{B}_n$ on such n-tuples is defined as follows:

$$\sigma_i(\tau_1, \dots, \tau_n) = (\tau_1, \dots, \tau_{i-1}, \tau_{i+1}, \tau_{i+1}\tau_i\tau_{i+1}, \tau_{i+2}, \dots, \tau_n),$$

where σ_i is the standard generator of the braid group $\mathbf{B}_n$. This action is also called *jumping with the transposition τ_i to the right, over the transposition τ_{i+1}*. Two n-tuples are *Hurwitz equivalent* if they belong to the same orbit of the Hurwitz action. We say that an n-tuple $(\tau_1, \dots, \tau_n)$ is *connected* if the transpositions τ_i generate the whole group Σ_d. This happens if the graph whose vertices are the numbers $1, \dots, d$ and edges are the transpositions τ_i is connected. The Hurwitz action takes a connected n-tuple to a connected n-tuple.

Hurwitz moves act on bases. If $\gamma_1, \dots, \gamma_n$ is a basis and if $x_1, \dots, x_{n-1}$ is the associated basic sequence of arcs, then the Hurwitz move σ_i takes the basis to its image under the action of the half-twist x_i. We have $x_i(\gamma_i) = \gamma_{i+1}$, $x_i(\gamma_{i+1}) = \gamma'_{i+1}$ and the other curves of the basis are fixed; see Figure 2. This move corresponds to jumping with γ_i to the right over γ_{i+1}. After the jump the curve γ_{i+1} appears at position i and the new curve γ'_{i+1} appears at number $i+1$. The inverse of this move, the image of the half-twist x_i^{-1}, corresponds to jumping with γ_{i+1} to the left over γ_i.

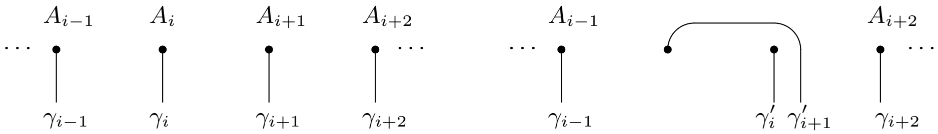

Fig. 2. A Hurwitz move. Jump with γ_i to the right over γ_{i+1}.

The path $\hat\gamma'_{i+1}$ is homotopic to $\hat\gamma_{i+1}^{-1}\hat\gamma_i\hat\gamma_{i+1}$, therefore the monodromy of the new curve γ'_{i+1} is equal to $\mu(\gamma_{i+1})\mu(\gamma_i)\mu(\gamma_{i+1})$ and the Hurwitz move σ_i on the basis induces the Hurwitz action σ_i on the monodromy sequence of the basis. The covering π is connected if and only if the monodromy sequence of a basis is connected.

If an n-tuple $t = (\tau_1, \dots, \tau_n)$ coincides with the monodromy sequence of a basis then the product $\tau_1 \dots \tau_n$ is equal to the total monodromy of the covering. Therefore for any k-tuple $t = (\tau_1, \dots, \tau_k)$ of transpositions we shall call the product $\tau_1 \dots \tau_k$ the *total monodromy of the k-tuple t*. The total monodromy of a k-tuple is preserved by the Hurwitz action.

REMARK 13. If two disjoint curves have the same monodromy then each of them is non-separating. Indeed we may choose a basis with the two curves at the beginning. Removing any of the curves does not change the connectivity of the graph corresponding to the transpositions in the monodromy sequence.

REMARK 14. There are three types of arcs. If the half-twist x is liftable, we call it of *type* 1; if x is not liftable but x^2 is liftable, we call x of *type* 2; if x and x^2 are not liftable, then x^3 is liftable and we call x of *type* 3. To see this we choose a curve α which meets x only in its end point. We denote by $\alpha_1 = x(\alpha)$ the image of α under the half-twist x, and we complete the curves α and α_1 to a basis. The other curves of the basis are fixed by x. One can check that if $\mu(\alpha) = \mu(\alpha_1)$ then the twist x is of type 1 (preserves the monodromy of the basis); if $\mu(\alpha)$ is disjoint from $\mu(\alpha_1)$ then the twist x is of type 2; and if the transpositions $\mu(\alpha)$ and $\mu(\alpha_1)$ have exactly one number in common then x is of type 3.

DEFINITION 15 (Standard sequence). Let τ be the total monodromy of a connected covering. We may write it as a product of disjoint cycles. By an *ordering* of τ we mean the choice of the order of the cycles $\tau = \nu_1 \dots \nu_s$ and the choice of the first number in each cycle. We then denote the numbers in the following way:

$$\tau = (a_{m_1}, a_{m_1-1}, \dots, a_1)(a_{m_2}, a_{m_2-1}, \dots, a_{m_1+1}) \dots \\ \dots (a_{m_s}, a_{m_s-1}, \dots, a_{m_{s-1}+1}).$$

The *standard connected sequence* of n transpositions (where the number $n \geq d + s - 2$ must have correct parity) corresponding to this ordering of τ is

$$(a_1, a_2), (a_2, a_3), \dots, (a_{m_1-1}, a_{m_1}), (a_{m_1}, a_{m_1+1}), (a_{m_1}, a_{m_1+1}), \\ (a_{m_1+1}, a_{m_1+2}), \dots, (a_{m_2-1}, a_{m_2}), \dots, (a_{m_s-1}, a_{m_s}), \dots, (a_{m_s-1}, a_{m_s}).$$

Any basis with this monodromy sequence is called a *standard basis* corresponding to the above ordering of τ.

Since we compose permutations from left to right, the product (the total monodromy) of this sequence is equal to τ. It takes a_{m_1} to a_{m_1-1} and so on. Because of the order of the transpositions we shall call $a_{m_{i-1}+1}$ the first number of cycle number i, and a_{m_i} the last number of cycle i though the permutation goes the other way. a_1 is called the *initial number* of the standard basis and $a_{m_s} = a_d$ is its *final number.*

In a standard sequence of transpositions there are $k-1$ distinct transpositions corresponding to a cycle of length k, followed by two equal connecting transpositions containing the last number of that cycle and the first number of the next one. The last transposition of the sequence may appear many times in order to complete the total number to n transpositions.

If the covering is quasi-connected then we remove the numbers corresponding to the trivial sheets from the total monodromy of the covering and we proceed as above.

If γ_i is a standard basis as above and if the last transposition in the monodromy sequence appears exactly $k \geq 1$ times then $k-2$ is called the *excess* of π.

Definition 16. A curve γ which lies in a subdisk D_1 of D *splits off number* p in D_1 if $\mu(\gamma)$ contains p and sheet number p is trivial over $D_1(\gamma)$ on the right side of γ. In this case, if β is a curve in D_1 which is disjoint from γ and lies on the right side of γ then $\mu(\beta)$ does not contain p.

Remark 17. If $\pi|D_1$ is quasi-connected and γ splits off p then $\pi|D_1(\gamma)$ is also quasi-connected. If $\mu(\gamma) = (p, q)$ and γ splits off p (on its right side) then γ splits off q on its left side.

Lemma 18. *All connected sequences of transpositions with a given length and a given product are Hurwitz equivalent.*

This is Lemma 1.2 in [MP] used in the proof of Proposition 2 above.

As a consequence we get the following proposition.

Proposition 19. *Suppose π is quasi-connected and we choose a particular ordering of the total monodromy $\tau = \nu_1 \ldots \nu_s$. Then there exists a standard basis corresponding to the chosen ordering of τ.*

Proof. We can start with any basis and its monodromy sequence of transpositions. The product of the sequence of transpositions is equal to the total monodromy τ of the covering. By Lemma 18 a suitable sequence of Hurwitz actions takes this sequence of transpositions to the chosen standard sequence of transpositions. By the properties of the Hurwitz moves the corresponding sequence of Hurwitz moves on the basis takes the basis to a standard basis (with the chosen monodromy sequence). ∎

Corollary 20. *Suppose π is connected.*

(1) *For any pair i, j with $1 \leq i < j \leq d$ there exists a curve with monodromy (i, j).*
(2) *For any number k there exists a standard basis with initial number k and there exists a standard basis with final number k.*

(3) *If a curve α splits off a number then there exists a standard basis which starts with α.*

Proof. Let τ be the total monodromy of π. If the pair (i, j) connects different cycles of τ we can choose the ordering of τ such that i, j are consecutive numbers of the corresponding standard sequence, and then the corresponding standard basis contains a curve with monodromy (i, j). If $i < j$ belong to the same cycle of τ we may assume that the cycle is $(k, k-1, \ldots, 1)$ and the part of the corresponding basis has monodromy sequence $(1, 2), (2, 3), \ldots, (k-1, k)$. Jumping with $(j-1, j)$ to the left several times we get the pair (i, j) and a curve with monodromy (i, j). The second statement of the corollary follows from Proposition 19. Now suppose that α is a curve, $\mu(\alpha) = (a, b)$ and α splits off a on the right side. Then over $D(\alpha)$ on the right side of α there is a trivial sheet number a and a non-trivial component connecting all other sheets. By the second statement there is a standard basis in $D(\alpha)$ with initial number b, and when we add α as the first curve we get a standard basis in D. ∎

Lemma 21.

(1) *Suppose $\pi : X \to D$ is a quasi-connected covering, γ is a curve with $\mu(\gamma) = (a, b)$, and the total monodromy of π is equal to τ. If (a, b) connects two cycles of τ then γ is non-separating and the genus of X is equal to the genus of $X(\gamma)$. If a, b belong to the same cycle of τ then either γ is separating* (*the only possibility if X has genus zero*), *or γ is non-separating and the genus of $X(\gamma)$ is one less than the genus of X.*
(2) *If $\pi : X \to D$ is a connected covering, then the excess is non-positive if and only if X is a planar surface* (*genus* 0).

Proof. Let X_1 denote the component of X which covers D non-trivially. The Euler characteristic of X_1 is $\chi(X_1) = 2 - 2g(X_1) - c$ where $g(X_1)$ is the genus of X_1 and c is the number of boundary components of X_1. The total monodromy τ of π corresponds to the lifting of the boundary ∂D. The lifting from the point B_i ends at the next point B_j along the same boundary component of ∂X in the clockwise order. In particular, numbers i and j are in the same cycle of τ if and only if B_i and B_j lie on the same boundary component of X and the number of cycles in τ, not counting the numbers of trivial sheets, is equal to c. Consider the preimage of the curve γ. One component of the preimage, call it λ, connects points B_a and B_b of the boundary. If (a, b) connects different cycles of τ then λ connects different boundary components. In this case λ does not separate X_1, we can connect the two sides of λ along the boundary, and when we cut X_1 along λ the genus does not change. If the pair (a, b) belongs to one cycle of τ then λ

connects points in the same boundary component. If λ does not separate X_1 and we cut along λ then the Euler characteristic goes up by 1, λ has one edge and two vertices and it splits into two edges and four vertices. Also c goes up by 1, therefore the genus goes down by 1. If $g(X_1) = 0$ then λ and γ must separate.

Consider statement (2). By the Riemann–Hurwitz formula the genus g of X satisfies $2-2g-c = d-n$. If the excess is non-positive then the number of cycles of τ is one more than the number of monodromy pairs which appear twice in the standard monodromy sequence, so it is equal to $c = n-d+2$. Then, by the above formula, $c = n-d+2-2g = n-d+2$ and $g = 0$. If the excess is 1 then the number of cycles drops by 1 and n increases by 1 and the genus becomes positive. A bigger excess increases n and increases c at most by one so the genus remains positive. This proves (2). ∎

2.2. Standard setup and notation. We consider a simple connected branched covering π of the unit disk D in the complex plane. The covering has degree d and has n branch values $A_1, \dots, A_n$. We choose the base point A_0 on the boundary of D in the lower half-plane. By Proposition 19 for a suitable numbering of points over A_0 there exists a basis α_i with the standard monodromy sequence $\mu(\alpha_i) = (k_i, k_i+1)$, $k_i \le k_{i+1} \le k_i+1$ and at most two consecutive k_i's are equal unless $k_i = d-1$. More explicitly the monodromy sequence consists of the pairs $(1,2), (2,3), \dots, (d-1,d)$, in this order, where each pair appears either once or twice except for $(d-1,d)$ which may appear many times.

After an isotopy we may assume that A_i's lie on the real axis, $A_i < A_{i+1}$ for $i = 1, \dots, n-1$ and α_i is a line segment connecting A_0 to A_i.

Definition 22. We define curves α'_i for $i = 1, \dots, n$, where $\alpha'_1 = \alpha_1$ and for $i > 1$, α'_i starts at A_0, crosses the real line at one point on the left side of A_1 and ends at A_i (see Figure 5).

Remark 23. The curve α'_i is non-separating if and only if the curve α_i is non-separating, which means that the pair $\mu(\alpha_i)$ appears twice (in a row) in the standard monodromy sequence. If α'_i is the first non-separating curve among α'_j (and if $i < d-1$) then the total monodromy τ of π has a cycle $(i, i-1, \dots, 1)$ and $\mu(\alpha'_i) = (1, i+1)$. For $s < i$ the curve α'_s is separating and $\mu(\alpha'_s) = (1, s+1)$. For any s if α'_s is separating then its monodromy $\mu(\alpha'_s) = (a,b)$ is contained in a cycle of τ and a is equal to the smallest number in the cycle. If α'_s is non-separating then its monodromy $\mu(\alpha'_s)$ consists of smallest numbers of two consecutive cycles of τ, except for the case when $\mu(\alpha_s) = (d-1,d)$ and the excess is odd. In the last case $\mu(\alpha'_s)$ consists of the smallest and the greatest number in the last cycle of τ. In particular if α'_i and α'_j are non-separating curves with different monodromies and $i < j$ then $\mu(\alpha'_i)$ connects two different cycles of τ and $\mu(\alpha'_j)$ lies in the last cycle of τ or connects a different pair of cycles of τ than $\mu(\alpha'_i)$. If α'_i and α'_j are consecutive non-separating curves among α'_s then $\mu(\alpha'_i)$ and $\mu(\alpha'_j)$ have a number in common.

DEFINITION 24. By the *index* of a curve β (respectively an arc z) we mean the minimum number of intersection points of β (respectively z) with the union of the curves $\alpha'_1 \cup \cdots \cup \alpha'_n$ (we may assume that there are only a finite number of intersection points and that the intersection is transverse) not counting the end points of β (respectively the end points of z).

If a curve β (respectively an arc z) meets α'_i (at a point different from A_0), let P_i be the first point of $\alpha'_i \cap \beta$ (respectively the first point of $\alpha'_i \cap z$) along α'_i, starting from the end point A_i of α'_i. Then the segment of α'_i from A_i to P_i is called the *segment of α'_i corresponding to β* (respectively the *segment of α'_i corresponding to z*).

REMARK 25. If a curve β meets only one curve α'_i and ends at A_i then it is isotopic to α'_i, which has index 0, because the complement of the other curves α'_j is isotopic to a disk with one distinguished point and all curves are isotopic to each other in such a disk. Also if an arc z meets only two curves α'_i and α'_j and connects the points A_i and A_j then it is isotopic to an arc of index 0, because the complement of the remaining curves α'_k is isotopic to a disk with two distinguished points and all arcs are isotopic to each other in such a disk.

DEFINITION 26. Let K be any subgroup of L^π. If γ is a curve then $L(\gamma)$ denotes the subgroup of the elements of L^π which leave γ invariant (up to isotopy), while $K(\gamma)$ denotes the subgroup of the elements of K which leave γ invariant.

An arc z is called *K-admissible* if the liftable powers of the half-twist z belong to K. A curve γ is called *K-admissible* if $L(\gamma) = K(\gamma)$. Two arcs (or two curves) are *K-equivalent* if one of them is equal to the image of the other under an element of K.

LEMMA 27. *If two arcs (or two curves) are K-equivalent and one of them is K-admissible then the other is also K-admissible. If one curve is non-separating then the other is also non-separating. If one curve splits off a number then the other splits off the same number.*

Proof. Let h be a homeomorphism, let z be an arc and let $z_1 = h(z)$. Then the half-twist with respect to z_1 is equal to hzh^{-1}. Indeed h^{-1} takes a regular neighborhood of z_1 to a regular neighborhood of z, the twist z makes a half-twist of the neighborhood and h takes the twisted neighborhood of z back to the neighborhood of z_1. If $h \in K$ and a power of the half-twist z is in K then the same power of the half-twist z_1 is in K. This proves the case of an arc.

Let γ be a curve and let $\gamma_1 = h(\gamma)$. Suppose $h \in K$ and γ is K-admissible, so that every liftable homeomorphism which leaves γ invariant belongs to K. Let f be a liftable homeomorphism which leaves γ_1 invariant. Then $h^{-1}fh$ leaves γ invariant and belongs to K, hence f also belongs to K and γ_1 is K-admissible.

If γ is a curve then one component of $\pi^{-1}(\gamma)$, call it λ, connects two points B_i and B_j in $\pi^{-1}(A_0)$. The points belong to a component X_1 of X. The curve γ is non-separating if λ does not separate X_1 and γ splits off a number if λ splits off a disk from X_1 which covers D trivially. If $\tilde{h}$ is the lifting of h to X which fixes $\pi^{-1}(A_0)$ pointwise then $\tilde{h}$ takes X_1 to X_1 and takes $X_1 - \lambda$ to $X_1 - \tilde{h}(\lambda)$. Consider in particular the case when γ splits off number i (on the right side). When we remove γ we shift the base point a little to the right, counterclockwise along ∂D. Then all points in $\pi^{-1}(A_0)$ move counterclockwise along ∂X (with respect to the orientation of X) and the point B_i moves to a point B_i' in the trivial component of X_1 split off by λ. The homeomorphism $\tilde{h}$ preserves ∂X so it preserves B_i' and takes the trivial component of $X_1 - \lambda$ to the trivial component of $X_1 - \tilde{h}(\lambda)$ and this component has the same number i. ∎

Definition 28 (Distinguished arcs). We distinguish some arcs in the standard setup. The arc $y_{i,j}$, $i < j$, lies in the lower half-plane except for its end points A_i and A_j. The arc $y_{i,i+1}$ is also called x_i. The arc x_i is of type 1 if $\mu(\alpha_i) = \mu(\alpha_{i+1})$. Otherwise x_i is of type 3. The arc $z_{i,j}$, $i < j$, lies in the upper half-plane except for its end points A_i and A_j.

If $i < j - 1$ and the arcs x_i and x_j are of type 1 then let $w_{i,j}$ denote the following arc: It starts at A_j and moves monotonically to the left. It lies below every A_k for which x_k is of type 1 and above any other A_k between A_{i+1} and A_j. It also lies below A_{i+1} and below A_i. Just after coming to the left of A_i it goes up and back, to the right. It passes above A_i and A_{i+1} and again below every A_k for which x_k is of type 1 and above any other A_k between A_{i+1} and A_j and above A_j, and ends at A_{j+1} (see Figure 3).

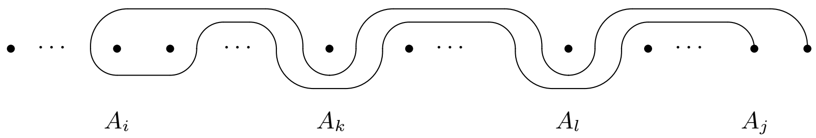

Fig. 3. The arc $w_{i,j}$. The arcs x_i, x_k, x_l, x_j are of type 1.

We also distinguish some *special* arcs. Suppose the excess of π is positive. Let m be the smallest number for which $\mu(\alpha_m) = (d-1, d)$. We denote by

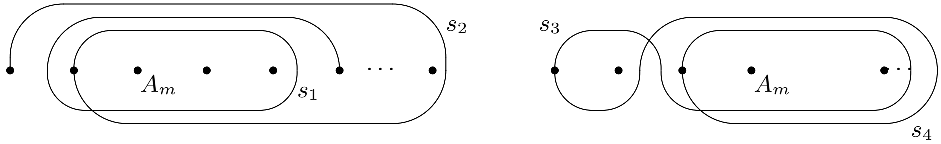

Fig. 4. Special arcs

s_1, s_2, s_3, s_4 the arcs in Figure 4. The arc s_1 exists if the excess is at least 2. It has type 1 and is *special*. The arcs s_2, s_3, s_4 are special if they are of type 2. This happens if the excess of π is odd and, for the arc s_2, the arcs x_{m-2} and x_{m-1} are of type 3, and for s_3 and s_4 the arcs x_{m-3} and x_{m-1} are of type 3 and x_{m-2} is of type 1. Observe that if $d \geq 4$ then s_1 is disjoint from α_1, and if $d \geq 5$ then all special arcs are disjoint from α_1.

DEFINITION 29 (The generating arcs and the group H). All arcs $y_{i,j}$, arcs $z_{i,j}$ of type 2, all arcs $w_{i,j}$ and arcs s_i which are special for π are called the *generating* arcs. We denote by H the subgroup of L^π generated by the minimal positive liftable powers of half-twists with respect to all generating arcs.

LEMMA 30. *Suppose $i+1 < j \leq m$. The arc $y_{i,j}$ is of type* 2 *if and only if there are at least two arcs x_k, x_l of type* 3 *between A_i and A_j (the end points may include A_i or A_j). The arc $z_{i,j}$ is of type* 2 *if and only if there is an arc x_k of type* 1 *strictly between A_i and A_j. Every arc $z_{i,j}$ is H-admissible.*

Proof. All statements follow immediately from the definitions except for the last one. Suppose $z_{i,j}$ is of type 3. We shall prove by induction on $j - i$ that $z_{i,j}$ is H-admissible. We may assume that x_i and x_{j-1} are of type 3, otherwise we can shorten the arc $z_{i,j}$. Then all arcs x_k between A_i and A_j are of type 3. Therefore all arcs $y_{i,k}$ with $i+1 < k \leq j$ are of type 2. Then $y^2_{i,i+2} y^2_{i,i+3} \cdots y^2_{i,j} y^{-2}_{i,j-1} y^{-2}_{i,j-2} \cdots y^{-2}_{i,i+2} z^3_{i+1,j}(z_{i,j}) = x_i$, hence $z_{i,j}$ is H-admissible. ∎

THEOREM 31 (Main Theorem). *If π is a connected simple covering of a disk D with a standard basis $\alpha_1, \ldots, \alpha_n$, then H is equal to the group of all liftable homeomorphisms.*

Theorem 31 is obvious for $d = 2$, since in this case all arcs are of type 1 and L^π is generated by the x_i's. The theorem was proven in [BW] for some coverings of degree $d = 3$ and in [WW] for the remaining case of coverings of degree $d = 3$ and all coverings of degree $d = 4$, and was proven for any d and $n = d - 1$ in [CW] and in [MP].

2.3. Non-connected coverings

REMARK 32. A quasi-connected covering can be treated in exactly the same way as a connected covering. To make the argument more similar to the connected case we may denote the number of sheets by $d+k$, where d is the number of sheets belonging to the non-trivial component and we may renumber the sheets (the points over A_0) in such a way that the sheets belonging to the non-trivial component are numbered from 1 to d and the trivial sheets are numbered from $d+1$ to $d+k$. Then the numbers $d+1$ to $d+k$ never enter the discussion. Theorem 31 is valid for quasi-connected coverings too.

We shall prove a theorem about generators of L^π for non-quasi-connected coverings. We describe first the corresponding standard setup.

Let $\pi : X \to D$ be a simple, non-quasi-connected covering of degree d with $n > 1$ branch values. Let V be a simple path separating D such that the branch values do not lie on V, there is at least one branch value on each side of V, and the branch values on different sides of V correspond to different connected components of X. Let $x_1, \ldots, x_{n-1}$ be a sequence of arcs in D such that x_j connects the branch values A_j and A_{j+1}, and that for some $1 \le k \le n-1$ the arcs $x_1, \ldots, x_{k-1}$ lie on one side of V, call it left side, the arcs $x_{k+1}, \ldots, x_{n-1}$ lie on the right side of V and x_k meets V in one point, not the end point of x_k. Up to an isotopy we may assume that D is the unit disk in the complex plane, that the points A_j and the arcs x_j lie on the real line and that V is a vertical line. We denote the part of D on the left side of V by D_1 and the part of D on the right side of V by D_2. We define curves α_i and α'_i and arcs $y_{p,q}$ and $z_{p,q}$ as in the standard setup for the connected case (see Figure 5).

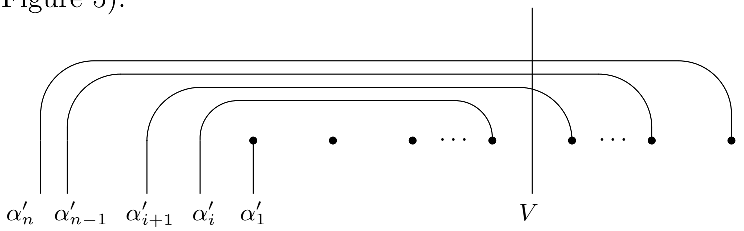

Fig. 5. The curves α'_i

We denote by K the subgroup of L^π generated by all liftable homeomorphisms which leave V invariant and by all liftable powers of half-twists with respect to arcs $y_{p,q}$.

THEOREM 33. *Let $\pi : X \to D$ be a simple, non-quasi-connected covering of degree d with $n > 1$ branch values. Assume the standard setup as above. Then $K = L^\pi$.*

Proof. We prove first, by induction on the index, that an arc z which connects branch values on different sides of V is K-admissible. If z has index zero then it is equal to some $y_{p,q}$ and is K-admissible. Suppose the index of z is positive. By Remark 25 there is a curve α'_i which meets z and does not

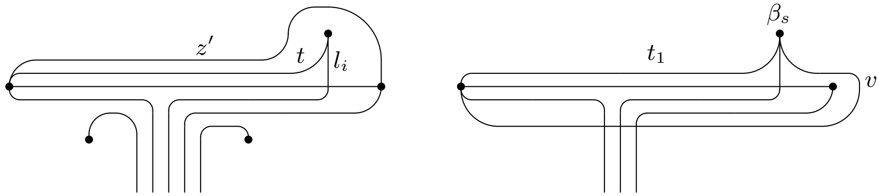

Fig. 6. Reducing the index of an arc

end at the end point of z. Let l_i be the segment of α'_i corresponding to z (see Definition 24 and Figure 6, left). Let t be the arc which moves along l_i and then along z to the end point of z which lies on the different side of V than the point A_i (see Figure 6, left). The arc t has index smaller than the index of z so it is K-admissible by the induction hypothesis. The arc t is of type 2, because the monodromies of curves which end on different sides of V are disjoint, and the arc $z' = t^2(z)$ or $z'' = t^{-2}(z)$ has index smaller than the index of z. This proves the claim.

Consider a liftable homeomorphism f and consider the curves $\alpha'_1, \dots, \alpha'_k$ in D_1. The map f permutes the branch values in D_1 and permutes the branch values in D_2 and moves the curve α'_i into D. Let $\gamma_i = f(\alpha'_i)$ for $i = 1, \dots, n$. If we disregard the branch values in D_2 we can isotope simultaneously the curves $\gamma_1, \dots, \gamma_k$ into D_1. Consider such an isotopy and consider the position of γ_i just before it passes to the other side of a point A_j in D_2. Consider an arc t which starts at the end point of γ_i on a suitable side of γ_i, moves parallel to γ_i and ends at the point A_j. This arc is K-admissible, by the previous claim, and passing with γ_i to the other side of A_j is the same as applying t^2 or t^{-2} to the curve γ_i. Thus there exists $h \in K$ such that hf takes each curve α'_i for $i \le k$ to a curve in D_1. The complement of the union of these new curves is isotopic to the disk D_2, relative to the branch values, so we may assume that hf also takes the curves α'_j for $j \ge k+1$ to curves in D_2. Now we may also assume that V is left invariant and thus $hf \in K$ and $f \in K$. ∎

3. Proof of Theorem 31. The proof is by induction on d.

The Induction Hypothesis. We assume that Theorem 31 is true for coverings of degree less than d and for coverings of degree d with less than n branch values. In particular if a connected covering has degree less than d or has degree d and less than n branch values then the group of liftable homeomorphisms is generated by liftable powers of half-twists.

We assume that $\pi : X \to D$ is a connected covering of degree d with n branch values and we assume the standard setup. We denote by τ the total monodromy of π.

For a connected covering we have $n \geq d-1$ and for $n = d-1$ Theorem 31 was proven in [CW] and in [MP]. Theorem 31 was also proven for $d \leq 4$ in [BW] and [WW] so we may assume $d \geq 5$ and $n \geq d$.

The idea of the proof is very simple. We choose a small number of special curves. We prove that the special curves are H-admissible. Then we prove (see Proposition 42) that if α is a special curve and $f \in L^\pi$ then $f(\alpha)$ is H-equivalent to α. This means that there exists $h \in H$ such that $hf(\alpha) = \alpha$. Since α is H-admissible we have $hf \in H$ and $f \in H$. This proves the theorem.

Remark 34. We often need to prove that a particular curve β is H-admissible. If β is non-separating or splits off a number then we can choose a standard basis in $D(\beta)$ and there are the distinguished arcs corresponding to this basis. It follows from the Induction Hypothesis that if all these distinguished arcs are H-admissible then all liftable homeomorphisms in $D(\beta)$ belong to H and β is H-admissible.

Lemma 35. *Let ρ be a simple covering. If β is a separating curve and γ is disjoint from β, then γ separates ρ if and only if γ separates $\rho(\beta)$. If ρ has r connected components then there are at least $d-r$ branch values, that is, $n \geq d-r$, and at most $d-r$ pairwise disjoint separating curves. Also $n = d-r$ if and only if there exist $d-r$ disjoint separating curves, if and only if every curve separates.*

Proof. Splitting ρ along one curve may increase the number of connected components of ρ at most by one. If β separates then $\rho(\beta)$ has one connected component more than ρ. If γ separates $\rho(\beta)$ then $\rho(\beta,\gamma)$ has two components more than π. If we split first along γ and then along β we get again a covering which has two more components than ρ so γ must separate ρ. Suppose now that γ does not separate $\rho(\beta)$, or in greater generality that γ is non-separating in some subdisk D_1 of D containing γ. Let λ be the component of $\rho^{-1}(\gamma)$ which connects two points on the boundary of X. Then the two sides of λ can be connected in $\rho^{-1}(D_1(\gamma))$, hence also in $X(\gamma)$, and γ is non-separating in D. Cutting D along any basis produces a trivial covering and d distinct components, therefore $n \geq d-r$. Any $d-r$ pairwise disjoint separating curves would split the covering into a trivial covering, which is impossible if $n > d-r$ since some branch points remain. If $n = d-r$ then cutting D along $d-r$ curves of any basis produces a trivial covering, so every curve of the basis separates, and every curve belongs to some basis. ∎

Lemma 36. *Consider a quasi-connected covering ρ over a disk D_1. Let β and γ be two curves in D_1 with the same monodromy. If β and γ are both non-separating or both split off the same number on the right side, or if every curve in D_1 is separating, then β and γ are L^ρ-equivalent.*

Proof. Let the degree of the non-trivial component of the covering be equal to d_1 and let the number of branch values be equal to n_1. If every curve is separating then, by Lemma 35, there are only $n_1 = d_1 - 1$ disjoint separating curves and X is a disk and the total monodromy τ_1 of ρ is a cycle of length d_1. The products $\mu(\beta)\tau_1 = \mu(\gamma)\tau_1$ split τ_1 in the same way into two cycles. We can choose bases which start with β and with γ respectively. The monodromy of any other curve of such a basis belongs to one cycle or to the other and monodromies from different cycles commute. We may change the bases by jumping so that all curves corresponding to the first cycle precede all curves corresponding to the second cycle. The total number of curves is $d_1 - 2$. The length of the cycles determines the number of curves corresponding to each cycle. This determines the monodromy sequence of the bases up to Hurwitz equivalence, by Lemma 18, where the Hurwitz moves do not involve the curves β and γ. The claim follows by Lemma 6.

If the curves are non-separating or split off one number then $\rho(\beta)$ and $\rho(\gamma)$ are quasi-connected and have the same total monodromy on the right side of β and γ respectively. By Lemma 18 they have bases with the same monodromy sequences. Together with β and γ we get bases in D_1 with the same monodromy, so, by Lemma 6, there exists $h \in L^\rho$ with $h(\beta) = \gamma$. ■

LEMMA 37. *All curves α_k' are H-admissible.*

Proof. Consider $k = 1$. Then $\pi(\alpha_1)$ is quasi-connected. The curves $\alpha_2, \ldots, \alpha_n$ form a standard basis in $D(\alpha_1)$ and all generating arcs corresponding to this basis are the generating arcs corresponding to the original basis, so they are H-admissible and α_1' is H-admissible by Remark 34.

Now consider $k = n$. Then $\pi(\alpha_n')$ is quasi-connected. The curves $\alpha_1, \ldots, \alpha_{n-1}$ form a standard basis in the complement of α_n'. All the generating arcs corresponding to this basis are also the generating arcs corresponding to the original basis so they are H-admissible, except possibly for special arcs. If α_n' is separating there are no special arcs, and if it is non-separating then $\pi(\alpha_n')$ has degree at least 5 and the special arcs are disjoint from α_1 and are H-admissible (see Definition 28).

Let $1 < k < n$. Suppose that α_k' is separating. We construct a standard setup in $D(\alpha_k')$ for non-quasi-connected coverings. The curve α_k plays the role of V, the arcs x_i, $i \neq k$, $i \neq k-1$, and the arc $y_{i-1,i+1}$ form a suitable sequence of arcs and the new arcs $y_{p,q}$ are equal to the old arcs $y_{p,q}$, possibly with different indices, and are H-admissible. Consider the corresponding group K. All arcs on each side of V are H-admissible, because they are disjoint from α_1' or from α_n', and they generate the groups of liftable homeomorphisms on each side of V, by the Induction Hypothesis. When we extend any such homeomorphism to the other side of V by the identity we get an element of H, and any liftable homeomorphism of D which leaves V invariant is a product of two such homeomorphisms (it preserves the monodromy on each side) and belongs to H. Therefore $K \subset H$ and $L(\alpha'_k) = K \subset H$.

Suppose now that α'_k is non-separating. Then x_k or x_{k-1} is of type 1. We may replace α'_k by an H-equivalent curve so we may assume x_k is of type 1. The curves α_i, $i \neq k$, form a standard basis in $D(\alpha'_k)$. Distinguished arcs of type $y_{p,q}$ and $w_{i,j}$ corresponding to the new basis are also distinguished for the original standard basis, hence are H-admissible. Let $\bar{z}_{i,j}$ denote the distinguished arc of type $z_{i,j}$ corresponding to the new basis. Some arcs $\bar{z}_{i,j}$ have a "dent", they go under the point A_k. If such an arc is of type 2 then there must be an arc x_l of type 1 between A_i and A_k or between A_k and A_j. But then the arc $z_{i,k}$ (respectively $z_{k,j}$) is of type 2 and $z_{i,k}^2(\overline{z}_{i,j}) = z_{i,j}$ (respectively $z_{k,j}^{-2}(\overline{z}_{i,j}) = z_{i,j}$) is H-admissible. Finally we may have new special arcs but since α'_i is non-separating the covering $\pi(\alpha'_i)$ is connected with more than four sheets and the special arcs are disjoint from α_1 (see Definition 28), so they are H-admissible. Therefore α'_k is H-admissible by Remark 34. ∎

LEMMA 38. *Every arc of type* 2 *and every arc of type* 3 *is H-admissible.*

Proof. We prove the lemma by induction on the index of the arcs. If an arc z is disjoint from some α'_i then it is H-admissible by the previous lemma. In particular z is H-admissible if it has index smaller than $n-2$. Let z be an arc of type 2 or of type 3. We may assume that z meets every curve α'_i. Consider segments l_i of the curves α'_i corresponding to the arc z (see Definition 24). They may lie on both sides of z. Consider the part γ of the curve α_1 from A_0 to the first intersection point with z. We consider disjoint curves which move along γ, turn left or right just before meeting z, move to the consecutive segments l_i (or to an end point of z), then move along l_i, possibly crossing z, and end at the end points of l_i's (see Figure 6, left). We call these curves $\beta_1, \dots, \beta_n$. They form a basis.

We first prove the claim for arcs of type 2. We may assume that z is of type 2 and of index k and that every arc of type 2 and of index smaller than k is H-admissible. The curves β_j and β_k, which end at end points of z, have monodromies (a,b) and (c,d), all letters distinct. We have at least five sheets. There must be a letter e different from a,b,c,d which appears in the monodromy (e,f) of some curve β_s. Then (e,f) is disjoint from (a,b) or from (c,d). The arc t connecting the end points of β_j (respectively β_k) and β_s along a suitable l_i (the index i depends on the end point A_i of the curve β_s) and a suitable part of z is of type 2 and has index smaller than k (see Figure 6, left). The arc $z' = t^2(z)$ or $z'' = t^{-2}(z)$ has index smaller than k (it misses the intersection of l_i with z) and therefore is H-admissible and z is H-admissible. It follows by induction that every arc of type 2 is H-admissible.

We now prove the claim for arcs of type 3. We assume that z is of type 3 and of index k and that every arc of type 2 and every arc of type 3 and index smaller than k is H-admissible. The curves β_j and β_k, which end at end points of z, have monodromies (a, b) and (a, c), where $b \neq c$. If there is a curve β_i with monodromy disjoint from (a, b) or from (a, c) we can reduce the index of z as in the previous case (arcs of type 2 are H-admissible). If not, then there must be a curve β_s with monodromy (a, d), where d is different from b and c. We now consider an arc v which starts at the end of β_s, goes around an end point of z (say, the right end point with respect to the arc l_i of the curve β_s as in Figure 6, right) and then all the way along z to the other end point of z. This arc has type 2 and is H-admissible. Consider the arcs t_1 and t_2 which connect the end point of β_s to the left end point of z (respectively the right end point of z). They are of type 3 and have indices smaller than k. Now $t_2^{-3}v^{-2}(z) = t_1$ has index smaller than z and thus z is H-equivalent to an arc of smaller index and is H-admissible. ■

Corollary 39. *Suppose that the curve β is disjoint from α_1 and that β is non-separating or β splits off a number. If α_1 is separating then β is H-admissible.*

Proof. If α_1 is separating then it splits off number 1 on the right side. By Corollary 20 there is a standard basis in $D(\beta)$ with the first curve α_1. Consider the set of distinguished arcs corresponding to this basis. The covering $\pi(\beta)$ is quasi-connected with the non-trivial component of degree at least 4. The special arc s_1 corresponding to the new basis is disjoint from α_1 (see Definition 28) and therefore is H-admissible. Other special arcs are of type 2, and since α_1 is separating all distinguished arcs which meet α_1 are of type 2 or 3. They are all H-admissible by Lemma 38. The remaining distinguished arcs are disjoint from α_1 and are H-admissible by Lemma 37. Therefore β is H-admissible by Remark 34. ■

In order to prove Theorem 31 we need to consider several cases which are listed in the next definition.

Definition 40. We define some *special* curves. We denote by m the smallest number for which $\mu(\alpha_m) = (d-1, d)$.

Case 1. The total monodromy τ has at least three disjoint cycles (including cycles of length 1). Then there are two special curves, the non-separating α'_i with the smallest index i and the next non-separating α'_j with $j > i + 1$.

Case 2. The total monodromy τ has at most two disjoint cycles and the excess is odd and positive. Then there is one special curve α'_m.

Case 3. The total monodromy τ has two disjoint cycles and the excess is non-positive. Then there is one special curve, the non-separating α'_i with the smallest index i. (If the excess is 0 then the curve α'_m is special.)

Case 4. The total monodromy τ has two disjoint cycles and the excess is even and positive. Then $m = d-1$. We define curves $\beta_{i,j}$ for $1 \leq i \leq d-2$ and $d-1 \leq j \leq n$. The curve $\beta_{i,j}$ starts between α_{i-1} and α_i (on the left side of α_1 if $i=1$), crosses the real axis, then turns right, stays in the upper half-plane and ends at A_j. In particular $\beta_{1,j} = \alpha'_j$ for $j \geq d-1$. The curve $\beta_{i,j}$ is H-equivalent to $\beta_{i,m}$. The curves $\beta_{i,m}$ are special.

The above cases exhaust all possibilities which we need to consider. If the excess is even then τ has at least two cycles. If the excess is -1 and there is only one cycle then $n = d-1$ and this case of Theorem 31 is already known.

Case 4 described in Definition 40 is most difficult and it requires additional preparation. If $j-d$ is odd we define a curve $\beta'_{i,j}$ symmetric to $\beta_{i,j}$. The curve $\beta'_{i,j}$ starts on the right side of α_j, crosses the real axis, then turns left, stays in the upper half-plane and ends at A_i. It also has monodromy (i,d). The next lemma shows that $\beta'_{i,j}$ is H-equivalent to $\beta_{i,j}$.

Lemma 41. *Assume Case 4 of Definition 40. Then the curve $\beta_{i,j}$ is H-admissible. Also, the curves $\beta_{i,d-1}$, $\beta'_{i,d-1}$ and $\beta'_{i,d+1}$ are H-equivalent. For all $1 \leq s,i \leq d-2$ and all $d-1 \leq j \leq n$ there exist curves δ and δ' which are disjoint from $\beta_{i,j}$, lie on different sides of $\beta_{i,j}$, are non-separating in $D(\beta_{i,j})$ and are H-equivalent to $\beta_{s,j}$.*

Proof. If $i=1$ then $\beta_{i,j} = \beta_{1,j} = \alpha'_j$, which is H-admissible by Lemma 37.

If $i > 1$ then $\beta_{i,j}$ is disjoint from α_1 and is non-separating, and α_1 is separating, so $\beta_{i,j}$ is H-admissible by Corollary 39.

Consider the curves $\beta_{s,d-1}$, $\beta'_{s,d-1}$ and $\beta'_{s,d+1}$. They have the same monodromy and are non-separating. If $s>1$ then the curves are non-separating in $D(\alpha_1)$, by Lemma 35, and therefore H-equivalent by Lemmas 36 and 37. If $s=1$ then the curves $\beta'_{1,d-1}$ and $\beta'_{1,d+1}$ lie in $D(\alpha_2)$. The curve α_2 splits off number 2 on the right side and is disjoint from α_1 so it is H-admissible by Corollary 39 and $\pi(\alpha_2)$ is quasi-connected. Our curves are non-separating in D and therefore are non-separating in $D(\alpha_2)$, by Lemma 35, and therefore are H-equivalent, by Lemma 36. The curves $\beta_{s,d-1}$ and $\beta'_{s,d-1}$ lie in $D(\alpha'_n)$ and are non-separating there, by Remarks 13 and 8 and Lemma 37, because α'_n and α'_{n-1} are disjoint, have the same monodromy and lie in the complement of each of the curves $\beta_{s,d-1}$ and $\beta'_{s,d-1}$. Therefore the curves $\beta_{s,d-1}$ and $\beta'_{s,d-1}$ are H-equivalent, by Lemmas 36 and 37.

We now prove the last claim of the lemma. We may replace $\beta_{i,j}$ by any curve $h(\beta_{i,j})$, where $h \in H$, and when we find curves δ and δ' for $h(\beta_{i,j})$, we can move them back by h^{-1}. Suppose that $s \le i$. Up to H-equivalence we may assume $j = d-1$. Then the curves $\beta_{s,j+1}$ and $\beta_{s,j+2}$ lie on the left side of $\beta_{i,j}$, are H-equivalent to $\beta_{s,j}$ and have the same monodromy, so they are non-separating in $D(\beta_{i,j})$, by Remark 13. We turn to the right side of $\beta_{i,j}$. For $s = i$ we may choose $j = d+1$ and then there are two curves $\beta_{s,d-1}$ and $\beta_{s,d}$ which are on the right side of $\beta_{i,j}$ and are H-equivalent to $\beta_{s,j}$ and are non-separating in $D(\beta_{i,j})$, by Remark 13. If $s < i$ then we choose $j = d-1$ and the curve $\beta'_{s,d+1}$ lies to the right of $\beta_{i,d-1}$. Now $\beta_{i,d-1}$ and $\beta_{i,d}$ lie to the left of $\beta'_{s,d+1}$ and have the same monodromy, so the curve $\beta_{i,j}$ is non-separating in $D(\beta_{s,d+1})'$, by Remark 13, and conversely $\beta'_{s,d+1}$ is non-separating in $D(\beta_{i,j})$, by Remark 8. Suppose now that $s > i$. We may choose $j = d+1$ and then $\beta_{s,d-1}$ and $\beta_{s,d}$ are disjoint, H-equivalent and lie on the right side of $\beta_{i,j}$ and are non-separating in $D(\beta_{i,j})$, by Remark 13. For the left side we choose $\beta'_{i,d+1}$, which is H-equivalent to $\beta_{i,j}$. Then $\beta_{s,d-1}$ and $\beta_{s,d}$ lie in $D(\beta'_{i,d+1})$ to the left of $\beta'_{i,d+1}$ and are non-separating there, by Remark 13. ∎

Theorem 31 follows from the next proposition.

Proposition 42. *Let β be a curve which is L^π-equivalent to a special curve σ. Then β is H-equivalent to σ.*

Remark 43. In the next lemmas we shall prove Proposition 42 under additional assumptions. We shall assume one or more of the following statements: the curve β is disjoint from some curve $\gamma = \alpha'_i$ (or, in Case 4, disjoint from some curve $\gamma = \beta_{j,m}$); β is non-separating in $D(\gamma)$; there exists a curve α which is H-equivalent to σ and is disjoint from γ and lies on the same side of γ as β. In the proof we may replace γ by a curve H-equivalent to it, $\gamma' = h(\gamma)$ for some $h \in H$, and consider the curve $h(\beta)$ instead of β.

Indeed if β is disjoint from γ then $h(\beta)$ is disjoint from γ'; if β lies on a particular side of γ, say left, then $h(\beta)$ lies on the left side of γ' if β is non-separating in $D(\gamma)$ then $h(\beta)$ is non-separating in $D(\gamma')$ because h is liftable and its lift preserves connected components of $\pi^{-1}(D(\gamma))$. Moreover, if we prove that $h(\beta)$ is H-equivalent to σ then β is also H-equivalent to σ.

Lemma 44. *Let β be a curve which is L^π-equivalent to a special curve σ and is disjoint from some curve $\gamma = \alpha'_i$ (or, in Case 4, disjoint from some curve $\gamma = \beta_{j,m}$). Suppose that there exists a curve α which is H-equivalent to σ and is disjoint from γ and lies on the same side of γ as β. If α and β are non-separating in $D(\gamma)$ then β is H-equivalent to σ.*

Proof. This follows from Lemmas 36, 37 and 41. ∎

Lemma 45. *Consider Case 4. Let β be a curve which is L^π-equivalent to a special curve $\beta_{j,m}$. Suppose β is disjoint from a curve γ and non-separating* *in* $D(\gamma)$, *where* γ *is either some* α'_i *or some* $\beta_{i,m}$. *Then* β *is* H*-equivalent to* $\beta_{j,m}$.

Proof. It suffices to prove that there exists a curve α such that α and β satisfy the assumptions of Lemma 44. Suppose first that $\gamma = \alpha'_i$ is separating ($i < d-1$, and $\mu(\alpha'_i) = (1, i+1)$). On the right side of α'_i, sheets number $1, \ldots, i$ are separated from sheets $i+1, \ldots, d$. We have $\mu(\beta_{j,m}) = (j, d)$ so if β lies on the right side of α'_i then $j \geq i+1$ and $\beta_{j,m}$ lies on the right side of α'_i. We may choose $\alpha = \beta_{j,m}$. The curves β and $\beta_{j,m}$ are non-separating and therefore they are also non-separating in $D(\alpha'_i)$, by Lemma 35.

On the left side of α'_i, sheets $2, 3, \ldots, i+1$ are separated from sheets $1, i+2, i+3, \ldots, d$. We see this when we jump with all the curves α_k, $k \neq i$, to the left of α'_i. If β lies on the left side of α'_i then $j = 1$ or $j \geq i+2$. The curve $\beta_{1,m} = \alpha'_m$ lies on the left side of α'_i and for $j = 1$ we may choose $\alpha = \beta_{1,m}$. For $j \geq i+2$ we choose the curve α in Figure 8. It has the same monodromy as $\beta_{j,m}$. The curves α'_n and α'_{n-1} lie on the left side of $\beta_{j,m}$ and of α ($n \geq m+3$). Therefore, by Remark 13, the curve α'_n is non-separating in $D(\alpha)$ and in $D(\beta_{j,m})$ and thus α is H-equivalent to $\beta_{j,m}$, by Remark 8 and Lemmas 36 and 37. The curves β and α are non-separating in $D(\alpha'_i)$ as in the previous case.

Suppose now that α'_i is non-separating. Then $i \geq m$. By Remark 43 we may replace α'_i by an H-equivalent curve. If β lies on the right side of α'_i we choose $i = m+2$ and $\alpha = \beta_{j,m}$. It is non-separating in $D(\alpha'_i)$ because $\beta_{j,m+1}$ also lies on the right side of α'_i. Suppose β lies on the left side of α'_i. If $j > 1$ we replace α'_i by $\beta'_{1,d+1}$, which is H-equivalent to α'_i, by Lemma 41. We may choose $\alpha = \beta_{j,m}$. It is non-separating in $D(\beta'_{1,d+1})$ because $\beta_{j,m+1}$ also lies on the left side of $\beta'_{1,d+1}$. Finally if $j = 1$ we choose $i = m$ and $\alpha = \beta_{j,m+1}$. It is non-separating in $D(\alpha'_i)$ because $\beta_{j,m+2}$ also lies on the left side of α'_i.

Suppose now that $\gamma = \beta_{k,m}$. By Lemma 41 there exists a curve δ which is disjoint from $\beta_{k,m}$, lies on the same side of $\beta_{k,m}$ as β, is non-separating in $D(\beta_{k,m})$ and is H-equivalent to $\beta_{j,m}$. We may choose $\alpha = \delta$. ∎

LEMMA 46. *Consider Cases* 1 *to* 3. *Let* β *be a curve which is* L^π*-equivalent to a special curve* α'_j *and is disjoint from some curve* α'_i. *In Case* 1 *we also assume that* $\mu(\alpha'_i) \neq \mu(\alpha'_j)$. *Then* β *is* H*-equivalent to* α'_j.

Proof. It suffices to prove that there exists a curve α such that α and β satisfy the assumptions of Lemma 44.

The curves β and α'_j are non-separating by the definition of special curves. If $z = z_{i,j}$ is of type 2, then it is H-admissible by Lemma 38 and there exists an arc x_k of type 1 strictly between A_i and A_j. Then the curve $z^2(\alpha'_j)$ or $z^{-2}(\alpha'_j)$ is disjoint from α'_i and lies on the other side of α'_i than α'_j so there exists a curve α which is H-equivalent to α'_j and lies on the correct side of α'_i. Moreover $\mu(\alpha'_j)$ connects different cycles of τ and $\mu(\alpha'_i)$ lies on one side of $\mu(\alpha'_j)$ (see Remark 23), so $\mu(\alpha'_j)$ connects different cycles of the total monodromy in $D(\alpha'_i)$ and the curves α and β are non-separating in $D(\alpha'_i)$, by Lemma 21.

Suppose that the arc $z_{i,j}$ is not of type 2 and α'_i is separating. Then α'_j is the nearest non-separating curve among the curves α'_k to the left or to the right of α'_i. The total monodromy τ has a cycle $(q, q-1, \ldots, p+1)$ and $\mu(\alpha'_i) = (p+1, t)$ for some $p+1 < t \leq q$. On the right side of α'_i there is a basis consisting of all curves α_s, $s \neq i$, and the corresponding monodromy sequence in $D(\alpha'_i)$ consists of pairs of consecutive numbers with the pair $(t-1, t)$ missing, so sheets $1, \ldots, t-1$ are disconnected from sheets $t, t+1, \ldots, d$.

When we jump with the curves α_s, $s \neq i$, to the left of α'_i the monodromy gets conjugated by $(p+1, t)$. In fact only pairs $(p, p+1), (p+1, p+2), (t, t+1)$ change into $(p, t), (p+2, t), (p+1, t+1)$. Therefore on the left side of α'_i sheets number $1, 2, \ldots, p, p+2, p+3, \ldots, t$ are separated from $p+1, t+1, t+2, \ldots, d$.

If $j > i$ then, by Remark 23, $\mu(\beta) = \mu(\alpha'_j) = (p+1, q+1)$ and there is no curve with such a monodromy on the right side of α'_i. Therefore β must be on the left side of α'_i and α'_j is also on the left side of α'_i so we can choose $\alpha = \alpha'_j$.

If $j < i$ then, by Remark 23, $\mu(\beta) = \mu(\alpha'_j) = (s, p+1)$ with $s < p+1$ and there is no curve with such a monodromy on the left side of α'_i. Therefore in this case β must be on the right side of α'_i and α'_j is also on the right side of α'_i so we can choose $\alpha = \alpha'_j$.

Since the curves β and α'_j are non-separating and α'_i is separating, we see by Lemma 35 that α and β are non-separating in $D(\alpha'_i)$ and we are done.

Suppose now that $z_{i,j}$ is not of type 2 and that α'_i is non-separating. Suppose $i < j$ and $\mu(\alpha'_i) \neq \mu(\alpha'_j)$. The curve α'_j lies on the left side of α_i. Each of the curves α'_i and α'_j has a neighbor which is H-equivalent to it. By Remark 43 we may replace α'_i by the neighbor and since we are looking for a curve α which is H-equivalent to α'_j we may replace α'_j by the neighbor. So we may assume that the arcs x_i and x_{j-1} are of type 1 (see Figure 7). Since $z_{i,j}$ is not of type 2 there are no other non-separating curves α'_k between α'_{i+1} and α'_{j-1} and the arc $w_{i,j-1}$ looks as in Figure 7, and is of type 1 and is H-admissible (see Definition 28). The curve $\delta = w_{i,j-1}(\alpha'_j)$ lies on the right side of α'_i and we can choose $\alpha = \delta$ or $\alpha = \alpha'_j$ on the same side as β. Moreover we may assume, by Remark 23, that τ has a cycle $(q, q-1, \ldots, p+1)$ and $\mu(\alpha'_i) = (p+1, q+1)$ and both numbers of $\mu(\alpha'_j)$ are greater than q. Therefore $\mu(\alpha'_i)$ connects different cycles in the total monodromy in $D(\alpha'_j)$, in $D(\alpha)$, and in $D(\beta)$, and α'_i is non-separating in these domains. By Remark 8 both α and β are non-separating in $D(\alpha'_i)$.

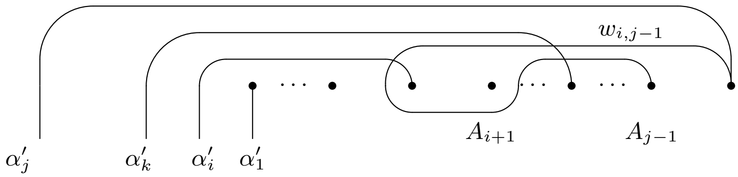

Fig. 7. Finding the curve δ on the other side of α'_i

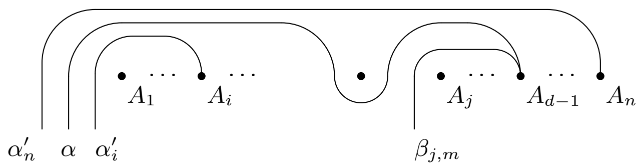

Fig. 8. Choosing the curve α in Case 4

The case when $z_{i,j}$ is not of type 2 and α'_i is non-separating and $i > j$ and $\mu(\alpha'_i) \neq \mu(\alpha'_j)$ follows by symmetry.

We now consider the case of $\mu(\alpha'_j) = \mu(\alpha'_i)$ (and we assume Case 2 or 3 or 4). In Case 2 we have three pairwise disjoint curves $\alpha'_m, \alpha'_{m+1}, \alpha'_{m+2}$ which are H-equivalent to α'_i and to α'_j. If β is on the left side of α'_i we choose $i = m$ and $\alpha = \alpha_{m+1}$. Then α is non-separating in $D(\alpha'_i)$ because there also exists α_{m+2} disjoint from α also on the left side of α'_i. If β is on the right side of α'_i we choose $i = m+2$ and $\alpha = \alpha_{m+1}$ and a similar argument works.

In Case 3 there are only two curves α'_j and α'_{j+1} disjoint from and H-equivalent to α'_i. We choose the left one for α'_i if β is on the right side, and we choose the other one for α. This time every curve in the complement of α'_i is separating and α and β have the same monodromy, so they are H-equivalent by Lemmas 36 and 37. The case of β on the left side follows by symmetry. ∎

Proof of Proposition 42 (by induction on the index of β). We fix a curve β which is L^π-equivalent to a special curve σ. Suppose β has index 0. We want to prove that β is H-equivalent to σ. There are $n-1$ curves α'_i disjoint from β. Some of them have monodromy different from $\mu(\beta)$, therefore in Cases 1 to 3 we are done by Lemma 46. In Case 4 we have $n \geq d+2$, hence, by Lemma 35, some of the curves α'_i are non-separating in $D(\beta)$. We are done by Lemma 45.

Now, assume that β has index $k > 0$ and that every curve γ which is L^π-equivalent to a special curve and has index less than k is H-equivalent to that special curve. By Lemma 36 if a curve γ has the same monodromy as a special curve and is non-separating then γ is L^π-equivalent to that special curve. Let $\mu(\beta) = (a, b)$.

We consider segments l_j of α'_j corresponding to the curve β (see Definition 24). They may lie on both sides of β. We consider disjoint curves which run along β on a suitable side, move to the consecutive segments l_i, then move along l_i, and end at the end points of l_i's (see Figure 9). We call these curves $\gamma_1, \ldots, \gamma_r$. The curves have indices smaller than the index of β, since they miss at least the intersection point of β with the l_i most distant from A_0 along β. There may also exist some curves α_i' disjoint from β (and disjoint from the curves γ_s). If we jump with all these curves to one side of β then we get a basis in $D(\beta)$. Since β is non-separating the monodromy of this basis forms a connected sequence of pairs and every number from 1 to d appears. It follows that also in the monodromy of the curves before jumping the sequence of pairs is connected and every number from 1 to d appears with the possible exception of a or b (but not both).

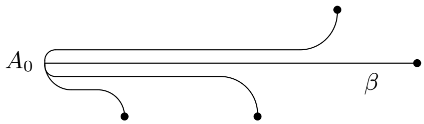

Fig. 9. The curves γ_s

If one of the curves γ_s has monodromy disjoint from (a,b) then the arc z connecting the end points of γ_s and β along β and along the suitable l_i is of type 2 and thus H-admissible. The curve $\beta' = t^2(\beta)$ or $\beta' = t^{-2}(\beta)$ has index smaller than k and is H-equivalent to β (see Figure 10, left). Then β is H-equivalent to a special curve by the induction hypothesis. So we may assume that the monodromy of each curve γ_j contains a letter a or a letter b.

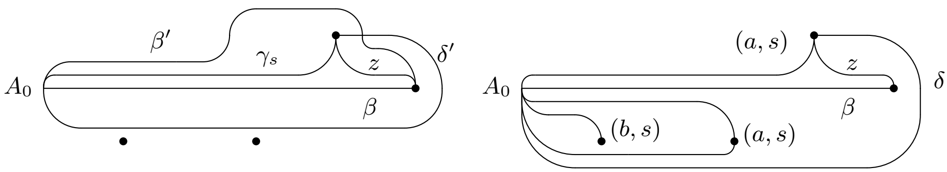

Fig. 10. Reducing the index of a curve, different cases

Lemma 47. *Suppose that some curve γ_s has monodromy (a,c) where $c \neq b$. Then there exists a curve δ' which is disjoint from β, has monodromy (b,c) and is H-equivalent to a curve of index smaller than k. If γ_s is non-separating in $D(\beta)$ then so is δ'.*

Proof. Consider Figure 10, left. Suppose γ_s ends at A_i. The arc z connecting the end of γ_s with the end of β along the arc l_i and a suitable part of β is of type 3, so it is H-admissible. The curve δ' is obtained by jumping with γ_s over β so it has monodromy (b,c) and $z^{-3}(\delta') = \beta'$ has index smaller than k, therefore it misses the intersection of β with l_i. The last claim of the lemma follows from Lemma 11. ■

We now pass to the induction step and we consider the different cases of Definition 40.

CASE 1. We have two special curves. They are consecutive non-separating curves with different monodromies among the curves α'_k. One of the special curves has monodromy $(a,b)=\mu(\beta)$. By Remark 23 the monodromy of the other curve has one number in common with (a,b), say its monodromy is equal to (b,c). Also the pairs (a,c) and (b,c) each connect different cycles of τ. Since β is non-separating, the number c must appear in the monodromy of some γ_s or some α'_i disjoint from β. In the second case $\mu(\alpha'_i)\neq\mu(\beta)$ and we are done by Lemma 46. In the first case either $\mu(\gamma_s)=(b,c)$ or, by Lemma 47, there exists a curve δ disjoint from β with monodromy (b,c). The curve is non-separating and therefore L^π-equivalent to a special curve and by the induction hypothesis it is H-equivalent to the special curve α'_k. Now β is H-equivalent to σ by Lemma 46.

In Cases 2 to 4 we may assume that every curve α'_i meets β, for otherwise we are done by Lemmas 45 and 46. Therefore the curves γ_s form a basis in $D(\beta)$ after we jump with all of them to one side of β. Since (after jumping) the monodromy sequence is connected and contains a and b and every pair contains either a or b, there must be a pair (a,b) or two pairs (a,s) and (b,s). Going back to the initial position we have a pair (a,b) or two pairs (a,s) and (b,s) on one side of β or two pairs (a,s) (equivalently (b,s)) on different sides of β (see Figure 10, right).

From this we shall produce a curve δ with monodromy (a,b) disjoint from β and H-equivalent to a curve of index less than k.

Suppose we have two curves on the same side of β (right side of β in Figure 10, right) with monodromy (b,s) and (a,s) respectively. Let l_i and l_j be the corresponding final segments of those curves and suppose that l_i is closer to A_0 than l_j. We choose a curve which moves along β on the side of l_i and l_j, goes around l_i, continues along β up to l_j, then moves along l_j up to its end (the inside curve in the lower part of Figure 10, right). The curve has monodromy (a,b) and has index smaller than k.

Suppose now that we have two curves with the same monodromy (a,s) on the opposite sides of β. Let l_i and l_j be the corresponding final segments of these curves and suppose that l_i is closer to A_0 than l_j. We choose a curve δ which moves along β on the side of l_i, goes around l_i, continues along β to its end, goes around β and goes back on the other side of β up to l_j, then moves along l_j up to its end (the curve δ on Figure 10, right, when we disregard the curve with monodromy (b,s)). Then $\mu(\delta)=(a,b)$. The arc z, which connects the end of β to the end of l_j along l_j and along a part of β, is of type 3 and is H-admissible, by Lemma 38. The curve $z^{-3}(\delta)$ (or $z^3(\delta)$) is H-equivalent to δ and has index smaller than k. It moves along β on the side of l_i, goes around l_i avoiding its intersection point with β, then crosses β to the other side, continues along β on the other side of β, goes around l_j avoiding its intersection point with β and continues to the end of β.

CASE 3. The monodromy (a, b) connects different cycles of τ, therefore the curve δ is non-separating and is H-equivalent to a special curve, by the induction hypothesis. Now β is H-equivalent to the special curve σ by Lemma 46.

CASE 2. The monodromy $\mu(\beta)$ contains the number d so we may assume $b = d$ and the pair (a, d) splits off number d or number a from τ, so the total monodromy in $D(\beta)$ has the cycle (d) if we consider the right side of β and has the cycle (a) if we consider the left side of β. In both cases $\mu(\delta)$ connects different cycles so δ is non-separating in $D(\beta)$ and thus non-separating in D. We proceed as in the previous case.

CASE 4. The curve δ constructed above may be separating so we proceed differently. The monodromy $\mu(\beta)$ contains the number d so we may assume $b = d$. There are $n - 1 > d$ curves γ_s so one of them must be non-separating in $D(\beta)$ (after jumping to one side), by Lemma 35, and therefore it must be non-separating in $D(\beta)$ before jumping, by Lemma 11. If $\mu(\gamma_s) = (s, d)$ then γ_s has the monodromy of a special curve and is H-equivalent to that special curve, by the induction hypothesis. Then β is H-equivalent to σ, by Lemma 45. Since $\mu(\gamma_s)$ contains a or d we may assume that $\mu(\gamma_s) = (s, a)$. By Lemma 47 there exists a curve δ which is disjoint from β, has monodromy (s, d) and is H-equivalent to a curve of index less than k. Moreover δ is non-separating in $D(\beta)$. We proceed as in the previous case.

This completes the proofs of Proposition 42 and of Theorem 31. ∎

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