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Finite-dimensional spaces in resolving classes

by

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Abstract. Using the theory of resolving classes, we show that if X is a CW complex of finite type such that $\operatorname{map}_*(X, S^{2n+1}) \sim *$ for all sufficiently large n, then $\operatorname{map}_*(X, K) \sim *$ for every simply-connected finite-dimensional CW complex K; and under mild hypotheses on $\pi_1(X)$, the same conclusion holds for *all* finite-dimensional complexes K. Since it is comparatively easy to prove the former condition for $X = B\mathbb{Z}/p$ (we give a proof in an appendix), this result can be applied to give a new, more elementary proof of the Sullivan conjecture.

Introduction. Haynes Miller proved the Sullivan conjecture (that the space of pointed maps from $B\mathbb{Z}/p$ to K is weakly contractible for all finite-dimensional CW complexes K) in the seminal paper [18]. The heart of Miller's proof is a herculean feat of pure algebra: he shows that the E_2-terms of certain Bousfield–Kan spectral sequences—involving homological algebra in the nonabelian category of unstable algebras over the Steenrod algebra—vanish. Around the same time, using simpler Massey–Peterson techniques and ordinary homological algebra of unstable *modules* over the Steenrod algebra, he showed that $\operatorname{map}_*(B\mathbb{Z}/p, S^{2n+1}) \sim *$ for all $n \geq 1$ [17]. Our goal is to prove by purely homotopy-theoretical methods that this easier result implies the full Sullivan conjecture.

Theorem 1. *Let X be a CW complex of finite type* [1]*. Then the following are equivalent:*

(1) $\operatorname{map}_*(X, S^n) \sim *$ *for all sufficiently large* $n \equiv 1 \bmod k$ *for some* k.
(2) $\operatorname{map}_*(X, K) \sim *$ *for all simply-connected finite-dimensional CW complexes* K.

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[1] X has *finite type* if it is homotopy equivalent to a CW complex with finitely many cells in each dimension.

Furthermore, if $\pi_1(X)$ *has no nontrivial perfect quotients* [2]*, then there is no need to restrict the fundamental group of* K.

Our proof relies heavily on the theory of *resolving classes*, introduced in the paper [20]. After some preliminaries, we give a streamlined and updated account of the basic theory of resolving classes, which we hope may be useful to other researchers. Once the theory is in place, the proof of Theorem 1 is accomplished in two steps, depending on whether or not K is simply-connected. In the final section, we briefly discuss some issues related to resolving classes and Theorem 1. For example, if $\Sigma X \not\sim *$, the condition $\mathrm{map}_*(X, K) \sim *$ for all simply-connected finite-dimensional complexes forces there to be nontrivial maps from X to certain infinite wedges of finite-dimensional complexes. We also offer some interesting problems and questions concerning the 'sphere codes' $\sigma(X) = \{n \mid \mathrm{map}_*(X, S^n) \sim *\}$.

For completeness, we include, in an appendix, a detailed outline of the proof of the following theorem, essentially due to Miller [17].

THEOREM 2. *Let* X *be a finite-type CW complex such that* $\widetilde{H}^*(X; \mathbb{Z}[1/p]) = 0$. *If* $H^*(X; \mathbb{F}_p)$ *is reduced and* $H^*(X; \mathbb{F}_p) \otimes J(n)$ *is injective for all* $n \geq 0$, *then* $\mathrm{map}_*(X, S^{2n+1}) \sim *$ *for all* $n \geq 1$.

Since $B\mathbb{Z}/p$ satisfies the conditions of Theorem 1, Theorem 2 implies the Sullivan conjecture.

1. Preliminaries

1.1. Notation for collections of spaces. Since we will use collections of spaces throughout this paper, it will be helpful to set up some basic notation for them. Our constructions are homotopy-respecting, so we tacitly close all of our collections under weak homotopy equivalence.

If $\mathcal{A}$ is a collection of spaces, then an expression of a space W as a *finite-type wedge* of spaces in $\mathcal{A}$ is a weak equivalence $W \sim \bigvee_{\mathcal{I}} A_i$ where $A_i \in \mathcal{A}$ for each $i \in \mathcal{I}$ and for each $n \in \mathbb{N}$ only finitely many of the spaces A_i are *not* n-connected. We say that W is a finite-type wedge of spaces in $\mathcal{A}$ if it has such an expression. For a finite-type wedge W of spaces in $\mathcal{A}$ that is $(n-1)$-connected but not n-connected (which we denote $\mathrm{conn}(W) = n-1$), define

$$s(W) = \min \left\{ k \;\middle|\; \begin{array}{l} W \text{ has an expression as a finite-type wedge of} \\ \text{spaces in } \mathcal{A} \text{ with all but } k \text{ summands } n\text{-connected} \end{array} \right\}.$$

We use the function s to impose a partial order on the collection of finite-

[2] Such groups are sometimes described as *hypoabelian*.

type wedges of spaces in $\mathcal{A}$: we say that $V < W$ if $\operatorname{conn}(W) < \operatorname{conn}(V)$ or if $\operatorname{conn}(V) = \operatorname{conn}(W)$ and $s(V) < s(W)$.

For collections $\mathcal{A}$ and $\mathcal{B}$, we write

$$\begin{aligned} \mathcal{A} \wedge \mathcal{B} &= \{A \wedge B \mid A \in A,\ B \in \mathcal{B}\}, \\ \Sigma\mathcal{A} &= \{\Sigma A \mid A \in \mathcal{A}\}, \\ \mathcal{A}^{\vee} &= \{\text{all finite-type wedges of spaces in } \mathcal{A}\}. \end{aligned}$$

Thus we say that $\mathcal{A}$ is *closed under suspension* if $\Sigma\mathcal{A} \subseteq \mathcal{A}$, that $\mathcal{A}$ is *closed under smash product* if $\mathcal{A} \wedge \mathcal{A} \subseteq \mathcal{A}$, and so on. Note that $(\mathcal{A}^{\vee})^{\vee} = \mathcal{A}^{\vee}$. If either $\mathcal{A}$ or $\mathcal{B}$ is closed under suspension, then so is $\mathcal{A} \wedge \mathcal{B}$.

1.2. Cone length. Let $\mathcal{A}$ be a collection of spaces. An $\mathcal{A}$-*cone decomposition* of *length* n for a map $f : X \to Y$ is a homotopy-commutative diagram

$$(\mathcal{D}) \qquad \begin{array}{ccccccccc} A_0 & & A_1 & & & & A_{n-1} & & \\ \downarrow & & \downarrow & & & & \downarrow & & \\ X_0 & \longrightarrow & X_1 & \longrightarrow & \cdots & \longrightarrow & X_{n-1} & \longrightarrow & X_n \\ \simeq\downarrow & & & & & & & & \downarrow\simeq \\ X & & & & \xrightarrow{\quad f \quad} & & & & Y \end{array}$$

in which $A_k \in \mathcal{A}$ for all k and each sequence $A_k \to X_k \to X_{k+1}$ is a cofiber sequence; if $f : X \to Y$ is a homotopy equivalence, then it has an $\mathcal{A}$-cone decomposition

$$\begin{array}{ccc} & X & \\ {\scriptstyle \mathrm{id}_X}\swarrow & & \searrow{\scriptstyle f} \\ X & \xrightarrow{\quad f \quad} & Y \end{array}$$

of length zero. The $\mathcal{A}$-*cone length* of f is

$$L_{\mathcal{A}}(f) = \inf\{\operatorname{length}(\mathcal{D}) \mid \mathcal{D} \text{ is an } \mathcal{A}\text{-cone decomposition of } f\}.$$

(Thus $L_{\mathcal{A}}(f) = \infty$ if f has no $\mathcal{A}$-cone decomposition.) The $\mathcal{A}$-*cone length* of a space X is $\operatorname{cl}_{\mathcal{A}}(X) = L_{\mathcal{A}}(* \to X)$.

1.3. Phantom maps. A *phantom map* is a pointed map $f : X \to Y$ from a CW complex X such that the restriction $f|_{X_n}$ of f to the n-skeleton is trivial for each n. We write $\operatorname{Ph}(X, Y) \subseteq [X, Y]$ for the set of pointed homotopy classes of phantom maps from X to Y. See [15] for an excellent survey on phantom maps.

If X is the homotopy colimit of a telescope diagram $\cdots \to X_{(n)} \to X_{(n+1)} \to \cdots$, then there is a short exact sequence of pointed sets

$$* \to \lim{}^1[\Sigma X_{(n)}, Y] \to [X, Y] \to \lim[X_{(n)}, Y] \to *,$$

and dually, if Y is the homotopy limit of a tower $\cdots \leftarrow Y_{(n)} \leftarrow Y_{(n+1)} \leftarrow \cdots$, then there is a short exact sequence

$$* \to \lim^1[X, \Omega Y_{(n)}] \to [X,Y] \to \lim[X, Y_{(n)}] \to *.$$

In the particular case of the expression of a CW complex X as the homotopy colimit of its skeleta or of a space Y as the homotopy limit of its Postnikov system, the kernels are the phantom sets.

We will be interested in showing that certain phantom sets $\mathrm{Ph}(X,Y)$ are trivial. One useful criterion is that if G is a tower of compact Hausdorff topological groups and continuous homomorphisms, then $\lim^1 G = *$ (see [15, Prop. 4.3]). This is used to prove the following lemma.

LEMMA 3. *Let $\cdots \leftarrow Y_{(s)} \leftarrow Y_{(s+1)} \leftarrow \cdots$ be a tower of spaces such that each homotopy group $\pi_k(Y_{(s)})$ is finite. If Z is of finite type, then* $\lim^1[Z, \Omega Y_{(s)}] = *$.

Proof. The homotopy sets $[Z_n, \Omega^j Y_{(s)}]$ are finite, and we give them the discrete topology, resulting in towers of compact groups and continuous homomorphisms. Fixing s and letting n vary, we find that $\lim^1_n[Z_n, \Omega^2 Y_{(s)}] = *$, and hence the exact sequence

$$0 \to \lim^1_n[Z_n, \Omega^2 Y_{(s)}] \to [Z, \Omega Y_{(s)}] \to \lim_n[Z_n, \Omega Y_{(s)}] \to 1$$

(of groups) reduces to an isomorphism $[Z, \Omega Y_{(s)}] \cong \lim_n[Z_n, \Omega Y_{(s)}]$. Since $[Z, \Omega Y_{(s)}]$ is an inverse limit of finite discrete spaces, it is compact and Hausdorff; and since the structure maps $Y_{(s)} \to Y_{(s-1)}$ induce maps of the towers that define the topology, the induced maps $[Z, \Omega Y_{(s)}] \to [Z, \Omega Y_{(s-1)}]$ are continuous. Thus $\lim^1_s[Z, \Omega Y_{(s)}] = *$. ∎

The Mittag-Leffler condition is another useful criterion for the vanishing of $\lim^1$. A tower of groups $\cdots \leftarrow G_n \leftarrow G_{n+1} \leftarrow \cdots$ is *Mittag-Leffler* if for each n the images $\mathrm{im}(G_{n+k} \to G_n)$ stabilize for large k, that is, there is a function $\kappa : \mathbb{N} \to \mathbb{N}$ such that $\mathrm{im}(G_{n+k} \to G_n) = \mathrm{im}(G_{n+\kappa(n)} \to G_n) \subseteq G_n$ whenever $k \geq \kappa(n)$.

PROPOSITION 4. *Let $\cdots \leftarrow G_n \leftarrow G_{n+1} \leftarrow \cdots$ be a tower of groups.*

(a) *If the tower is Mittag-Leffler, then* $\lim^1 G_n = *$.
(b) *If each G_n is a countable group, then the converse holds: if* $\lim^1 G_n = *$*, then the tower is Mittag-Leffler* [15, Thm. 4.4].

Importantly, the Mittag-Leffler condition does not refer to the algebraic structure of the groups G_n. This observation plays a key role in the following result (cf. [16, §3]).

PROPOSITION 5. *Let X be a CW complex of finite type, and let Y_1 and Y_2 be countable CW complexes with $\Omega Y_1 \simeq \Omega Y_2$. Then* $\mathrm{Ph}(X, Y_1) = *$ *if and only if* $\mathrm{Ph}(X, Y_2) = *$.

Proof. The homotopy equivalence $\Omega Y_1 \simeq \Omega Y_2$ gives levelwise bijections $\{[\Sigma X_n, Y_1]\} \cong \{[X_n, \Omega Y_1]\} \cong \{[X_n, \Omega Y_2]\} \cong \{[\Sigma X_n, Y_2]\}$ of towers of sets. Since X is of finite type and Y_1, Y_2 are countable CW complexes, these towers are towers of countable groups. Now the triviality of $\mathrm{Ph}(X, Y_1)$ implies that the first tower is Mittag-Leffler; but then all four towers must be Mittag-Leffler, and the result follows. ■

We end our account of phantom maps with a criterion for the vanishing of phantom maps into countable wedges of spheres.

Proposition 6. *If Z is rationally trivial* ([3]) *and of finite type, then*

$$\mathrm{Ph}\Big(Z, \bigvee_{i=1}^{\infty} S^{n_i}\Big) = *.$$

Proof. The Hilton–Milnor theorem implies that there is a *weak* product of spheres $P = \prod_\alpha S^{m_\alpha}$ such that $\Omega(\bigvee_{i=1}^{\infty} S^{n_i}) \simeq \Omega P$ (that is, P is the (homotopy) colimit of the diagram of finite subproducts of the categorical product). By Proposition 5, it suffices to show that $\mathrm{Ph}(Z, P) = *$.

Since the skeleta of Z are compact, every map $\Sigma Z_k \to P$ factors through a finite subproduct of P, so $[\Sigma Z_k, P]$ is a weak product $\prod_\alpha [\Sigma Z_k, S^{m_\alpha}]$. Because Z is rationally trivial and of finite type, we have $\mathrm{Ph}(Z, S^m) \cong \lim^1 [\Sigma Z_k, S^m] = *$ for each m [15]. These are towers of countable groups, so they must all be Mittag-Leffler. Write $\lambda(n, m)$ for the first k for which the images

$$\mathrm{im}([(\Sigma Z)_{n+k}, S^m] \to [(\Sigma Z)_n, S^m])$$

stabilize. Since $\lambda(n, m) = 0$ for $m > n + 1$, the set $\{\lambda(n, m) \mid m \ge 0\}$ is finite, and we define $\kappa(n)$ to be its maximum. Now it is clear that the images

$$\mathrm{im}\Big(\prod_\alpha [\Sigma Z_{n+k}, S^{m_\alpha}] \to \prod_\alpha [\Sigma Z_n, S^{m_\alpha}]\Big)$$

are independent of k for $k \ge \kappa(n)$. Thus the tower $\{\prod_\alpha [\Sigma Z_k, S^{n_\alpha}]\}$ is Mittag-Leffler, $\mathrm{Ph}(Z, P) = *$, and the proof is complete. ■

2. Resolving classes. We are interested in the condition $\mathrm{map}_*(X, Y) \sim *$. Since this can happen only for path-connected X, we tacitly assume that X is path-connected; thus $\mathrm{map}_*(X, Y) = \mathrm{map}_*(X, Y_\star)$, where $Y_\star$ is the basepoint component of Y, so we may assume that Y is path-connected too, if we like.

2.1. Basic definitions. Let $\mathfrak{X}$ and $\mathfrak{Y}$ both denote the classes of all collections of spaces (we intend to use $\mathfrak{X}$ for domains and $\mathfrak{Y}$ for targets). We

([3]) That is, $\widetilde{H}^*(Z; \mathbb{Q}) = 0$.

define functions

$$\Phi : \mathfrak{X} \to \mathfrak{Y} \quad \text{and} \quad \Theta : \mathfrak{Y} \to \mathfrak{X}$$

by the rules

$$\Phi(\mathcal{X}) = \{Y \mid \mathrm{map}_*(X,Y) \sim * \text{ for all } X \in \mathcal{X}\},$$
$$\Theta(\mathcal{Y}) = \{X \mid \mathrm{map}_*(X,Y) \sim * \text{ for all } Y \in \mathcal{Y}\}.$$

The maps Φ and Θ are a *Galois connection* between $\mathfrak{X}$ and $\mathfrak{Y}$, and hence establish a bijection between $\mathrm{im}(\Theta)$ and $\mathrm{im}(\Phi)$.

A collection $\mathcal{C} \in \mathrm{im}(\Theta)$ is a *strongly closed class*; in particular, $\Theta \circ \Phi(\{X\})$ is the Bousfield class $\langle X \rangle$ (studied by A. K. Bousfield, E. D. Farjoun, W. Chachólski, and others). The collections $\mathcal{R} \in \mathrm{im}(\Phi)$ have received less attention; we call them *resolving kernels* (4).

It follows formally from the definition that a strongly closed class is closed under weak equivalence, pointed homotopy colimits and extensions by cofibrations; dually, resolving kernels are closed under weak equivalence, pointed homotopy limits and extensions by fibrations.

We call a class $\mathcal{R} \in \mathfrak{Y}$ (which we assume to be closed under weak equivalence) a *resolving class* if it is closed under pointed homotopy limits, and a *strong resolving class* if it is also closed under extensions by fibrations.

2.2. Closure properties for resolving classes. The power of the theory of closed classes is founded on a few duality-violating theorems, such as the Zabrodsky lemma and E. D. Farjoun's theorem relating the fiber of an induced map of homotopy colimits to the 'pointwise fibers'. Similarly, resolving classes are useful tools by virtue of three formally implausible closure properties. These properties are best expressed in terms of collections of spaces rather than one space at a time; throughout this section we write $\mathcal{A}$ and $\mathcal{B}$ to denote collections of simply-connected spaces.

The first result concerns the closure of resolving kernels under the formation of wedges. This is a minor extension of [20, Prop. 7] using an argument due to W. Dwyer.

THEOREM 7. *Let $\mathcal{R}$ be a resolving kernel. If $\mathcal{A} \wedge \mathcal{A} \subseteq \mathcal{A}$ and $\Sigma\mathcal{A} \subseteq \mathcal{R}$, then $\Sigma\mathcal{A}^{\vee} \subseteq \mathcal{R}$.*

Proof. If $W \in \Sigma\mathcal{A}^{\vee}$ then we can write $W = \Sigma A \vee \Sigma B$, where $B \in \mathcal{A}$ and $\Sigma A < W$ (in the partial order defined in Section 1.1). The homotopy fiber W_1 of the quotient map $\Sigma A \vee \Sigma B \to \Sigma B$ is easily seen to be homotopy

(4) Thus $\Phi(\{X\}) = \mathrm{im}(P_X)$ and $\Theta \circ \Phi(\{X\}) = \ker(P_X)$, where P_X denotes the X-*nullification* functor (see [6, 5]).

equivalent to

$$W_1 \simeq \Sigma A \rtimes \Omega\Sigma B = \Sigma A \wedge (\Omega\Sigma B)_+ \simeq \Sigma A \wedge \Big(\bigvee_{k=0}^{\infty} B^{\wedge k}\Big),$$

where the 0-fold smash product $B^{\wedge 0}$ is S^0; see [8] for a proof (or [20]). Since $\mathcal{A}$ is closed under smash products, it follows that $W_1 \in \Sigma\mathcal{A}^\vee$, and the displayed wedge decomposition demonstrates that $W_1 < W$. Repeating this process yields a tower

$$W \leftarrow W_1 \leftarrow W_2 \leftarrow \cdots \leftarrow W_n \leftarrow W_{n+1} \leftarrow \cdots$$

in which $W_n \in \Sigma\mathcal{A}^\vee$ and $W_{n+1} < W_n$ for each n. It follows that the connectivity of W_n increases without bound and so $\operatorname{holim} W_n \sim *$.

To complete the proof, we use the fact that $\mathcal{R} = \Phi(\mathcal{X})$ is a resolving kernel. Let $X \in \mathcal{X}$ and observe that since $\operatorname{map}_*(X, \Sigma B) \sim *$ for all $B \in \mathcal{A}$, the induced maps

$$\operatorname{map}_*(X, W_{n+1}) \to \operatorname{map}_*(X, W_n)$$

are weak equivalences for all n; hence

$$\operatorname{map}_*(X, W) \sim \operatorname{holim}_n \operatorname{map}_*(X, W_n) \sim \operatorname{map}_*(X, \operatorname{holim}_n W_n) \sim *,$$

so $W \in \mathcal{R}$. ∎

Next we investigate suspension in resolving classes.

THEOREM 8. *Let $\mathcal{R}$ be a resolving class. If $\Sigma\mathcal{A}^\vee \subseteq \mathcal{R}$ then $\mathcal{A}^\vee \subseteq \mathcal{R}$.*

Proof. To prove the theorem, it suffices to show that if X is simply-connected and $\bigvee_1^k \Sigma X \in \mathcal{R}$ for all k, then $X \in \mathcal{R}$. This is a consequence, known to Barratt in the 1950s, of the generalization of the Blakers–Massey theorem to n-ads of simply-connected spaces, proved in [1, 22].

Start with n copies of the inclusion $X \hookrightarrow CX$ and build a strongly cocartesian n-cube by repeatedly forming (homotopy) pushouts. The result is an n-cube with each entry (except X) a wedge of copies of ΣX. Remove X from the cube and form the homotopy limit, $Y_{(n)} \in \mathcal{R}$. The n-ad excision theorem implies that the natural comparison map $X \to Y_{(n)}$ is an n-equivalence. These cubes map to one another, leading to a morphism of towers

$$\begin{array}{ccccccccc}
\cdots = & X & = & X & = \cdots = & X & = & X \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots \to & Y_{(n+1)} & \to & Y_{(n)} & \to \cdots \to & Y_{(1)} & \to & Y_{(0)}
\end{array}$$

in which the vertical maps become ever more highly connected as n increases. The homotopy limit Y is then both in $\mathcal{R}$ and weakly equivalent to X. ∎

The last of our three main theorems on resolving classes concerns their closure under certain extensions by cofibrations.

THEOREM 9. *Let $\mathcal{R}$ be a strong resolving class with $\Sigma\mathcal{A}^\vee, \Sigma\mathcal{B}^\vee \subseteq \mathcal{R}$. Assume that $\mathcal{A} \wedge \mathcal{A} \subseteq \mathcal{A}$, $\Sigma\mathcal{A} \subseteq \mathcal{A}$ and that $\mathcal{A} \wedge \Sigma\mathcal{B} \subseteq \Sigma\mathcal{B}^\vee$. If X sits in a cofiber sequence*

$$B \to X \to A$$

with $A \in \mathcal{A}^\vee$ and $B \in \mathcal{B}^\vee$, then $X \in \mathcal{R}$.

Proof. The proof depends on a decomposition of the homotopy fiber of a principal cofibration: if $P \to X \to \Sigma Q$ is a cofiber sequence, then the suspension of the homotopy fiber F of $X \to \Sigma Q$ is a half-smash product

$$\Sigma F \simeq \Omega\Sigma Q \ltimes \Sigma P \simeq \Big(\bigvee_{n=0}^{\infty} Q^{\wedge n}\Big) \wedge \Sigma P$$

(see [20, Prop. 4] for a proof; recall that $Q^{\wedge 0} = S^0$).

Applying this to the cofiber sequence $\bigvee_1^k \Sigma B \to \bigvee_1^k \Sigma X \to \bigvee_1^k \Sigma A$ we conclude that the homotopy fiber F_k satisfies

$$\Sigma F_k \simeq \Big(\bigvee_{n=0}^{\infty} \Big(\bigvee_1^k A\Big)^{\wedge n}\Big) \wedge \Sigma B.$$

Therefore $\Sigma F_k \in \mathcal{A}^\vee \wedge \Sigma\mathcal{B}^\vee \subseteq \Sigma\mathcal{B}^\vee \subseteq \mathcal{R}$, so Theorem 8 implies that $F_k \in \mathcal{R}$. Since $\mathcal{R}$ is closed under extension by fibrations, $\bigvee_1^k \Sigma X \in \mathcal{R}$ for each k; then Theorem 8 implies $X \in \mathcal{R}$. ∎

Notice that, like the proof of Theorem 7, this argument requires that we work with collections rather than individual spaces. It is not enough to know that $B \in \mathcal{R}$; rather, we need to know that a vast array of related spaces are all in $\mathcal{R}$.

2.3. Cone length in resolving classes. We finish this section by observing that Theorem 9 implies a closure property for strong resolving classes best expressed in terms of cone length.

Recall that throughout this section, the collections $\mathcal{A}$ and $\mathcal{B}$ are assumed to contain only simply-connected spaces.

THEOREM 10. *Let $\mathcal{R}$ be a strong resolving class with $\Sigma\mathcal{A}^\vee, \Sigma\mathcal{B}^\vee \subseteq \mathcal{R}$. Assume that $\mathcal{A} \wedge \mathcal{A} \subseteq \mathcal{A}$, $\Sigma\mathcal{A} \subseteq \mathcal{A}$ and that $\mathcal{A} \wedge \Sigma\mathcal{B} \subseteq \Sigma\mathcal{B}^\vee$. If there is a map $f : B \to K$ such that $B \in \mathcal{B}^\vee$ and $L_{\mathcal{A}^\vee}(f) < \infty$, then $K \in \mathcal{R}$.*

Proof. Write $\mathcal{B}_n$ for the collection of all spaces K such that there is a map $f : B \to K$ with $B \in \mathcal{B}^\vee$ and $L_{\mathcal{A}^\vee}(f) \le n$. The hypotheses imply that $\mathcal{A} \wedge \Sigma\mathcal{B}_n \subseteq \Sigma\mathcal{B}_n^\vee$ for each n. We will prove that each $\mathcal{B}_n \subseteq \mathcal{R}$ by induction on n.

First of all, $\mathcal{B}_0 = \mathcal{B}^\vee \subseteq \mathcal{R}$ by Theorem 8. Now suppose that $\mathcal{B}_n \subseteq \mathcal{R}$, and let $K \in \mathcal{B}_{n+1}$. The last step in an $\mathcal{A}^\vee$-cone decomposition for $f : B \to K$ gives a cofiber sequence

$$A_n \to K_n \to K \to \Sigma A_n$$

with $K_n \in \mathcal{B}_n$ and $A_n \in \mathcal{A}^\vee$. Therefore Theorem 9 implies that $K \in \mathcal{R}$, so $\mathcal{B}_{n+1} \subseteq \mathcal{R}$, and the induction is complete. ∎

If we set $\mathcal{A} = \{*\}$ in Theorem 10 (or even Theorem 9), we recover Theorem 8 (which, of course, was used in the proof of Theorem 9). Taking $\mathcal{B} = \{*\}$, on the other hand, we derive the following corollary.

COROLLARY 11. *If $\mathcal{R}$ is a strong resolving class with $\Sigma\mathcal{A}^\vee \subseteq \mathcal{R}$, $\mathcal{A} \wedge \mathcal{A} \subseteq \mathcal{A}$ and $\Sigma\mathcal{A} \subseteq \mathcal{A}$, then $\mathcal{R}$ contains every simply-connected space K with $\mathrm{cl}_{\mathcal{A}^\vee}(K) < \infty$.*

3. Proof of Theorem 1. Now we apply the theory of resolving classes to prove Theorem 1. We begin with two reductions.

COROLLARY 12. *Let $\mathcal{R}$ be a resolving kernel.*

(a) *If $\{S^{nk+1} \mid n \geq n_0\} \subseteq \mathcal{R}$, then $\mathcal{R}$ contains all finite-type wedges of simply-connected finite complexes.*

(b) *If $\mathcal{R}$ contains all simply-connected wedges of spheres, then $\mathcal{R}$ contains all wedges of simply-connected finite-dimensional spaces.*

Proof. We begin our proof of (a) by showing that $\mathcal{R}$ contains all simply-connected finite-type wedges of spheres. Since $\mathcal{A} = \{S^{nk} \mid n \geq n_0\}$ is closed under smash product and $\Sigma\mathcal{A} \subseteq \mathcal{R}$, we may apply Theorem 7 to conclude $\Sigma\mathcal{A}^\vee \subseteq \mathcal{R}$. Repeated application of Theorem 8 implies $\bigvee_1^m S^n \in \mathcal{R}$ for all $m \in \mathbb{N}$ and all $n \geq 2$. Then Theorems 7 and 8 give the result.

Now let $\mathcal{F}$ denote the collection of all simply-connected finite complexes. Since every space $K \in \Sigma\mathcal{F}$ has finite cone length with respect to the collection of simply-connected finite-type wedges of spheres, Corollary 11 implies that $\Sigma\mathcal{F} \subseteq \mathcal{R}$. Since $\mathcal{F} \wedge \mathcal{F} \subseteq \mathcal{F}$ and $\Sigma\mathcal{F} \subseteq \mathcal{F}$, Theorem 7 implies that $\Sigma\mathcal{F}^\vee \subseteq \mathcal{R}$. Theorem 8 shows that $\mathcal{F}^\vee \subseteq \mathcal{R}$, proving (a).

For (b), observe that the collection of all simply-connected wedges of spheres is closed under suspension, smash and finite-type wedge, so Corollary 11 implies that $\mathcal{R}$ contains every 2-connected finite-dimensional space. Theorems 7 and 8, applied to the collection of all simply-connected finite-dimensional spaces, complete the proof. ∎

Next we establish a simple lemma characterizing the rational homotopy type of spaces like the ones considered in Theorem 1.

LEMMA 13. *If $\mathrm{map}_*(X, S^n) \sim *$ for infinitely many values of n, then $\widetilde{H}^*(X; \mathbb{Q}) = 0$.*

Proof. Since ΣX also satisfies the hypotheses and $\widetilde{H}^*(X;\mathbb{Q}) = 0$ if and only if $\widetilde{H}^*(\Sigma X;\mathbb{Q}) = 0$, we may assume that X is simply-connected. To show that $\widetilde{H}^k(X;\mathbb{Q}) = 0$, choose n such that $k \le n-2$ and $\mathrm{map}_*(X, S^n) \sim *$. The map $\ell_* : [X, \Omega^{n-k}S^n] \to [X, \Omega^{n-k}S^n_{\mathbb{Q}}]$ induced by the rationalization of spaces $\ell : \Omega^{n-k}S^n \to \Omega^{n-k}S^n_{\mathbb{Q}}$ is rationalization of abelian groups. But $\mathrm{map}_*(X, S^n) \sim *$ implies $[X, \Omega^{n-k}S^n] = 0$, so $[X, \Omega^{n-k}S^n_{\mathbb{Q}}] = 0$. Since every rational loop space splits as a product of Eilenberg–Mac Lane spaces, $K(\mathbb{Q}, k)$ is a retract of $\Omega^{n-k}S^n_{\mathbb{Q}}$ and $\widetilde{H}^k(X;\mathbb{Q}) = [X, K(\mathbb{Q}, k)] = 0$. ∎

Proof of Theorem 1. Let X be a space satisfying the hypotheses of Theorem 1, and suppose that K is a simply-connected finite-dimensional CW complex. Since the resolving kernel

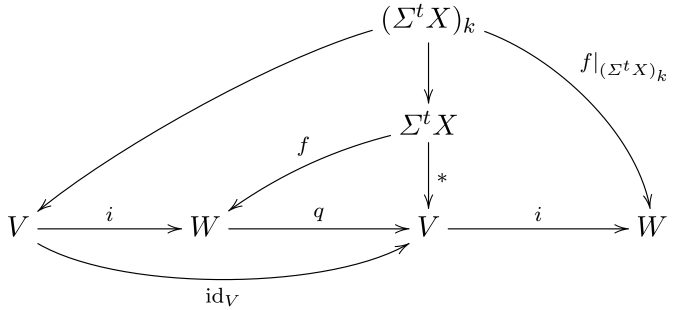
$$\mathcal{R} = \{K \mid \mathrm{map}_*(X, K) \sim *\}$$

contains S^{nk+1} for all sufficiently large n, Corollary 12(a) guarantees that $\mathcal{R}$ contains all finite-type wedges of simply-connected finite complexes. According to Corollary 12(b), it suffices to show that

$$[\Sigma^t X, W] = \pi_t(\mathrm{map}_*(X, W)) = 0$$

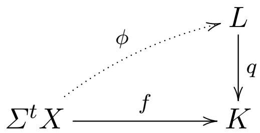
for all $t \ge 0$, where $W = \bigvee_{i\in\mathcal{I}} S^{n_i}$ is a simply-connected wedge of spheres. So we choose a typical map $f : \Sigma^t X \to W$ and attempt to show it is trivial.

Since X has finite type, each skeleton $(\Sigma^t X)_k$ is compact so $f((\Sigma^t X)_k)$ is contained in a finite subwedge $V \subseteq W$. It follows that f factors through the inclusion of a *countable* subwedge of W, and so we may as well assume in retrospect that W is itself a countable wedge. Furthermore, we know from Corollary 12(a) that $\mathrm{map}_*(X, V) \sim *$, so there is a homotopy commutative diagram



in which q is the collapse map to V and i is the inclusion. This shows that $f|_{(\Sigma^t X)_k} \simeq *$ for every k, and hence that f is a phantom map. The conclusion $f \simeq *$ follows from Lemma 13 and Proposition 6. Thus $\mathrm{map}_*(X, K) \sim *$ if K is simply-connected.

Finally allow the possibility that K is not simply-connected and assume that $\pi_1(X)$ has no nontrivial perfect quotients. Write $G = \mathrm{im}(\pi_1(f))$ and consider the covering $q : L \to K$ corresponding to the subgroup $G \subseteq \pi_1(K)$. There is a lift ϕ in the diagram



which induces a surjection on fundamental groups. If $G = \{1\}$, then L is simply-connected and finite-dimensional, and $\phi \simeq *$ by the simply-connected part of Theorem 1. If $G \neq \{1\}$, then it is not perfect, so there is a nontrivial map $u : L \to K(A, 1)$ for some abelian group A (u can be chosen so that $u_* : \pi_1(L) \to A$ is abelianization). Thus ϕ is nonzero on cohomology and so its suspension $\Sigma\phi : \Sigma^{t+1} X \to \Sigma L$ is also nontrivial, contradicting the simply-connected part of Theorem 1. ■

4. Discussion

4.1. Some comments on theorems. Corollary 11 implies a bit more than is actually stated. Since the collection $\mathrm{cl}_{<\infty}(\mathcal{A}^\vee)$ of spaces K with finite $\mathcal{A}^\vee$-cone length is closed under suspension and smash, we find that

$$\mathrm{cl}_{<\infty}(\mathcal{A}^\vee) \subseteq \mathrm{cl}_{<\infty}((\mathrm{cl}_{<\infty}(\mathcal{A}^\vee))^\vee) \subseteq \mathrm{cl}_{<\infty}(\mathrm{cl}_{<\infty}((\mathrm{cl}_{<\infty}(\mathcal{A}^\vee))^\vee)^\vee) \subseteq \cdots \subseteq \mathcal{R}.$$

These are genuine improvements: for example, they imply that for X as in Theorem 1,

$$\mathrm{map}_*\Big(X, \bigvee_{n=2}^{\infty} (\Omega S^{n+1})_{n^2}\Big) \sim *$$

(the subscript n^2 denotes dimension of a CW skeleton); in this example, the target has infinite Lusternik–Schnirelmann category and hence infinite cone length with respect to any collection $\mathcal{A}$.

Perhaps the reader is thinking that the insistence on *finite-type* wedges in Theorems 7 and 8 is simply a matter of expediency—that ‘finite-type’ could be eliminated if we tried hard enough. But it is not true that a space X satisfying the conditions of Theorem 1 satisfies the condition $\mathrm{map}_*(X, W) \sim *$ for *all* wedges of finite complexes W; indeed all such spaces X that are not killed by suspension *must* have nontrivial maps into certain wedges of finite complexes.

THEOREM 14. *If X is as in Theorem 1 and $\Sigma X \nsim *$, then the universal phantom map $\Theta_X : X \to \bigvee_{n=1}^{\infty} \Sigma X_n$ is nontrivial.*

Proof. Just as in the proof of Theorem 1, we can show that every map from ΣX to a wedge of finite complexes must be a phantom map. It is shown in [10, Thm. 2] that if $\Theta_X \simeq *$ then ΣX is a retract (up to homotopy) of a wedge of finite complexes. Thus $\Theta_X \simeq *$ simultaneously implies that $\mathrm{id}_{\Sigma X}$ is a phantom map and that ΣX is not the domain of any nontrivial phantom maps. ∎

The conclusion $\mathrm{map}_*(X, K) \sim *$ for non-simply-connected spaces K in Theorem 1 can be deduced more generally. It is valid provided no homomorphism from $\pi_1(X)$ to $\pi_1(K)$ can have a nontrivial perfect group as its image. This is the case, for example, if $\pi_1(K)$ is hypoabelian, regardless of the structure of $\pi_1(X)$. The restriction on fundamental groups cannot be entirely dispensed with, however. Any nontrivial acyclic 2-dimensional complex X satisfies $\mathrm{map}_*(X, K) \sim *$ for all finite-dimensional spaces K with hypoabelian fundamental groups, but $\mathrm{map}_*(X, X) \nsim *$. Such spaces also demonstrate the need for the hypothesis $\Sigma X \nsim *$ in Theorem 14.

4.2. The sphere code of a space. Corollary 12(a) suggests an interesting array of questions. Define the *sphere code* of a space X to be the set

$$\sigma(X) = \{n \in \mathbb{N} \mid \mathrm{map}_*(X, S^n) \sim *\}.$$

(This can be extended to resolving classes: $\sigma(\mathcal{R}) = \{n \mid S^n \in \mathcal{R}\}$.) We have shown in Corollary 12(a) that if $\sigma(X)$ contains an infinite arithmetic sequence of the form $\{nk + 1 \mid n \geq n_0\}$, then $\sigma(X) = \mathbb{N}$. What else can be said of it?

We offer only a few simple observations, followed by some questions.

PROPOSITION 15. *Let X and Y be CW complexes.*

(a) *If $2 \in \sigma(X)$, then $1 \in \sigma(X)$; if $4 \in \sigma(X)$, then $3 \in \sigma(X)$; if $8 \in \sigma(X)$, then $7 \in \sigma(X)$.*
(b) *If X is p-local (p is an odd prime) and $2n \in \sigma(X)$, then $2n - 1 \in \sigma(X)$* ([5]).
(c) *$\sigma(X \vee Y) = \sigma(X) \cap \sigma(Y)$.*
(d) *$\sigma(X \wedge Y) \supseteq \sigma(X) \cup \sigma(Y)$.*

We omit the proof and offer a few questions about sphere codes.

(1) If $\sigma(X) \neq \mathbb{N}$, can $\sigma(X)$ contain an infinite arithmetic sequence?
(2) Can $\sigma(X)$ be infinite without being all of $\mathbb{N}$?
(3) Is it possible to classify the sphere codes of spaces? Is there a space X such that $\sigma(X) = \{1 \text{ and all primes}\}$?
(4) There is a closure operation for subsets $N \subseteq \mathbb{N}$ given by $\overline{N} = \sigma(\mathcal{R})$, where $\mathcal{R} = \Theta(\{S^n \mid n \in N\})$; can it be described numerically?

5. Appendix: Reduction from algebra to topology. The following theorem gives the basic algebraic input for the proof of the Sullivan conjecture (see Section 5.1 for notation and terminology).

([5]) Thus if $\{nk + 1 + \epsilon_k \mid k \geq k_0,\ \epsilon_k \in \{0, 1\}\} \subseteq \sigma(X)$, then $\sigma(X) = \mathbb{N}$.

THEOREM 16 (Miller, Carlsson). *The unstable $\mathcal{A}_p$-module $\widetilde{H}^*(B\mathbb{Z}/p;\mathbb{F}_p)$ is reduced and $\widetilde{H}^*(B\mathbb{Z}/p;\mathbb{F}_p)\otimes J(n)$ is injective for all $n\geq 0$.*

We will not prove this here ([6]). Rather, we show how these algebraic properties guarantee that $\mathrm{map}_*(B\mathbb{Z}/p, S^{2n+1}) \sim *$ for $n \geq 1$. We begin by reviewing some preliminary material on the category $\mathcal{U}$ of unstable $\mathcal{A}_p$-algebras. Then we give a brief account of Massey–Peterson towers and finally derive from Theorem 16 that $\mathrm{map}_*(B\mathbb{Z}/p, S^{2n+1}) \sim *$ for all n.

5.1. Unstable modules over the Steenrod algebra. The cohomology functor $H^*(\,?\,;\mathbb{F}_p)$ takes its values in the category $\mathcal{U}$ of unstable modules and their homomorphisms. An *unstable module* over the Steenrod algebra $\mathcal{A}_p$ is a graded $\mathcal{A}_p$-module M satisfying $P^I(x)=0$ if $e(I)>|x|$, where $e(I)$ is the excess of I and $|x|$ is the degree of $x\in M$. We begin with some basic algebra of unstable modules, all of which is (at least implicitly) in [19].

Suspension of modules. An unstable module $M\in\mathcal{U}$ has a *suspension* $\Sigma M\in\mathcal{U}$ given by $(\Sigma M)^n = M^{n-1}$. The functor $\Sigma:\mathcal{U}\to\mathcal{U}$ has a left adjoint Ω and a right adjoint $\widetilde{\Sigma}$ ([7]). A module M is called *reduced* if $\widetilde{\Sigma}M=0$.

Projective and injective unstable modules. In the category $\mathcal{U}$, there are free modules $F(n)=\mathcal{A}_p/E(n)$, where $E(n)$ is the smallest left ideal containing all Steenrod powers P^I with excess $e(I)>n$. It is easy to see that the assignment $f\mapsto f([1])$ defines natural isomorphisms

$$\mathrm{Hom}_{\mathcal{U}}(F(n),M)\xrightarrow{\cong} M^n.$$

This property defines $F(n)$ up to natural isomorphism, and shows that $F(n)$ deserves to be called a *free* module on a single generator of dimension n. More generally, the free module on a set $X=\{x_\alpha\}$ with $|x_\alpha|=n_\alpha$ is (up to isomorphism) the sum $\bigoplus F(n_\alpha)$ (see [19, §1.6] for details).

A graded $\mathbb{F}_p$-vector space M is of *finite type* if $\dim_{\mathbb{F}_p}(M^k)<\infty$ for each k. Since $\mathcal{A}_p$ is of finite type, so is $F(n)$.

The functor which takes $M\in\mathcal{U}$ and returns the dual $\mathbb{F}_p$-vector space $(M^n)^*$ is representable: there is a module $J(n)\in\mathcal{U}$ and a natural isomorphism

$$\mathrm{Hom}_{\mathcal{U}}(M,J(n))\xrightarrow{\cong}\mathrm{Hom}_{\mathbb{F}_p}(M^n,\mathbb{F}_p).$$

Since finite sums of vector spaces are also finite products, these functors are exact, so the module $J(n)$ is an injective object in $\mathcal{U}$.

5.2. The functor $\overline{\tau}$. In [19, Thm 3.2.1] it is shown that for any module $H\in\mathcal{U}$, the functor $H\otimes_{\mathcal{A}_p}?$ has a left adjoint, denoted $(?:H)_{\mathcal{U}}$. Fix a

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([6]) A proof can be found in [19, Lem. 2.6.5 & Thm. 3.1.1].

([7]) $\widetilde{\Sigma}M$ is the largest suspension module contained in M.

module H (to stand in for $\widetilde{H}^*(X;\mathbb{F}_p)$) and write $\overline{\tau}$ for the functor $(\,?\,:H)_{\mathcal{U}}$; this is intended to evoke the standard notation $\overline{T}$ for the special case $H = \widetilde{H}^*(B\mathbb{Z}/p;\mathbb{F}_p)$.

LEMMA 17. *Let $H \in \mathcal{U}$ be a reduced unstable module of finite type and suppose that $H \otimes J(n)$ is injective in $\mathcal{U}$ for every $n \geq 0$. Then:*

(a) *$\overline{\tau}$ is exact.*
(b) *$\overline{\tau}$ commutes with suspension.*
(c) *If M is free and of finite type, then so is $\overline{\tau}(M)$.*
(d) *If $H^0 = 0$, then $\overline{\tau}(M) = 0$ for any finite module $M \in \mathcal{U}$.*
(e) *If $H^0 = 0$, then $\mathrm{Ext}^s_{\mathcal{U}}(H, \Sigma^{s+t}M) = 0$ for all $s,t \geq 0$ and all finite modules $M \in \mathcal{U}$.*

Proof. These results are covered in Sections 3.2 and 3.3 of [19]. Specifically, parts (a) and (b) are proved as in [19, Thm. 3.2.2 & Prop. 3.3.4]. Parts (c) and (d) may be proved following [19, Lem. 3.3.1 & Prop. 3.3.6], but since there are some changes needed, we prove those parts here.

Write $d_k = \dim_{\mathbb{F}_p}(H^k)$; then there are natural isomorphisms

$$\mathrm{Hom}_{\mathcal{U}}(\overline{\tau}(F(n)), M) \cong \mathrm{Hom}_{\mathcal{U}}(F(n), H \otimes M) \cong \mathrm{Hom}_{\mathcal{U}}\Big(\bigoplus_{i+j=n} F(i)^{\oplus d_j}, M\Big),$$

proving (c) in the case of a free module on one generator. Since $\overline{\tau}$ is a left adjoint, it commutes with colimits (and sums in particular), and we derive the full statement of (c).

If $H^0 = 0$, then $d_0 = 0$ and $\overline{\tau}(F(n))$ is a sum of free modules $F(k)$ with $k < n$. Since $F(0) = \mathbb{F}_p$, we see that $\overline{\tau}(\mathbb{F}_p) = 0$; then (a), together with the fact that $\overline{\tau}$ commutes with colimits, implies that $\overline{\tau}(M) = 0$ for all trivial modules M. Finally, any finite module M has filtration all of whose subquotients are trivial, and (d) follows.

To prove (e), let $P_* \to M \to 0$ be a free resolution of M in $\mathcal{U}$. Parts (a), (c) and (d) together imply that $\overline{\tau}(P_*) \to 0 \to 0$ is a free resolution of 0, so

$$\begin{aligned}\mathrm{Ext}^s_{\mathcal{U}}(M, \Sigma^{s+t}H) &= \mathrm{Ext}^s_{\mathcal{U}}(M, H \otimes \Sigma^{s+t}\mathbb{F}_p)) = H^s(\mathrm{Hom}(P_*, H \otimes \Sigma^{s+t}\mathbb{F}_p)) \\ &\cong H^s(\mathrm{Hom}(\overline{\tau}(P_*), \Sigma^{s+t}\mathbb{F}_p)) = \mathrm{Ext}^s_{\mathcal{U}}(0, \Sigma^{s+t}\mathbb{F}_p) = 0. \ \blacksquare\end{aligned}$$

5.3. Massey–Peterson towers. Cohomology of spaces has more structure than just that of an unstable $\mathcal{A}_p$-module: it has a cup product which makes $H^*(X;\mathbb{F}_p)$ into an *unstable algebra* over $\mathcal{A}_p$. The category of unstable algebras is denoted $\mathcal{K}$.

The forgetful functor $\mathcal{K} \to \mathcal{U}$ has a left adjoint $U : \mathcal{U} \to \mathcal{K}$. A space X is said to have *very nice* cohomology if $H^*(X) \cong U(M)$ for some unstable module M of finite type.

Since $U(F(n)) \cong H^*(K(\mathbb{Z}/p, n))$, there is a contravariant functor K which carries a free module F to a generalized Eilenberg–Mac Lane space (usually abbreviated GEM) $K(F)$ such that $H^*(K(F)) \cong U(F)$. If F is free, then so is ΩF, and $K(\Omega F) \simeq \Omega K(F)$.

LEMMA 18. *For any X, $[X, K(F)] \cong \mathrm{Hom}_{\mathcal{U}}(F, \widetilde{H}^*(X))$.*

It is shown in [17, 11, 14] that if $H^*(Y) \cong U(M)$ and $P_* \to M \to 0$ is a free resolution in $\mathcal{U}$, then Y has a *Massey–Peterson tower*

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & Y_s & \longrightarrow & Y_{s-1} & \longrightarrow \cdots \longrightarrow & Y_1 & \longrightarrow & Y_0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow \\
& K(\Omega^s P_{s+1}) & & K(\Omega^{s-1} P_s) & & K(\Omega P_2) & & K(P_1)
\end{array}$$

in which

(1) $Y_0 = K(P_0)$,
(2) each homotopy group $\pi_k(Y_s)$ is a finite p-group,
(3) the limit of the tower is the p-completion $Y_p^\wedge$,
(4) each sequence $Y_s \to Y_{s-1} \to K(\Omega^{s-1} P_s)$ is a fiber sequence, and
(5) the compositions $\Omega K(\Omega^{s-1} P_s) \to Y_s \to K(\Omega^s P_{s+1})$ can be naturally identified with $K(\Omega^s d_{s+1})$, where $d_{s+1} : P_{s+1} \to P_s$ is the differential in the given free resolution.

We use Massey–Peterson towers to give a criterion for the vanishing of homotopy sets.

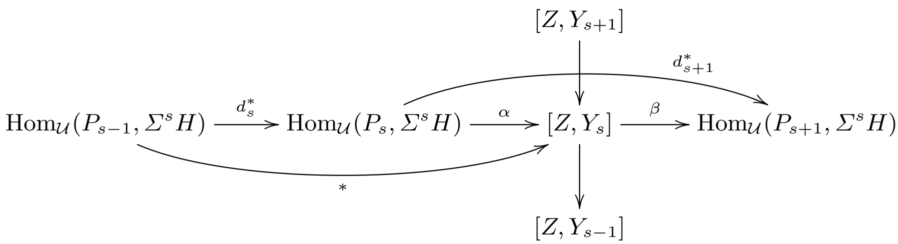
THEOREM 19. *Suppose Y is a simply-connected CW complex with $H^*(Y) = U(M)$ for some finite $M \in \mathcal{U}$ and Z is a CW complex of finite type with $\widetilde{H}^*(Z; \mathbb{Z}[1/p]) = 0$. If $\mathrm{Ext}^s_{\mathcal{U}}(M, \Sigma^s \widetilde{H}^*(Z)) = 0$ for all $s \geq 0$, then $[Z, Y] = *$.*

Proof. According to [17, Thm. 4.2], the natural map $Y \to Y_p^\wedge$ induces a bijection $[Z, Y] \xrightarrow{\cong} [Z, Y_p^\wedge]$, so it suffices to show $[Z, Y_p^\wedge] = *$. Since $H^*(Y) = U(M)$, Y has a Massey–Peterson tower, whose homotopy limit is $Y_p^\wedge$. Let f_s be the composite $Z \to Y \to Y_s$; we will show by induction that $f_s \simeq *$ for all s.

Since Y_0 is a GEM, f_0 is determined by its effect on cohomology; and since $\mathrm{Hom}_{\mathcal{U}}(M, \widetilde{H}^*(Z)) = \mathrm{Ext}^0_{\mathcal{U}}(M, \Sigma^0 \widetilde{H}^*(Z)) = 0$, f_0 is trivial on cohomology, and hence trivial. Inductively, suppose f_{s-1} is trivial. We have the following situation:

$$\begin{tikzcd}
K(\Omega^s P_{s-1}) \arrow[r] \arrow[rr, bend left, "K(\Omega^s d_s)"] & \Omega Y_{s-1} \arrow[r] \arrow[rr, bend right, "*"'] & K(\Omega^s P_s) \arrow[r] \arrow[rr, bend left, "K(\Omega^s d_{s+1})"] & Y_s \arrow[r] \arrow[d] & K(\Omega^s P_{s+1}) \\
& & & Y_{s-1} &
\end{tikzcd}$$

Now apply $[Z, ?\,]$ to this diagram and observe that Lemma 18, together with the isomorphism $\mathrm{Hom}_{\mathcal{U}}(\Omega^s P, H) \cong \mathrm{Hom}_{\mathcal{U}}(P, \Sigma^s H)$ (with $H = \widetilde{H}^*(Z)$), gives



Exactness at $[Z, Y_s]$ implies that the homotopy class $[f_s]$ is equal to $\alpha(g_s)$ for some $g_s \in \mathrm{Hom}_{\mathcal{U}}(P_s, \Sigma^s H)$. Since $[f_s]$ is in the image of the vertical map from $[Z, Y_{s+1}]$, it is in the kernel of β, so $d^*_{s+1}(g_s) = \beta([f_s]) = 0$; in other words, g_s is a cycle representing an element $[g_s] \in \mathrm{Ext}^s_{\mathcal{U}}(M, \Sigma^s H)$. Since $\mathrm{Ext}^s_{\mathcal{U}}(M, \Sigma^s H) = 0$, we conclude that there is a $g_{s-1} \in \mathrm{Hom}_{\mathcal{U}}(P_{s-1}, \Sigma^s H)$ such that $g_s = d^*_s(g_{s-1})$. Therefore $[f_s] = \alpha(d^*_s(g_{s-1})) = [*]$.

Since every map $f : Z \to Y$ is trivial on composition to Y_s for each s, the exact sequence $* \to \lim^1[Z, \Omega Y_s] \to [Z, Y^\wedge_p] \to \lim[Z, Y_s] \to *$ reduces to a surjection $\lim^1[Z, \Omega Y_s] \to [Z, Y^\wedge_p]$, and Lemma 3 finishes the proof. ∎

5.4. Maps from $B\mathbb{Z}/p$ to odd spheres. We are finally able to establish Theorem 2, which, by virtue of Theorem 16, implies the weak contractibility of the space of maps from $B\mathbb{Z}/p$ to odd spheres.

Proof of Theorem 2. Write $H = \widetilde{H}^*(X; \mathbb{F}_p)$; thus $H \in \mathcal{U}$ is a reduced module of finite type and $H \otimes J(n)$ is injective for all n. Since $H^*(S^{2n+1}) = U(\Sigma^{2n+1}\mathbb{F}_p)$, the result follows from applying Lemma 17(e) and Theorem 19 to the spaces $Z = \Sigma^t X$ for $t \geq 0$. ∎

Corollary 20. $\mathrm{map}_*(B\mathbb{Z}/p, S^{2n+1}) \sim *$ *for all* $n \geq 1$.

Proof. Lemma 13 and Theorem 16 imply that we may take $X = B\mathbb{Z}/p$ in Theorem 2. ∎

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